A Bernoulli problem with non constant gradient boundary constraint

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A BERNOULLI PROBLEM WITH NON CONSTANT GRADIENT BOUNDARY CONSTRAINT

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ABSTRACT. We present in this paper a result about existence and convexity of solutions to a free boundary problem of Bernoulli type, with non constant gradient boundary constraint depending on the outer unit normal. In particular we prove that, in the convex case, the existence of a subsolution guarantees the existence of a classical solution, which is proved to be convex.

1. INTRODUCTION

Consider an annular condenser with a constant potential difference equal to one, such that one of the two plates is given and the other one has to be determined in such a way that the intensity of the electrostatic field is constant on it. If $\Omega \setminus \overline{K}$ represents the condenser, whose plates are Ω and K (with $\overline{K} \subseteq \Omega$), and u is the electrostatic potential, it holds $\Delta u = 0$ in $\Omega \setminus \overline{K}$ and $|Du| = constant$ on either $\partial\Omega$ or ∂K, depending on which of them represents the unknown plate.

This gives rise to the classical Bernoulli problems (interior and exterior), where the involved differential operator is the Laplacian Δ, which expresses the linearity of the electrical conduction law. However, some physical situations can be better modeled by general power flow laws, then yielding to the p-Laplacian as governing operator. Moreover, one can consider the possibility to have a non constant prescribed intensity of the electric field on the free boundary. In particular, as the intensity of the electrostatic field $\overrightarrow{E}$ on an equipotential surface is related to its outer unit normal vector, through the curvature of that surface, one can assume $|\overrightarrow{E}|$ to depend on the outer unit normal vector $\nu(x)$ of the unknown boundary. In view of these considerations, we deal here with the following problem.

Given a domain in $\Omega \subseteq \mathbb{R}^N$, a real number $p > 1$ and a smooth function $g : S^{N-1} \to \mathbb{R}$ such that

$$c \leqslant g(\mathsf{v}) \leqslant C \quad \text{for every } \mathsf{v} \in S^{N-1}, \tag{1.1}$$

for some $C > c > 0$, find a function u and a domain K, contained in Ω, such that

$$\begin{cases} \Delta_p u(x) = 0 & \text{in } \Omega \setminus \overline{K}, \\ u = 0 & \text{on } \partial\Omega, \\ u = 1, & \text{on } \partial K, \\ |Du(x)| = g(\nu(x)), & \text{on } \partial K, \end{cases} \tag{1.2}$$

where $\nu(x) = \nu_K(x)$ is the outer unit normal to ∂K at $x \in \partial K$.

Here an later Δ_p is the p-Laplace operator for $p > 1$, that is

$$\Delta_p u = \mathrm{div}(|Du|^{p-2} Du)\,.$$

If u is a solution to (1.2) we will tacitly continue u by 1 in K throughout the paper, so that a solution u to (1.2) is defined, and continuous, in the whole Ω.

The boundary condition $|Du| = \tau$ has to be understood in a classical way:

$$\lim_{\substack{y \to x \\ y \in \Omega \setminus \overline{K}}} |Du(y)| = |Du(x)|.$$

Moreover, in the convex case, that is when Ω is a convex set, we are allowed to consider classical solutions (justified by [18], since K inherits the convexity of Ω, as shown later).

Notice that, given K in (1.2), the function u is uniquely determined since it represents the capacitary potential of $\Omega \setminus \overline{K}$; on the other hand, given the function u the free boundary ∂K is determined as $\partial K = \partial\{x \in \mathbb{R}^N : u(x) \geqslant 1\}$. Hence, we will speak of *a solution* to (1.2) referring indifferently to the sets K or to the corresponding

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potential function u (or to both) and we will indicate the class of solutions as $\mathscr{F}(\Omega, g)$, where Ω is the given domain and g is the gradient boundary datum.

The original interior Bernoulli problem corresponds to the case $p = 2$, that is the Laplace operator, with constant gradient boundary constraint $g(\nu(x)) \equiv \tau$. In general, given a domain $\Omega \subseteq \mathbb{R}^N$, and $\tau > 0$, the classical interior Bernoulli problem consists in finding a domain K, with $\overline{K} \subseteq \Omega$ and a function u such that

$$\begin{cases} \Delta_p u(x) = 0 & \text{in } \Omega \setminus \overline{K}, \\ u = 0 & \text{on } \partial\Omega, \\ u = 1, & \text{on } \partial K, \\ |Du| = \tau, & \text{on } \partial K. \end{cases} \tag{1.3}$$

Easy examples show that Problem (1.3), and hence Problem (1.2), need not have a solution for every given domain Ω and for every positive constant τ. Many authors consider the classical problem, both from the side of the existence and geometric properties of the solution. In particular we recall the pioneering work of Beurling [6] and several other contributions as [1, 3, 12, 11, 17]. The treatment of the nonlinear case is more recent and mainly due to Henrot and Shahgholian (see for instance [14, 15]; see also [5, 8, 13, 20] and references therein). The uniqueness problem has been solved later in [9] for $p = 2$ and [8] for $p > 1$. Here we summarize some of the known results.

(1.4) Let $\Omega \subseteq \mathbb{R}^N$ be a convex C^1 bounded domain. There exists a positive constant $\Lambda_p = \Lambda_p(\Omega)$, named *Bernoulli constant*, such that Problem (1.3) has a solution if and only if $\tau \geqslant \Lambda_p$; in such a case there is at least one which is $C^{2,\alpha}$ and convex. In particular for $\tau = \Lambda_p$ the solution is unique.

In this paper we consider Problem (1.2) in the convex case, that is when the given domain is a convex set, and we prove that the convexity is inherited by the unknown domain without making additional assumptions on the function g. More precisely, let us indicate by $\mathscr{F}^-(\Omega, g(\nu))$ the class of the so called *subsolution* to Problem (1.2); essentially, v and K are subsolutions if v solves

$$\begin{cases} \Delta_p v \geqslant 0 & \text{in } \Omega \setminus \overline{K} \\ v = 0 & \text{on } \partial\Omega \\ v = 1, \ |Dv(x)| \leqslant g(\nu(x)) & \text{on } \partial K\,; \end{cases}$$

(see Section 2.4 for more details).

Our main theorem is the following.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a convex C^1 domain, and $g : S^{N-1} \to \mathbb{R}$ be a continuous function such that (1.1) holds. If $\mathscr{F}^-(\Omega, g(\nu))$ is non empty, then there exists a C^1 convex domain K with $\overline{K} \subseteq \Omega$ such that the p-capacitary potential u of $\Omega \setminus \overline{K}$ is a classical solution to the interior Bernoulli problem (1.2).*

The idea of a non constant boundary gradient condition has been developed in the literature by many authors, who considered the case of a space variable dependent constraint, $a : \Omega \to (0, +\infty)$. We refer to [3, 4, 21] for a functional approach, and to [1, 2, 16] for the subsolution method. In particular, an analogous result to Theorem 1.1 has been proved in [16] where the authors considered a Bernoulli problem with non constant gradient boundary datum $a(x)$. For a given convex domain $\Omega \subseteq \mathbb{R}^N$, and a positive function $a : \Omega \to (0, \infty)$, such that

$$c \leqslant a(x) \leqslant C, \text{ for every } x \in \Omega,$$

for some $C > c > 0$, with

$$\frac{1}{a} \text{ convex in } \Omega, \tag{1.5}$$

they consider the problem

$$\begin{cases} \Delta_p u(x) = 0 & \text{in } \Omega \setminus \overline{K}, \\ u = 0 & \text{on } \partial\Omega, \\ u = 1, & \text{on } \partial K, \\ |Du(x)| = a(x) & \text{on } \partial K. \end{cases} \tag{1.6}$$

and they proved that, if a subsolution to the problem exists, then there exists a classical solution and moreover the convexity of the given domain transfers to the free boundary.

Notice that in Problem (1.6) the function a is required to be given in the whole Ω, while in (1.2) the function g is defined only on the unit sphere S^{N-1}. Moreover, while in Problem (1.6) the convexity property (1.5) is required for the boundary constraint a, in Problem (1.2) no additional assumptions on g are needed.

2. Preliminaries

2.1. **Notations.** In the N-dimensional Euclidean space, $N \geqslant 2$, we denote by $|\cdot|$ the Euclidean norm; for $K \subseteq \mathbb{R}^N$, we denote by $\overline{K}$ its closure and by ∂K its boundary, while $\mathsf{conv}(K)$ is its convex hull. $\mathcal{H}^m$ indicates the m-dimensional Hausdorff measure. We denote by $B(x_0, r)$ the ball in $\mathbb{R}^N$ of center x_0 and radius $r > 0$: $B(x_0, r) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$; in particular B denotes the unit ball $B(0, 1)$ and we set $\omega_N = \mathcal{H}^N(B)$. Let us define

$$S^{N-1} = \partial B = \{x \in \mathbb{R}^N : |x| = 1\};$$

hence $\mathcal{H}^{N-1}(S^{N-1}) = N\omega_N$.

We set

$$\Lambda_m = \{\lambda = (\lambda_1, ..., \lambda_m) \mid \lambda_i \geqslant 0, \sum_{i=1}^{m} \lambda_i = 1\}.$$

Given an open set $\Omega \subseteq \mathbb{R}^N$, and a function u of class $C^2(\Omega)$, $Du = (u_{x_1}, \ldots, u_{x_N})$ and $D^2u = (u_{x_i x_j})_{i,j=1}^N$ denote its gradient and its Hessian matrix respectively.

2.2. **Quasi-concave and Q^2_- functions.** An upper semicontinuous function $u : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ is said *quasi-concave* if it has convex superlevel sets, or, equivalently, if

$$u((1-\lambda)x_0 + \lambda x_1) \geqslant \min\{u(x_0), u(x_1)\},$$

for every $\lambda \in [0, 1]$, and every $x_0, x_1 \in \mathbb{R}^N$. If u is defined only in a proper subset Ω of $\mathbb{R}^n$, we extend u as $-\infty$ in $\mathbb{R}^n \setminus \Omega$ and we say that u is quasi-concave in Ω if such an extension is quasi-concave in $\mathbb{R}^N$. Obviously, if u is concave then it is quasi-concave.

By definition a quasi-concave function determines a family of monotone decreasing convex sets; on the other hand, a continuous family of monotone decreasing convex sets, whose boundaries completely cover the first element, can be seen as the family of super-level sets of a quasi-concave function.

We use a local strengthened version of quasi-concavity, which was introduced and studied in [19]: let u be a function defined in an open set $\Omega \subset \mathbb{R}^n$; we say that u is a Q^2_- function at a point $x \in \Omega$ (and we write $u \in Q^2_-(x)$) if:

1. u is of class C^2 in a neighborhood of x;
2. its gradient does not vanish at x;
3. the principal curvatures of $\{y \in \mathbb{R}^n \mid u(y) = u(x)\}$ with respect to the normal $-\frac{Du(x)}{|Du(x)|}$ are positive at x.

In other words, a C^2 function u is Q^2_- at a regular point $\bar{x}$ if its level set $\{x : u(x) = u(\bar{x})\}$ is a regular convex surface (oriented according to $-Du$), whose Gauss curvature does not vanish in a neighborhood of $\bar{x}$. By $u \in Q^2_-(\Omega)$ we mean $u \in Q^2_-(x)$ for every $x \in \Omega$.

2.3. **Quasi concave envelope.** If u is an upper semicontinuous function, we denote by u^* its *quasi-concave envelope*. Roughly speaking, u^* is the function whose superlevel sets are the closed convex hulls of the corresponding superlevel sets of u. It turns out that u^* is also upper semicontinuos.

Let us indicate by $\Omega(t)$ the superlevel set of u of value t, i.e.

$$\Omega(t) = \{x \in \mathbb{R}^N \mid u(x) \geqslant t\},$$

and let $\Omega^*(t) = \overline{\mathsf{conv}(\Omega(t))}$. Then u^* is the function defined by its superlevel sets in the following way:

$$\Omega^*(t) = \{x \in \mathbb{R}^N \mid u^*(x) \geqslant t\} \qquad \text{for every } t \in \mathbb{R},$$

that is

$$u^*(x) = \sup\{t \in \mathbb{R} \mid x \in \Omega^*(t)\}.$$

Equivalently, as shown in [10],

$$u^*(x) = \max\left\{\min\{u(x_1), ..., u(x_{N+1})\} \,:\, x_i \in \overline{\Omega \setminus \overline{K}}, \exists\lambda \in \Lambda_{N+1},\ x = \sum_{i=1}^{N+1} \lambda_i x_i\right\}.$$

Notice that u^* is the smallest upper semicontinuous quasi-concave function greater than u, hence in particular $u^* \geqslant u$. Moreover, if u satisfies $\Delta_p u = 0$ in a convex ring $\Omega \setminus \overline{K}$ (that is Ω, K convex with $\overline{K} \subseteq \Omega$), then it holds $\Delta_p u^* \geqslant 0$ in $\Omega \setminus \overline{K}$ in the viscosity sense (see for instance [10]).

2.4. **Subsolutions.** In his pioneering work [6], Beurling introduced the notion of sub-solution for the classical Problem (1.3). This concept was further developed by Acker [1] and then generalized by Henrot and Shahgholian [15, 16] to the case $p > 1$, both for constant and for non constant gradient boundary constraint.

Following the same idea, let us introduce the class of sub-solutions to the generalized Bernoulli Problem (1.2). Let Ω be a subset of $\mathbb{R}^N$; $\mathscr{F}^-(\Omega, g)$ is the class of functions v that are Lipschitz continuous on $\overline{\Omega}$ and such that

$$\begin{cases} \Delta_p v \geqslant 0 & \text{in } \{v < 1\} \cap \Omega \\ v = 0 & \text{on } \partial\Omega \\ |Dv(x)| \leqslant g(v(x)) & \text{on } \partial\{v < 1\} \cap \Omega\,. \end{cases} \tag{2.1}$$

If $v \in \mathscr{F}^-(\Omega, g)$ we call it a *subsolution*.

As in the definition of solutions, we say that a set K is a *subsolution*, and we possibly write $K \in \mathscr{F}^-(\Omega, \tau)$ or $(v, K) \in \mathscr{F}^-(\Omega, \tau)$, if $K = \{x \in \Omega \,:\, v(x) \geqslant 1\}$ for some $v \in \mathscr{F}^-(\Omega, \tau)$.

In the standard case $g \equiv \tau$, for some positive constant τ, it is known that the class of subsolutions and that of solutions are equivalent, indeed in [15] is proved that, if Ω is a C^1 convex domain, and $\mathscr{F}^-(\Omega, \tau)$ is not empty, then there exists a classical solution to (1.3). In particular it is proved that

$$\widetilde{K}(\Omega, \tau) = \bigcup_{C \in \mathscr{F}^-(\Omega,\tau)} C, \qquad \widetilde{u} = \sup_{v \in \mathscr{F}^-(\Omega,\tau)} v,$$

solve Problem (1.3) and hence, recalling (1.4), it follows as a trivial consequence:

$$\boldsymbol{\Lambda}_p(\Omega) = \inf\{\tau \,:\, \mathscr{F}(\Omega, \tau) \neq \emptyset\} = \inf\{\tau \,:\, \mathscr{F}^-(\Omega, \tau) \neq \emptyset\}.$$

Regarding the proof of Theorem 1.1, it is clear that an analogous relation between subsolutions and solutions hold true also in the non constant case, that is:

$$\widetilde{K}(\Omega, g) = \bigcup_{C \in \mathscr{F}^-(\Omega,g)} C, \qquad \widetilde{u} = \sup_{v \in \mathscr{F}^-(\Omega,g)} v,$$

solve Problem (1.2) and they are said *maximal solution* to (1.2).

3. Proof of the main result

In order to give a proof of Theorem 1.1, some preliminary steps are needed; they are collected in the following propositions and lemmas.

Proposition 3.1. *Let Ω be a regular C^1 convex subset of $\mathbb{R}^N$; let $u_0, u_1 \in \mathscr{F}^-(\Omega, g)$ with $K_0 = \{u_0 = 1\}$ and $K_1 = \{u_1 = 1\}$. Define $K = K_0 \cup K_1$, $K^* = \overline{\mathrm{conv} K}$. Then $v \in \mathscr{F}^-(\Omega, g)$, where v is the p-capacitary potential of $\Omega \setminus K^*$. Moreover*

$$|Dv(x)| \leqslant g(\nu_{\Omega(t)}(y_x)),$$

for every $x \in \Omega \setminus K^*$ and $y_x \in \partial K^*$ such that $\nu_{K^*}(y_x) = -Dv(x)/|Dv(x)| = \nu_{\Omega(t)}(x)$, being $\Omega(t)$ the superlevel set of v of level $t = v(x)$.

Proof. Let u^* be the quasi-concave envelope of $u = \max\{u_0, u_1\}$; it satisfies in the viscosity sense

$$\begin{cases} \Delta_p u^* \geqslant 0 & \text{in } \Omega \setminus K^* \\ u^* = 0 & \text{on } \partial\Omega \\ u^* = 1 & \text{on } \partial K^*, \end{cases}$$

and hence, by the viscosity comparison principle,

$$|Dv| \leqslant |Du^*| \text{ on } \partial K^*. \tag{3.1}$$

Consider $y \in \partial K^*$; then either $y \in \partial K^* \cap \partial K$ or $y \in \partial K^* \setminus \partial K$.

Assume $y \in \partial K^* \cap \partial K$, so that $\nu_K(y) = \nu_{K^*}(y)$. Then either $y \in \partial K_0$, or $y \in \partial K_1$ and hence $|Du^*(y)| \leqslant |Du_0(y)|$ or $|Du_1(y)|$; however in both the cases

$$|Dv(y)| \leqslant |Du_i(y)| \leqslant g\big(\nu_K(y)\big) = g\big(\nu_{K^*}(y)\big),$$

as $u_0, u_1 \in \mathscr{F}^-(\Omega, g)$.

Now assume $y \in \partial K^* \setminus \partial K$. By Proposition 3.1 in [10] there exist $x_1, ..., x_N \in \partial(K_0 \cup K_1)$ such that $x_1, ..., x_l \in \partial K_0$, $x_{l+1}, ..., x_N \in \partial K_1$ (with $0 \leqslant l \leqslant N$) and $\lambda \in \Lambda_N$ such that

$$\nu_{K_0}(x_i) = \nu_K(x_i) \text{ parallel to } \nu_{K_1}(x_j) = \nu_K(x_j) \text{ parallel to } \nu_{K^*}(y),$$

for $i = 1, ..., l$, $j = l, ..., N$ and $y = \sum_{i=1}^N \lambda_i x_i$. Moreover thanks to Proposition 2.2 in [7] it holds

$$|Du^*(y)| = \left(\sum_{k=1}^N \frac{\lambda_k}{|Du_{i_k}(x_k)|}\right)^{-1} \leqslant \left(\sum_{k=1}^N \frac{\lambda_k}{g(\nu_{K_{i_k}}(x_k))}\right)^{-1} = \left(\sum_{k=1}^N \frac{\lambda_k}{g(\nu(x))}\right) = g(\nu(x)),$$

where $i_k \in \{0, 1\}$. Hence, by (3.1), $v \in \mathscr{F}^-(\Omega, g)$.

Notice that, as Ω, K^* are convex, the function v is quasi-concave, in particular, thanks Lewis's result [18], $v \in Q^2_-(\Omega \setminus K^*)$. For every $x \in \Omega \setminus K^*$, let $\nu_{\Omega(t)}(x)$ be the outer unit normal vector to the superlevel set $\{v(y) \geqslant v(x)\}$; hence by Lemma 4.1 in [8], it holds

$$|Dv(x)| \leqslant g(\nu_{K^*}(y_x)),$$

where $y_x \in \partial K^*$ is such that $\nu_{K^*}(y_x) = \nu_{\Omega(t)}(x)$. □

For the sake of completeness we rewrite here two lemmas in [16] which are particularly useful in the proof of Theorem 1.1.

Lemma 3.2 ([16]). *Let $D_R = \{x_1 < 1\} \setminus B_R$, where $B_R = B(x_R, R)$ and $x_R = (-R, 0, \dots, 0)$. Assume $l > 0$ and let u_R solve*

$$\begin{cases} \Delta_p u = 0 & \text{in } D_R \\ u = l & \text{on } \{x_1 = 1\} \\ u = 0 & \text{on } \partial B_R. \end{cases}$$

Then for any $\varepsilon > 0$ there exists R sufficiently large such that $|Du_R| \leqslant l + \varepsilon$ on ∂B_R.

Lemma 3.3 ([16]). *Let u be the p-capacitary potential of the convex ring $\Omega \setminus \overline{K}$, with $|Du| \leqslant C$ uniformly in $\Omega \setminus \overline{K}$. Then any converging blow-up sequence*

$$u_{r_j}(x) = \frac{1}{r_j}\ \big(1 - u(r_j x)\big)\,,$$

at any boundary points gives a linear function $u_0 = \alpha x_1^+$, after suitable rotation and translation, where $\alpha = |Du(O)|$ and O indicates the origin.

Following the idea of the proof of Theorem 1.2 in [16], now we present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us consider $u = \sup\{v\ : v \in \mathscr{F}^-(\Omega, g)\}$, and let u_n be a maximizing sequence. Notice that, thanks to Proposition 3.1, we can assume $\{u_n\}$ to be an increasing sequence of the p-capacitary potentials of convex rings $\Omega \setminus \overline{K_n}$, with $|Du_n(x)| \leqslant g(\nu_{K_n}(x))$ on ∂K_n for every n. Let K be the increasing limit of K_n; hence K is convex and, as uniform limit of p-harmonic functions, u is the p-capacitary potential of $\Omega \setminus \overline{K}$, with $|Du(x)| \leqslant g(\nu_K(x))$ on ∂K.

We need to show that in fact $|Du(x)| = g(\nu_K(x))$ and we will prove it by contradiction, constructing a function $w \in \mathscr{F}^-(\Omega, g)$ such that $w \geqslant u$ with $w > u$ at some point. Let us remind that $\nu(x)$ indicates the outer unit normal vector to ∂K at x.

Let us assume by contradiction that there exists a point $y \in \partial K$ such that

$$\alpha = |Du(y)| < g(\nu(y))$$

and assume y to be the origin O with outer unit normal ν parallel to the first axis. Let δ be such that

$$\alpha + 3\delta < g(\nu). \tag{3.2}$$

By Lemma 3.3 the sequence

$$u_{r_j} = \frac{1}{r_j}\left(1 - u(r_j x)\right),$$

converges to $u_0(x) = \alpha x_1^+$, hence for every $\eta > 0$,

$$u(x) > 1 - \alpha x_1^+ - \eta r_j, \tag{3.3}$$

if r_j is small enough, for $x = (x_1, ..., x_N) \in B(O, r_j)$.

Consider

$$w_R(x) = w_{R,\varepsilon}(x) = \left(\alpha + \frac{\delta}{2}\right)\left(\frac{u_R - \varepsilon}{\alpha + \delta/2 - \varepsilon}\right)^+,$$

where u_R is as in Lemma 3.2 and $l = \alpha + \delta/2$. Then there exist $\varepsilon_0, R_0 > 0$ such that for $\varepsilon \leqslant \varepsilon_0$ and $R \geqslant R_0$,

$$|Dw_R| \leqslant \alpha + 2\delta, \text{ on } \partial\{u_R \leqslant \varepsilon\} = \{w_R = 0\}. \tag{3.4}$$

Moreover there exist $\delta_1, \delta_2 > 0$ such that

$$w_R > \alpha x_1^+ + \delta_2 \quad \text{on } \partial B(O, 1) \cap \{x_1 > -\delta_1\},$$

in particular we can fix δ_1 small enough such that $\{u_R = \varepsilon\} \cap \partial B(0, 1) \subseteq \{x_1 > -2\delta_1\}$, and choose

$$0 < \delta_2 = 2 \inf\{u_R(x) - \alpha {x_1}^+ \ : \ x \in \partial B(O, 1) \cap \{x_1 > -\delta_1\}\}.$$

Let $\tilde{w}(x) = 1 - r_j w_R(x/r_j)$; notice that, as u_R is quasi-convex, then $\tilde{w}$ is quasi-concave. Moreover for r_j sufficiently small, recalling (3.3) it holds

$$\tilde{w} < 1 - \alpha x_1^+ - \delta_2 r_j < u \quad \text{on } \partial B(O, r_j).$$

Define

$$w(x) = \begin{cases} \max\{u(x), \tilde{w}(x)\} & \text{in } B(O, r_j), \\ u(x) & \text{in } \mathbb{R}^N \setminus B(O, r_j), \end{cases}$$

and $W = \{\tilde{w} = 1\} = r_j\{w_R = 1\}$; observe that on $\partial B(O, r_j)$, $w = \tilde{w}$. By (3.2) and (3.4), for every $x \in W$ it holds

$$|D\tilde{w}(x)| \leqslant \alpha + 2\delta < g(\nu) - \delta.$$

Notice that $\{u_R = 0\} = \partial B(x_R, R)$ and for every $x \in \partial\{u_R = 0\}$ it holds

$$\lim_{R\to\infty} \nu_{B_R}(x) = \nu = (1, 0, ..., 0).$$

Moreover $\lim_{\varepsilon\to 0}\{u_R = \varepsilon\} = \{u_R = 0\}$ as limit in the Hausdorff metric of convex sets. Hence, by continuity, for sufficiently large R and sufficiently small ε, we have

$$|g(\nu) - g(\nu_W(z))| \leqslant \delta,$$

for every $z \in W \cap B(O, r_j)$, and hence,

$$|D\tilde{w}(x)| < g(\nu) - \delta \leqslant g(\nu_W(x)),$$

for every $x \in W \cap B(o, r_J)$.

Then $w \in \mathscr{F}^-(\Omega, g)$ and, since $w > u$ at some points, we get a contradiction with the maximality of u. Therefore $|Du| = g(\nu)$ on ∂K. □

4. Final remarks

Remark 4.1. In the non constant case no characterization of functions g for which $\mathscr{F}^-(\Omega, g)$ is not empty are known. However in some trivial case the existence or non-existence of a solution can be easily deduced by the characterization of the existence for the standard problem in (1.4). Indeed if g satisfies

$$\min_{\nu \in S^{N-1}} g(\nu) \geqslant \mathbf{\Lambda}_p(\Omega), \qquad \text{then} \quad \mathscr{F}(\Omega, \mathbf{\Lambda}_p(\Omega)) \subseteq \mathscr{F}^-(\Omega, g),$$

and hence $\mathscr{F}^-(\Omega, g) \neq \emptyset$; on the other hand, if

$$M = \max_{\nu \in S^{N-1}} g(\nu) < \mathbf{\Lambda}_p(\Omega), \qquad \text{then} \quad \mathscr{F}^-(\Omega, g) \subseteq \mathscr{F}^-(\Omega, M) = \emptyset,$$

and hence problem (1.2) has no solutions.

Remark 4.2 (Concavity property of Bernoulli Problems (1.2))**.** As in the classical case, geometric properties for the maximal solutions to (1.2) can be proved. Indeed, following the argument in [8], it is possible to define a combination of the Bernoulli Problems (1.2) in the Minkowski sense and to prove that Problem (1.2) has a concave behaviour with respect to this combination. More precisely: fix $\lambda \in [0, 1]$; let Ω_0, Ω_1 be two given convex domains and $g_0, g_1 : S^{N-1} \to \mathbb{R}^+$ two continuous functions (which both stay far away from zero). We define Ω_λ as the Minkowski combination of Ω_0, Ω_1, that is

$$\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1 = \{z = (1-\lambda)x_0 + \lambda x_1 \mid x_0 \in \Omega_0, x_1 \in \Omega_1\},$$

and g_λ as the harmonic mean of g_0 and g_1, that is

$$\frac{1}{g_\lambda(\nu)} = \frac{(1-\lambda)}{g_0(\nu)} + \frac{\lambda}{g_1(\nu)}.$$

Consider Problem (1.2) for Ω_0, g_0 and Ω_1, g_1, respectively; we define their *combined problem* of ratio λ the Bernoulli problem of the type (1.2), with given set Ω_λ and gradient boundary constraint $g_\lambda(\nu)$. Following the proof of Proposition 7.1 in [8] we can prove that if $\mathscr{F}^-(\Omega_i, g_i)$, $i = 0, 1$, are non empty sets, then so is $\mathscr{F}^-(\Omega_\lambda, g_\lambda)$. More precisely let $(\widetilde{K}(\Omega_i, g_i), u_i)$ be the maximal solutions, for $i = 0, 1$ and let u_λ be the Minkowski combination of u_0 and u_1 of ratio λ, that is

$$\{u_\lambda \geqslant t\} = (1-\lambda)\{u_0 \geqslant t\} + \lambda\{u_1 \geqslant t\};$$

(see for instance [8] for more detailed definitions and properties). The function u_λ belongs to $\mathscr{F}^-(\Omega_\lambda, g_\lambda)$ and hence, by Theorem 1.1, Problem (1.2) for Ω_λ and g_λ admits a solution $(\widetilde{K}(\Omega_\lambda, g_\lambda), \tilde{u}_\lambda)$ which satisfies

$$(1-\lambda)\widetilde{K}(\Omega_0, g_0(\nu)) + \lambda\widetilde{K}(\Omega_1, g_1(\nu)) \subseteq \widetilde{K}(\Omega_\lambda, g_\lambda).$$

Remark 4.3 (A flop in the unbounded case)**.** It could be natural to try to extend Theorem 1.1 to the unbounded case with an approximation method considering a sequence of given domains $\Omega_R = \Omega \cap B(O, R)$ as R grows. As the sequence $\{\Omega_R\}$ is monotone increasing by comparison principle $\widetilde{K}(\Omega_R, g)$ also increases and hence it converges to a convex set. Unfortunately, this approach fails in the limit process as it turns out that in fact $\widetilde{K}(\Omega_R, g)$ converges to the given set Ω which means that the limit of maximal solutions degenerates.

More precisely assume for simplicity $\Omega = \mathbb{R}^N$, so that $\Omega_R = B_R = B(O, R)$ (or, analogously $\Omega = H^-$ the half space $\{x_N \leqslant 0\}$ and take $\Omega_R = B_R = B(x_R, R)$, where $x_R = (0, ..., 0, -R)$). If R is sufficiently large, then the Bernoulli constant of B_R, $\mathbf{\Lambda}_p(B_R) = C_N/R$ (see [8] for example) is smaller than c_0 and hence

$$B_r \subseteq \widetilde{K}(B_R, c_0) \subseteq \widetilde{K}(B_R, g(\nu)),$$

where $B_r = B(O, r)$ is the unique solution to Problem (1.2) corresponding to $\Omega = B_R$ and $g(\nu) \equiv \mathbf{\Lambda}_p(B_R)$.

Hence for sufficiently large R, $\mathscr{F}^-(B_R, g)$ is not empty and Theorem 1.1 gives a sequence of quasi-concave p-capacitary potentials $\{u^R\}$ which solve Problem (1.2) in $\Omega_R \setminus \overline{K_R}$, where $K_R = B_r$. By easy computations one can check that $r = R/c_N$, for some constant c_N depending on the dimension and hence, the sequence of interior domains $\{K_R\}_{R>0}$ is not bounded for R which tends to infinity. This implies that the limit of the maximal solutions (K_R, u^R) is not the solution to the limit problem.

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