Continuation from a flat to a round Earth model in the coplanar orbit transfer problem

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Abstract

In this article we focus on the problem of minimization of the consumption for the coplanar orbit transfer problem. This problem is usually solved numerically by a shooting method, based on the application of the Pontryagin Maximum Principle, however the shooting method is known to be hard to initialize and the convergence is difficult to obtain due to discontinuities of the optimal control. Several tricks are known in order to overcome that problem, however in this article we introduce a new approach based on the following preliminary remark. When considering a 2D flat Earth model with constant gravity, the optimal control problem of passing from an initial configuration to some final configuration by minimizing the consumption can be very efficiently solved, and the solution leads to an extremely efficient algorithm. Based on that, we propose a continuous deformation from this flat Earth model to a modified flat Earth model that is diffeomorphic to the usual round Earth model. The resulting numerical continuation process thus provides a new way to solve the problem of minimization of the consumption for the coplanar orbit transfer problem.

Keywords: orbit transfer problem; optimal control; Pontryagin Maximum Principle; shooting method; continuation.

1 Introduction

The fuel efficient orbit transfer of a satellite is a widely studied problem, [13, 7]. We can distinguish mainly between two formulations of this problem. The first one considers that the vehicle produces instantaneous change

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of velocity and is referred to as the impulse orbit transfer, [14, 15, 19]. The second formulation takes into account the fact that all engines have a limited thrust and that the vehicle's dynamics has to be continuous in the position and velocity coordinates, [12]. In this continuous approach, we also separate the high-thrust and the low-thrust transfer, depending on the available acceleration.

In this paper, we focus on the high thrust orbit transfer that we furthermore restrict to be coplanar. This problem naturally writes as an optimal control problem. There exists various numerical methods to solve such a problem, and we usually separate them in two classes: direct and indirect methods. Direct methods (e.g. surveyed in [4]) consist in discretizing the optimal control problem in order to rewrite it as a parametric optimization problem. Then a nonlinear large scale optimization solver is applied. The advantage of this approach is that it is straightforward and is usually quite robust. The main drawback is that those methods are computationally demanding and that they are not very accurate when compared with the indirect approach. Indirect methods are based on the Pontryagin Maximum Principle (PMP, see [17]) that is a set of necessary conditions for a candidate trajectory and control strategy to be optimal. The idea is to use those necessary conditions to reduce the search of a solution to the search of the zero of the so-called shooting function (indirect methods are also called shooting methods in this context). The advantage is that shooting methods are very fast when they converge and that they produce high accuracy solutions. Their main drawback is that they typically use Newton-like algorithm to look for the zero of the shooting function and thus, they are hard to properly initialize. We can also mention mixed method that use a discretization of the PMP necessary conditions and then apply large-scale optimization solver (see e.g. [5]).

Because of its fast convergence and high accuracy, we will turn to a shooting method to solve the coplanar orbit transfer problem. There already exist some methods to cope with the initialization drawback of this method. In [3], the authors use the impulse transfer solution to provide a good initial guess to the shooting algorithm. This method is based on the fact that limited thrust orbit transfer try to mimic impulse transfer, as outlined in [15, 9]. However this approach is only valid for nearly circular initial and final orbits. In [16], a multiple shooting method, parameterized by the number of thrust arcs, is used to solve an Earth-Mars transfer. In [10, 11], a differential continuation method linking the minimization of the L^2-norm of the control to the minimization of the consumption is used to solve the low-thrust orbit transfer around the Earth. However this approach is not

adapted for high-thrust transfer.

In this article we propose a novel way to initialize a shooting method for high-thrust coplanar orbit transfer with fixed final time. It is based on a continuation method starting with the solving of a simplified transfer on a flat Earth model and then continuously adding curvature to end up with the model we want to solve. Note that we restrict ourselves to fixed final time problems since it has already been numerically shown that the continuous transfer with maximization of the final mass does not have any solution (see [16, 9]).

This paper is organized as follows. First, we explicit the optimal control problem we want to solve, along with the necessary conditions given by the PMP. Then we introduce the simplified flat Earth model and modify it so as to introduce curvature and make it diffeomorphic to the round Earth model. The next section presents the continuation procedure and explains how to pass from the simplified model to the targeted optimal control problem. A refined analysis is then carried out to provide a robust algorithm to solve the simplified flat Earth model. Finally, we give a numerical example in which we solve an orbit transfer from an SSO initial orbit to a circular final one. Since our approach involves diffeomorphic changes of coordinates, we explain in the appendix the impact of a change of coordinates onto the set of adjoint vectors of the PMP.

1.1 The round Earth model and the optimal control problem

The model that we use for the coplanar orbit transfer problem is the following. Assume that the Earth is spherical with center O and consider an inertial geocentric frame $(O, \vec{i}, \vec{j}, \vec{k})$. Since we consider the coplanar orbit transfer problem, we assume that whole trajectory lies in the plane $O + \mathbb{R}\vec{i} + \mathbb{R}\vec{j}$. The satellite is modeled as a mass point $M(t)$, with $\overrightarrow{OM}(t) = r(t)\vec{e}_r$, where $(\vec{e}_r, \vec{e}_\varphi)$ denotes the usual Frénet frame defined by

$$\vec{e}_r = \sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}, \quad \vec{e}_\varphi = \cos\varphi\,\vec{i} - \sin\varphi\,\vec{j}.$$

It is subject to the central gravitational field $g(r) = \frac{\mu}{r^2}$, where μ is the Earth gravitational constant, and to the thrust $\overrightarrow{T}(t) \in \mathbb{R}^2$. The mass of the satellite is denoted by $m(t)$. The vehicle follows the two-dimensional controlled Kepler equation

$$\frac{d^2\overrightarrow{OM}}{dt}(t) = -g(r(t))\vec{e}_r + \frac{\overrightarrow{T}(t)}{m(t)}, \quad \dot{m}(t) = -\beta\|\overrightarrow{T}(t)\|, \tag{1}$$

where $\beta > 0$ is the inverse of the thruster exhaust velocity. Moreover, the control $\overrightarrow{T}(\cdot)$ must satisfy the constraint

$$\|\overrightarrow{T}(\cdot)\| \leqslant T_{\max}, \tag{2}$$

where $T_{\max}$ is the maximal allowed thrust, and $\|\ \|$ denotes the usual Euclidean norm.

Instead of cartesian coordinates, we next use polar coordinates whose definition is recalled next. Recall that $r(t) = \|\overrightarrow{OM}(t)\|$, and set

$$v(t) = \left\| \frac{d\overrightarrow{OM}}{dt}(t) \right\| = \sqrt{\dot{r}(t)^2 + r(t)^2 \dot{\varphi}(t)^2}.$$

Define the flight path angle $\gamma(t)$ by

$$\frac{d\overrightarrow{OM}}{dt}(t) = v(t)(\sin\gamma(t)\,\vec{e}_r + \cos\gamma(t)\,\vec{e}_\varphi).$$

Define the coordinates $q = (r, \varphi, v, \gamma, m)$, with (r, φ) the polar coordinates of the satellite, v its speed and γ the slope of the velocity vector. Then, the control system (1) is written in spherical coordinates as

$$\begin{aligned}
\dot{r}(t) &= v(t)\sin\gamma(t) \\
\dot{\varphi}(t) &= \frac{v(t)}{r(t)}\cos\gamma(t) \\
\dot{v}(t) &= -g(r(t))\sin\gamma(t) + \frac{T_{\max}}{m(t)}u_1(t) \\
\dot{\gamma}(t) &= \left(\frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)}\right)\cos\gamma(t) + \frac{T_{\max}}{m(t)v(t)}u_2(t) \\
\dot{m}(t) &= -\beta T_{\max}\|u(t)\|
\end{aligned} \tag{3}$$

where the normalized control $u(t) = (u_1(t), u_2(t))$ satisfies $T(t) = u(t)T_{\max}$ and the constraint

$$\|u(t)\| \leqslant 1, \tag{4}$$

for almost every t. The optimal control problem under consideration then consists in steering the control system (3) from an initial configuration

$$r(0) = r_0,\ \varphi(0) = \varphi_0,\ v(0) = v_0,\ \gamma(0) = \gamma_0,\ m(0) = m_0, \tag{5}$$

to some final configuration that is either of the form

$$r(t_f) = r_f,\ v(t_f) = v_f,\ \gamma(t_f) = \gamma_f, \tag{6}$$

or of the form

$$\begin{aligned}\xi_{K_f} &= \frac{v(t_f)^2}{2} - \frac{\mu}{r(t_f)} - K_f = 0,\\ \xi_{e_f} &= \sin(\gamma)^2 + \left(1 - \frac{r(t_f)v(t_f)^2}{\mu}\right)^2 \cos(\gamma)^2 - e_f^2 = 0.\end{aligned} \tag{7}$$

The conditions (6) state that the satellite has to enter a specified orbit at a given point of it. The conditions (7) only prescribe the final orbit, of energy $K_f < 0$ (for it to be elliptic) and eccentricity e_f, but not the entry point on this orbit. Note that for both final conditions the orientation of the final orbit is not prescribed ($\varphi(t_f)$ is free). The criterion to consider is the maximization of the final mass $m(t_f)$. As mentioned in [9, 16], this problem does not have a solution for free final time, so we assume t_f to be fixed. In what follows, this optimal control problem is referred to as (**OCP**).

According to the Pontryagin Maximum Principle, every optimal trajectory $q(\cdot)$ of (**OCP**), associated with a control $u(\cdot)$ on $[0, t_f]$, is the projection of an extremal $(q(\cdot), p(\cdot), p^0, u(\cdot))$, where $p(\cdot) : [0, t_f] \to \mathbb{R}^5$ is an absolutely continuous mapping called *adjoint vector*, p^0 is a non-positive real number, with $(p(\cdot), p^0) \neq (0, 0)$, and there holds

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t), p^0, u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t), p^0, u(t)), \tag{8}$$

for almost every $t \in [0, t_f]$, where the Hamiltonian is defined by

$$\begin{aligned}H(q, p, p^0, u) = {} & p_r v \sin\gamma + p_\varphi \frac{v}{r}\cos\gamma + p_v\left(-g(r)\sin\gamma + \frac{T_{\max}}{m}u_1\right)\\ & + p_\gamma\left(\left(\frac{v}{r} - \frac{g(r)}{v}\right)\cos\gamma + \frac{T_{\max}}{mv}u_2\right) - p_m\beta T_{\max}\|u\|,\end{aligned}$$

with $p = (p_r, p_\varphi, p_v, p_\gamma, p_m)$. This yields the adjoint equations

$$\begin{aligned}\dot{p}_r &= p_\varphi\frac{v}{r^2}\cos\gamma - \frac{2}{r}p_v g(r)\sin\gamma + \frac{1}{r}p_\gamma\left(\frac{v}{r} - 2\frac{g(r)}{v}\right)\cos\gamma\\ \dot{p}_\varphi &= 0\\ \dot{p}_v &= -p_r\sin\gamma - \frac{1}{r}p_\varphi\cos\gamma - p_\gamma\left(\frac{1}{r} + \frac{g(r)}{v^2}\right)\cos\gamma + p_\gamma\frac{T_{\max}}{mv^2}u_2\\ \dot{p}_\gamma &= -p_r v\cos\gamma + p_\varphi\frac{v}{r}\sin\gamma + p_v g(r)\cos\gamma + p_\gamma\left(\frac{v}{r} - \frac{g(r)}{v}\right)\sin\gamma\\ \dot{p}_m &= \frac{T_{\max}}{m^2}\left(p_v u_1 + \frac{p_\gamma}{v}u_2\right)\end{aligned}$$

Moreover, the maximization condition

$$H(q(t), p(t), p^0, u(t)) = \max_{\|w\| \leqslant 1} H(q(t), p(t), p^0, w) \tag{9}$$

holds almost everywhere on $[0, t_f]$, and this quantity is constant since the dynamics is autonomous. Note that if t_f were to be free, we would also have that $H(t_f) = 0$. Furthermore, one has the transversality conditions, that depend on the chosen final configuration. For (6) we simply have

$$p_\varphi(t_f) = 0, \quad p_m(t_f) = -p^0. \tag{10}$$

For (7), the conditions (10) hold as well, and additionally the vector $(p_r(t_f), p_v(t_f), p_\gamma(t_f))$ is a linear combination of the gradients (with respect to (r, v, γ)) of the two relations (7). This writes as

$$\partial_r \xi_{K_f}(p_\gamma \partial_v \xi_{e_f} - p_v \partial_\gamma \xi_{e_f}) + \partial_v \xi_{K_f}(p_r \partial_\gamma \xi_{e_f} - p_\gamma \partial_r \xi_{e_f}) = 0, \tag{11}$$

where the expression has to be evaluated at the final time t_f.

The extremal $(x(\cdot), p(\cdot), p^0, u(\cdot))$ is said *normal* whenever $p^0 \neq 0$, and in that case it is usual to normalize the adjoint vector so that $p^0 = -1$; otherwise it is said *abnormal*. The controllability aspects of the orbit transfer problem have been studied in [6, 7, 8, 10], and it has been proved in these references that the abnormal case cannot occur for our terminal conditions (5) and (6) or (7). Hence, from now on we assume that $p^0 = -1$.

Define the so-called switching function $\Phi(\cdot)$ along a given extremal by

$$\Phi(t) = \frac{1}{m(t)} \sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}} - \beta p_m(t),$$

for every $t \in [0, t_f]$. Then, we infer from the maximization condition (9) that $u(t) = (u_1(t), u_2(t)) = (0, 0)$ whenever $\Phi(t) < 0$, and

$$u_1(t) = \frac{p_v(t)}{\sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}}, \quad u_2(t) = \frac{p_\gamma(t)}{v(t)\sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}},$$

whenever $\Phi(t) > 0$. Note that these formulas are well defined since the functions $p_v(\cdot)$ and $p_\gamma(\cdot)$ do not vanish simultaneously identically on any subinterval[1]. Note that the extremal control cannot be determined from the

[1]Indeed otherwise, it would follow from (9) and (9) that $p_r(\cdot)$ and $p_\varphi(\cdot)$ vanish identically as well, and then the vanishing of the Hamiltonian would imply that $p_m(\cdot) = -p^0 = 0$, which raises a contradiction.

maximization condition in the case the switching function $\Phi(\cdot)$ vanishes on a subinterval of $[0, t_f]$. The non-occurrence of this singular case must be checked from the numerical simulations.

Based on these necessary conditions, recall that the (single) shooting method consists in finding a zero of the shooting function S defined as follows. Given $t_f > 0$ and $p_0 \in \mathbb{R}^5$, denote by $(q(t, p_0), p(t, p_0))$ the extremal solution of (8), starting from the initial condition $q(0)$ given by (5) and from $p(0) = p_0$. Then, the shooting function is defined by

$$S(t_f, p_0) = \begin{pmatrix} r(t_f, p_0) - r_f \\ v(t_f, p_0) - v_f \\ \gamma(t_f, p_0) - \gamma_f \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix} \text{ or } \begin{pmatrix} \xi_{K_f}(p_0) \\ \xi_{e_f}(p_0) \\ \text{eq. (11)} \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix}, \tag{12}$$

depending on the chosen final conditions. The (single) shooting method thus consists in combining any numerical method for integrating a differential equation with a Newton-like method in order to determine a zero of the shooting function S.

As mentioned formerly, it is difficult to obtain convergence of this method, due to a difficulty of initialization and to the discontinuities of the control. However, we observe that, when assuming that the Earth is flat and the gravity is constant, the corresponding optimal control problem can be easily solved, in a very explicit way. We next introduce this very simplified model and explain our idea of passing continuously to the round Earth model.

1.2 Simplified flat Earth model

The movement of a vehicle in a flat Earth model with constant gravity is governed by the control system

$$\begin{aligned} \dot{x}(t) &= v_x(t) \\ \dot{h}(t) &= v_h(t) \\ \dot{v}_x(t) &= \frac{T_{\max}}{m(t)} u_x(t) \\ \dot{v}_h(t) &= \frac{T_{\max}}{m(t)} u_h(t) - g_0 \end{aligned} \tag{13}$$

$$\dot{m}(t) = -\beta T_{\max} \sqrt{u_x(t)^2 + u_h(t)^2}$$

where $x(t)$ denotes the downrange, $h(t)$ is the altitude, $v_x(t)$ is the horizontal component of the speed, $v_h(t)$ is the vertical component of the speed, and the control $(u_x(\cdot), u_h(\cdot))$ satisfies the constraint

$$u_x(\cdot)^2 + u_h(\cdot)^2 \leqslant 1. \tag{14}$$

The constant g_0 stands for the gravity $g_0 = \frac{\mu}{r_T^2}$ at zero altitude, with r_T the Earth radius. We denote by $(\mathbf{OCP})_{\mathbf{flat}}$ the optimal control problem of maximizing the final mass $m(t_f)$ for the control system (13), with the initial and final conditions

$$x(0) = x_0,\ h(0) = h_0,\ v_x(0) = v_{x0},\ v_h(0) = v_{h0},\ m(0) = m_0, \tag{15}$$

$$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0. \tag{16}$$

If we had to make a connection to the round Earth model, these final conditions would be similar to (6) rather than (7). Furthermore, we do not have to assume t_f fixed and can include it in the parameter to optimize.

It happens that $(\mathbf{OCP})_{\mathbf{flat}}$ can be explicitly, analytically solved by applying the Pontryagin Maximum Principle. This is the object of Section 2 further, and this resolution leads to an extremely efficient algorithm based on a shooting method whose initialization is obvious. Based on that observation, it is tempting to try to use this efficient resolution in order to guess a good initialization for the shooting method applied to (**OCP**). To this aim, the idea is to use a continuation process by introducing parameters such that, when one makes these parameters evolve continuously, one passes from the flat Earth model to the initial round Earth model. Since the coordinates of the flat Earth model are cartesian, and the coordinates of the round Earth model are polar, this will of course require, at the end of the process, a change of coordinates.

Before going into more details we can make one preliminary remark. In the continuation process, the gravity constant g_0 must be obviously deformed in order to end up with the gravity model $g(r)$. However there is a serious difference between the flat Earth model, the gravity being constant or not, and the round Earth model: indeed, in the round Earth model, periodic trajectories (with $u = 0$) do exist, namely keplerian orbits. Whereas in the flat Earth model there do not exist any trajectories with a zero control having a periodic altitude $h(t)$, due to the presence of the gravity term. This obvious but important remark leads to the idea of deforming the flat Earth model by introducing some new term into the dynamics, so that there may exist some horizontal trajectories with zero control. Moreover, we would

like this modified model to be equivalent, up to some change of coordinates, to the round Earth model.

This modified flat Earth model is derived in the next subsection, by defining a change of coordinates that is flattening circular orbits into horizontal trajectories, and then computing the control system from this change of coordinates.

1.3 Modified flat Earth model

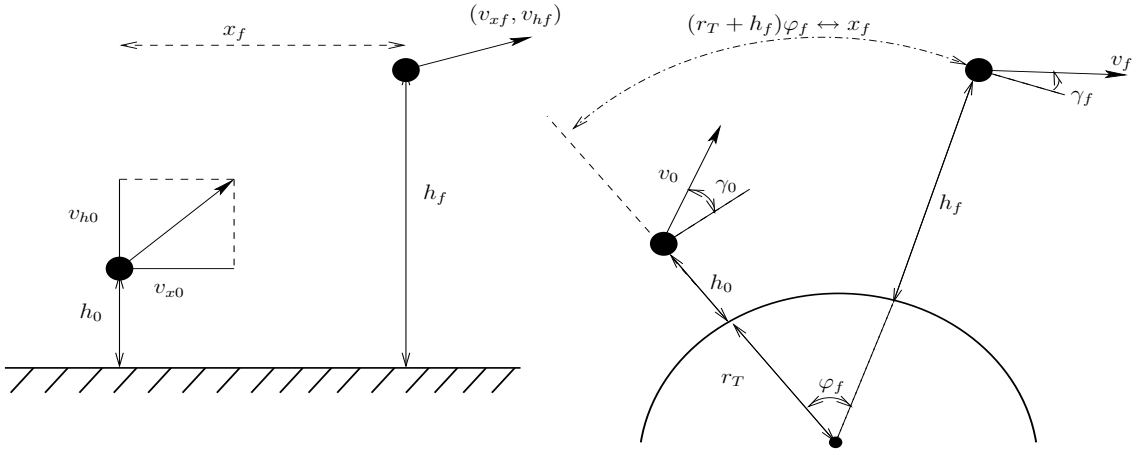

Figure 1: Correspondence between flat and round Earth coordinates.

Starting from the polar coordinates (r, φ, v, γ) of the round Earth model, let us define some cartesian coordinates (x, h, v_x, v_h). The idea of mapping circular orbits to horizontal trajectories leads to define $x = r\varphi$. Then, $x(t)$ represents the curvilinear abscissa of the point $M(t)$. The altitude is logically defined by $h = r - r_T$, where r_T denotes the radius of the Earth. Using the geometric definition of the angle γ, finally, one is led to define $v_x = v\cos\gamma$ and $v_y = v\sin\gamma$. Summing up, we consider the change of coordinates

$$\left\{\begin{array}{rcl} x &=& r\varphi \\ h &=& r - r_T \\ v_x &=& v\cos\gamma \\ v_h &=& v\sin\gamma \end{array}\right. \iff \left\{\begin{array}{rcl} r &=& r_T + h \\ \varphi &=& \frac{x}{r_T+h} \\ v &=& \sqrt{v_x^2 + v_h^2} \\ \gamma &=& \arctan\frac{v_h}{v_x} \end{array}\right. \tag{17}$$

and denote by F the corresponding diffeomorphism, such that $F(x, h, v_x, v_h) = (r, \varphi, v, \gamma)$. For the control, the transformation from spherical to cartesian

coordinates is

$$\begin{pmatrix} u_x \\ u_h \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{18}$$

Applying this change of coordinates to the control system (3) now leads, after easy computations, to the control system

$$\begin{aligned} \dot{x}(t) &= v_x(t) + v_h(t)\frac{x(t)}{r_T + h(t)} \\ \dot{h}(t) &= v_h(t) \\ \dot{v}_x(t) &= \frac{T_{\max}}{m(t)}u_x(t) - \frac{v_x(t)v_h(t)}{r_T + h(t)} \\ \dot{v}_h(t) &= \frac{T_{\max}}{m(t)}u_h(t) - g(r_T + h(t)) + \frac{v_x(t)^2}{r_T + h(t)} \\ \dot{m}(t) &= -\beta T_{\max}\sqrt{u_x(t)^2 + u_h(t)^2} \end{aligned} \tag{19}$$

This modified formulation in cartesian coordinates is strictly equivalent to the initial formulation in spherical coordinates. Therefore it still represents the true round Earth transfer problem. Especially it admits the keplerian orbits as periodic trajectories with a null thrust. Comparing this modified formulation wit the flat Earth formulation, we have two differences: the first one is of course the gravity term, which is constant in the simplified model (13); the second difference is the presence of new terms in the dynamics of x, v_x and v_h. These new terms can be seen as corrective terms in the flat Earth model, which make possible in particular the existence of periodic trajectories (with no thrust).

1.4 The continuation procedure

To pass from the simplified flat Earth model (13) to the modified flat Earth model (19), we introduce two parameters. One of them permits to pass continuously from the constant gravity term to the varying gravity term, and the other introduces continuously the corrective terms. In brief, we

consider the family of control systems

$$\begin{aligned}
\dot{x}(t) &= v_x(t) + \lambda_2 v_h(t)\frac{x(t)}{r_T + h(t)} \\
\dot{h}(t) &= v_h(t) \\
\dot{v}_x(t) &= \frac{T_{\max}}{m(t)}u_x(t) - \lambda_2\frac{v_x(t)v_h(t)}{r_T + h(t)} \\
\dot{v}_h(t) &= \frac{T_{\max}}{m(t)}u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2\frac{v_x(t)^2}{r_T + h(t)} \\
\dot{m}(t) &= -\beta T_{\max}\sqrt{u_x(t)^2 + u_h(t)^2}
\end{aligned} \tag{20}$$

parameterized by the parameters λ_1 and λ_2, themselves varying between 0 and 1. For $\lambda_1 = \lambda_2 = 0$, one recovers the simplified flat Earth model (13), and for $\lambda_1 = \lambda_2 = 1$ one recovers the modified flat Earth model (19), which is diffeomorphic to the initial round Earth model (3).

Now, for all $(\lambda_1, \lambda_2) \in [0,1]^2$, denote by $(\mathbf{OCP})_{\lambda_1,\lambda_2}$ the optimal control problem of steering the system (20) from (15) to (16) and maximizing the final mass $m(t_f)$. Since the problem $(\mathbf{OCP})_{1,1}$ does not have a solution for free final time, we decide to fix the final time t_f for (λ_1, λ_2) in $\{1\} \times (0,1]$. The value that we choose for t_f is the optimal final time of $(\mathbf{OCP})_{(1,0)}$. This is not restrictive since numerical simulations show that the shooting method is relatively robust with respect to changes on the fixed t_f.

As before, the application of the Pontryagin Maximum Principle leads to a shooting problem, as follows. For every optimal trajectory $X(\cdot) = (x(\cdot), h(\cdot), v_x(\cdot), v_h(\cdot), m(\cdot))$ of $(\mathbf{OCP})_{\lambda_1,\lambda_2}$, associated with a control $U(\cdot) = (u_x(\cdot), u_h(\cdot))$ on $[0, t_f]$ there exists an absolutely continuous mapping $P(\cdot) = (p_x(\cdot), p_h(\cdot), p_{v_x}(\cdot), p_{v_h}(\cdot), p_m(\cdot)) : [0, t_f] \to \mathbb{R}^5$ and $p^0 \leqslant 0$, satisfying $(p(\cdot), p^0) \neq (0,0)$, such that

$$\begin{aligned}
\dot{X}(t) &= \frac{\partial H_{\lambda_1,\lambda_2}}{\partial P}(X(t), P(t), p^0, U(t)), \\
\dot{P}(t) &= -\frac{\partial H_{\lambda_1,\lambda_2}}{\partial X}(X(t), P(t), p^0, U(t)),
\end{aligned} \tag{21}$$

for almost every $t \in [0, t_f]$, where the Hamiltonian is defined by

$$\begin{aligned}
H_{\lambda_1,\lambda_2}(X, P, p^0, U) = p_x\left(v_x + \lambda_2 v_h\frac{x}{r_T + h}\right) + p_h v_h + p_{v_x}\left(\frac{T_{\max}}{m}u_x - \lambda_2\frac{v_x v_h}{r_T + h}\right) \\
+ p_{v_h}\left(\frac{T_{\max}}{m}u_h - \frac{\mu}{(r_T + \lambda_1 h)^2} + \lambda_2\frac{v_x^2}{r_T + h}\right) - p_m\beta T_{\max}\|u\|.
\end{aligned}$$

In coordinates, the adjoint equations are

$$\begin{aligned}
\dot{p}_x &= -p_x \frac{\lambda_2 v_h}{r_T + h} \\
\dot{p}_h &= p_x \frac{\lambda_2 x v_h}{(r_T + h)^2} - p_{v_x} \frac{\lambda_2 v_x v_h}{(r_T + h)^2} - p_{v_h} \left(\frac{2\mu\lambda_1}{(r_T + \lambda_1 h)^3} - \frac{\lambda_2 v_x^2}{(r_T + h)^2} \right) \\
\dot{p}_{v_x} &= -p_x + p_{v_x} \frac{\lambda_2 v_h}{r_T + h} - 2 p_{v_h} \frac{\lambda_2 v_x}{r_T + h} \\
\dot{p}_{v_h} &= -p_x \frac{\lambda_2 x}{r_T + h} - p_h + p_{v_x} \frac{\lambda_2 v_x}{r_T + h} \\
\dot{p}_m &= \frac{T_{\max}}{m^2} (p_{v_x} u_x + p_{v_h} u_h)
\end{aligned}$$

Moreover, the maximization condition

$$H_{\lambda_1,\lambda_2}(X(t), P(t), p^0, U(t)) = \max_{\|W\| \leqslant 1} H(X(t), P(t), p^0, W) \tag{22}$$

holds almost everywhere on $[0, t_f]$, and this quantity is equal to 0 for $(\lambda_1, \lambda_2) \in [0,1] \times \{0\}$ since the final time t_f is free and the dynamics is autonomous. Furthermore, one has the transversality conditions

$$p_x(t_f) = 0, \quad p_m(t_f) = -p^0. \tag{23}$$

The conditions (7) can be taken into account thanks to another continuation from the transversality conditions (23) to the transversality conditions (11) expressed in this cartesian reference frame.

Defining the switching function $\Phi(\cdot)_{\lambda_1,\lambda_2}$ as

$$\Phi_{\lambda_1,\lambda_2}(t) = \frac{1}{m(t)} \sqrt{p_{v_x}(t)^2 + p_{v_h}(t)^2} - \beta p_m(t),$$

for every $t \in [0, t_f]$, one has $U(t) = (u_x(t), u_h(t)) = (0,0)$ whenever $\Phi(t) < 0$, and

$$u_x(t) = \frac{p_{v_x}(t)}{\sqrt{p_{v_x}(t)^2 + p_{v_h}(t)^2}}, \quad u_h(t) = \frac{p_{v_h}(t)}{\sqrt{p_{v_x}(t)^2 + p_{v_h}(t)^2}},$$

whenever $\Phi(t) > 0$. Note that these formulas are well defined since the functions $p_{v_x}(\cdot)$ and $p_{v_h}(\cdot)$ do not vanish simultaneously identically on any subinterval[2].

[2]Indeed otherwise, it would follow from (22) that $p_x(\cdot)$ vanishes identically as well, and then from (22) that $p_x(\cdot)$ also vanishes identically; then, the vanishing of the Hamiltonian would imply that $p_m(\cdot) = -p^0 = 0$, which raises a contradiction.

Assuming $p^0 = -1$ (see comments for the round Earth model), the continuation method (see [2]) consists in solving a series of shooting problems for sequences of parameters λ_1 and λ_2, with the starting point $\lambda_1 = \lambda_2 = 0$. At each step, the previous solution is used as an initial guess for the shooting problem. For every couple (λ_1, λ_2), the shooting function S_{λ_1,λ_2} is defined as follows. Given $t_f > 0$ and $P_0 \in \mathbb{R}^5$, denote by $(X(t, P_0), P(t, P_0))$ the extremal solution of (21), starting from the initial condition $X(0)$ given by (5) up to the change of coordinates (17), and from $P(0) = P_0$. Then, the shooting function is defined by

$$S_{\lambda_1,\lambda_2}(P_0, t_f) = \begin{pmatrix} h(t_f, P_0) - h_f \\ v_h(t_f, P_0) \\ v_x(t_f, P_0) - v_{xf} \\ p_x(t_f, P_0) \\ p_m(t_f, P_0) - 1 \\ H(t_f) \text{ if } (\lambda_1, \lambda_2) \in [0, 1] \times \{0\} \end{pmatrix}. \tag{24}$$

Note that in the case where t_f is fixed, the shooting function has only five components.

The continuation procedure consists in solving iteratively by a Newton-like method the equation $S_{\lambda_1,\lambda_2}(P_0, t_f) = 0$ for a sequence of parameters (λ_1, λ_2), starting from $(0, 0)$ and ending at $(1, 1)$. The starting point $(0, 0)$ of the continuation process corresponds to the simplified flat Earth model, and hence we initialize the procedure with the solution of that simplified model, as detailed in Section 2.

This algorithmic procedure provides a way of solving (**OCP**) without any a priori knowledge on the optimal solution. The price to pay is that, instead of solving only one optimal control problem, one has to solve a series of $(\mathbf{OCP})_{\lambda_1,\lambda_2}$. However the whole procedure is time-efficient since the shooting method relies on a Newton-like method. Our procedure provides a way for bypassing the difficulty due to the initialization of the shooting method when applied directly to (**OCP**). Numerical simulations are given in Section 3.

One item remains however to be explained in our procedure. Indeed, the continuation process above leads, provided it has converged, to the solution of $(\mathbf{OCP})_{1,1}$, which corresponds to the control system (19). As explained in Section 1.3, this control system is equivalent to the initial control system (3) via the change of coordinates (17) and (18). Hence, we must explain how the change of coordinates must act onto the adjoint vector, so as to recover the adjoint vector in the initial coordinates $(p_r, p_\varphi, p_v, p_\gamma, p_m)$. Recalling that F

denotes the diffeomorphism defined by (17), we claim that

$$ {}^t(p_r, p_\varphi, p_v, p_\gamma) = {}^t dF(x, h, v_x, v_h)^{-1} \cdot {}^t(p_x, p_h, p_{v_x}, p_{v_h}), \tag{25} $$

i.e., one passes from the adjoint vector in cartesian coordinates to polar coordinates by applying the transpose of the inverse of the differential of F. In coordinates, this yields, after easy computations,

$$ \begin{aligned} p_r &= \frac{x}{r_T + h} p_x + p_h \\ p_\varphi &= (r_T + h) p_x \\ p_v &= \cos\gamma\, p_{v_x} + \sin\gamma\, p_{v_h} \\ p_\gamma &= v(-\sin\gamma\, p_{v_x} + \cos\gamma\, p_{v_h}) \end{aligned} \tag{26} $$

This claim follows from a general fact recalled separately, in the Appendix (Section 5), for the sake of clarity. Note that p_m remains unchanged.

2 Analysis of the optimal control problem with the simplified flat Earth model

For this simplified optimal control problem, we consider similar terminal conditions as (5) and (6) except that we impose γ_f to be zero and we express them in cartesian coordinates. The terminal conditions are given by

$$ \begin{aligned} &x(0) = x_0,\ h(0) = h_0,\ v_x(0) = v_{x0},\ v_h(0) = v_h(0),\ m(0) = m_0 \\ &x(t_f) \text{ free},\ h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0,\ m(t_f) \text{ free}, \end{aligned} \tag{27} $$

Contrarily to the initial (**OCP**), we leave t_f free because the simplified problem does not allow orbits and we thus have a solution for free final time.

We assume moreover that

$$ h_f > h_0 + \frac{v_{h0}^2}{2g_0}. \tag{28} $$

Note that (28) discards the non interesting case of a zero consumption trajectory. This assumption is satisfied in practice.

2.1 Application of the Pontryagin Maximum Principle

Denoting the adjoint variables by $p = (p_x, p_h, p_{v_x}, p_{v_h}, p_m)$, the Hamiltonian is

$$ H = p_x v_x + p_h v_h + p_{v_x} \frac{T_{\max}}{m} u_x + p_{v_h} \left(\frac{T_{\max}}{m} u_h - g_0 \right) - \beta T_{\max} p_m \sqrt{u_x^2 + u_h^2}, $$

and hence the adjoint equations are

$$\dot{p}_x = 0,\ \dot{p}_h = 0,\ \dot{p}_{v_x} = -p_x,\ \dot{p}_{v_h} = -p_h,\ \dot{p}_m = \frac{T_{\max}}{m^2}(p_{v_x}u_x + p_{v_h}u_h).$$

According to the boundary conditions (27), the transversality conditions yield $p_x(t_f) = 0$ and $p_m(t_f) = -p^0$ (with $p^0 \leqslant 0$). It follows that $p_x(\cdot)$ is identically equal to 0, that $p_h(\cdot)$ is constant (denoted p_h in what follows), that $p_{v_x}(\cdot)$ is constant (denoted p_{v_x} in what follows), and that $p_{v_h}(t) = -p_h t + p_{v_h}(0)$. Since the final time t_f is free and the system is autonomous, we infer that $H = 0$ along any extremal.

Setting $\Phi(t) = \frac{1}{m(t)}\sqrt{p_{v_x}^2 + p_{v_h}(t)^2} - \beta p_m(t)$ for every $t \in [0, t_f]$, we infer from the maximization condition of the Hamiltonian that $u(t) = (u_x(t), u_h(t)) = (0,0)$ whenever $\Phi(t) < 0$, and

$$u_x(t) = \frac{p_{v_x}}{\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}}, \quad u_h(t) = \frac{p_{v_h}(t)}{\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}},$$

whenever $\Phi(t) > 0$.

2.2 Analysis of extremal equations

First of all, notice that $\dot{p}_m(t) = \frac{\|u(t)\|T_{\max}}{m(t)^2}\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}$, hence $p_m(\cdot)$ is nondecreasing. More precisely, $p_m(\cdot)$ is increasing whenever $\Phi(\cdot) > 0$ and constant whenever $\Phi(\cdot) < 0$.

Lemma 1. *The function $t \mapsto \sqrt{p_{v_x}^2 + p_{v_h}(t)^2}$ does not vanish identically on any subinterval of $[0, t_f]$.*

Proof. The argument goes by contradiction. If $p_{v_x} = 0$ and $p_{v_h}(\cdot) = 0$ on a subinterval I, then derivating with respect to t yields $p_h = 0$, and then by Cauchy uniqueness $p_{v_h}(\cdot) = 0$ on $[0, t_f]$. Hence, $p_m(\cdot)$ is constant, equal to $-p^0$. Besides, since $H = 0$, we infer that $p^0\|u(\cdot)\| = 0$ on $[0, t_f]$. Since $h_f > h_0$ (using (28)), the thrust $\|u(\cdot)\|$ cannot be identically equal to 0 on $[0, t_f]$, and hence $p^0 = 0$. We have proved that $(p(\cdot), p^0) = (0, 0)$, which is a contradiction with the Pontryagin Maximum Principle. □

This lemma implies in particular that the formulas for the extremal controls above are well defined. Moreover, it follows from easy computations that the function $t \mapsto \Phi(t)$ is almost everywhere two times derivable, and

(recall that $p_x = 0$)

$$\dot{\Phi}(t) = \frac{-p_h p_{v_h}(t)}{m(t)\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}}, \tag{29}$$

$$\ddot{\Phi}(t) = \frac{\beta\|u(t)\|}{m(t)}\dot{\Phi}(t) - \frac{m(t)}{\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}}\dot{\Phi}(t)^2 + \frac{p_h^2}{m(t)\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}} \tag{30}$$

Lemma 2. *The function $t \mapsto \Phi(t)$ is constant if and only if $p_h = 0$.*

Proof. If $p_h = 0$ then it follows from (29) that $\dot{\Phi} = 0$. Conversely, if $\dot{\Phi} = 0$ then $p_h p_{v_h} = 0$, and derivating with respect to time yields $p_h = 0$. □

Lemma 3. *The function $t \mapsto \Phi(t)$ does not vanish identically on any subinterval of $[0, t_f]$.*

Proof. The argument goes by contradiction. If $\Phi(\cdot) = 0$ on a subinterval I, then, from Lemma 2, $p_h = 0$, and then using the adjoint equations, $p_{v_h}(\cdot)$ is constant. Moreover, there holds $0 = H = -p_{v_h} g_0$, and hence $p_{v_h}(\cdot) = 0$ on $[0, t_f]$. In particular, this yields $u_h(\cdot) = 0$ on $[0, t_f]$, and hence $\dot{v}_h = -g_0$ and $v_h(t) = v_{h0} - g_0 t$. From (27), $v_h(t_f) = 0$, hence $t_f = \frac{v_{h0}}{g_0}$. Besides, integrating $\dot{x}_2 = v_h$, one gets $h(t) = h_0 + v_{h0}t - \frac{g_0}{2}t^2$, and thus $h(t_f) = h_0 + \frac{v_{h0}^2}{2g_0}$. From (27) one has $h(t_f) = h_f$, and we get a contradiction with (28). □

This lemma shows that the singular case where the extremal controls cannot be inferred directly from the maximization condition does not occur.

Lemma 4. *If $p_h = 0$ then the thrust $\|u(\cdot)\|T_{\max}$ is constant on $[0, t_f]$, equal to $T_{\max}$. In other words, in that case the thrust is always maximal and there is no switching.*

Proof. If $p_h = 0$ then, from Lemma 2, $\Phi(\cdot)$ is constant, and from Lemma 3 this constant Φ cannot be equal to 0. If $\Phi < 0$, then $u = 0$ on $[0, t_f]$, which is not possible since the thrust cannot be identically zero (this would contradict, for instance, the fact that $h_f > h_0$). Hence, $\Phi > 0$ and therefore $\|u(\cdot)\| = 1$ on $[0, t_f]$. □

Lemma 5. *If $p_h \neq 0$ then*

- *either $\Phi(\cdot)$ is increasing on $[0, t_f]$,*
- *or $\Phi(\cdot)$ is decreasing on $[0, t_f]$,*
- *or $\Phi(\cdot)$ has a unique minimum on $[0, t_f]$, is decreasing before that minimum and then increasing.*

Proof. If $p_h \neq 0$ then, from Lemma 2, $\Phi(\cdot)$ is not constant, hence $\dot{\Phi}(\cdot)$ is not identically equal to 0. If $\dot{\Phi}(\cdot)$ does not vanish on $[0, t_f]$, then $\Phi(\cdot)$ is strictly monotone, and this yields the two first cases of the result. If $\dot{\Phi}(\cdot)$ vanishes at some point t_1 of $[0, t_f]$, then, using (30), for every $t_1 \in [0, t_f]$ such that $\dot{\Phi}(t_1) = 0$ there must hold $\ddot{\Phi}(t_1) > 0$ (since $p_h \neq 0$), and therefore this point is a local minimum. This reasoning shows that every extremum of $\Phi(\cdot)$ is a local minimum. It follows that the function $\dot{\Phi}(\cdot)$ cannot vanish more than one time, otherwise there would exist another local minimum, and hence there should then exist a local maximum between those two minima; but this is a contradiction since every extremum of $\Phi(\cdot)$ is a local minimum. Therefore, the third point of the lemma follows. □

Lemmas 4 and 5 imply that the thrust $\|u(\cdot)\|T_{\max}$ of the optimal trajectory is either constant, equal to $T_{\max}$, or has exactly one switching (and in that case, passing either from 0 to $T_{\max}$, or from $T_{\max}$ to 0), or has exactly two switchings and passes from $T_{\max}$ to 0 and then from 0 to $T_{\max}$. Actually, we next prove that the latter possibility cannot occur, and finally derive the following result.

Theorem 1. *The optimal trajectory of* $(\mathbf{OCP})_{\mathbf{flat}}$ *is a succession of at most two arcs with a control modulus* $\|u(\cdot)\|$ *being either equal to 1 or to* 0*. More precisely, the thrust* $\|u(\cdot)\|T_{\max}$ *is*

- *either constant on* $[0, t_f]$ *and equal to* $T_{\max}$*,*
- *or of the type* $T_{\max} - 0$*,*
- *or of the type* $0 - T_{\max}$*.*

Proof. Let us first start with the following lemma.

Lemma 6. *If the modulus of the control* $\|u(\cdot)\|$ *has at least one switching on* $[0, t_f]$ *then* $p^0 \neq 0$*.*

Proof of Lemma 6. The argument goes by contradiction. If $p^0 = 0$ then $p_m(t_f) = 0$. Since $p_m(\cdot)$ is nondecreasing, it follows that $p_m(t) \leqslant 0$ for every $t \in [0, t_f]$. From Lemma 1, $\sqrt{p_{v_x}^2 + p_{v_h}(\cdot)^2}$ does not vanish identically on any subinterval, hence $\Phi(\cdot) = \frac{1}{m(\cdot)}\sqrt{p_{v_x}^2 + p_{v_h}(\cdot)^2} - \beta p_m(\cdot) > 0$ on $[0, t_f]$, and therefore $\|u(\cdot)\| = 1$ on $[0, t_f]$. This contradicts the assumption of having one switching. □

To prove the theorem, one has to prove that the strategy $T_{\max} - 0 - T_{\max}$ for the thrust cannot occur. The argument goes by contradiction. Assume

that the thrust $\|u(\cdot)\|T_{\max}$ is of this type, and denote by $t_1 < t_2$ the two switching times.

Let us first prove that the minimum of $\Phi(\cdot)$ is reached at $\bar{t} = \frac{p_{v_h}(0)}{p_h}$. Using (29), if $\dot{\Phi}(\bar{t}) = 0$ then there must hold $p_h p_{v_h}(\bar{t}) = 0$. Since $\Phi(\cdot)$ is not constant, one has $p_h \neq 0$, hence $p_{v_h}(\bar{t}) = 0$. Integrating the differential equation satisfied by $p_{v_h}(\cdot)$, one gets $\bar{t} = \frac{p_{v_h}(0)}{p_h}$. By definition, this minimum is reached within the interval $(0, t_f)$. In particular, we deduce that

$$0 < t_1 < \bar{t} = \frac{p_{v_h}(0)}{p_h} < t_2 < t_f. \tag{31}$$

On $[t_1, t_2]$, one has $\|u(t)\| = 0$, hence in particular $m(\cdot)$ and $p_m(\cdot)$ are constant on this interval, and thus $m(t_1) = m(t_2)$ and $p_m(t_1) = p_m(t_2)$. Since the function $t \mapsto \Phi(t) = \frac{\sqrt{p_{v_x}^2 + (p_{v_h}(0) - p_h t)^2}}{m(t)} - \beta p_m(t)$ vanishes by definition at t_1 and t_2, it follows that

$$\sqrt{p_{v_x}^2 + (p_{v_h}(0) - p_h t_1)^2} = \sqrt{p_{v_x}^2 + (p_{v_h}(0) - p_h t_2)^2},$$

and hence $|p_{v_h}(0) - p_h t_1| = |p_{v_h}(0) - p_h t_2|$. Since $t_1 \neq t_2$, we infer that $t_2 = 2\frac{p_{v_h}(0)}{p_h} - t_1$.

Note that the latter equality means that the graph of $\Phi(\cdot)$ on the interval $[t_1, t_2]$ is symmetric with respect to the point $\bar{t} = \frac{p_{v_h}(0)}{p_h}$ where the minimum is reached.

Using the fact that $H = 0$ along an extremal and that $p_x = 0$, one gets

$$p_h v_h(t) + \|u(t)\| T_{\max} \Phi(t) - g_0 p_{v_h}(t) = 0, \tag{32}$$

for every $t \in [0, t_f]$. In particular, at $t = t_f$, one gets $g_0 p_{v_h}(t_f) = T_{\max}\Phi(t_f) > 0$, which implies $p_{v_h}(t_f) > 0$. Since $p_{v_h}(\cdot)$ is affine and vanishes at $\bar{t}$, we get that $p_h < 0$, $p_{v_h}(\cdot) \leqslant 0$ on $[0, \bar{t}]$ and $\geqslant 0$ on $[\bar{t}, t_f]$.

Now, note that $u_h(\cdot)$ has the same sign as $p_{v_h}(\cdot)$ during the thrust arcs, so it is negative on $[0, t_1]$ and positive on $[t_2, t_f]$. In particular, we have that

$$\forall t \in [0, t_2], \dot{v}_h(t) \leqslant -g_0.$$

We can also deduce that $v_{h0} > 0$ otherwise we couldn't have $v_h(\bar{t}) = 0$. Let $\tilde{t} = min(v_{h0}/g_0, t_f) \in [\bar{t}, t_f]$. Then

$$\begin{aligned} h(\tilde{t}) &\leqslant h_0 + \int_0^{\tilde{t}} (v_{h0} - g_0 t)\ dt + max\left(0, \int_0^{\tilde{t} - v_{h0}/g_0} (v_{h0} - g_0 t)\ dt\right) \\ &\leqslant h_0 + \frac{v_{h0}^2}{2g_0}. \end{aligned}$$

Furthermore, $u_h(\cdot)$ is positive and increasing on $[t_2, t_f]$, and $v_{hf} = 0$, so $v_h(\cdot)$ is non positive on $[\bar{t}, t_f]$. We then necessarily have

$$\forall t \in [\bar{t}, t_f],\ h(t) \geqslant h_f$$

In particular, we need $h(\tilde{t}) \geqslant h_f$. This leads to

$$h_0 + \frac{v_{h0}^2}{2g_0} \geqslant h_f.$$

This raises a contradiction with assumption (28). □

Since the strategy where the thrust is maximal all along the flight is also a minimum time strategy and is not cost efficient, we next focus on the strategy $T_{\max} - 0$ (with one switching). Note however that the former strategy can be viewed as a particular case of the latter one. The study of the strategy $0 - T_{\max}$ is similar.

2.3 Refined analysis of the strategy $T_{\max} - 0$, and algorithmic procedure

Assume that we are in the case where the thrust has one switching, denoted t_1, with $0 < t_1 < t_f$, and is of the form $T_{\max} - 0$.

Lemma 7. *There holds* $t_f = \frac{p_{v_h}(0)}{p_h}$ *and* $p_h v_h(t_1) + p_h g_0 t_1 = g_0 p_{v_h}(0)$. *Moreover,* $p_h > 0$, $sign(p_{v_x}) = sign(v_{xf} - v_{x0})$ *and* $p_{v_h}(0) > 0$.

Proof. First of all, note that the identity (32) still holds in that case.

On $[t_1, t_f]$, one has $\|u(\cdot)\| = 0$, and hence, from (32), $p_h v_h(\cdot) = g_0 p_{v_h}(\cdot)$. Taking $t = t_f$ yields $p_{v_h}(t_f) = 0$ since $v_h(t_f) = 0$ from the boundary conditions (27). Since $p_{v_h}(t) = -p_h t + p_{v_h}(0)$, it follows that $t_f = \frac{p_{v_h}(0)}{p_h}$ (note that $p_h \neq 0$ from Lemma 2). Moreover, for every $t \in [t_1, t_f]$, one has $\dot{v}_h(t) = -g_0$, hence $v_h(t) = v_h(t_1) - g_0(t - t_1)$. Since $p_{v_h}(t) = -p_h t + p_{v_h}(0)$, we infer that $p_h v_h(t_1) + p_h g_0 t_1 = g_0 p_{v_h}(0)$.

Since $t_f = \frac{p_{v_h}(0)}{p_h}$, necessarily p_h and $p_{v_h}(0)$ have the same sign. Let us prove by contradiction that, actually, $p_{v_h}(0) > 0$. If $p_{v_h}(0) < 0$ then, since the function $p_{v_h}(\cdot)$ is affine and since $p_{v_h}(t_f) = 0$, there should hold $p_{v_h}(t) \leqslant 0$ for every $t \in [0, t_f]$. Hence, $u_h(t) < 0$ and $\dot{v}_h(t) \leqslant -g_0$ on $[0, t_f]$. Integrating, we would obtain $h(t) \leqslant h_0 + v_{h0} t - \frac{g_0}{2} t^2 \leqslant h_0 + \frac{v_{h0}^2}{2g_0}$. At the final time t_f, this would contradict (28). We thus conclude that $p_h > 0$ and $p_{v_h}(0) > 0$.

For every $t \in [0, t_1[$, one has $\dot{v}_x(t) = \frac{T_{\max}}{m(t)} u_x(t)$ with $u_x(t)$ having the same sign than p_{v_x}, and for every $t \in]t_1, t_f]$, one has $\|u(\cdot)\| = 0$ and hence $v_x(\cdot)$ remains constant. Then, we directly have $sign(v_{xf} - v_{x0}) = sign(u_x(\cdot)) = sign(p_{v_x})$. □

Lemma 7 enables to significantly simplify the application of the single shooting method to that case. We next explain the construction of this simplified algorithmic procedure. A priori, when applying the single shooting method, we have 5 unknowns, namely:

- the initial adjoint vector $(p_h, p_{v_x}, p_{v_h}(0), p_m(0), p^0)$, defined up to a multiplicative scalar[3] (note that $p^0 \neq 0$ from Lemma 6),
- the final time t_f,

and 5 equations:

$$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0,\ p_m(t_f) = -p^0,\ H(t_f) = 0.$$

Recall that the adjoint vector (completed with p^0) is defined up to a multiplicative scalar, and instead of choosing the usual normalization $p^0 = -1$, since there holds $p_h > 0$, we rather choose to normalize the adjoint vector so that $p_h = 1$. Since the variable p^0 is only used here to tune the equation $p_m(t_f) = -p^0$, we can therefore forget about the variable p^0 and the equation $p_m(t_f) = -p^0$. This is a first simplification.

Now there remain 4 unknowns, $(p_{v_x}, p_{v_h}(0), p_m(0))$ and t_f, and 4 equations:

$$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0,\ H(t_f) = 0.$$

Note that the knowledge of the value of $p_m(0)$ permits to determine the switching function $\Phi(\cdot)$ and hence the switching time t_1. It is therefore possible to replace the unknown $p_m(0)$ with the new unknown t_1. Hence, from now on we have the 4 unknowns $(p_{v_x}, p_{v_h}(0), t_1, t_f)$, and the 4 previous equations.

Taking into account the fact that $p_h = 1$, one can see from the previous computations and from Lemma 7 that the system of equations

$$v_h(t_f) = 0,\ H(t_f) = 0,$$

is equivalent to the system of equations

$$t_f = p_{v_h}(0),\ v_h(t_1) + g_0 t_1 = g_0 p_{v_h}(0).$$

[3]Note that we will not use, here, the usual normalization of the adjoint vector leading to choose $p^0 = -1$ whenever $p^0 \neq 0$.

The final time t_f being then directly determined by the value of $p_{v_h}(0)$, we can reduce the problem to 3 unknowns $(p_{v_x}, p_{v_h}(0), t_1)$ and 3 equations:

$$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_1) + g_0 t_1 = g_0 p_{v_h}(0).$$

We finally end up with the following simplified algorithmic procedure.

Algorithmic procedure

- Initialization: choose some values $p_{v_x} > 0$ and $p_{v_h}(0) > 0$.
- Integrate numerically the differential equations in order to compute $(h(t), v_x(t), v_h(t))$ on $[0, t_1]$, and stop the integration at t_1 satisfying $v_h(t_1) + g_0 t_1 = g_0 p_{v_h}(0)$
- Set $t_f = p_{v_h}(0)$. On the interval $[t_1, t_f]$, one computes explicitly

  $$v_x(t) = \text{Cste} = v_x(t_1),\ h(t) = h(t_1) + v_h(t_1)(t - t_1) - \frac{g_0}{2}(t - t_1)^2.$$

  Then, solve the system

  $$h(t_f) = h_f,\ v_x(t_f) = v_{xf},$$

  with a Newton-like method.

Note that it is possible to compute explicit expressions of $h(t)$ and of $v_x(t)$ on the whole interval $[0, t_f]$, however it happens that, from the numerical point of view, this does not save time and the procedure described above is more efficient.

The above algorithm is very easy to carry out and happens to be extremely efficient. The convergence is obtained instantaneously for almost every random choice of initialized values of $p_{v_x} > 0$ and $p_{v_h}(0) > 0$ (on a standard MacBook Pro dual core machine, in `Matlab`).

This simple code is used as a first step in our continuation procedure described in Section 1.4.

The algorithm provides a solution in terms of $(t_1, t_f, p_{v_x}, p_{v_h}(0))$. Then, the unknowns $(t_f, \bar{p}_x(0), \bar{p}_h(0), \bar{p}_{v_x}(0), \bar{p}_{v_h}(0), \bar{p}_m(0))$ of the shooting function (24) associated with this simplified model are computed by

$$\begin{aligned}
&\bar{p}_x(0) = 0,\ \bar{p}_h(0) = \frac{m_0 - \beta T_{\max} t_1}{\beta \sqrt{p_{v_x}^2(0) + (p_{v_h}(0) - t_1)^2}},\\
&\bar{p}_{v_h}(0) = p_{v_h}(0)\bar{p}_h(0),\ \bar{p}_{v_x}(0) = p_{v_x}(0)\bar{p}_{v_h}(0),\\
&\bar{p}_m(0) = \frac{1}{\beta T_{\max}} \left(\bar{p}_h(0) v_{h0} + \frac{T_{\max}}{m_0} \sqrt{\bar{p}_{v_x}^2(0) + \bar{p}_{v_h}^2(0)} - g_0 \bar{p}_{v_h}(0) \right).
\end{aligned} \tag{33}$$

3 Numerical simulations

In this section we provide numerical simulations of the algorithmic procedure described in Section 1.4, which consists in solving, in a continuation process, a series of shooting problems initialized with the simple algorithm introduced in Section 2.3. Since the latter code converges without any tricky initialization, we thus get a way of solving $(\mathbf{OCP})_{0,0}$ without any a priori knowledge on the optimal solution.

Consider $(\mathbf{OCP})_{0,0}$ with the initial conditions

$$x_0 = 0,\ h_0 = 200\ km,\ v_0 = 5.5\ km/s,\ \gamma_0 = 2^\circ,\ m_0 = 40000\ kg,$$

and the final conditions and problem parameters

$$h_f = 800\ km,\ v_f = 7.5\ km/s,\ \gamma_f = 0,\ T_{\max} = 170\ kN,\ Isp = 450\ s.$$

Note that this corresponds nearly to a circular final orbit ($v_f \approx \sqrt{\mu/(r_T + h_f)}$, $\gamma_f = 0$), so one might want to introduce the transversality conditions (11) once $(\mathbf{OCP})_{1,1}$ is solved.

Using the code developed in Section 2.3 and the transformation (33), we directly get the zero of the shooting function associated to $(\mathbf{OCP})_{0,0}$

$$(t_{f,(0,0)}, p_{0,0}(0)) \approx (1433\ s, 0, 0.755, 72.688, 1082.328, -0.137),$$

associated to the final mass $m_{0,0}(t_f) \approx 1676\ kg$.

This solution is used as the starting point to the continuation from $(\mathbf{OCP})_{0,0}$ to $(\mathbf{OCP})_{1,0}$, the problem for the flat Earth model with variable gravity. This leads to the following zero of the shooting function

$$(t_{f,(1,0)}, p_{1,0}(0)) \approx (1483\ s, 0, 3.851, 69.818, 2198.465, -0.236),$$

associated with the final mass $m_{1,0}(t_f) \approx 1505\ kg$.

We should note that $m_{1,0}(t_f) < m_{0,0}(t_f)$ seems counterintuitive since the variable gravity is always lower than g_0. However, the gravity does not only tend to decelerate the vehicle; it also helps to flatten the trajectory in order to reach $v_h(t_f) = 0$.

At this point, we switch from $(\mathbf{OCP})_{1,0}$ with free final time t_f to $(\mathbf{OCP})_{1,0}$ with a fixed final time. As mentioned in Section 1.4 this simply means that the shooting function has one less unknown and thus one less relation to satisfy at the end point of the extremal flow. This final time is $t_f \approx 1483\ s$. The solution $p_{1,0}(0)$ is then used to initialize the continuation from $(\mathbf{OCP})_{1,0}$ to $(\mathbf{OCP})_{1,1}$. This leads to the solution

$$p_{1,1}(0) \approx (0, 12.219, 6824.539, 5230.033, 0.310),$$

associated with the final mass $m_{1,1}(t_f) \approx 18922\ kg$.

First we notice that the final mass of $(\mathbf{OCP})_{1,1}$ is far better than the ones of $(\mathbf{OCP})_{0,0}$ and $(\mathbf{OCP})_{1,0}$. This could be expected because in $(\mathbf{OCP})_{1,1}$ the vehicle can use the centripetal forces that allow it to park on an intermediary orbit in between two thrust arcs.

Figure 2 shows the zero path of the shooting function from $(\mathbf{OCP})_{1,0}$ to $(\mathbf{OCP})_{1,1}$.

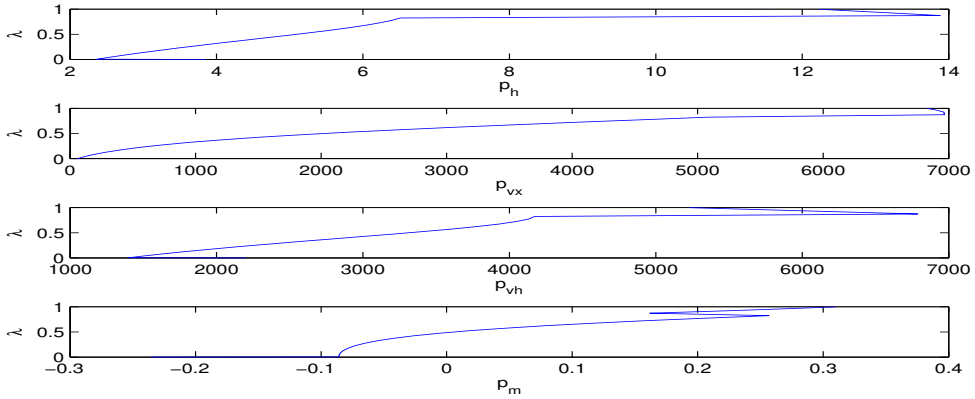

Figure 2: evolution of the shooting function unknowns ($p_h, p_{v_x}, p_{v_h}, p_m$ (abscissa) with respect to homotopic parameter λ_2 (ordinate).

We can see that this zero path does not seem continuous. This is explained by the appearance of a new thrust arc during the continuation, around $\lambda_2 \approx 0.01$, that is right at the start of the continuation. However, this does not prevent the continuation to converge.

Figure 3 compares the trajectory and control strategy of $(\mathbf{OCP})_{1,0}$ and $(\mathbf{OCP})_{1,1}$. On this figure we see that the solution of $(\mathbf{OCP})_{1,0}$ is clearly not acceptable because its altitude becomes negative. However $(\mathbf{OCP})_{1,0}$ is only a fictive problem and there is no need to only accept collision free trajectory until we solve $(\mathbf{OCP})_{1,1}$. The main difference between the two control strategies is that $(\mathbf{OCP})_{1,0}$ (and $(\mathbf{OCP})_{0,0}$) only has one thrust arc while $(\mathbf{OCP})_{1,1}$ has two. Furthermore, the fact that the mass consumption is directly proportional to the thrust duration explains that $m_{1,1}(t_f)$ could be expected to be better than $m_{1,0}(t_f)$.

From the solution of $(\mathbf{OCP})_{1,1}$, it is possible to find other solutions with

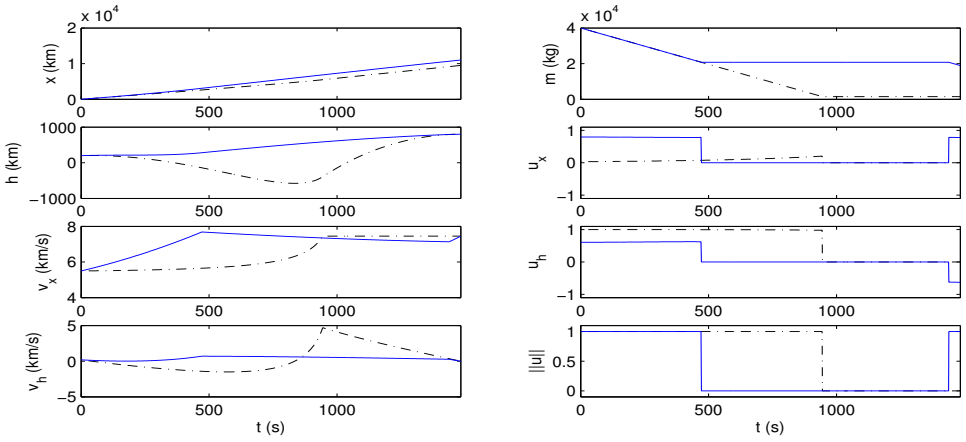

Figure 3: Trajectory and control strategy of $(\mathbf{OCP})_{1,0}$ (dashed) and $(\mathbf{OCP})_{1,1}$ (plain).

different final time t_f. This can help to find solutions with better final mass than the one obtained here. For example, we can easily find a solution for $t_f = 2000\ s.$ with a corresponding final mass $m_f \approx 20050\ kg.$

Also, since our final orbit is not strictly circular it can be interesting to introduce the final condition of the form (7) rather than (6). This can be done thanks to another continuation directly on the final conditions and transversality conditions.

The whole procedure is time-efficient and only takes a few seconds on a standard personal computer, without any coding trick.

4 Conclusion

We have given an algorithmic procedure to solve the problem of minimization of the consumption for the coplanar orbit transfer problem by a shooting method approach, without any a priori knowledge on the optimal solution and hence on the way to initialize the shooting method. Our trick relies on the preliminary remark that, when studying the same problem within a simplified flat Earth model with constant gravity, the optimal control problem can be explicitly solved, and the solution leads to an extremely efficient algorithm that does not need any tricky initialization. Based on that remark, we have proposed a continuous deformation of this simplified model to the initial model (up to some change of coordinates), introducing continuously

corrective terms into the flat Earth model. From the algorithmic point of view, the procedure then consists in solving a series of shooting problems, starting from the simplified flat Earth model which is easy to initialize, and ending up with the sought solution. The whole procedure is time-efficient and provides a way for bypassing the difficulty due to the initialization of the shooting method when it is applied directly to the initial problem.

Many questions remain open and from this point of view our work should be considered as preliminary. A first question is to investigate whether this procedure is systematically efficient, for any possible coplanar orbit transfer. Up to now we did not make any exhaustive tests, however it is very probable that one may encounter some difficulties, as in any continuation process, due to the intricate topology of the space of possible continuation paths, this space being not always arcwise connected. Indeed, the flat Earth model only has one thrust arc while the round Earth model has two or more. Another question is to extend our study to the three-dimensional case, the final objective for an enterprise as *Astrium Space Transportation* being to have available a reliable and efficient tool to realize any possible orbit transfer without having to spend much time on the initialization of the algorithm.

5 Appendix: action of a change of coordinates onto the adjoint vector

To understand how a change of coordinates acts onto the adjoint vector, it is useful to come back to the geometric meaning of the Pontryagin Maximum Principle, recalling its intrinsic character.

Let M (resp. N) be a smooth manifold of dimension n (resp. m). Consider on M the control system $\dot{x}(t) = f(x(t), u(t))$, where $f : M \times N \longrightarrow TM$ is smooth, TM is the usual tangent bundle, and the controls are bounded measurable functions taking their values in a subset U of N. Let M_0 and M_1 be two subsets of M. Consider the optimal control problem of determining a trajectory $x(\cdot)$ solution of the control system, associated with a control $u(\cdot)$ on $[0, t_f]$, so that $x(0) \in M_0$, $x(t_f) \in M_1$, and minimizing a cost function $C(t_f, u) = \int_0^{t_f} f^0(x(t), u(t))dt$, where $f^0 : M \times N \longrightarrow \mathbb{R}$ is smooth, and the final time t_f may be fixed or not. According to the Pontryagin Maximum Principle, if $x(\cdot)$ is optimal then there exists $p^0 \leqslant 0$ and an absolutely continuous mapping $p(\cdot)$ on $[0, t_f]$ (adjoint vector) satisfying $(p(\cdot), p^0) \neq (0, 0)$ and $p(t) \in T^*_{x(t)}M$, such that

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), p^0, u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)),$$

almost everywhere on $[0, t_f]$, where $H(x, p, p^0, u) = \langle p, f(x, u)\rangle + p^0 f^0(x, u)$ is the Hamiltonian, and $H(x(t), p(t), p^0, u(t)) = M(x(t), p(t), p^0)$, almost everywhere on $[0, t_f]$, where $M(x(t), p(t), p^0) = \max_{v\in U} H(x(t), p(t), p^0, v)$. If the final time t_f is not fixed, there holds moreover $M(x(t), p(t), p^0) = 0$, for every $t \in [0, t_f]$. If M_0 and M_1 (or just one of both) are regular submanifolds of M, then the adjoint vector can be chosen so that $p(0) \perp T_{x(0)}M_0$ and $p(t_f) \perp T_{x(t_f)}M_1$ (transversality conditions).

Settled in such a way on a manifold, we recall that the Pontryagin Maximum Principle is *intrinsic*, i.e., its statement does not depend on the specific choice of coordinates. This intrinsic version has been proved e.g. in [1].

Let now M_1 (resp. N_1) be another smooth manifold of dimension n (resp. m), and let $\phi : M \to M_1$ (resp. $\psi : N \to N_1$) be a diffeomorphism. Then, it is well known in differential geometry that the differential $d\phi$ maps diffeomorphically the tangent bundle TM into TM_1, and that the transpose (also called adjoint) of its inverse ${}^t d\Phi^{-1}$ maps diffeomorphically the cotangent bundle T^*M into T^*M_1. From this remark and from the intrinsic character of the Pontryagin Maximum Principle, we derive the following claim.

Let $x_1(t) = \phi(x(t))$ and $u_1(t) = \psi(u(t))$. The trajectory $x_1(\cdot)$, associated to the control $u_1(\cdot)$, is solution of the control system $\dot{x}_1(t) = f_1(x_1(t), u_1(t)) = d\phi(x(t)).f(\phi^{-1}(x_1(t), \psi^{-1}(u_1(t))$, and corresponds to $x(\cdot)$ via the change of coordinates ϕ on the state and ψ on the control. Then, the adjoint vector $p_1(\cdot)$ associated with the trajectory $x(\cdot)$ is given by

$$p_1(\cdot) = {}^t d\phi(x(\cdot))^{-1} p(\cdot). \tag{34}$$

The formula (34) may of course be proved directly, without any geometric insight, by using Cauchy uniqueness arguments.

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