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# Minimal Repair under Step-Stress Test

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## Abstract

In the one- and multi-sample cases, in the context of life-testing reliability experiments, we introduce minimal repair processes under a simple step-stress test, based on exponential distributions and an associated cumulative exposure model, and then develop likelihood inference for such a model.

**Keywords and Phrases:** Minimal repair times, Record values, Nonhomogeneous Poisson process, Maximum likelihood estimation, Conditional moment generating function, Gamma distribution, Mixtures.

## 1 Introduction

We consider the sequence of failure times of a technical system in the sense of minimal repair. In such a scheme, with respect to just one component, upon failure, this component will instantaneously be repaired, and by this, put into the condition immediately prior to its failure. The times to repair are considered to be low and so are neglected. As we will indicate below, the minimal repair model is also used as an approximate description of complex systems.

Let  $Z_i$  denote the operating time between the  $(i-1)^{\text{th}}$  and  $i^{\text{th}}$  repairs of some component, so that  $X_{(i)} = \sum_{j=1}^i Z_j$  corresponds to the  $i^{\text{th}}$  repair time,  $i \in \mathbb{N}$ . Moreover, let  $F$  be the continuous distribution function of  $Z_1$ , and furthermore  $Z_j$ , conditioned on  $\sum_{i=1}^{j-1} Z_i = z$ , be distributed according to  $\frac{F(\bullet+z)-F(z)}{1-F(z)}$ ,  $j = 2, 3, \dots$ , which is simply the distribution  $F$  truncated on the left at  $z$ .

Then, the minimal repair times form a Markov chain, and we find for  $x, t > 0$ ,

$$(1.1) \quad \begin{aligned} P(X_{(i+1)} - X_{(i)} > x | X_{(i)} = t_i) &= P(X_{(i+1)} > x + t_i | X_{(i)} = t_i) \\ &= P\left(Z_{i+1} > x \mid \sum_{j=1}^i Z_j = t_i\right) = \frac{1 - F(x + t_i)}{1 - F(t_i)}, \quad i \in \mathbb{N}. \end{aligned}$$

These transition probabilities coincide with those of record values; see, for example, Arnold et al. (1998, p. 11) and Kamps (1995, p. 32). Hence, the minimal repair times possess the same joint distribution as record values based on  $F$  (which are usually denoted by  $X_{L(1)}, X_{L(2)}, \dots$ ) as well as epoch times of some nonhomogeneous Poisson process (NHPP); see Gupta and Kirmani (1988).

A formal definition of the operating times  $Z_1, Z_2, \dots$  can be given based on a sequence  $(Y_i)_{i \in \mathbb{N}}$  of iid random variables with continuous distribution function  $F$  via

$$Z_1 = Y_1, \quad Z_j = F^{-1} \left[ F(Y_j) \left\{ 1 - F \left( \sum_{i=1}^{j-1} Z_i \right) \right\} + F \left( \sum_{i=1}^{j-1} Z_i \right) \right] - \sum_{i=1}^{j-1} Z_i, \quad j \geq 2;$$

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see Kamps (1995, p. 45).

Another interpretation of minimal repair is to say that, successively, the failed component is replaced by a component of equal age in contrast to the model of a renewal process. A general scheme of life tests with replacement of failed items and censoring has been presented by Fairbanks et al. (1982). For further details on minimal repair processes we also refer to Kirmani and Gupta (1995).

The term minimal repair was introduced first by Barlow and Hunter (1960). Other terms in vogue are “bad-as-old” model (Ascher 1968) and “age-persistence” model (Balaban 1978). For further details on terminology and pertinent references, we refer the readers to Ascher and Feingold (1984, p. 51-52, 83-86). Moreover, in this reference, several concrete applications are outlined while dealing with reliability of automobiles and aircrafts, and inferential procedures are also addressed. Some other applications can be found in Balaban and Singpurwalla (1984). For more recent papers on bad-as-old models and minimal repair models as well as for an overview, we refer to Finkelstein (2004), Kirmani and Gupta (1995), Langseth and Lindqvist (2006), Lugtigheid et al. (2008), Raqab and Asadi (2008), and Wang and Pham (2006). A different way of viewing a minimal repair process is to consider iterations of the so-called relevation transform (see Krakowski 1973, Baxter 1982, Lau and Prakasa Rao 1990, and Cramer and Kamps 2003). Hence, the model of record values, the analysis of occurrence times of some NHPP, the iterative use of the relevation transform as well as the minimal repair model all are equivalent distributionwise. Results derived for any of these models may therefore be used for the situation under consideration.

For a review of more advanced models for imperfect maintenance and repairable systems, one may consult Pham and Wang (1996), Wang and Pham (2006) and Lindqvist (2006), and also the references contained therein. For stochastic ordering results in the context of records and minimal repair, stochastic comparisons in terms of epoch times of NHPP, and some results on prediction intervals in terms of record values, one may refer to Khaledi and Shojaei (2007), Belzunce and Shaked (2001), Belzunce et al. (2003), and Raqab and Balakrishnan (2008).

Up to now, we have discussed successive minimal repair of a single particular component within a system. However, considering just one component which is successively minimally repaired is not a practical situation. Ascher and Feingold (1984, p. 51) have explained the use of a minimal repair modelling in their discussion on probabilistic modelling with NHPP. Understanding minimal repair as described above, we may likewise argue in terms of occurrence times of some corresponding NHPP as mentioned above. In particular, such a modelling may be appropriate when considering successive repairs of a system when only a very small fraction of components is either repaired or replaced by new components. In these cases, it is reasonable to assume that, upon restart, the reliability of the (complex) system after some (minimal) repair is approximately the same as it was immediately prior to its failure. We also refer to Love and Guo (1991) for a justification of using a bad-as-old model on the system level. Methods of accelerated life-testing (cf. Bagdonavicius and Nikulin 2002, Meeker and Escobar 1998, Nelson 1990), in particular step-stress methods, are a common approach in life-time experiments and are applied in general to reduce experimental time, when technical systems tend to have quite long life times. Under normal operating conditions, life-time tests would be time consuming and expensive. Therefore, an accelerated testing is adopted, wherein experimental units are exposed to increasing stress levels higher than the normal one. Moreover, since the number of minimal repair times or records that occur would be fairly small [see Arnold et al. (1998, p. 24)], a problem arises in making inference based on lifetimes observed from such experiments. We consider a Type-II censored experiment which terminates as soon as the  $r^{\text{th}}$  failure is observed for some  $r$ . Thus, the mean time to the  $r^{\text{th}}$  failure of the system under test may be quite large, even too large to complete the experiment within a reasonable period of time.

By applying the step-stress set-up, we develop a methodology to shorten experimental time in life-tests for (complex) systems, in the sense that, successively upon failures of components, the system is restored to operating status by repairing or replacing respective components and so

may be regarded as having been minimally repaired. For details on step-stress models, we refer to Nelson (1990), Gouno and Balakrishnan (2001), Bagdonavicius and Nikulin (2002), Gouno (2006), Balakrishnan et al. (2007) and Balakrishnan (2009). We consider here the simple step-stress set-up which means that there is only one change in the stress levels; however, the results can be generalized to the case of multiple stress levels as well.

Besides the consideration of a step-stress experiment as one form of a planned experiment, one may also think of situations facing an unavoidable change in the underlying life-time distribution of the test units during an experiment at some change point  $\tau$  for some technical reason.

Let  $X_{(1)}, X_{(2)}, \dots$  denote minimal repair times (or records, respectively) from some absolutely continuous distribution function  $F$  with density function  $f$ . Then, the joint density function of  $X_{(1)}, \dots, X_{(r)}$ , for some  $r \in \mathbb{N}$ , is given by [see Arnold et al. (1998, p. 10) and Kamps (1995, p. 31)]

$$f^{X_{(1)}, \dots, X_{(r)}}(x_1, \dots, x_r) = \left( \prod_{i=1}^{r-1} \frac{f(x_i)}{1 - F(x_i)} \right) f(x_r), \quad 0 \leq x_1 \leq \dots \leq x_r.$$

In the simple step-stress model, we assume to start the lifetime experiment at the first stress level with an underlying distribution function  $F_1$  and then to switch to the second stress level at some pre-fixed time  $\tau > 0$  with an underlying distribution function  $F_2$ , where

$$F_i(t) = 1 - \exp \left\{ - \frac{t - \mu}{\theta_i} \right\}, \quad t \geq \mu \geq 0, \quad \theta_i > 0, \quad i = 1, 2.$$

By applying the cumulative exposure model which chooses  $s$  such that  $F_1(\tau) = F_2(s)$ , we have the distribution function  $G$  to be [Nelson (1990)]

$$(1.2) \quad G(t) = \begin{cases} F_1(t), & \mu \leq t < \tau \\ F_2(s + t - \tau) = 1 - \exp \left\{ - \frac{t - \tau}{\theta_2} - \frac{\tau - \mu}{\theta_1} \right\}, & t \geq \tau \end{cases}$$

for some pre-fixed  $\tau > \mu$  and  $s = \frac{\theta_2}{\theta_1}(\tau - \mu) + \mu$ . The corresponding density function  $g$  is given by

$$g(t) = \begin{cases} \frac{1}{\theta_1} \exp \left\{ - \frac{t - \mu}{\theta_1} \right\}, & \mu \leq t < \tau \\ \frac{1}{\theta_2} \exp \left\{ - \frac{t - \tau}{\theta_2} - \frac{\tau - \mu}{\theta_1} \right\}, & t \geq \tau \end{cases},$$

and the hazard rate  $g/(1 - G)$  is consequently

$$\frac{g(t)}{1 - G(t)} = \begin{cases} \frac{1}{\theta_1}, & \mu \leq t < \tau \\ \frac{1}{\theta_2}, & t \geq \tau \end{cases}.$$

Here, we consider minimal repair times  $X_{(1)}, X_{(2)}, \dots$  in the step-stress context based on the cumulative exposure distribution  $G$  in (1.2), and for convenience we denote the survival function  $1 - G$  by  $\bar{G}$ .

In order to increase precision of inferential procedures, one may wish to combine different step-stress experiments which were conducted at different locations or at different times or even under different testing conditions. For this kind of meta-analysis we present MLEs to handle multi-sample situations.

The paper is organized as follows. Some preliminary results are stated and proved in Section 2. Section 3 presents the maximum likelihood estimates (MLEs) of the model parameters  $\theta_1$  and  $\theta_2$  in the case of one-sample simple step-stress minimal repair system, as well as their exact distributions, conditional on the fact that the MLEs exist. In Section 4, the multi-sample case is introduced and dealt with and the corresponding results are developed. Specifically, the MLEs  $\hat{\theta}_1$  and  $\hat{\theta}_2$  and their exact conditional distributions (by means of their conditional moment generating functions) are derived in this section.

## 2 Preliminaries

Let  $(X_{(i)})_{i \in \mathbb{N}}$  be a sequence of minimal repair times (or record values or epoch times of some NHPP) based on a continuous distribution function  $G$ . Moreover, let  $R$  denote the random number of minimal repairs before time  $\tau > 0$ , i.e.,

$$R = \rho \iff X_{(\rho)} \leq \tau < X_{(\rho+1)}, \quad \rho \in \mathbb{N}_0, \quad \text{with } X_{(0)} \equiv 0.$$

The following theorem states that  $R$  is distributed as Poisson. This result, in fact, holds true for any continuous distribution function  $G$ .

**Theorem 2.1.** *With the above notation, we have  $R \sim \text{Poisson}(-\log \bar{G}(\tau))$ , i.e.,*

$$P(X_{(j)} \leq \tau < X_{(j+1)}) = \frac{1}{j!} \bar{G}(\tau) (-\log \bar{G}(\tau))^j, \quad j \in \mathbb{N}_0, \quad \tau > 0.$$

*Proof.* Since record values possess a nice distributional structure, we use it to establish the required result. It is known (cf. Arnold et al., 1998) that  $X_{(j)} \stackrel{d}{=} G^{-1}(1 - \prod_{i=1}^j U_i)$ ,  $j \in \mathbb{N}$ , where  $U_i$  are iid  $\sim \text{Unif}(0, 1)$ . Moreover, if  $Y_{(j)}$  denote record values from a standard exponential distribution, then  $Y_{(j)} \stackrel{d}{=} \sum_{i=1}^j Z_i$ , where  $Z_i$  are iid  $\text{Exp}(1)$ . Using these, we find, with  $\tilde{t} = -\log \bar{G}(\tau)$ ,

$$\begin{aligned} P(X_{(j)} \leq \tau < X_{(j+1)}) &= P\left(G^{-1}\left(1 - \prod_{i=1}^j U_i\right) \leq \tau < G^{-1}\left(1 - \prod_{i=1}^{j+1} U_i\right)\right) \\ &= P(Y_{(j)} \leq -\log \bar{G}(\tau) < Y_{(j+1)}) = P\left(\sum_{i=1}^j Z_i \leq \tilde{t} < \sum_{i=1}^{j+1} Z_i\right) = P(N(\tilde{t}) = j) = \frac{\tilde{t}^j}{j!} e^{-\tilde{t}}, \quad j \in \mathbb{N}_0, \end{aligned}$$

where  $N(\cdot)$  denotes the Poisson-process associated with  $(\sum_{i=1}^j Z_i)_{j \in \mathbb{N}}$ . □

Furthermore, it is well-known [see Arnold et al. (1998, p. 11)] that for minimal repair times (or records)  $X_{(1)}, X_{(2)}, \dots$  based on some continuous distribution function  $G$ , the distribution of such quantities, conditioned on a previous one, is distributed as the unconditioned minimal repair times from  $G$  truncated on the left. More precisely,

$$(2.1) \quad P^{X_{(\rho+1)}, \dots, X_{(r)} | X_{(\rho)} = y} \equiv P^{Y_{(1)}, \dots, Y_{(r-\rho)}}, \quad \rho < r,$$

where  $Y_{(1)}, Y_{(2)}, \dots$  are minimal repair times based on distribution function  $H$  with  $H(z) = \frac{G(z) - G(y)}{1 - G(y)}$ ,  $z \geq y$ . A similar result, which is also valid for an arbitrary  $G$ , holds true by conditioning on the number  $\rho$  of minimal repairs up to time  $\tau$ , i.e., conditioning on the event  $X_{(\rho)} \leq \tau < X_{(\rho+1)}$ .

**Theorem 2.2.** *Let  $X_{(1)}, X_{(2)}, \dots$  be minimal repair times (or record values) based on some absolutely continuous distribution function  $G$  with density function  $g$ . Then,*

$$P^{X_{(r)} | X_{(\rho)} \leq \tau < X_{(\rho+1)}} \equiv P^{Y_{(r-\rho)}}, \quad \rho \leq r - 1,$$

where  $Y_{(r-\rho)}$  is the  $(r - \rho)^{\text{th}}$  minimal repair time based on  $G$  truncated on the left at  $\tau$ , i.e., based on distribution function  $H(z) = \frac{G(z) - G(\tau)}{1 - G(\tau)}$ ,  $z \geq \tau$ .

*Proof.* We first consider the joint distribution of  $X_{(r)}$  and the event  $X_{(\rho)} \leq \tau < X_{(\rho+1)}$ ,  $\rho < r - 1$ :

$$\begin{aligned} P(X_{(r)} \leq \tau < X_{(\rho+1)}) &= \int_{-\infty}^{\infty} P(X_{(r)} \leq x, X_{(\rho)} \leq \tau < X_{(\rho+1)} \mid X_{(\rho)} = y) dP^{X_{(\rho)}}(y) \\ &= \int_{-\infty}^{\tau} P(X_{(r)} \leq x, X_{(\rho+1)} > \tau \mid X_{(\rho)} = y) dP^{X_{(\rho)}}(y) = \int_{-\infty}^{\tau} \tilde{P}(y) dP^{X_{(\rho)}}(y), \end{aligned}$$

say. Then, upon using (2.1), we have

$$\tilde{P}(y) = P(Y_{(r-\rho)} \leq x, Y_{(1)} > \tau) = \int_{\tau}^x \int_{\tau}^v f^{Y_{(1)}, Y_{(r-\rho)}}(u, v) du dv;$$

since the joint density of  $Y_{(1)}$  and  $Y_{(r-\rho)}$  is [Arnold et al. (1998, p. 11) and Kamps (1995, p. 68)]

$$\begin{aligned} f^{Y_{(1)}, Y_{(r-\rho)}}(u, v) &= \frac{1}{(r-\rho-2)!} \left( \log \frac{\bar{H}(u)}{\bar{H}(v)} \right)^{r-\rho-2} \frac{h(u)}{\bar{H}(u)} h(v) \\ &= \frac{1}{(r-\rho-2)!} \left( \log \frac{\bar{G}(u)}{\bar{G}(v)} \right)^{r-\rho-2} \frac{g(u)}{\bar{G}(u)} \frac{g(v)}{\bar{G}(v)}, \end{aligned}$$

we obtain

$$\tilde{P}(y) = \frac{1}{(r-\rho-1)!} \frac{1}{\bar{G}(y)} \int_{\tau}^x \left( \log \frac{\bar{G}(\tau)}{\bar{G}(v)} \right)^{r-\rho-1} g(v) dv$$

Thus, by interchanging the integrals, we find

$$\begin{aligned} &\int_{-\infty}^{\tau} \tilde{P}(y) dP^{X_{(\rho)}}(y) \\ &= \frac{1}{(\rho-1)!(v-\rho-1)!} \int_{-\infty}^{\tau} \int_{\tau}^x \left( \log \frac{\bar{G}(\tau)}{\bar{G}(v)} \right)^{r-\rho-1} g(v) \frac{1}{\bar{G}(y)} (-\log \bar{G}(y))^{\rho-1} g(y) dv dy \\ &= \frac{(-\log \bar{G}(\tau))^{\rho}}{\rho!(r-\rho-1)!} \int_{\tau}^x \left( \log \frac{\bar{G}(\tau)}{\bar{G}(v)} \right)^{r-\rho-1} g(v) dv. \end{aligned}$$

From this expression and Theorem 2.1, the distribution function of  $X_{(r)}$ , conditioned on  $R = \rho$ , is

$$P(X_{(r)} \leq x \mid X_{(\rho)} \leq \tau < X_{(\rho+1)}) = \frac{1}{(r-\rho-1)!} \frac{1}{\bar{G}(\tau)} \int_{\tau}^x \left( \log \frac{\bar{G}(\tau)}{\bar{G}(v)} \right)^{r-\rho-1} g(v) dv$$

and the conditional density is

$$f^{X_{(r)} \mid X_{(\rho)} \leq \tau < X_{(\rho+1)}}(x) = \frac{1}{(r-\rho-1)!} \left( -\log \frac{\bar{G}(x)}{\bar{G}(\tau)} \right)^{r-\rho-1} \frac{g(x)}{\bar{G}(x)}, \quad x > \tau,$$

which is incidentally the same as the density of the  $(r-\rho)^{\text{th}}$  minimal repair time based on  $G$  truncated on the left at  $\tau > 0$ .

If  $\rho = r - 1$ , then by the Markovian property of records [see Arnold et al. (1998) and Kamps (1995, p. 32)], we have

$$\tilde{P}(y) = P(\tau < X_{(\rho+1)} \leq x \mid X_{(\rho)} = y) = \frac{G(x) - G(y)}{1 - G(y)} - \frac{G(\tau) - G(y)}{1 - G(y)} = \frac{G(x) - G(\tau)}{1 - G(y)},$$

and consequently

$$\begin{aligned} P(X_{(r)} \leq x, X_{(r-1)} \leq \tau < X_{(r)}) &= \frac{1}{(r-2)!} \int_{-\infty}^{\tau} \frac{G(x) - G(\tau)}{1 - G(y)} (-\log[1 - G(y)])^{r-2} g(y) dy \\ &= \frac{1}{(r-1)!} [G(x) - G(\tau)] [-\log \bar{G}(\tau)]^{r-1}. \end{aligned}$$

Thus, we obtain

$$P(X_{(r)} \leq x \mid X_{(r-1)} \leq \tau < X_{(r)}) = \frac{G(x) - G(\tau)}{1 - G(\tau)}, \quad x > \tau,$$

which is same as the distribution function of  $Y_1 \equiv Y_{(1)}$  based on  $G$  truncated on the left at  $\tau$ . Hence, the theorem.  $\square$

Based on the cumulative exposure model and underlying exponential distributions, the conditional distribution of  $X_{(r)}$ , given  $R = r$ , turns out to be a power function distribution, which does not depend on the model parameters  $\theta_1$  and  $\theta_2$ .

**Lemma 2.3.** *Let the minimal repair times  $X_{(1)}, X_{(2)}, \dots$  be based on  $G$  in (1.2). Then, we have*

$$P(X_{(j)} \leq x \mid X_{(j)} \leq \tau < X_{(j+1)}) = \left( \frac{x - \mu}{\tau - \mu} \right)^j, \quad j \in \mathbb{N}, \quad x \leq \tau.$$

*Proof.* For  $x \leq \tau$ , upon using the expression of the joint density of two records in Arnold et al. (1998, p. 11) or Kamps (1995, p. 68), we have

$$\begin{aligned} P(X_{(j)} \leq x \mid X_{(j)} \leq \tau < X_{(j+1)}) &= \frac{1}{P(R = j)} P(X_{(j)} \leq x, X_{(j+1)} > \tau) \\ &= \frac{1}{P(R = j)} \int_{\mu}^x \int_{\tau}^{\infty} \frac{1}{(j-1)!} \frac{g(u)}{\bar{G}(u)} (-\log \bar{G}(u))^{j-1} g(v) dv du = \frac{1}{P(R = j)} \frac{(x - \mu)^j}{j! \theta_1^j} \exp \left\{ -\frac{\tau - \mu}{\theta_1} \right\}. \end{aligned}$$

Then, the assertion follows readily by applying Theorem 2.1.  $\square$

### 3 One-sample case

We now suppose that we have a sample of observations of minimal repair times  $X_{(1)}, \dots, X_{(r)}$  based on distribution function  $G$  in (1.2). Given a number of  $\rho$  observations before time  $\tau$ , i.e., given  $R = \rho$ ,  $1 \leq \rho < r$ , the likelihood function becomes

$$L(\theta_1, \theta_2) = \frac{1}{\theta_1^\rho \theta_2^{r+\rho}} \exp \left\{ -\frac{x_r - \tau}{\theta_2} - \frac{\tau - \mu}{\theta_1} \right\}, \quad \mu \leq x_1 < \dots < x_\rho \leq \tau < x_{\rho+1} < \dots < x_r.$$

The above likelihood function yields the conditional MLEs as given in the following theorem.

**Theorem 3.1.** *For the exponential cumulative exposure model in (1.2), the conditional MLEs, conditioned on  $R = \rho$  for some  $1 \leq \rho < r$ , of  $\theta_1$  and  $\theta_2$  are given by*

$$(3.1) \quad \hat{\theta}_1 = \frac{\tau - \mu}{\rho} \quad \text{and} \quad \hat{\theta}_2 = \frac{x_r - \tau}{r - \rho}.$$



Obviously,  $\hat{\theta}_1$  depends only on the number of minimal repairs up to time  $\tau$ ; hence, the distribution of  $\frac{\tau-\mu}{R}$ , conditioned on  $1 \leq R \leq \rho - 1$ , is discrete. Further, as in the case of a sample of records based on an exponential distribution, the MLE of  $\theta_2$  only depends on the largest observation  $x_r$  (cf. Arnold et al. 1998, pp. 122-123).

*Remark 3.2.* (i) Given  $R = r$ , i.e., all observations are smaller than  $\tau$ , the likelihood function is independent of  $\theta_2$ . Hence, a MLE of  $\theta_2$  does not exist. The MLE of  $\theta_1$  in this case is given by  $\hat{\theta}_1 = \frac{x_r - \mu}{r}$ ;

(ii) Given  $R = 0$ , i.e., all observations are larger than  $\tau$ , the likelihood function is a monotonic increasing function of  $\theta_1$ , and so MLE of  $\theta_1$  does not exist. The MLE of  $\theta_2$  in this case is given by  $\hat{\theta}_2 = \frac{x_r - \tau}{r}$ .

Now, we shall derive the distributions of  $\hat{\theta}_1$  and  $\hat{\theta}_2$ , conditioned on  $R \in \{1, \dots, r-1\}$ , which simply means conditioned on the event that the MLEs  $\hat{\theta}_1$  and  $\hat{\theta}_2$  both exist. For this purpose, let us first denote

$$(3.2) \quad p_\rho = P(R = \rho) = \frac{1}{\rho!} \bar{G}(\tau) (-\log \bar{G}(\tau))^\rho = \frac{1}{\rho!} \left(\frac{\tau - \mu}{\theta_1}\right)^\rho \exp\left\{-\frac{\tau - \mu}{\theta_1}\right\}.$$

Then,  $P(1 \leq R \leq r-1) = \sum_{\rho=1}^{r-1} p_\rho = p$ , say.

*Remark 3.3.* Let  $x_\rho = \frac{\tau - \mu}{\rho}$ ,  $1 \leq \rho \leq r-1$ . Then,

$$P(\hat{\theta}_1 = x_\rho | 1 \leq R \leq r-1) = \frac{1}{p} \sum_{\rho=1}^{r-1} P(\hat{\theta}_1 = x_\rho, R = \rho) = \frac{p_\rho}{p}, \quad 1 \leq \rho \leq r-1.$$

The conditional distribution of  $\hat{\theta}_2$  turns out to be a mixture of gamma distributions as presented below.

**Theorem 3.4.** *The conditional distribution of  $\hat{\theta}_2$  in (3.1) is given by*

$$P(\hat{\theta}_2 \leq x | 1 \leq R \leq r-1) = \sum_{\rho=1}^{r-1} \frac{p_{r-\rho}}{p} F^{\Gamma(\rho, \frac{\theta_2}{\rho})}(x), \quad x > 0,$$

with  $p$  and  $p_j$ ,  $1 \leq j \leq r-1$ , as defined in (3.2), where  $F^{\Gamma(n, \theta)}$  denotes the distribution function of a gamma distribution with parameter  $n$  and  $\theta$ , i.e.,

$$F^{\Gamma(n, \theta)}(x) = 1 - e^{-x/\theta} \sum_{j=0}^{n-1} \frac{(x/\theta)^j}{j!}, \quad x > 0,$$

and the corresponding density function is

$$f^{\Gamma(n, \theta)}(x) = \frac{\theta^{-n}}{(n-1)!} x^{n-1} e^{-x/\theta}, \quad x > 0.$$

*Proof.* Since  $P(\hat{\theta}_2 \leq x | 1 \leq R \leq r-1) = \frac{1}{p} \sum_{\rho=1}^{r-1} P(\hat{\theta}_2 \leq x | R = \rho) P(R = \rho)$ , we have upon applying

Theorem 2.2:

$$P(\hat{\theta}_2 \leq x | 1 \leq R \leq r-1) = \frac{1}{p} \sum_{\rho=1}^{r-1} P(Y_{(r-\rho)} \leq x') P(R = \rho),$$

where  $x' = x(r - \rho) + \tau$ , and the minimal repair times  $Y_{(1)}, \dots, Y_{(r-1)}$  are based on the distribution function  $H(z) = \frac{G(z) - G(\tau)}{1 - G(\tau)}$ ,  $z \geq \tau$ , and corresponding density function  $h(z)$ .

For  $z \geq \tau$ , we find

$$\bar{H}(z) = \frac{\bar{G}(z)}{\bar{G}(\tau)} = \exp \left\{ -\frac{z - \tau}{\theta_2} \right\}.$$

Since the density of  $\frac{Y_{(r-\rho)} - \tau}{r - \rho}$  is given by

$$\frac{1}{(r - \rho - 1)!} (-\log \bar{H}(x'))^{r - \rho - 1} h(x') (r - \rho) = \frac{1}{(r - \rho - 1)!} \left( \frac{r - \rho}{\theta_2} \right)^{r - \rho} x^{r - \rho - 1} \exp \left\{ -\frac{r - \rho}{\theta_2} x \right\},$$

which is a gamma density with parameters  $r - \rho$  and  $\frac{\theta_2}{r - \rho}$ , and  $P(R = \rho) = p_\rho$ , we derive

$$P(\hat{\theta}_2 \leq x \mid 1 \leq R \leq r - 1) = \sum_{\rho=1}^{r-1} \frac{p_\rho}{p} F^{\Gamma(r-\rho, \frac{\theta_2}{r-\rho})}(x) = \sum_{\rho=1}^{r-1} \frac{p_{r-\rho}}{p} F^{\Gamma(\rho, \frac{\theta_2}{\rho})}(x),$$

which establishes the required result.  $\square$

It is worth noting that in the case of the usual step-stress experiment under cumulative exposure model and exponential lifetimes, the conditional distribution of the MLE of  $\theta_2$  is also a mixture of gamma distributions, but with different mixing coefficients; see Balakrishnan et al. (2007, p. 38).

From the conditional distributions of  $\hat{\theta}_1$  and  $\hat{\theta}_2$  presented above, conditional moments can be readily found. In particular,  $\hat{\theta}_2$  turns out to be conditionally unbiased.

*Remark 3.5.* We have:

$$\begin{aligned} (i) \quad E(\hat{\theta}_1 \mid 1 \leq R \leq r - 1) &= \frac{\tau - \mu}{p} \exp \left\{ -\frac{\tau - \mu}{\theta_1} \right\} \sum_{\rho=1}^{r-1} \frac{1}{\rho!} \left( \frac{\tau - \mu}{\theta_1} \right)^\rho, \\ (ii) \quad E(\hat{\theta}_2 \mid 1 \leq R \leq r - 1) &= \theta_2, \quad E(\hat{\theta}_2^2 \mid 1 \leq R \leq r - 1) = \frac{\theta_2^2}{p} \sum_{\rho=1}^{r-1} \frac{\rho+1}{\rho} p_{r-\rho}, \\ (iii) \quad Var(\hat{\theta}_2 \mid 1 \leq R \leq r - 1) &= \frac{\theta_2^2}{p} \sum_{\rho=1}^{r-1} \frac{p_{r-\rho}}{\rho}. \end{aligned}$$

## 4 Multi-sample case

The motivation for considering the multi-sample situation is two-fold. On the one hand, we may have data from two or more minimal repair experiments with possibly different numbers of observations and different change points, in which case the MLEs based on a larger number of observations is in the sense of a meta-analysis. On the other hand, in the planning phase of a life-testing experiment with minimal repair schemes, one may intend to control or minimize the non-existence probabilities of the MLEs since if there is no observation at one of the stress levels, the corresponding MLEs of  $\theta_1$  or  $\theta_2$  do not exist (see Theorem 3.1 and Remark 3.2).

In this multi-sample set-up, let

- $(X_{(i)}^{(k)})_{i \in \mathbb{N}}$  denote the minimal repair times in the  $k^{\text{th}}$  sample ( $1 \leq k \leq s$ ), where the samples are assumed to be independent; further, in the  $k^{\text{th}}$  sample, the stress level changes at time  $\tau_k$  and that these change points may be different for different samples;
- Consider Type-II censored samples with  $r_k$  observations in the  $k^{\text{th}}$  sample, and let the corresponding observations be denoted by  $(x_i^{(k)})_{1 \leq i \leq r_k}$ , for  $1 \leq k \leq s$ ;
- In each sample, analogous to the one-sample situation, the minimal repair times are based on the cumulative exposure distribution  $G$  in (1.2) with  $\tau$  replaced by  $\tau_k$  for the  $k^{\text{th}}$  sample.

Moreover, let  $R_{1k}$  denote the number of minimal repairs before time  $\tau_k$  in the  $k^{\text{th}}$  sample for  $1 \leq k \leq s$ , and let indicators  $I_k$  and  $J_k$  be defined by  $I_k = \mathbf{1}_{\{R_{1k} \geq 1\}}$  and  $J_k = \mathbf{1}_{\{R_{1k} < r_k\}}$ . A realization of  $R_{1k}$  is denoted by  $r_{1k}$ .

Obviously, the implication  $I_k = 0 \implies J_k = 1$  holds true, and that  $J_k = 0$  implies  $R_{1k} = r_k$ , and  $I_k = 0$  implies  $R_{1k} = 0$ .

**Theorem 4.1.** *In the multi-sample simple step-stress model as described above, with the minimal repair times in each sample being based on  $G$  in (1.2), with  $\tau$  replaced by  $\tau_k$  in the  $k^{\text{th}}$  sample, the unique MLEs of  $\theta_1$  and  $\theta_2$  are given by*

$$\hat{\theta}_1 = \left( \sum_{k=1}^s R_{1k} \right)^{-1} \sum_{k=1}^s \{(1 - J_k)(X_{(\tau_k)}^{(k)} - \tau_k) + \tau_k - \mu\}$$

if there exists a  $k_1 \in \{1, \dots, s\}$  with  $R_{1k_1} \geq 1$ , and

$$\hat{\theta}_2 = \left( \sum_{k=1}^s (r_k - R_{1k}) \right)^{-1} \sum_{k=1}^s J_k (X_{(\tau_k)}^{(k)} - \tau_k)$$

if there exists a  $k_2 \in \{1, \dots, s\}$  with  $R_{1k_2} \leq r_{k_2} - 1$ . Consequently, in the multi-sample case, the MLEs of both  $\theta_1$  and  $\theta_2$  exist iff there is at least one observation under the first stress level (i.e., one observation before one of the  $\tau_k$ 's) and at least one observation under the second stress level (i.e., one observation after one of the  $\tau_k$ 's).

*Proof.* Let  $L_1^{(k)}$ ,  $L_{1,2}^{(k)}$  and  $L_2^{(k)}$  denote the likelihood functions given  $R_{1k} = r_k$ ,  $1 \leq R_{1k} < r_k$  and  $R_{1k} = 0$ , respectively. In case the arguments of  $I_k$  and  $J_k$  are deterministic, we use the notations  $i_k$  and  $j_k$ , respectively. Then, the joint likelihood function is given by

$$\begin{aligned} L(\theta_1, \theta_2; x_{r_k}^{(k)}, 1 \leq k \leq s) &= \prod_{k=1}^s \left[ (1 - j_k) L_1^{(k)} + i_k j_k L_{1,2}^{(k)} + (1 - i_k) L_2^{(k)} \right] \\ &= \prod_{k=1}^s \left[ (1 - j_k) \frac{1}{\theta_1^{r_k}} \exp \left\{ -\frac{x_{r_k}^{(k)} - \mu}{\theta_1} \right\} \right. \\ &\quad \left. + i_k j_k \frac{1}{\theta_1^{r_k} \theta_2^{r_k - r_{1k}}} \exp \left\{ -\frac{x_{r_k}^{(k)} - \tau_k}{\theta_2} - \frac{\tau_k - \mu}{\theta_1} \right\} + (1 - i_k) \frac{1}{\theta_2^{r_k}} \exp \left\{ -\frac{x_{r_k}^{(k)} - \tau_k}{\theta_2} - \frac{\tau_k - \mu}{\theta_1} \right\} \right] \\ &= \theta_1^{-\sum_{k=1}^s r_{1k}} \theta_2^{-\sum_{k=1}^s (r_k - r_{1k})} \\ &\quad \times \prod_{k=1}^s \left[ (1 - j_k) \exp \left( -\frac{x_{r_k}^{(k)} - \mu}{\theta_1} \right) + (i_k j_k + 1 - i_k) \exp \left( -\frac{x_{r_k}^{(k)} - \tau_k}{\theta_2} - \frac{\tau_k - \mu}{\theta_1} \right) \right] \\ &= \theta_1^{-\sum_{k=1}^s r_{1k}} \theta_2^{-\sum_{k=1}^s (r_k - r_{1k})} \exp \left[ -\sum_{k=1}^s \left\{ (1 - j_k) \frac{x_{r_k}^{(k)} - \mu}{\theta_1} + j_k \left( \frac{x_{r_k}^{(k)} - \tau_k}{\theta_2} + \frac{\tau_k - \mu}{\theta_1} \right) \right\} \right] \end{aligned}$$

upon using the fact that  $i_k j_k + 1 - i_k = j_k$ . Hence, the log-likelihood function is given by

$$\begin{aligned} l(\theta_1, \theta_2; x_{r_k}^{(k)}, 1 \leq k \leq s) &= - \left( \sum_{k=1}^s r_{1k} \right) \log \theta_1 - \left( \sum_{k=1}^s (r_k - r_{1k}) \right) \log \theta_2 \\ &\quad - \sum_{k=1}^s \left\{ (1 - j_k) \frac{x_{r_k}^{(k)} - \mu}{\theta_1} + j_k \left( \frac{x_{r_k}^{(k)} - \tau_k}{\theta_2} + \frac{\tau_k - \mu}{\theta_1} \right) \right\} \end{aligned}$$

Equating the partial derivatives of  $l$  with respect to  $\theta_1$  and  $\theta_2$  to zero readily yields the (possible) MLEs of  $\theta_1$  and  $\theta_2$  to be

$$\hat{\theta}_1 = \frac{1}{\sum_{k=1}^s r_{1k}} \sum_{k=1}^s \left\{ (1 - j_k)(x_{r_k}^{(k)} - \tau_k) + (\tau_k - \mu) \right\} \quad \text{and} \quad \hat{\theta}_2 = \frac{\sum_{k=1}^s j_k(x_{r_k}^{(k)} - \tau_k)}{\sum_{k=1}^s (r_k - r_{1k})},$$

where  $\hat{\theta}_1$  exists iff  $\sum_{k=1}^s r_{1k} > 0$ , i.e., iff  $\exists k : r_{1k} \geq 1$ , and  $\hat{\theta}_2$  exists iff  $\sum_{k=1}^s (r_k - r_{1k}) > 0$ , i.e., iff  $\exists k : r_{1k} < r_k$ .

Inspecting the second derivatives, it can be shown that  $(\hat{\theta}_1, \hat{\theta}_2)$  is the global maximum if  $\exists k : r_{1k} \geq 1$  and if  $\exists j : r_{1j} < r_j$ , and so the MLEs become unique in this case.  $\square$

We shall now derive the conditional moment generating functions of the MLEs  $\hat{\theta}_1$  and  $\hat{\theta}_2$ . For simplicity in notation, we shall use  $R_1 = \sum_{k=1}^s R_{1k}$  to denote the total number of observations under the first stress level, and  $\tilde{r} = \sum_{k=1}^s r_k$  for the total number of observations in the  $s$  samples altogether.

Since the random variables  $R_{11}, \dots, R_{1s}$  are independent and, according to Theorem 2.1, distributed as Poisson, the distribution of  $R_1$  is

$$R_1 \sim \text{Poisson} \left( \sum_{k=1}^s -\log \bar{G}(\tau_k) \right).$$

Since  $\bar{G}(\tau_k) = \bar{F}_1(\tau_k) = \exp\{-\frac{\tau_k - \mu}{\theta_1}\}$ , we readily have  $R_1 \sim \text{Poisson}(\tilde{r})$  with  $\tilde{r} = \frac{1}{\theta_1} \sum_{k=1}^s (\tau_k - \mu)$ .

Hence,  $P(R_1 = l) = \frac{\tilde{r}^l}{l!} e^{-\tilde{r}} = \pi_l$ , say,  $l \in \mathbb{N}_0$ . Moreover, let  $\pi = \sum_{l=1}^{\tilde{r}-1} \pi_l$ .

The conditional moment generating functions of  $\hat{\theta}_1$  and  $\hat{\theta}_2$ , under the condition  $1 \leq R_1 \leq \tilde{r} - 1$  ensuring the existence of both  $\hat{\theta}_1$  and  $\hat{\theta}_2$ , are as given in the following theorem.

**Theorem 4.2.** *The conditional moment generating functions of the MLEs  $\hat{\theta}_1$  and  $\hat{\theta}_2$  presented in Theorem 4.1 are as follows (for  $t \geq 0$ ):*

(i)

$$E(e^{t\hat{\theta}_1} | 1 \leq R_1 \leq \tilde{r} - 1) = \frac{1}{\pi} \underbrace{\sum_{l=1}^{\tilde{r}-1} \sum_{r_{11}=0}^{r_1} \dots \sum_{r_{1s}=0}^{r_s}}_{\sum_{k=1}^s r_{1k}=l} \left( \prod_{k=1}^s P(R_{1k} = r_{1k}) \right) \times \exp \left\{ \frac{t}{l} \sum_{k=1}^s (\tau_k - \mu) \right\} \prod_{k=1}^s \left( \frac{r_k! \exp\{-\frac{t}{l}(\tau_k - \mu)\}}{(\tau_k - \mu)^{r_k-1} (-\frac{t}{l})^{r_k}} \right)^{1-j_k} A_k^{1-j_k},$$

$$\text{where } P(R_{1k} = r_{1k}) = \frac{\lambda_k^{r_{1k}}}{r_{1k}!} e^{-\lambda_k}, \lambda_k = \frac{\tau_k - \mu}{\theta_1} \text{ and } A_k = 1 - \exp \left\{ \frac{t}{l} (\tau_k - \mu) \sum_{j=0}^{r_k-1} \frac{(-\frac{t}{l}(\tau_k - \mu))^j}{j!} \right\};$$

$$(ii) \quad E(e^{t\hat{\theta}_2} | 1 \leq R_1 \leq \tilde{r} - 1) = \sum_{l=1}^{\tilde{r}-1} \frac{\pi_l}{\pi} h_l(t),$$

where  $h_l(t) = (1 - \frac{t\theta_2}{\tilde{r}-l})^{-(\tilde{r}-l)}$  is the moment generating function of  $\Gamma(\tilde{r}-l, \frac{\theta_2}{\tilde{r}-l})$ , the gamma distribution with parameters  $\tilde{r}-l$  and  $\frac{\theta_2}{\tilde{r}-l}$ .

*Proof.* (i) Recalling that  $\hat{\theta}_1 = \frac{1}{R_1} \sum_{k=1}^s \left\{ (1 - J_k)(X_{(r_k)}^{(k)} - \tau_k) + \tau_k - \mu \right\}$ , we have for  $l \in \{1, \dots, \tilde{r}-1\}$

$$E(e^{t\hat{\theta}_1} | R_1 = l) = \frac{1}{\pi_l} \underbrace{\sum_{r_{11}=0}^{r_1} \dots \sum_{r_{1s}=0}^{r_s}}_{\sum_{k=1}^s r_{1k}=l} \left( \prod_{k=1}^s P(R_{1k} = r_{1k}) \right) \exp \left\{ \frac{t}{l} \sum_{k=1}^s (\tau_k - \mu) \right\}$$

$$\times E \left( \prod_{k=1}^s \exp \left\{ \frac{t}{l} (1 - J_k) (X_{(r_k)}^{(k)} - \tau_k) \right\} \mid R_{11} = r_{11}, \dots, R_{1s} = r_{1s} \right),$$

where the latter expected value can be expressed as

$$\prod_{k=1}^s E \left( \exp \left\{ \frac{t}{l} (1 - J_k) (X_{(r_k)}^{(k)} - \tau_k) \right\} \mid R_{1k} = r_{1k} \right) = \prod_{k=1}^s E^{1-j_k} \left( \exp \left\{ \frac{t}{l} (X_{(r_k)}^{(k)} - \tau_k) \right\} \mid R_{1k} = r_{1k} \right).$$

Since

$$E^{1-j_k} \left( \exp \left\{ \frac{t}{l} (X_{(r_k)}^{(k)} - \tau_k) \right\} \mid R_{1k} = r_{1k} \right) = \left( e^{-\frac{t\tau_k}{l}} \right)^{1-j_k} E^{1-j_k} \left( e^{\frac{t}{l} X_{(r_k)}^{(k)}} \mid R_{1k} = r_k \right)$$

and

$$\begin{aligned} E \left( e^{\frac{t}{l} X_{(r_k)}^{(k)}} \mid R_{1k} = r_k \right) &= \int_{\mu}^{\tau_k - \mu} e^{\frac{t}{l} x} r_k \left( \frac{x - \mu}{\tau_k - \mu} \right)^{r_k - 1} \frac{1}{\tau_k - \mu} dx = e^{\frac{t\mu}{l}} \frac{r_k}{(\tau_k - \mu)^{r_k - 1}} \int_0^{\tau_k - \mu} e^{\frac{t}{l} x} x^{r_k - 1} dx \\ &= e^{\frac{t\mu}{l}} \frac{r_k}{(\tau_k - \mu)^{r_k - 1}} \frac{(r_k - 1)!}{(-t/l)^{r_k}} \int_0^{\tau_k - \mu} \frac{(-t/l)^{r_k}}{(r_k - 1)!} x^{r_k - 1} e^{\frac{t}{l} x} dx = \frac{r_k! e^{\frac{t\mu}{l}}}{(\tau_k - \mu)^{r_k - 1} (-\frac{t}{l})^{r_k}} A_k, \end{aligned}$$

with  $A_k = 1 - e^{\frac{t}{l}(\tau_k - \mu)} \sum_{j=0}^{r_k - 1} \frac{(-\frac{t}{l}(\tau_k - \mu))^j}{j!}$ , we immediately find

$$\begin{aligned} E(e^{t\hat{\theta}_1} \mid R_1 = l) &= \frac{1}{P(R_1 = l)} \sum_{r_{11}=0}^{r_1} \dots \sum_{r_{1s}=0}^{r_s} \left( \prod_{k=1}^s P(R_{1k} = r_{1k}) \right) \\ &\quad \times \exp \left\{ \frac{t}{l} \sum_{k=1}^s (\tau_k - \mu) \right\} \prod_{k=1}^s \left( \frac{r_k! \exp(-\frac{t}{l}(\tau_k - \mu))}{(\tau_k - \mu)^{r_k - 1} (-\frac{t}{l})^{r_k}} \right)^{1-j_k} A_k^{1-j_k}. \end{aligned}$$

Thus the assertion follows by noting that

$$E(e^{t\hat{\theta}_1} \mid 1 \leq R_1 \leq \tilde{r} - 1) = \sum_{l=1}^{\tilde{r}-1} \frac{\pi_l}{\pi} E(e^{t\hat{\theta}_1} \mid R_1 = l).$$

(ii) Next, we have

$$\begin{aligned} E(e^{t\hat{\theta}_2} \mid R_1 = l) &= E \left( \exp \left\{ \frac{t}{\tilde{r} - l} \sum_{k=1}^s J_k (X_{(r_k)}^{(k)} - \tau_k) \right\} \mid R_1 = l \right) \\ &= \frac{1}{\pi_l} \sum_{r_{11}=0}^{r_1} \dots \sum_{r_{1s}=0}^{r_s} \left( \prod_{k=1}^s P(R_{1k} = r_{1k}) \right) \\ &\quad \times E \left( \exp \left\{ \frac{t}{\tilde{r} - l} \sum_{k=1}^s J_k (X_{(r_k)}^{(k)} - \tau_k) \right\} \mid R_{11} = r_{11}, \dots, R_{1s} = r_{1s} \right). \end{aligned}$$

Now, the latter expected value equals

$$\prod_{k=1}^s E \left( \exp \left\{ \frac{t}{\tilde{r} - l} J_k (X_{(r_k)}^{(k)} - \tau_k) \right\} \mid R_{1k} = r_{1k} \right) = \prod_{k=1}^s \left[ E \left( \exp \left\{ \frac{t}{\tilde{r} - l} (X_{(r_k)}^{(k)} - \tau_k) \right\} \mid R_{1k} = r_{1k} \right) \right]^{j_k}$$

$$= \prod_{k=1}^s \left( \exp \left\{ -\frac{t\tau_k}{\tilde{r}-l} \right\} \right)^{j_k} E^{j_k} \left( \exp \left\{ \frac{t}{\tilde{r}-l} X_{(r_k)}^{(k)} \right\} \mid R_{1k} = r_{1k} \right)$$

Upon applying Theorem 2.2, this conditional expected value equals  $E(\exp\{\frac{t}{\tilde{r}-l} Y_{(r_k-r_{1k})}\})$ , wherein the minimal repair time  $Y_{(r_k-r_{1k})}$  is based on  $H_k(x) = \frac{G(x)-G(\tau_k)}{1-G(\tau_k)}$ ,  $x \geq \tau_k$ . Hence, by the definition of  $G$  in (1.2), we have

$$H_k(x) = \frac{\exp(-\frac{\tau_k-\mu}{\theta_1}) - \exp(-\frac{x-\tau_k}{\theta_2} - \frac{\tau_k-\mu}{\theta_1})}{\exp(-\frac{\tau_k-\mu}{\theta_1})} = 1 - \exp\left(-\frac{x-\tau_k}{\theta_2}\right), \quad x \geq \tau_k.$$

Thus, for  $0 \leq t < \frac{1}{\theta_2}$ , the moment generating function of  $Y_{(r)}$  based on  $H$  is given by

$$E(e^{tY_{(r)}}) = \frac{1}{(r-1)!} \theta_2^{-r} \int_{\tau_k}^{\infty} e^{tx} (x-\tau_k)^{r-1} e^{-\frac{x-\tau_k}{\theta_2}} dx = \frac{e^{t\tau_k}}{(1-t\theta_2)^r}.$$

Using the fact that  $j_k(r_k - r_{1k}) = r_k - r_{1k}$ , we arrive at

$$E(e^{t\hat{\theta}_2} | R_1 = l) = \frac{1}{\pi_l} \underbrace{\sum_{r_{1,1}=0}^{r_1} \cdots \sum_{r_{1,s}=0}^{r_s}}_{\sum_{k=1}^s r_{1k}=l} \left( \prod_{k=1}^s P(R_{1k} = r_{1k}) \right) \left( 1 - \frac{t\theta_2}{\tilde{r}-l} \right)^{-(\tilde{r}-l)}.$$

Setting  $t = 0$  and observing that the latter factor does not depend on the summation variables any more, we obtain

$$E\left(e^{t\hat{\theta}_2} | 1 \leq R_1 \leq \tilde{r}-1\right) = \sum_{l=1}^{\tilde{r}-1} \frac{\pi_l}{\pi} E\left(e^{t\hat{\theta}_2} | R_1 = l\right) = \sum_{l=1}^{\tilde{r}-1} \frac{\pi_l}{\pi} \left( 1 - \frac{t\theta_2}{\tilde{r}-l} \right)^{-(\tilde{r}-l)}.$$

□

*Remark 4.3.* The conditional distribution of  $\hat{\theta}_1$  is discrete if  $j_k = 0$  for all  $k \in \{1, \dots, s\}$ ; otherwise, it is a continuous distribution. As in the one-sample situation, the distribution of  $\hat{\theta}_2$  is a mixture of gamma distributions with  $\pi_l/\pi$ ,  $1 \leq l \leq \tilde{r}-1$ , as the mixing proportions. From the form of the conditional moment generating function of  $\hat{\theta}_2$ , we immediately obtain its moments as given below.

$$\begin{aligned} \text{Remark 4.4.} \quad E\left(\hat{\theta}_2 | 1 \leq R_1 \leq \tilde{r}-1\right) &= \theta_2, \quad E\left(\hat{\theta}_2^2 | 1 \leq R_1 \leq \tilde{r}-1\right) = \theta_2^2 \sum_{l=1}^{\tilde{r}-1} \frac{\pi_l}{\pi} \left( 1 + \frac{1}{\tilde{r}-l} \right), \\ \text{Var}\left(\hat{\theta}_2 | 1 \leq R_1 \leq \tilde{r}-1\right) &= \theta_2^2 \sum_{l=1}^{\tilde{r}-1} \frac{1}{\tilde{r}-l} \frac{\pi_l}{\pi}. \end{aligned}$$

Finally, we compare the conditional variances of the MLEs of  $\theta_2$  in the one-sample case (viz.,  $\hat{\theta}_2^{(1)}$ ) with  $\tilde{r} = \sum_{k=1}^s r_k$  observations and change point  $\tau$  and in the multi-sample case (viz.,  $\hat{\theta}_2^{(s)}$ ) with  $s$  independent samples having  $r_1, \dots, r_s$  observations and change points  $\tau_1, \dots, \tau_s$ . Note that the comparison is proper in this case since the same number of observations are present in both sampling situations.

**Lemma 4.5.** *With the above notation, we have*

$$\text{Var}(\hat{\theta}_2^{(1)} | 1 \leq R \leq \tilde{r}-1) \geq \text{Var}(\hat{\theta}_2^{(s)} | 1 \leq R_1 \leq \tilde{r}-1) \iff \sum_{k=1}^s (\tau_k - \mu) \leq \tau - \mu.$$

*Proof.* With  $\lambda = \frac{\tau - \mu}{\theta_1}$ , we have from Remark 3.5 that

$$\text{Var} \left( \hat{\theta}_2^{(1)} \mid 1 \leq R \leq \tilde{r} - 1 \right) = \theta_2^2 \sum_{i=1}^{\tilde{r}-1} \frac{1}{\tilde{r}-i} \frac{\frac{\lambda^i}{i!}}{\sum_{j=1}^{\tilde{r}-1} \frac{\lambda^j}{j!}},$$

while from Remark 4.4 that

$$\text{Var} \left( \hat{\theta}_2^{(s)} \mid 1 \leq R_1 \leq \tilde{r} - 1 \right) = \theta_2^2 \sum_{i=1}^{\tilde{r}-1} \frac{1}{\tilde{r}-i} \frac{\frac{\tilde{\tau}^i}{i!}}{\sum_{j=1}^{\tilde{r}-1} \frac{\tilde{\tau}^j}{j!}}.$$

Hence,

$$\text{Var} \left( \hat{\theta}_2^{(1)} \mid 1 \leq R \leq \tilde{r} - 1 \right) - \text{Var} \left( \hat{\theta}_2^{(s)} \mid 1 \leq R_1 \leq \tilde{r} - 1 \right) = \theta_2^2 \left( \sum_{j=1}^{\tilde{r}-1} \frac{\lambda^j}{j!} \right)^{-1} \left( \sum_{j=1}^{\tilde{r}-1} \frac{\tilde{\tau}^j}{j!} \right)^{-1} A,$$

where

$$A = \sum_{i=1}^{\tilde{r}-1} \frac{1}{\tilde{r}-i} \frac{1}{i!} \sum_{j=1}^{\tilde{r}-1} \frac{1}{j!} (\lambda^i \tilde{\tau}^j - \tilde{\tau}^i \lambda^j) = \sum_{i=1}^{\tilde{r}-1} \sum_{j=1}^{\tilde{r}-1} \frac{1}{i! j!} \frac{i-j}{(\tilde{r}-i)(\tilde{r}-j)} \lambda^i \tilde{\tau}^j = \sum_{i=1}^{\tilde{r}-1} \sum_{\substack{j=1 \\ i \neq j}}^{\tilde{r}-1} a_{ij} (i-j) \lambda^i \tilde{\tau}^j$$

with  $a_{ij} = \frac{1}{i!(\tilde{r}-i)j!(\tilde{r}-j)}$  ( $= a_{ji}$ ). We can then express

$$A = \sum_{i=1}^{\tilde{r}-1} \sum_{j=i+1}^{\tilde{r}-1} a_{ij} (i-j) \lambda^i \tilde{\tau}^i (\tilde{\tau}^{j-i} - \lambda^{j-i}),$$

from which the assertion follows.  $\square$

## 5 Concluding remarks

In this paper, we have discussed a special form of accelerated life-testing, viz., the simple step-stress scheme in order to analyze data from minimal repair processes based on exponential distributions. In the one-sample as well as multi-sample cases, we have derived explicit expressions for the unique maximum likelihood estimators of the model parameters and have further derived their exact distributions as well. From these distributional results, further inferential procedures such as confidence intervals and tests of hypotheses may be developed.

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