HAL open science

On the functional limits for sums of a function of partial sums

Kamil M. Kosiński

▶ To cite this version:

Kamil M. Kosiński. On the functional limits for sums of a function of partial sums. Statistics and Probability Letters, 2009, 79 (13), pp.1522. 10.1016/j.spl.2009.03.011 . hal-00542572

HAL Id: hal-00542572

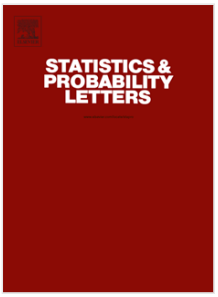
https://hal.science/hal-00542572v1

Submitted on 3 Dec 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Accepted Manuscript



On the functional limits for sums of a function of partial sums

Kamil M. Kosiński

PII: S0167-7152(09)00115-1
DOI: 10.1016/j.spl.2009.03.011
Reference: STAPRO 5378

To appear in: *Statistics and Probability Letters*

Received date: 23 November 2008
Revised date: 6 March 2009
Accepted date: 6 March 2009

Please cite this article as: Kosiński, K.M., On the functional limits for sums of a function of partial sums. *Statistics and Probability Letters* (2009), doi:10.1016/j.spl.2009.03.011

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

ACCEPTED MANUSCRIPT

On the functional limits for sums of a function of partial sums

Kamil M. Kosiński*,[1]

Wydział Matematyki, Informatyki i Mechaniki, Uniwersytet Warszawski, Warszawa, Poland

Abstract

We derive a functional central limit theorem (fclt) for normalised sums of a function of the partial sums of independent and identically distributed random variables. In particular, we show, using a technique presented in Huang and Zhang (Electron. Comm. Probab. 12, 51–56), that the result from Qi (Statist. Probab. Lett. 62, 93–100), for normalised products of partial sums, can be generalised in this fashion to a fclt.

Key words: Limit distributions, Product of sums, Stable laws

1. Introduction

While considering limiting properties of sums of records, Arnold and Villaseñor (1998) obtained the following version of the central limit theorem (clt) for a sequence (X_n) of independent, identically distributed (iid) exponential random variables (rv's) with the mean equal one:

$$\left(\prod_{k=1}^{n} \frac{S_k}{k}\right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}} \quad \text{as } n \to \infty,$$

where $S_n = \sum_{k=1}^{n} X_k$ and $\mathcal{N}$ is a standard normal random variable.

Later Rempała and Wesołowski (2002) extended such a clt to general iid positive rv's (X_n). Namely, provided that $\mathbb{E}X_1^2 < \infty$,

$$\left(\prod_{k=1}^{n} \frac{S_k}{k\mu}\right)^{\gamma/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}} \quad \text{as } n \to \infty, \tag{1}$$

where $\mu = \mathbb{E}X_1$ and $\gamma = \mu/\sigma$ with $\sigma^2 = \operatorname{Var} X_1 > 0$.

This result was generalised by Qi (2003) by assuming that the underlying distribution of X_1 is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$. In this case

$$\left(\prod_{k=1}^{n} \frac{S_k}{k\mu}\right)^{\mu/a_n} \xrightarrow{d} e^{(\Gamma(\alpha+1))^{1/\alpha}\mathcal{L}} \quad \text{as } n \to \infty, \tag{2}$$

where $\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x}\, dx$ and the sequence a_n is taken such that

$$\frac{S_n - n\mu}{a_n} \xrightarrow{d} \mathcal{L},$$

where $\mathcal{L}$ is one of the stable distributions with index $\alpha \in (1, 2]$. Lu and Qi (2004) obtained a similar result in the case $\alpha = 1$ with $\mathbb{E}|X_1| < \infty$. In a paper by Huang and Zhang (2007) it is shown that (1) follows from the weak invariance principle and the whole result can be reformulated to a functional theorem.

*Corresponding author.

Email addresses: `kosinski@mimuw.edu.pl` (Kamil M. Kosiński)

[1]This research was done while the author stayed at the Vrije Universiteit, Amsterdam.

ACCEPTED MANUSCRIPT

Let (S_n) be *any* nondecreasing sequence of positive rv's (which do not have to be a sequence of partial sums). Suppose there exists a standard Wiener process $(W(t))_{t\geq 0}$ and two positive constants μ and σ such that

$$\frac{S_{[nt]} - [nt]\mu}{\sigma\sqrt{n}} \xrightarrow{d} W(t) \quad \text{in} \quad D[0,1] \quad \text{as } n \to \infty, \tag{3}$$

and

$$\sup_n \frac{\mathbb{E}|S_n - n\mu|}{\sqrt{n}} < \infty. \tag{4}$$

Then the result from Huang and Zhang (2007) states that

$$\left(\prod_{k=1}^{[nt]} \frac{S_k}{k\mu}\right)^{\gamma/\sqrt{n}} \xrightarrow{d} \exp\left\{\int_0^t \frac{W(x)}{x}\,dx\right\} \quad \text{in} \quad D[0,1] \quad \text{as } n \to \infty, \tag{5}$$

where $\gamma = \mu/\sigma$.

For example, if (X_n) are iid positive rv's with mean μ and variance σ^2 and S_n are the partial sums, then (3) is satisfied by the invariance principle (c.f. Billingsley, 1999, Theorem 14.1), and (4) follows from the Cauchy-Schwarz inequality. Then, one can check that (5) implies (1).

The purpose of this paper is to show that the technique used by Huang and Zhang (2007) can be utilised to obtain a similar result for the rv's in the domain of attraction of a stable law with index $\alpha \in (1,2]$. Furthermore, we will set our discussion in a more general setting. It is straightforward that (1), and analogously (2), is a simple corollary from

$$\frac{\sum_{k=1}^n f(S_k/k) - b_n}{a_n} \xrightarrow{d} \mathcal{N},$$

if one sets $f(x) = \log x$ and chooses the sequences a_n, b_n properly.

2. Preliminaries

The following theorem is well known and can be easily found in the literature (see e.g. Bingham et al., 1987).

Theorem 1 (Stability Theorem)**.** *The general stable law with index $\alpha \in (0,2]$ is given by a characteristic function of one of the following forms:*

1. $\phi(t) = \exp(-\sigma^\alpha |t|^\alpha (1 - i\beta(\operatorname{sgn} t)\tan\frac{1}{2}\pi\alpha) + i\mu t)$, $\alpha \neq 1$,
2. $\phi(t) = \exp(-\sigma|t|(1 + i\beta(\operatorname{sgn} t)\frac{2}{\pi}\log|t|) + i\mu t)$, $\alpha = 1$,
3. $\phi(t) = \exp(-\sigma^2 t^2/2 + i\mu t)$, $\alpha = 2$

with $\beta \in [-1,1]$, $\mu \in \mathbb{R}$ and $\sigma > 0$.

From the above theorem one can see that every stable law with index $\alpha \in (0,2)$ can be parametrized by four parameters and written as $S_\alpha(\sigma,\beta,\mu)$. We distinguish the case $\alpha = 2$ because otherwise $S_2(\sigma,\beta,\mu) \stackrel{d}{=} \mathcal{N}(\mu, 2\sigma^2)$ and β plays no role, moreover one would like to think of $S_2(1,\beta,0)$ as $\mathcal{N}$ not $\mathcal{N}(0,2)$.

Let (X_n) be a sequence of iid rv's, set $S_n = \sum_{k=1}^n X_k$ and assume X_1 is in the domain of attraction of a stable law with index $\alpha \in (1,2]$. Note that for such X_1 we have $\mathbb{E}|X_1| < \infty$. Recall that a sequence of iid rv's (X_n) is said to be in the domain of attraction of a stable law $S_\alpha(\sigma,\beta,\mu)$, if there exists constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} S_\alpha(\sigma,\beta,\mu). \tag{6}$$

Clearly by scaling it suffices to let $\sigma = 1$ and $\mu = 0$, hence only the parameters α and β are unaltered by scaling. In case $\alpha = 2$ as mentioned above, we understand $S_2(1,\beta,0)$ as $\mathcal{N}$. Moreover, in case $\alpha \in (1,2]$, the sequence b_n can be taken equal to μn. On how to choose a_n we refer to (Whitt, 2002, Theorem 4.5.1). The choice is irrelevant for our discussion, the only fact that plays the crucial role is the fact that if (6) holds, then $a_n = n^{1/\alpha}L(n)$, where L is slowly varying.

ACCEPTED MANUSCRIPT

Furthermore, in addition to stable clt (6), there is convergence in distribution

$$S_n(t) := \frac{S_{[nt]} - [nt]\mu}{a_n} \xrightarrow{d} \mathcal{L}(t) \quad \text{in} \quad D[0,1], \tag{7}$$

where $\mathcal{L}$ is a standard (α,β)-stable Lévy motion, with

$$\mathcal{L}(t) \stackrel{d}{=} t^{1/\alpha} S_\alpha(1,\beta,0) \stackrel{d}{=} S_\alpha(t,\beta,0)$$

and as before, for $\alpha = 2$, $\mathcal{L}$ is a standard Wiener process (c.f. Whitt, 2002, Theorem 4.5.3).

3. Main Results

Theorem 2. *Let (X_n) be a sequence of iid rv's in the domain of attraction of the stable law $S_\alpha(1,\beta,0)$ with $\alpha \in (1,2]$, so that* (6) *holds for some sequence a_n and $b_n = n\mu$, where $\mu = \mathbb{E}X_1$. Let f be a real function defined on an interval I such that $\mathbb{P}(X_1 \in I) = 1$ and $f'(\mu)$ exists. Then, as $n \to \infty$*

$$\frac{1}{a_n} \sum_{k=1}^{[nt]} (f(S_k/k) - f(\mu)) \xrightarrow{d} f'(\mu) \int_0^t \frac{\mathcal{L}(x)}{x}\, dx \quad in \quad D[0,1], \tag{8}$$

where S_k denotes the k-th partial sum.

Because $\mathcal{L}(x)$ is cádlág, it has at most countably many discontinuity points, so the integral on the right hand side of (8) exists and is finite almost surely if

$$\int_0^1 \frac{|\mathcal{L}(x)|}{x}\, dx < \infty \quad \text{a.s.}$$

To ensure this, note that for a positive nondecreasing function h we have

$$\int_0^t \frac{|\mathcal{L}(x)|}{x}\, dx \le \sup_{0\le s\le t} \left\{ \frac{|\mathcal{L}(s)|}{h(s)} \right\} \int_0^t \frac{h(x)}{x}\, dx.$$

Setting $h(x) = x^\gamma$ with $\gamma \in (0, 1/\alpha)$ we get $\int_0^1 \frac{h(x)}{x}\, dx < \infty$ and

$$\sup_{0\le s\le t} \left\{ \frac{|\mathcal{L}(s)|}{h(s)} \right\} \to 0 \text{ a.s. as } \quad t \to 0,$$

by Khintchine's Theorem (see e.g., Barndorff-Nielsen et al., 2001, Theorem 2.1). This guaranties the existence of the integral in (8) as well as implies that

$$\sup_{0\le s\le t} \left| \int_0^t \frac{\mathcal{L}(x)}{x}\, dx \right| = 0 \text{ a.s. as } \quad t \to 0,$$

a fact that is going to be used later in the proof.

ACCEPTED MANUSCRIPT

Remark 1. Observe that

$$\begin{aligned}
\int_0^t \frac{\mathcal{L}(x)}{x}\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{t}{n}\frac{n}{tk}\mathcal{L}\left(\frac{tk}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{1}{k}\left(\mathcal{L}\left(\frac{ti}{n}\right)-\mathcal{L}\left(\frac{t(i-1)}{n}\right)\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\mathcal{L}\left(\frac{ti}{n}\right)-\mathcal{L}\left(\frac{t(i-1)}{n}\right)\right)\sum_{k=i}^{n}\frac{1}{k}\\
&\overset{d}{=} \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\sum_{k=i}^{n}\frac{1}{k}\right)^{\alpha}\right)^{1/\alpha} S_\alpha\left(\frac{t}{n},\beta,0\right)\\
&\overset{d}{=} S_\alpha(t,\beta,0)\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{n}\left(\sum_{k=i}^{n}\frac{1}{k}\right)^{\alpha}\right)^{1/\alpha}\\
&= S_\alpha(t,\beta,0)\left(\int_0^1(-\log x)^{\alpha}\,dx\right)^{1/\alpha}\\
&= S_\alpha(t,\beta,0)\left(\Gamma(\alpha+1)\right)^{1/\alpha}.
\end{aligned}$$

If X_1 is a positive rv, then the limiting stable law has $\beta = 1$. Setting $f(x) = \mu\log(x/\mu)$, Theorem 2 yields

$$\left(\prod_{k=1}^{n}\frac{S_k}{k\mu}\right)^{\mu/a_n} \xrightarrow{d} \exp\left(S_\alpha\left(\Gamma(\alpha+1),1,0\right)\right), \quad \text{as } n\to\infty.$$

which is the result (2) obtained by Qi (2003).

Remark 2. If $\mathbb{E}X_1^2 < \infty$, then X_1 is in the domain of attraction of normal distribution $\mathcal{N}$ and $a_n \sim \sigma\sqrt{n}$, where $\sigma^2 = \mathrm{Var}(X_1) > 0$. If furthermore X_1 is positive, then setting $\gamma = \mu/\sigma$

$$\left(\prod_{k=1}^{[nt]}\frac{S_k}{k\mu}\right)^{\gamma/\sqrt{n}} \xrightarrow{d} \exp\left(\int_0^t\frac{W(x)}{x}\,dx\right), \quad \text{in} \quad D[0,1] \quad \text{as } n\to\infty,$$

which coincides with the result (5) by Huang and Zhang (2007).

Before proceeding to the proof of the main theorem, we need a technical lemma.

Lemma 3. *Under the assumptions of Theorem 2*

$$\sum_{k=1}^{n}\frac{\mathbb{E}|S_k-k\mu|}{k} = O(a_n).$$

Proof. Note that

$$\sum_{k=1}^{n}\frac{\mathbb{E}|S_k-k\mu|}{k} \le \sup_{k\le n}\left\{\mathbb{E}\frac{|S_k-k\mu|}{a_k}\right\}\sum_{k=1}^{n}\frac{a_k}{k}.$$

By Theorem 6.1 in DeAcosta and Giné (1979)

$$\mathbb{E}\frac{|S_n-n\mu|}{a_n} = O(1). \tag{9}$$

Now, for a regularly varying function $A > 0$ with index $\gamma > -1$, its easy to see that

$$\sum_{k\le x}A(k) \sim \int_1^x A(t)dt \sim \frac{1}{1+\gamma}xA(x) \quad \text{as } x\to\infty \quad \text{if } \gamma > -1,$$

ACCEPTED MANUSCRIPT

where the last asymptotic equivalence follows from the Karamata's Theorem (c.f. Bingham et al., 1987, Theorem 1.5.8). Recall that a_n is slowly varying with index $1/\alpha > 0$, this implies

$$\sum_{k=1}^{n} \frac{a_k}{k} = O(a_n),$$

and proves the Lemma. □

Now we may proceed to the proof of the main theorem. The proof follows the steps of the proof of (5) in Huang and Zhang (2007).

Proof of Theorem 2. Expand f in the neighbourhood of μ, then

$$\frac{1}{a_n}\sum_{k=1}^{[nt]}(f(S_k/k) - f(\mu)) = \frac{f'(\mu)}{a_n}\sum_{k=1}^{[nt]}(S_k/k - \mu) + \frac{1}{a_n}\sum_{k=1}^{[nt]}(S_k/k - \mu)r(S_k/k), \tag{10}$$

where $r(x) \to 0$ as $x \to \mu$. Note that $\mathbb{E}|X_1| < \infty$ so by the SLLN $r(S_k/k) \to 0$ a.s.. It now follows from Lemma 3 that

$$\sup_{0\le t\le 1}\left|\frac{1}{a_n}\sum_{k=1}^{[nt]}(S_k/k - \mu)r(S_k/k)\right| \le \frac{1}{a_n}\sum_{k=1}^{n}\frac{|S_k - k\mu|}{k}|r(S_k/k)| = o_{\mathbb{P}}(1).$$

So, according to (10) it suffices to show that, as $n \to \infty$

$$Y_n(t) := \frac{1}{a_n}\sum_{k=1}^{[nt]}\frac{S_k - k\mu}{k} \xrightarrow{d} \int_0^t \frac{\mathcal{L}(x)}{x}\,dx, \quad \text{in} \quad D[0,1]. \tag{11}$$

Let

$$H_\varepsilon(f)(t) = \begin{cases} \int_\varepsilon^t \frac{f(x)}{x}\,dx, & \varepsilon < t \le 1 \\ 0, & 0 \le t \le \varepsilon \end{cases}$$

and

$$Y_{n,\varepsilon}(t) = \begin{cases} \frac{1}{a_n}\sum_{k=[n\varepsilon]+1}^{[nt]}\frac{S_k - k\mu}{k}, & \varepsilon < t \le 1 \\ 0, & 0 \le t \le \varepsilon, \end{cases}$$

It is obvious that

$$\sup_{0\le t\le 1}\left|\int_0^t \frac{\mathcal{L}(x)}{x}\,dx - H_\varepsilon(\mathcal{L})(t)\right| = \sup_{0\le t\le \varepsilon}\left|\int_0^t \frac{\mathcal{L}(x)}{x}\,dx\right| \to 0 \text{ a.s. as } \varepsilon \to 0 \tag{12}$$

and

$$\mathbb{E}\max_{0\le t\le 1}\left|Y_n(t) - Y_{n,\varepsilon}(t)\right| \le \frac{1}{a_n}\sum_{k=1}^{[n\varepsilon]}\frac{\mathbb{E}|S_k - k\mu|}{k} \le C\varepsilon^{1/\alpha} \tag{13}$$

by the same argumentation as in the proof of Lemma 3.

On the other hand, it is easily seen that, for n large enough such that $n\varepsilon \ge 1$,

$$\begin{aligned}
\sup_{\varepsilon\le t\le 1}\left|\sum_{k=[n\varepsilon]+1}^{[nt]}\frac{S_k - k\mu}{k} - \int_{n\varepsilon}^{nt}\frac{S_{[x]} - [x]\mu}{x}\,dx\right| &= \sup_{\varepsilon\le t\le 1}\left|\int_{[n\varepsilon]+1}^{[nt]+1}\frac{S_{[x]} - [x]\mu}{[x]}\,dx - \int_{n\varepsilon}^{nt}\frac{S_{[x]} - [x]\mu}{x}\,dx\right| \\
&\le \left|\int_{n\varepsilon}^{[n\varepsilon]+1}\frac{S_{[x]} - [x]\mu}{x}\,dx\right| + \sup_{\varepsilon\le t\le 1}\left|\int_{nt}^{[nt]+1}\frac{S_{[x]} - [x]\mu}{x}\,dx\right| \\
&\quad + \sup_{\varepsilon\le t\le 1}\left|\int_{[n\varepsilon]+1}^{[nt]+1}(S_{[x]} - [x]\mu)\left(\frac{1}{x} - \frac{1}{[x]}\right)dx\right| \\
&\le \max_{k\le n}|S_k - k\mu| \sup_{\varepsilon\le t\le 1}\left(\frac{2}{n\varepsilon} + \frac{2}{nt} + \frac{1}{n\varepsilon}\right) \\
&\le 5\max_{k\le n}|S_k - k\mu|/(n\varepsilon) = O_{\mathbb{P}}(a_n/n) = o_{\mathbb{P}}(1),
\end{aligned}$$

ACCEPTED MANUSCRIPT

by noticing that $\max_{k\le n}|S_k - k\mu|/a_n \xrightarrow{d} \sup_{0\le t\le 1}|\mathcal{L}(t)|$ according to (7). So

$$\frac{1}{a_n}\sum_{k=[n\varepsilon]+1}^{[nt]}\frac{S_k - k\mu}{k} = \frac{1}{a_n}\int_{n\varepsilon}^{nt}\frac{S_{[x]}-[x]\mu}{x}\,dx + o_{\mathbb{P}}(1) = \int_{\varepsilon}^{t}\frac{S_n(x)}{x}\,dx + o_{\mathbb{P}}(1)$$

uniformly in $t \in [\varepsilon, 1]$. Notice that $H_\varepsilon(\cdot)$ is a continuous mapping on the space $D[0,1]$. Using the continuous mapping theorem (c.f. Theorem 2.7. of Billingsley (1999)) it follows that

$$Y_{n,\varepsilon}(t) = H_\varepsilon(S_n)(t) + o_{\mathbb{P}}(1) \xrightarrow{d} H_\varepsilon(\mathcal{L})(t) \quad \text{in} \quad D[0,1] \quad \text{as } n\to\infty. \tag{14}$$

Combining (12)-(14) yields (11) by Theorem 3.2 of Billingsley (1999). □

4. Extensions

To prove Lemma 3, we have only used the property (9) (which is in fact the condition (4)) and the fact that a_n varies regularly with a positive index. The proof of Theorem 2 was based on the convergence (7) and the fact that $S_k/k \to \mu$ a.s.. All those conditions are satisfied when S_k is defined to be the partial sum of a sequence of iid rv's in the domain of attraction of a stable law with index greater than one. However, we do not need to assume anything about S_k and only require that it satisfies the aforementioned conditions. This leads to

Theorem 4. *Let (S_k) be a sequence of random variables. Suppose there exists an (α,β)-stable Lévy process $(\mathcal{L}(t))_{t\ge 0}$, a constant μ and a sequence a_n such that as $n\to\infty$*

$$\frac{S_{[nt]}-[nt]\mu}{a_n} \xrightarrow{d} \mathcal{L}(t) \quad \textit{in} \quad D[0,1],$$

where a_n can be written as $a_n = n^{1/\alpha}L(n)$ with $\alpha\in(1,2]$ and L slowly varying. In addition, suppose that

$$\sup_n \frac{\mathbb{E}|S_n - n\mu|}{a_n} = O(1), \tag{15}$$

and $S_n/n \to \mu$ a.s., then, as $n\to\infty$

$$\frac{1}{a_n}\sum_{k=1}^{[nt]}(f(S_k/k) - f(\mu)) \xrightarrow{d} f'(\mu)\int_0^t \frac{\mathcal{L}(x)}{x}\,dx \quad \textit{in} \quad D[0,1],$$

for any real function f defined on an interval I such that $\mathbb{P}(S_k/k\in I) = 1$ for all k, provided that $f'(\mu)$ exists.

In their paper, Huang and Zhang showed that if (S_k) is a nondecreasing (in fact we only need monotonicity) sequence satisfying (15), then $S_k/k \to \mu$ a.s.. Thus, Theorem 4 is an extension of the result (5) from Huang and Zhang (2007).

5. Acknowledgments

The author is grateful to dr. R. Latała for supervision of his work and would also like to thank the referees for pointing out some subtleties in the previous version of this paper.

References

Arnold, B.C., Villaseñor, J.A., 1998. The asymptotic distribution of sums of records. Extremes 1:3, 351-363.
Billingsley, P., 1999. Convergence of Probability Measures. 2nd edition, John Wiley & Sons, Inc., New York.
Bingham, N.H., Goldie, C.M., Teugels, J.L., 1987. Regular Variation. Cambridge University, New York.
DeAcosta, A., Giné, E., 1979. Convergence of moments and related functionals in the general central limit theorem in Banach spaces. Z. Wahrsch. Verw. Gebiete. 48, 213–231.
Huang, W., Zhang, L.-X., 2007. A note on the invariance principle of the product of sums of random variables. Electron. Comm. Probab. 12, 51–56.
Lu, X., Qi, Y., 2004. A note on asymptotic distribution of products of sums. Statist. Probab. Lett. 68, 407–413.
Barndorff-Nielsen, O.E., Mikosch, T., Resnick, S.I., 2001. Lévy Processes: Theory and Applications. Birkhäuser, Berlin.
Rempała, G., Wesołowski, J., 2002. Asymptotics for products of sums and U-statictics. Electron. Comm. Probab. 7, 45-54.
Qi, Y., 2003. Limit distributions for products of sums. Statist. Probab. Lett. 62, 93–100.
Whitt, W., 2002. Stochastic-Process Limits: an introduction to stochastic-process limits and their application to queues. Springer