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Georgi K. Mitov, Kosto V. Mitov, Nikolay M. Yanev

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Georgi K. Mitov [a] Kosto V. Mitov [b,*] Nikolay M. Yanev [c]

[a] *Department of Probability and Statistics*
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
1113 Sofia, BULGARIA. email:

[b] *Faculty of Aviation, National Military University*
5856 D. Mitropolia, Pleven, BULGARIA. email: kmitov@af-acad.bg

[c] *Department of Probability and Statistics*
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
1113 Sofia, BULGARIA. email:

Abstract

Bienaymé-Galton-Watson branching processes subordinated to a continuous time random index are considered. The branching processes are assumed to be critical with finite or infinite offspring variance. The indexing process is assumed to be a renewal one with finite or infinite mean of the interarrival times. Under these conditions we prove the asymptotic formulas for the first two moments and for the probability of non-extinction. We also obtain proper limiting distributions under suitable normalization.

Key words: branching process; random index; renewal process; probability of extinction; limit theorems.
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1 Introduction

A randomly indexed branching process was introduced by Epps (1996) for modeling of daily stock prices as an alternative of the geometric Brownian motion. Epps considered a Bienaymé-Galton-Watson (BGW) branching process indexed by a Poisson process, assuming four particular discrete offspring distributions. Under these conditions Epps obtained the asymptotic behavior of the moments, submitted certain estimates of the parameters of the process, and made the calibration of the model using real data from the NYSE. Later, Dion and Epps (1999) compared, by simulations, several types of estimates of the parameters of this process under the same assumptions as in Epps (1996). Mitov and Mitov (2007) derived a formula for pricing European Call Options if the price of the underlying

* Corresponding author: Faculty of Aviation, 5856 D. Mitropolia, Pleven, BULGARIA. email: kmitov@af-acad.bg

asset follows this process. These applications are possible because of the extra randomness, induced from the subordination to the Poisson random time. In general a random time change usually provides new phenomena in many stochastic models (for example, summation of independent random variables, extremal processes, continuous time random walk, etc.). This is a motivation to investigate the BGW processes with random time in a general situation. Assuming that the BGW process has a general offspring distribution, and the continuous-time random index is an ordinary renewal process, we study the asymptotic behavior of the moments, the probability of non-extinction and limiting distributions under the appropriate normalization. We consider the critical case, which is usually the most interesting. The subcritical case was studied in Mitov and Mitov (2006).

The paper is organized as follows. The basic definitions, assumptions and equations are given in Section 2. The asymptotic behavior of the moments and probabilities of non-extinction is investigated in Section 3 (Theorems 3-5). Limiting distributions are obtained in Section 4 (Theorems 6, 7, and 9). Finally, in Section 5 we discuss the results and propose some new possible generalizations and applications.

2 Definitions and assumptions

Let be given two independent sets of random elements on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$:

(i) The set $X = \{X_i(n), n = 1, 2, \ldots; i = 1, 2, \ldots\}$ of i.i.d. nonnegative integer valued random variables (r.v.) with a probability generating function (p.g.f.)$f(s) = \mathbf{E}(s^{X_i(n)}) = \sum_{k=0}^{\infty} p_k s^k, \quad s \in [0, 1]$.

(ii) The set $J = \{J_1, J_2, \ldots\}$ of positive i.i.d. r.v. with the cumulative distribution function (c.d.f.) $F(x) = \mathbf{P}\{J_n \leq x\}$.

The classical BGW branching process can be defined as follows

$$Z_0 = 1, \text{ a.s. }, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_i(n+1), \quad n = 0, 1, 2, \ldots.$$

The p.g.f. of Z_n, $f_n(s) = \mathbf{E}(s^{Z_n}|Z_0 = 1), |s| \leq 1$, is the $n-$fold iteration of $f(s)$; that is $f_n(s) = f(f_{n-1}(s))$, $f_1(s) = f(s)$, $f_0(s) = s$.

Define also the ordinary renewal process $S_0 = 0, \quad S_n = \sum_{j=1}^{n} J_j, \quad n = 0, 1, \ldots,$ and let for $t \geq 0$, $N(t) = \max\{n \geq 0 : S_n \leq t\}$, be the corresponding counting process. Denote the renewal function $H(t) = \mathbf{E}(N(t)) = \sum_{n=0}^{\infty} F^{*n}(t), \quad t \geq 0$, and let $P_k(t) = \mathbf{P}\{N(t) = k\}, \quad k = 0, 1, 2, \ldots$.

Definition 1 *The continuous time process* $\{Y(t), t \geq 0\}$ *defined by*

$$Y(0) = 1, \quad Y(t) = Z_{N(t)}, \quad t > 0.$$

is called a randomly indexed BGW branching process.

Remark 2 *If $N(t)$ is a Poisson process then $Y(t)$ will be a continuous time Markov chain. Otherwise, the process $Y(t), t \geq 0$, will be non-Markovian. One can interpret $Y(t)$ as an age-dependent branching process, governed by the offspring p.g.f. $f(.)$ and the particle life c.d.f. $F(.)$, where all particles in a given generation have the same life-time and they give birth of their daughters simultaneously. Thus the life-times of the particles are dependent, but the numbers of offspring are independent. Recall that in the classical Bellman-Harris process both the life span and the offspring of every particle are independent and does not depend from the other particles.*

Using the law of the total probability and the independence of Z_n and $N(t)$, we obtain the p.g.f. of the process $Y(t), t \geq 0$,

$$\Phi(t;s) = \mathbf{E}(s^{Y(t)}|Y(0)=1) = \sum_{k=0}^{\infty} P_k(t)\mathbf{E}(s^{Z_k}|Z_0=1) = \sum_{k=0}^{\infty} P_k(t)f_k(s). \tag{1}$$

Substituting $s=0$ in (1) we have

$$\mathbf{P}\{Y(t)=0|Y(0)=1\} = \Phi(t;0) = \sum_{k=1}^{\infty} P_k(t)f_k(0). \tag{2}$$

Further, we will suppose that the following "Branching" conditions hold:

(B) The offspring p.g.f. satisfies

$$f(s) = s + (1-s)^{1+\beta}\mathcal{L}\left(\frac{1}{1-s}\right), \tag{B1}$$

where $\beta \in (0,1]$ and $\mathcal{L}(x)$ is a function slowly varying at infinity (s.v.f.). This condition means that the process is critical ($f'(1)=1$) with an infinite or finite variance. The variance will be finite if $\beta = 1$ and $\mathcal{L}(t) \to b$ as $t \to \infty$ for some $0 < b < \infty$. In this case the offspring variance is $2b = f''(1) \in (0,\infty)$ and $f(s)$ takes the following form

$$f(s) = s + b(1-s)^2 + o((1-s)^2),\ s \uparrow 1. \tag{B2}$$

These branching properties will be combined with the following "Renewal" conditions:

(R) The c.d.f. $F(t)$ is continuous, $F(0+)=0$, and either

$$0 < \mu = \int_0^{\infty} x dF(x) < \infty \tag{R1}$$

or

$$1 - F(x) = x^{-\alpha}L(x), \tag{R2}$$

where $\alpha \in [0,1)$ and $L(x)$ is a s.v.f. as $x \to \infty$.

Example 1 *A non trivial example for slowly varying function is $(\log x)^{\delta}$ for any fixed $\delta > 0$. Then, an example of p.g.f. satisfying condition (B1) is*

$$f(s) = s + (1-s)^{1+\beta}\log\left(\frac{1}{1-s}\right),$$

which is the p.g.f. of a power series distribution

$$p_k = C_\beta k^{-(2+\beta)} \log k, \quad k \geq 2,$$

where C_β is a normalizing constant.

The condition (B2) is satisfied by the most commonly used discrete distributions like geometric, negative binomial, Poisson, etc., and also by the mixture distributions used in Epps (1996).

Condition (R2) is satisfied, for example, by the following distribution function

$$F(x) = 1 - (1+x)^{-\alpha}(1 + \alpha \log(1+x)), \quad x \geq 0.$$

3 Moments and probability for non-extinction

Evidently the second moments exist only if the offspring variance is finite, i.e. if (B2) holds. The asymptotic behavior of the moments is given in the following theorem.

Theorem 3 *Assume $f'(1) = 1$. Then* $\mathbf{E}(Y(t)) = 1, \quad t \geq 0.$

(i) If additionally (B2) and (R1) hold then

$$Var(Y(t)) \sim \frac{2bt}{\mu}, \quad \text{as } t \to \infty.$$

(ii) If additionally (B2) and (R2) hold then

$$Var(Y(t)) \sim C_\alpha \frac{t^\alpha}{L(t)}, \quad \text{as } t \to \infty,$$

where $C_\alpha = (\Gamma(1+\alpha)\Gamma(1-\alpha))^{-1}$.

Proof. Differentiating the equation (1) with respect to s and substituting $s = 1$, we obtain

$$\Phi'_s(t;1) = \sum_{k=0}^{\infty} P_k(t) f'_k(1), \qquad \Phi''_{ss}(t;1) = \sum_{k=0}^{\infty} P_k(t) f''_k(1) \tag{3}$$

whenever the derivatives in the above equations are finite.

Since the process $\{Z_n\}$ is critical it is well known (see e.g. Athreya and Ney (1972) or Sevastyanov (1971)) that $f'_n(1) = 1$ for every n. Then from the first equation in (3) it follows that

$$\mathbf{E}(Y(t)) = \Phi'_s(t;1) = \sum_{n=0}^{\infty} \mathbf{P}\{N(t) = n\} = 1, \quad t \geq 0.$$

If additionally (B2) holds then $f''_n(1) = 2bn, \ n = 0, 1, 2, \ldots$ (see e.g. Athreya and Ney (1972) or Sevastyanov (1971)). Then from the second equation in (3) one gets for $t \geq 0$,

$$Var(Y(t)) = \mathbf{E}(Y(t)(Y(t)-1)) = \Phi''_{ss}(t;1) = \sum_{n=0}^{\infty} P_n(t) 2bn = 2b\mathbf{E}(N(t)) = 2bH(t).$$

We complete the proofs of both cases (i) and (ii), applying the renewal theorem for $H(t)$ when the inter-arrival periods have a finite or infinite mean (see e.g. Bingham et. al (1987), §8.6.2, eqs. (8.6.4), (8.6.6) or Feller (1971), Ch.14, §3). □

The following two theorems represent the asymptotic behavior of the probability for non-extinction.

Theorem 4 *Suppose that (R1) is satisfied.*

(i) If additionally (B2) holds and $t^2(1-F(t)) \to 0, \ \ t \to \infty$, *then*

$$\mathbf{P}\left\{Y(t) > 0 | Y(0) = 1\right\} \sim \frac{\mu}{bt}, \ t \to \infty.$$

(ii) If additionally (B1) holds and $t^{1/\beta+1}(1-F(t))/M(t) \to 0, \ t \to \infty$, *then*

$$\mathbf{P}\{Y(t) > 0 | Y(0) = 1\} \sim \left(\frac{\mu}{t}\right)^{1/\beta} M(t), \ t \to \infty,$$

where $M(.)$ *is a s.v.f. such that (see Slack (1968))*

$$\beta M^{\beta}(t)\mathcal{L}(t^{-1/\beta}M(t)) \to 1, \ \ t \to \infty. \tag{4}$$

Proof. Using (2), $f_0(0) = 0$, and $f_n(0) = \sum_{k=1}^{n}(f_k(0) - f_{k-1}(0))$ for $n \geq 1$, one gets

$$\Phi(t;0) = \sum_{n=1}^{\infty} P_n(t) \sum_{k=1}^{n}(f_k(0) - f_{k-1}(0)) = \sum_{k=1}^{\infty}(f_k(0) - f_{k-1}(0))F^{*k}(t) \tag{5}$$

and for $R(t) = \mathbf{P}\{Y(t) > 0 | Y(0) = 1\}$,

$$R(t) = 1 - \Phi(t;0) = 1 - \sum_{k=1}^{\infty}(f_k(0) - f_{k-1}(0))F^{*k}(t) = \sum_{k=1}^{\infty}(1 - f_k(0))P_n(t). \tag{6}$$

Define the function $A(x) = 1 - f_{[x]}(0)$, where $[x]$ is the integer part of x. Evidently, $A(n) = 1 - f_n(0) \downarrow 0, n \to \infty$. Since (6) we can consider $R(t)$ as a weighted renewal function (see e.g. Omey and Teugels (2002) or Omey and Mallor (2006))

$$R(t) = \sum_{n=0}^{\infty} A(n)P_n(t), \ \ t \geq 0. \tag{7}$$

(i) Let $\xi > 0$ be fixed. Using the well-known Kolmogorov result (see e.g. Athreya and Ney (1972) or Sevastyanov (1971))

$$1 - f_n(0) = \mathbf{P}\{Z_n > 0 | Z_0 = 1\} \sim \frac{1}{bn}, \quad n \to \infty \tag{8}$$

we have

$$\limsup_{x\to\infty} \frac{A(x)}{A(x(1+\xi))} = \limsup_{x\to\infty} \frac{1 - f_{[x]}(0)}{1 - f_{[x(1+\xi)]}(0)} \leq \limsup_{x\to\infty} \frac{b[x(1+\xi)]}{b[x]} = 1 + \xi.$$

Hence,

$$\lim_{\xi\downarrow 0}\left(\limsup_{x\to\infty}\frac{A(x)}{A(x(1+\xi))}\right)=\lim_{\xi\downarrow 0}(1+\xi)=1.$$

Under the conditions of the theorem we also have $t^{\rho}(1-F(t))\to 0,\ t\to\infty$ for $\rho=2$ and (see (8)) $\liminf_{x\to\infty} xA(x)=\lim_{x\to\infty} x(1-f_{[x]}(0))=1/b>0$. Now we are able to apply Theorem 4 from Omey and Teugels (2002) to the weighted renewal function (7)

$$R(t)=\sum_{n=0}^{\infty}A(n)P_n(t)\sim A(t/\mu)=1-f_{[t/\mu]}(0)\sim\frac{\mu}{bt},\ t\to\infty.$$

This completes the proof of the case (i).

(ii) Let $\xi>0$ be fixed. Using the asymptotic formula (see Slack (1968))

$$1-f_n(0)\sim n^{-1/\beta}M(n),\quad n\to\infty, \tag{9}$$

and the properties of slowly varying functions one obtains

$$\begin{aligned}&\limsup_{x\to\infty}\frac{A(x)}{A(x(1+\xi))}=\limsup_{x\to\infty}\frac{1-f_{[x]}(0)}{1-f_{[x(1+\xi)]}(0)}\\ &\le\limsup_{x\to\infty}\frac{[x(1+\xi)]^{-1/\beta}M([x(1+\xi)])}{[x]^{-1/\beta}M([x])}\le\lim_{x\to\infty}\frac{x^{-1/\beta}(1+\xi)^{-1/\beta}}{(x-\{x\})^{-1/\beta}}=(1+\xi)^{-1/\beta}.\end{aligned}$$

Therefore,

$$\lim_{\xi\downarrow 0}\left(\limsup_{x\to\infty}\frac{A(x)}{A(x(1+\xi))}\right)=\lim_{\xi\downarrow 0}(1+\xi)^{-1/\beta}=1.$$

Under the conditions of the theorem, for $\rho=1/\beta+1>1$ one has

$$(t^{\rho}/M(t))(1-F(t))\to 0,\quad t\to\infty.$$

Using the properties of regularly varying functions and (9) we obtain

$$\liminf_{x\to\infty}\frac{x^{1/\beta}}{M(x)}A(x)=\lim_{x\to\infty}\frac{x^{1/\beta}}{M(x)}(1-f_{[x]}(0))=\lim_{x\to\infty}\frac{x^{1/\beta}}{M(x)}[x]^{-1/\beta}M([x])=1>0.$$

Now we are able to apply again Theorem 4 by Omey and Teugels (2002) to (7) in order to obtain

$$R(t)=\sum_{n=0}^{\infty}A(n)P_n(t)\sim A(t/\mu)=1-f_{[t/\mu]}(0)\sim(t/\mu)^{-1/\beta}\,M(t),\quad t\to\infty.$$

The theorem is proved. □

Theorem 5 *Suppose that (R2) is satisfied.*

(i) If additionally (B2) holds then as $t\to\infty$,

$$\mathbf{P}\{Y(t)>0|Y(0)=1\}\sim\begin{cases}b^{-1}L(t)\log(1/L(t)), & \alpha=0,\\ \alpha b^{-1}t^{-\alpha}L(t)\log t, & 0<\alpha<1.\end{cases}$$

(ii) If additionally (B1) holds and $K = \sum_{n=1}^{\infty}(1 - f_n(0)) \in (0, \infty)$ *then*

$$\mathbf{P}\{Y(t) > 0 | Y(0) = 1\} \sim KL(t)t^{-\alpha}, \qquad as\ t \to \infty.$$

Proof. Note first that $\Phi(t; 0) \uparrow 1$, as $t \to \infty$ and $\Phi(0; 0) = 0$ (see equation (5)). Therefore $\Phi(t; 0)$ can be considered as a c.d.f. on $(0, \infty)$. Taking the Laplace-Stieltjes transform from both sides of the equation (5) and after some simple calculations one obtains

$$1 - \hat{\Phi}(\lambda) = (1 - \hat{\phi}(\lambda)) \sum_{l=0}^{\infty}(1 - f_l(0))\hat{\phi}(\lambda)^l, \tag{10}$$

where $\hat{\Phi}(\lambda) = \int_0^{\infty} e^{-\lambda t} d\Phi(t; 0)$ and $\hat{\phi}(\lambda) = \int_0^{\infty} e^{-\lambda t} dF(t)$.

(i) Using (R2) and a Tauberian theorem for Laplace transforms (see e.g. Bingham et. al (1987), Corollary 8.1.7, or Feller (1971), Ch.13, §5, (5.21), (5.22)) we obtain

$$1 - \hat{\phi}(\lambda) \sim \lambda^{\alpha}\Gamma(1 - \alpha)L\left(1/\lambda\right), \quad \text{as } \lambda \downarrow 0. \tag{11}$$

Since (8) one has $\sum_{l=0}^{n}(1 - f_l(0)) \sim b^{-1} \log n$, as $n \to \infty$. Then using a Tauberian theorem for power series (see Feller (1971), Ch. 13, §5, Th.5) we get

$$\sum_{l=0}^{\infty}(1 - f_l(0))s^l \sim b^{-1} \log\left(1/(1 - s)\right) \text{ as } s \uparrow 1.$$

Since $F(t)$ is a proper c.d.f., then $\hat{\phi}(\lambda) \uparrow 1$ as $\lambda \downarrow 0$. Hence (see (11))

$$\sum_{l=0}^{\infty}(1 - f_l(0))\hat{\phi}(\lambda)^l \sim b^{-1} \log\left(1/(1 - \hat{\phi}(\lambda))\right) \sim \begin{cases} b^{-1}\alpha \log(1/\lambda), & 0 < \alpha < 1, \\ b^{-1} \log(L(1/\lambda)), & \alpha = 0, \end{cases}$$

as $\lambda \downarrow 0$. This relation, (11) and (10) imply

$$1 - \hat{\Phi}(\lambda) \sim \begin{cases} b^{-1}\alpha\lambda^{\alpha}\Gamma(1 - \alpha)L\left(1/\lambda\right) \log(1/\lambda), & 0 < \alpha < 1, \\ b^{-1}L(1/\lambda) \log(1/L(1/\lambda)), & \alpha = 0, \end{cases}$$

as $\lambda \downarrow 0$. Applying again the same Tauberian theorem for Laplace transforms we complete the proof of the case (i).

(ii) If $\sum_{l=0}^{\infty}(1 - f_l(0)) = K \in (0, \infty)$ then using the fact that $\hat{\phi}(\lambda) \uparrow 1$ as $\lambda \downarrow 0$ it is not difficult to obtain that

$$\sum_{l=0}^{\infty}(1 - f_l(0))\hat{\phi}(\lambda)^l \to K \text{ as } \lambda \downarrow 0.$$

From this relation, (11) and (10) the proof follows. □

4 Limiting distributions

Further on we will use the notation $X|_{Y>0}$ to point out that we consider the r.v. X under the condition $\{Y > 0\}$. We denote $\xrightarrow{d}$ the convergence in distribution.

Theorem 6 *Assume that the conditions (R1) and (B2) hold and* $t^2(1-F(t)) \to 0,\ t \to \infty$. *Then*

$$R(t)Y(t)|_{Y(t)>0} \xrightarrow{d} \mathcal{E},$$

where $\mathcal{E}$ *has the standard exponential distribution (i.e.* $\mathbf{P}\{\mathcal{E} > x\} = e^{-x}$*).*

Proof. Using (1) we can represent $1 - \Phi(t; s)$ as follows

$$1 - \Phi(t; s) = \sum_{k=1}^{\infty}(1 - f_k(s))P_n(t) = \sum_{n=0}^{\infty} P_n(t)A(n; s), \quad s \in [0, 1),\ t \geq 0, \tag{12}$$

where $A(n; s) := 1 - f_n(s)$. For $A(n; s)$ we have (see e.g. Sevastyanov (1971) or Athreya and Ney (1972))

$$A(n; s) = \frac{1 - s}{1 + bn(1 - s)}(1 + \alpha(n; s))$$

for $s \in [0, 1)$ and $n \geq 0$, where $\alpha(n; s) \to 0,\ n \to \infty$ uniformly in $s \in [0, 1)$. Then we define

$$A(x; s) = 1 - f_{[x]}(s) = \frac{1 - s}{1 + b[x](1 - s)}(1 + \alpha([x]; s))$$

for $s \in [0, 1)$ and $x \geq 0$. Let $\xi > 0$ be fixed. We will prove that uniformly in $s \in [0, 1)$,

$$\lim_{\xi \downarrow 0}\left(\limsup_{x\to\infty} \frac{A(x; s)}{A(x(1 + \xi); s)}\right) = 1. \tag{13}$$

We have

$$\frac{A(x; s)}{A(x(1 + \xi); s)} = \frac{1 + b[x(1 + \xi)](1 - s)}{1 + b[x](1 - s)} \times \frac{(1 + \alpha([x]; s))}{(1 + \alpha([x(1 + \xi)]; s))}.$$

For the function $g(s) = \dfrac{1 + b[x(1 + \xi)](1 - s)}{1 + b[x](1 - s)}$ one gets $g'_s(s) = \dfrac{b[x] - b[x(1 + \xi)]}{(1 + b[x](1 - s))^2} \leq 0$, because $\xi > 0$. Therefore, $g(s)$ is non-increasing in $s \in [0, 1)$. Then for every $s \in [0, 1)$,

$$1 \leq \frac{1 + b[x(1 + \xi)](1 - s)}{1 + b[x](1 - s)} \leq \frac{1 + b[x(1 + \xi)]}{1 + b[x]}.$$

These inequalities imply

$$1 \leq \limsup_{x\to\infty} \frac{1 + b[x(1 + \xi)](1 - s)}{1 + b[x](1 - s)} \leq \limsup_{x\to\infty} \frac{1 + b[x(1 + \xi)]}{1 + b[x]} = 1 + \xi.$$

On the other hand the convergence

$$\lim_{x\to\infty} \frac{(1 + \alpha([x]; s))}{(1 + \alpha([x(1 + \xi)]; s))} = 1$$

is also uniform in $s \in [0, 1)$. Hence

$$1 \leq \limsup_{x\to\infty} \frac{A(x; s)}{A(x(1 + \xi); s)} = 1 + \xi$$

uniformly in $s \in [0,1)$. The last relation proves (13).

For the sequence $\{nA(n;s),\ n=1,2,\ldots\}$ one gets

$$\liminf_{n\to\infty} nA(n;s) = \lim_{n\to\infty} \frac{n(1-s)}{1+bn(1-s)}(1+\alpha(n;s)) = \frac{1}{b} > 0,$$

uniformly in $s \in [0,1)$. Now we conclude that uniformly in $s \in [0,1)$ the following relation holds

$$\liminf_{x\to\infty} xA(x;s) = \frac{1}{b} > 0. \tag{14}$$

Using (13) and (14) we can apply Theorem 4 from Omey and Teugels (2002) to the weighted renewal function (12) and to obtain that uniformly in $s \in [0,1)$,

$$1-\Phi(t,s) \sim A(t/\mu;s) \sim \frac{1-s}{1+b[t/\mu](1-s)},\ \ t\to\infty. \tag{15}$$

Let $\lambda > 0$ be fixed. Using the asymptotic $1-e^{-x} \sim x, x \to 0$, relation (15) and the asymptotic of $R(t) = 1-\Phi(t,0)$ (see Theorem 4 (i)) we obtain that

$$1-\mathbf{E}(e^{-\lambda R(t)Y(t)}|Y(t)>0) = \frac{1-\Phi(t,e^{-\lambda R(t)})}{R(t)} \sim \frac{\lambda R(t)}{1+b[t/\mu]\lambda R(t)}\frac{1}{R(t)} \to \frac{\lambda}{1+\lambda}.$$

Therefore

$$\lim_{t\to\infty} \mathbf{E}(e^{-\lambda R(t)Y(t)}|Y(t)>0) = \frac{1}{1+\lambda},\ \ \lambda > 0.$$

The continuity theorem for Laplace transforms completes the proof. □

Theorem 7 *Assume that the conditions (R1) and (B1) hold and*

$$t^{1+1/\beta}(1-F(t)) \to 0,\ \ t\to\infty.$$

Then

$$R(t)Y(t)|_{Y(t)>0} \xrightarrow{d} \mathcal{E}_\beta,$$

where $\mathcal{E}_\beta$ has Laplace transform $\mathbf{E}(e^{-\lambda\mathcal{E}_\beta}) = 1-\lambda(1+\lambda^\beta)^{-1/\beta},\ \ \lambda > 0$.

Proof. Under the condition (B1) there exists the invariant measure with p.g.f. $U(s)$ for the branching process Z_n. The function $U(s)$ is analytic in $|s|<1$. If we denote by $V(x)$ the inverse function of $U(1-s)$ then (see Slack (1968))

$$V(x) = x^{-1/\beta}M(x),\ \ V(0)=1,$$

where the slowly varying function $M(x)$ is defined by (4). It is clear that $V(x)$ is non increasing in $[0,\infty)$. Further the following representation holds for $s \in [0,1)$(see Slack (1968))

$$A(n;s) = 1-f_n(s) = V(n+U(s))$$

for $s \in [0,1)$ and $n \geq 0$. Define the function $A(x;s) = V([x]+U(s))$ for $s \in [0,1)$ and $x \geq 0$. Following a similar method as in the proof of Theorem 6 we will check the conditions of Theorem 4 (Omey and Teugels (2002)) for the weighted renewal function (12).

Let $\xi > 0$ be fixed. We will prove that uniformly in $s \in [0,1)$,

$$\lim_{\xi\downarrow 0}\left(\limsup_{x\to\infty} \frac{A(s,x)}{A(s;x(1+\xi))}\right) = 1. \tag{16}$$

Note first that

$$\frac{A(x;s)}{A(x(1+\xi);s)} = \left(\frac{([x(1+\xi)]+U(s))}{([x]+U(s))}\right)^{1/\beta} \times \frac{M([x]+U(s))}{M([x(1+\xi)]+U(s))}. \tag{17}$$

For the function $g(s) = ([x(1+\xi)]+U(s))/([x]+U(s))$ one gets

$$g_s'(s) = (U'(s)([x]-[x(1+\xi)]))\,([x]+U(s))^{-2} \le 0,$$

because $\xi > 0$ and $U(s)$ is non-decreasing, hence $U'(s) \ge 0$. Therefore $g(s)$ is non-increasing in $s \in [0,1)$. Then for every $s \in [0,1)$,

$$1 \le \left(\frac{([x(1+\xi)]+U(s))}{([x]+U(s))}\right)^{1/\beta} \le \left(\frac{([x(1+\xi)]+U(0))}{([x]+U(0))}\right)^{1/\beta}.$$

Therefore

$$1 \le \limsup_{x\to\infty}\left(\frac{([x(1+\xi)]+U(s))}{([x]+U(s))}\right)^{1/\beta} \le \limsup_{x\to\infty}\left(\frac{([x(1+\xi)]+U(0))}{([x]+U(0))}\right)^{1/\beta} = 1+\xi.$$

For the second factor in (17) we obtain by the properties of s.v.f. that

$$\limsup_{x\to\infty}\frac{M([x]+U(s))}{M([x(1+\xi)]+U(s))} = 1,$$

uniformly in $s \in [0,1)$. Combining (17), and the last two relations we obtain

$$1 \le \limsup_{x\to\infty}\frac{A(s,x)}{A(s;x(1+\xi))} = 1+\xi$$

uniformly in $s \in [0,1)$, which proves (16).

Let us consider the functions

$$g(n;s) = V(n)^{-1}A(n;s) = \frac{n^{1/\beta}}{M(n)}V(n+U(s)), \quad n = 0,1,2,\ldots.$$

$$\liminf_{n\to\infty} V(n)^{-1}A(n;s) = \lim_{n\to\infty}\frac{n^{1/\beta}}{M(n)}(n+U(s))^{-1/\beta}M(n+U(s)) = 1,$$

uniformly in $s \in [0,1)$.

No we conclude that uniformly in $s \in [0,1)$ the following relation holds

$$\liminf_{x\to\infty} V(x)^{-1}A(x;s) = 1. \tag{18}$$

Using (16) and (18) and applying Theorem 4 (Omey and Teugels (2002)) to the weighted renewal function (12) one can obtain that uniformly in $s \in [0,1)$,

$$1-\Phi(t,s) \sim A(t/\mu,s) \sim V([t/\mu]+U(s)),\ t\to\infty. \tag{19}$$

Let $\lambda > 0$ be fixed. Using the asymptotic $1-e^{-x} \sim x,\ \ x \to 0$, (19) and the asymptotic of $1-\Phi(t,0)$ (see Theorem 4 (ii)) we obtain that, as $t \to \infty$,

$$1-\mathbf{E}(e^{-\lambda Y(t)R(t)}|Y(t)>0) = \frac{1-\Phi(t,e^{-\lambda R(t)})}{R(t)} \sim \frac{V([t/\mu]+U(e^{-\lambda R(t)}))}{R(t)} \to \frac{\lambda}{(1+\lambda^{\beta})^{1/\beta}}.$$

Here we use the relations (see Slack (1968))

$$V([t/\mu] + U(e^{-\lambda R(t)})) = 1 - f_{[t/\mu]}(e^{-\lambda R(t)})$$

and

$$\lim_{n\to\infty} \frac{1 - f_n(e^{-\lambda(1-f_n(0))})}{1 - f_n(0)} = \frac{\lambda}{(1+\lambda^\beta)^{1/\beta}}.$$

Therefore

$$\lim_{t\to\infty} \mathbf{E}(e^{-\lambda Y(t) R(t)} | Y(t) > 0) = 1 - \frac{\lambda}{(1+\lambda^\beta)^{1/\beta}}, \quad \lambda > 0,$$

The continuity theorem for Laplace transforms completes the proof. □

In order to formulate and prove the last limit theorem we need some preliminaries. Let us denote by T the time to extinction of the process Z_n, $n = 0, 1, 2, \ldots$, i.e.

$$Z_0 = 1 > 0, Z_n > 0, n = 2, 3, \ldots, T-1, Z_T = 0.$$

T is a positive integer valued random variable with distribution $\mathbf{P}\{T > n\} = \mathbf{P}\{Z_n > 0\}, n = 1, 2, \ldots$. If the condition (B1) with $\beta < 1$ is satisfied then $\mathbf{E}(T) = \sum_{n=1}^{\infty} \mathbf{P}\{T > n\} < \infty$. Further, it is clear that $\mathbf{E}(\min\{T, y\}) < \infty$ for every $y \geq 0$ and $\mathbf{E}(\min\{T, y\}) \uparrow \mathbf{E}(T) < \infty$, as $y \to \infty$.

Therefore,

$$\pi(y) = \mathbf{E}(\min\{T, [y]+1\})/\mathbf{E}(T), \quad y \geq 0$$

is a proper distribution on $[0, \infty)$.

Lemma 8 *Assume the conditions (R2) and (B1) with $\beta < 1$. Then*

$$\lim_{t\to\infty} \mathbf{P}\{N(t) < y | Z_{N(t)} > 0\} = \mathbf{E}(\min\{T, [y]+1\})/\mathbf{E}(T).$$

Proof. Note first that the events $\{Z_{N(t)} > 0\}$ and $\{N(t) < T\}$ are equivalent. Then, for the distribution $\pi_t(y) = \mathbf{P}\{N(t) \leq y | Z_{N(t)} > 0\}, t \geq 0, y \geq 0$ we have that

$$\pi_t(y) = \mathbf{P}\{N(t) \leq y | N(t) < T\} = \mathbf{P}\{N(t) \leq y, N(t) < T\}/\mathbf{P}\{N(t) < T\}.$$

Using the properties of the renewal sequence one can write $\mathbf{P}\{N(t) < T\} = \mathbf{P}\{S_T > t\}$. It is well known that if $\mathbf{E}(T) < \infty$ and (R2) holds then $\mathbf{P}\{S_T > t\} \sim \mathbf{E}(T)\mathbf{P}\{X_1 > t\}, t \to \infty$. Therefore,

$$\mathbf{P}\{N(t) < T\} \sim \mathbf{E}(T)\mathbf{P}\{X_1 > t\}, \quad t \to \infty.$$

Further we have in a similar way that

$$\mathbf{P}\{N(t) \leq y, N(t) < T\} = \mathbf{P}\{N(t) < [y]+1, N(t) < T\}$$

$$= \mathbf{P}\{N(t) < \min(T, [y]+1)\} = \mathbf{P}\{S_{\min\{T,[y]+1\}} > t\}$$
$$\sim \mathbf{E}(\min\{T, [y]+1\})\mathbf{P}\{X_1 > t\}, \quad t \to \infty.$$

Therefore one gets that as $t \to \infty$,

$$\pi_t(y) \to \mathbf{E}(\min\{T, [y]+1\})/\mathbf{E}(T), \text{ for } y \geq 0.$$

□

Theorem 9 *Assume the conditions (R2) and (B1) with $\beta < 1$. Then*

$$Y(t)|_{Y(t)>0} \xrightarrow{d} Y_\infty, \quad t \to \infty,$$

where

$$\mathbf{P}\{Y_\infty \le x\} = \int_0^\infty \mathbf{P}\{Z_{[y]} \le x | Z_{[y]} > 0\} d\pi(y).$$

Proof. Let $x > 0$ be fixed. Then, using the independence of Z_n and $N(t)$ one gets

$$\begin{aligned}
&\mathbf{P}\left\{Y(t) \le x | Y(t) > 0\right\} = \mathbf{P}\left\{Z_{N(t)} \le x | Z_{N(t)} > 0\right\} \\
&= \mathbf{P}\left\{Z_{N(t)} \le x, Z_{N(t)} > 0\right\} / \mathbf{P}\{Z_{N(t)} > 0\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\left\{Z_n \le x, Z_n > 0, N(t) = n\right\} / \mathbf{P}\{Z_{N(t)} > 0\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\left\{Z_n \le x | Z_n > 0\right\} \mathbf{P}\{N(t) = n | Z_{N(t)} > 0\} \\
&= \int_0^\infty \mathbf{P}\left\{Z_{[y]} \le x | Z_{[y]} > 0\right\} d\pi_t(y).
\end{aligned}$$

Finally from Lemma 8 one can see that

$$\lim_{t\to\infty} \int_0^\infty \mathbf{P}\left\{Z_{[y]} \le x | Z_{[y]} > 0\right\} d\pi_t(y) = \int_0^\infty \mathbf{P}\left\{Z_{[y]} \le x | Z_{[y]} > 0\right\} d\pi(y),$$

which completes the proof of the theorem. □

5 Conclusion remarks

The considered process can be compare with the classical Bellman-Harris branching process. Note that in the randomly indexed branching process all particles in a given generation have the same life span and they give birth of their daughters simultaneously (i.e. the life-times of the particles are dependent, but the numbers of offspring are independent). On the other hand, in the Bellman-Harris branching process both the life span and the offspring of every particle are independent and the evolutions of the particles are also independent.

The obtained limit theorems show that in case where the inter-arrival times have a finite mean the limit distributions are similar to those for critical Bellman-Harris branching processes under the relevant non-random normalization (Theorems 6 and 7).

In case when the time to extinction has finite mean ($\beta \in (0, 1)$) the limiting distribution is new (Theorem 9). It does not take place for the classical branching processes (as one can check in the well-known monographs of Athreya and Ney (1972) or Sevastyanov (1971)).

An open problem is to obtain the limiting distribution of the process in case where the interarrival times have infinite mean and the branching process has finite variance, i.e. the tail the distribution of the time to extinction is $\sim C/n$, $n \to \infty$. Let us note, that in this case, the asymptotic behavior of the probability for non-extinction is new (Theorem 5).

The proposed model can be generalized in several directions. First of all, instead of a classical BGW branching process one can consider more general models with random migration components investigated in Yanev and Mitov (1985), Yanev and Yanev (1996) and Yanev and Yanev (1997). On the other hand, one can investigate the case when the processes start with an increasing random number of ancestors as in Dion and Yanev (1997). Of course, the index process can be considered also in a more general situation (see e.g. Mitov and Yanev (2001)).

An interesting application of the randomly indexed branching processes could be realized in the Cell Biology similarly to the model of Crump and Mode (1969). Note that in our case the sister cells have identical life spans (i.e. with correlation 1). Probably this model will be interesting for further applications in PCR processes.

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