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SLOPES OF TILINGS

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ABSTRACT. We study here slopes of periodicity of tilings. A tiling is of slope θ if it is periodic along direction θ but has no other direction of periodicity.

We characterize in this paper the set of slopes we can achieve with tilings, and prove they coincide with recursively enumerable sets of rationals.

1. Introduction

The model of tilings was introduced by Wang [14] to study fragments of the first order theory. This model is described by geometrical local properties, deciding whether a given tile can be placed on a given cell based only on its surrounding neighbours.

While the definition of tilings is deceptively simple, they exhibit complex behaviours. As an example, the most basic problem (decide if a given tiling system can tile the plane) is undecidable [2]. This is due to both a straightforward encoding of Turing machines in tilings [3, 4, 13] and to the existence of so-called *aperiodic* tiling systems [8, 12], that can tile the plane but in no periodic way.

In this paper we explore the periodic behaviour of tiling systems. Periodic tilings have nice closure properties, in the sense that the image of a periodic point by a shift-preserving morphism (i.e. a block map) is again a periodic point. As a consequence, understanding the structure of the periodic points of a tiling system is a first step to decide when some tiling system embeds in another, or when two tilings systems are “isomorphic” (more accurately conjugate [9])

In dimension one, the question boils down to determine for a tiling system τ the set of integers n so that there is a valid tiling by τ of period (exactly) n . This question was answered successfully: Using automata theory, a complete characterization of the set of integers we can obtain this way was obtained [9].

The question is more delicate in two dimensions. We might break it down in two parts: Given a tiling system τ ,

- For which n is there a tiling of horizontal and vertical period n ?
- For which direction θ is there a tiling which is periodic only along direction θ ?

Key words and phrases: tilings, formal languages, foundations of computing.

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The authors gave an answer to the first question in [6]: Sets of integers we can obtain correspond to the complexity class **NE**. We deal in this paper with the second question, characterizing the set of *slopes* we can obtain by tiling systems.

While the answer in dimension one involves finite automata theory, it turns out that the good tool to solve the problem in higher dimensions is computability theory. The undecidability of the domino problem (deciding if a tiling system tiles the plane) is indeed not an anomaly: many combinatorial aspects of tilings can only be fully comprehended by means of recursivity theory arguments [1, 5, 10].

Along these lines, we will prove here the following theorem:

Theorem 1.1. *The sets of slopes of tilings are exactly the recursively enumerable sets of rationals.*

As a consequence, one might for example build a tiling system which admits slopes arbitrary close to 0, but does not admit 0 as a slope.

This paper is organized as follows. We first give the definition of tiling systems, and an encoding of Turing machines that will be used later. Then we proceed to the proof of the theorem. The main part of this paper is a construction, for any recursively enumerable set R , of a tiling system with R as a set of slopes.

2. Definitions

2.1. Tilings

Usually when considering tiling systems, Wang rules are used. We use here a generalization that is equivalent in terms of expressivity but makes the constructions easier.

While Wang rules consider only adjacent tiles only, our rules may consider an arbitrary large (but finite) neighborhood of tiles.

A *tiling* of \mathbb{Z}^2 with a finite set of tiles T is a mapping $c : \mathbb{Z}^2 \rightarrow T$. A *pattern* of neighborhood $N \subseteq \mathbb{Z}^2$ is a mapping from N to T . A pattern is finite if N is finite. A *tiling system* is a pair (T, F) , where F is a finite set of finite patterns. A tiling c is said to be *valid* if and only if none of the patterns of F ever appear in c . Since the number of forbidden patterns is finite, we could specify the rules by *allowed* patterns as well. We give an example of such a tiling system with the tiles of figure 1a and the forbidden patterns of figure 1b. The allowed tilings are shown in figure 2.



Figure 1: The set of tiles (a) and the forbidden patterns (b).

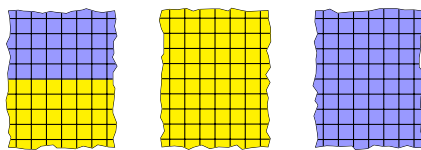


Figure 2: The only valid tilings of the system.

2.2. (a)periodicity

A tiling c is *periodic of period* $v = (v_x, v_y) \in \mathbb{Z}^2$ if for all points $x, y \in \mathbb{Z}$, $c(x, y) = c(x+v_x, y+v_y)$. The *direction* of a vector $v \neq (0, 0)$ is $\theta = v_y/v_x \in \mathbb{Q} \cup \{\infty\}$ with the convention $\theta = \infty$ if $v_x = 0$.

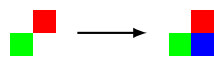
A tiling is *periodic along a direction* θ if it is periodic of period $v \neq (0, 0)$ and v is of direction θ .

For a given tiling c , there are three cases:

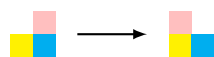
- Either c is periodic of period v, w and v, w are of different directions. In this case, the tiling c is *biperiodic*: there exists an integer $n \in \mathbb{N}$ (the period) so that $c(x, y) = c(x + n, y) = c(x, y + n)$, and as a consequence c is periodic along all directions $\theta \in \mathbb{Q} \cup \{\infty\}$
- c is periodic along one direction θ only. In this case, we will call θ the *slope* of c .
- c has no nonzero vector of periodicity. c is then called aperiodic.

The set of slopes of a tiling system τ , noted \mathcal{S}_τ , is the set of the slopes of all valid tilings by τ . As an example, the first tiling in fig.2 is periodic of vector $(1, 0)$ (hence of slope 0) and the two other tilings are biperiodic (hence have no slope). As a consequence, $\mathcal{S}_\tau = \{0\}$ for this example. Using rotated versions of this elementary tiling system, we can produce for each $\theta \in \mathbb{Q} \cup \{\infty\}$ a tiling system τ so that $\mathcal{S}_\tau = \{\theta\}$.

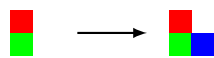
A tiling system is *aperiodic* if and only if it tiles the plane but all valid tilings are aperiodic. Such tiling systems have been shown to exist [2] and are at the core of the undecidability of the *domino problem* (decide whether a given tiling system admits a valid tiling). J. Kari [7] gave such a tiling system with an interesting property: determinism. A tiling system is *NW-deterministic* (for *North-West*) if it is given by forbidden patterns of shape \square and given two tiles respectively at the north and west of a given cell, there is at most one tile that can be put in this cell so that the finite pattern is valid. The mechanism is shown below:



If we modify the forbidden patterns of this tiling system in the following way :



a tile will be forced by the one on its west and on its northwest, we will call this East-determinism :



East-determinism has the interesting property that if we set a whole column of the plane then the whole half plane on its east will be determined by it. Moreover, this tiling system is also aperiodic (the tilings are skewed versions of the original one; diagonal lines are transformed into columns).

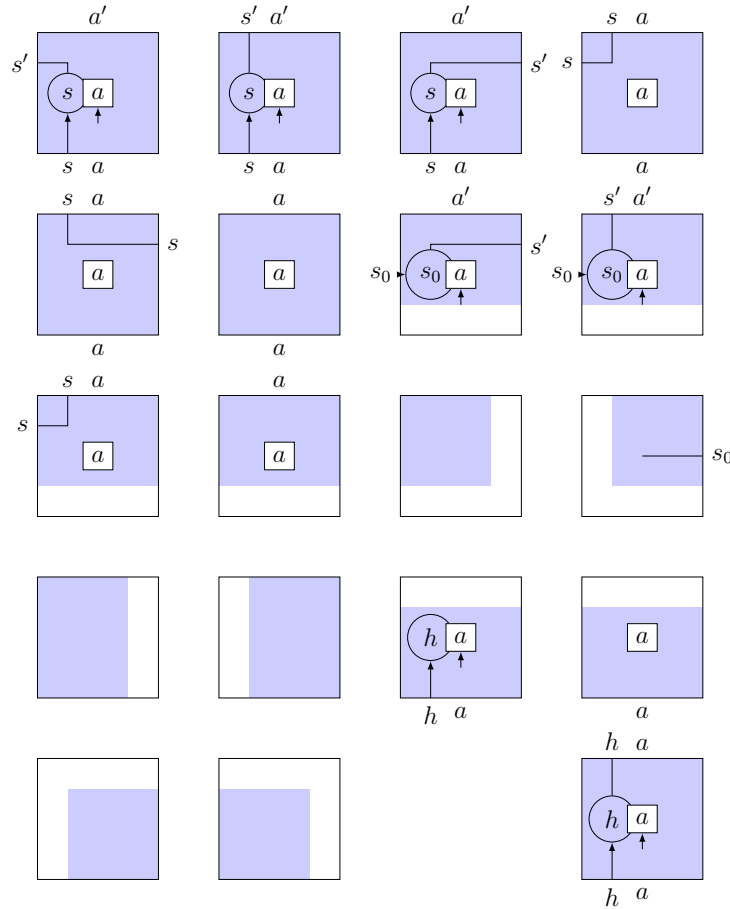


Figure 3: A tiling system, given by Wang tiles, simulating a Turing machine : the states are in the circles and the tape is in the rectangles.

2.3. Computability

The undecidability of the domino problem [2] hinted earlier also comes from a straightforward encoding of Turing machines into tilings. We provide here such an encoding for future reference.

For a given Turing machine M , consider the tiling system τ_M presented in figure 3. The tiling system is given by *Wang tiles*, i.e., we can only glue two tiles together if they coincide on their common edge. We now give some details on the picture:

- s_0 in the tiles is the initial state of the Turing machine.
- The first tile corresponds to the case where the Turing machine, given the state s and the letter a chose to go to the left and to change from s to s' , writing a' . The two other tiles are similar.
- h represents a halting state. Note that the only states that can appear in the last step of a computation (before a border appears) are halting states.

This tiling system τ_M has the following property: there is an accepting path for the word u in time (less than) t using space (less than) w if and only if we can tile a rectangle of size $(w + 2) \times t$ with white borders, the first row containing the input.

3. The sets of slopes are recursively enumerable

We say that a subset S of $\mathbb{Q} \cup \{\infty\}$ is *recursively enumerable* if there exists a Turing machine M that on input $(p, q) \in \mathbb{Z}^2 \neq (0, 0)$ halts if and only if $q/p \in S$.

$$\begin{aligned} \theta \in S &\implies \forall (p, q), q/p = \theta, M \text{ halts on } (p, q) \\ \theta \notin S &\implies \forall (p, q), q/p = \theta, M \text{ does not halt on } (p, q) \end{aligned}$$

The exact definition is irrelevant as all reasonable definitions will give rise to the same class. An alternative interesting definition is as follows: A set S is recursively enumerable if there exists a Turing machine M so that

$$\theta \in S \iff \exists (p, q), q/p = \theta \wedge M \text{ halts on } (p, q)$$

Using a known projection technique to go down to dimension 1, we prove here:

Lemma 3.1. *For any tiling system τ , S_τ is recursively enumerable.*

Proof. We first give a procedure to decide if there is a tiling which is $(n, 0)$ -periodic. Let k be an integer bigger than the size of any forbidden pattern in τ .

If w is a pattern of support $[0, n-1] \times [0, l]$ for some l , we write $w^{\mathbb{Z}}$ for the pattern of support $\mathbb{Z} \times [0, l]$ defined by $w_{i,j}^{\mathbb{Z}} = w_{(i \bmod n), j}$, that is for the horizontal repetition of w .

Let V be the set of all patterns w of size $n \times k$ so that $w^{\mathbb{Z}}$ is correctly tiled. Consider this a directed graph G , where there is an edge from v to w if and only if $(v \otimes w)^{\mathbb{Z}}$ is correctly tiled, where $v \otimes w$ denotes the pattern of size $n \times 2k$ obtained by putting w above v .

It is then clear that tilings of period $(n, 0)$ correspond to biinfinite walks on this graph, so that there exists a tiling of period $(n, 0)$ if and only if there exists a cycle in the graph G . Furthermore, there exist a tiling of period $(n, 0)$ which is not biperiodic if and only if we can find two distinct cycles C_1, C_2 in the graph so that C_2 is accessible from C_1 . All the construction is clearly algorithmic.

Now for a given (p, q) we use the same procedure, where w is a pattern of size $|p| \times k|q|$ and $w^{\mathbb{Z}}$ is of support $\{(i + np, j + nq), i \leq |p|, j \leq k|q|\}$ and defined by $w_{i+np, j+nq}^{\mathbb{Z}} = w_{i, j}$.

The following algorithm gives then the expected result: Starting from a given (p, q) , test all (p', q') so that $q'/p' = q/p$ to see if there exists a tiling which is (p', q') -periodic but not biperiodic. ■

4. The recursively enumerable sets are sets of slopes

Lemma 4.1. *For any recursively enumerable set $R \subseteq \mathbb{Q} \cup \{\infty\}$, there exists a tiling system τ , such that $\mathcal{S}_\tau = R$.*

Proof. We use for this proof techniques similar to [6]. We will construct for each Turing machine M , corresponding to a recursively enumerable set R , a tiling system τ whose slopes are exactly the rationals θ accepted by M . We assume that M takes θ as an input under the form (p, q) in binary and that its input depends only on q/p .

We will first build a tiling system τ that has as slopes $\{\theta \in R \mid 0 < \theta < 1\}$. The other cases are treated in the same way and the final tiling system is the disjoint union of the tiling systems treating each case. The special cases $\theta = 0$, $\theta = \infty$, and $\theta = \pm 1$ will be shortly discussed later on.

For the particular case where $p > q > 0$ we want to enforce the fact that when a tiling of the plane has exactly one direction of periodicity, this direction of periodicity has to be accepted by the Turing machine M . The tiling τ_M will enforce the skeleton described in figure 4, where each square encodes the computation by M proving that the slope θ is accepted. For this, we need the size of the square to be arbitrarily large independently of θ , so that the computation of M has enough time to accept. This skeleton in itself could be biperiodic, we will then color the background of each square to ensure the existence of tilings with only one direction of periodicity.

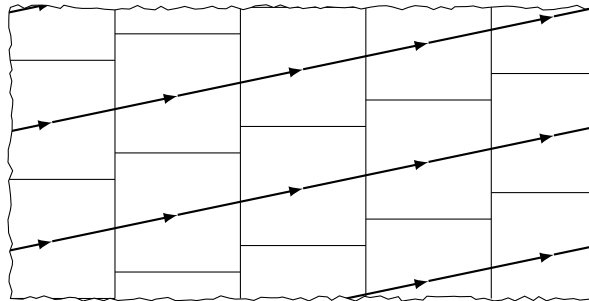


Figure 4: Skeleton of the tiling : when the tiling is periodic, the squares appear and each of them is the shifted version of its lower left neighbor. Inside the squares we will encode the Turing machine.

In order to enforce this skeleton, we will use several layers (or components), each of them having their own aim, and impose some constraints on how the layers may combine. We give here $\tau_M = C \times R \times W \times S \times P \times T_M \times A$ where :

- C will allow us to make the rows and columns,
- R to make the squares,
- W to force the periodicity vector and to write the input for the Turing machine,
- S to force the aperiodic background of the squares to be the same,
- P will reduce the size of the input,
- T_M will code the Turing machine M ,
- A will allow slopes of unique periodicity to appear.

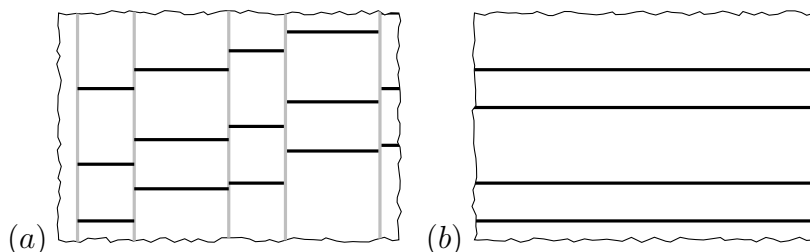


Figure 5: Valid periodic tilings are formed of columns of vertical breaking tiles (a) or of rows of horizontal breaking tiles (b). Between two columns of vertical breaking tiles there can be rows of horizontal breaking tiles.

We will now proceed to the details of the proof, by giving each component and explaining what it enforces.

Component C: The first component is made of an **East-deterministic** aperiodic set of tiles that we will call white tiles (the white background of figure 4), and we add two sets of tiles the horizontal breaking tiles $\{\blacksquare\}$ and the vertical breaking tiles $\{\boxtimes, \boxtimes, \blacksquare, \square\}$ (the horizontal and vertical lines of figure 4). The rules are simple :

- on the left of a \blacksquare there can only be a \blacksquare or a \boxtimes ,
- on the right of a \blacksquare there can only be a \blacksquare or a \boxtimes ,
- above and below a \blacksquare , there can only be a white,
- above a \boxtimes can only be a \blacksquare ,
- above a \blacksquare can only be a \boxtimes or a \blacksquare ,
- above a \boxtimes can only be a \square ,
- above a \square can only be a \boxtimes or a \square .

To put it in a nutshell, it means that horizontal breaking tiles forms rows that can only be broken by vertical breaking tiles, and vertical breaking tiles can only form columns that cannot be broken.

In a periodic tiling, we cannot have a quarter of plane filled with white (aperiodic tiles). As a consequence, periodic tilings at this stage are necessarily formed by a white background broken *infinitely many times* by horizontal or vertical breaking tiles.

One more rule we add is that the rules on white tiles "jump" over the black tiles. That is to say if we remove a black row, then the white tiles have to glue themselves together correctly. The valid tilings at this stage are represented on figure 5.

Component R: . The next component will force the apparition of squares between two columns of vertical breaking tiles and prevent several infinite rows of horizontal breaking tiles to appear. This layer is made of the set of tiles $\{\blacksquare, \blacksquare, \blacksquare, \blacksquare, \blacksquare, \blacksquare, \blacksquare\}$, the rules applied on this layer are given by Wang tiles. We superimpose the rules as follows :

- \blacksquare can only be superimposed to \blacksquare, \square ,
- \blacksquare can only be superimposed to \blacksquare ,
- \blacksquare goes on \boxtimes , and \blacksquare goes on \boxtimes ,
- $\blacksquare, \blacksquare, \blacksquare$ are superimposed to the white tiles.

Figure 6 shows how this component R forces rows of black tiles to appear between two gray columns. The distance between these black rows is exactly

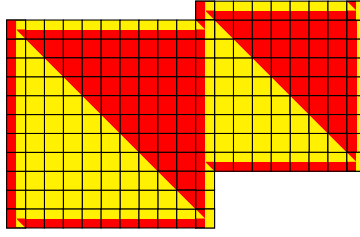
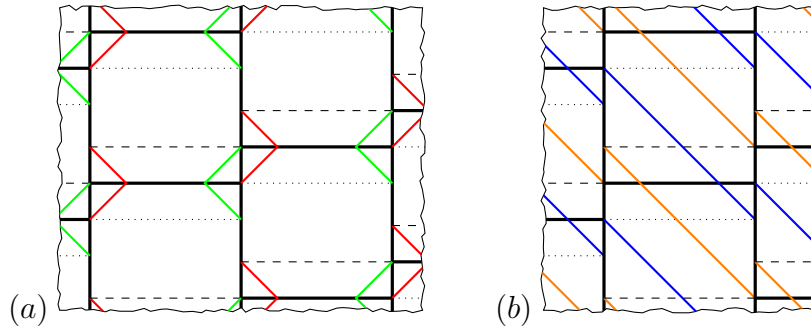
Figure 6: Component R forces squares.

Figure 7: The dotted row (resp. dashed) corresponds to the prolongation on the right (resp. left) of the black cells. In (a) the signals sent from the extremities of the rows forming the square forces the offset between rectangles of three neighboring columns to be exactly the same for any of them. In (b) the signals sent from the extremities force the distance between columns to be identical.

the distance between the gray columns thus black rows and gray columns form squares. At this stage the valid periodic tilings cannot be formed of only rows of black tiles anymore.

Component W : What this component does is that it synchronises the offsets between squares of two neighboring columns, and forces all columns to be at equal distance of their two neighboring columns, for all of them. As a side effect, it also writes the offset between two squares (which we call q) in each square. In order to do that, what we do is that we prolongate the black rows of each column into their direct neighbors with two new layers, one for the left and one for the right. The end of the black row then sends a diagonal signal which changes its direction when it collides with the projected lines of the neighbors and its collision with the column has to coincide with the projection of the other column. Figure 7.a shows how this mechanism works. The collision of the signal sent on the right extremity of the black lines marks the end of the input q on each square. We add two other sublayers to make the white rows of same width. The first one sends a signal from the left extremity of a black line which has to meet the next column at the exact point of the extension of the square. The second one does the same for the right extremity. Figure 7.b shows these signals.

Component S : This component is meant to synchronize the aperiodic backgrounds of all the squares. In order to do that, we only need to transmit the

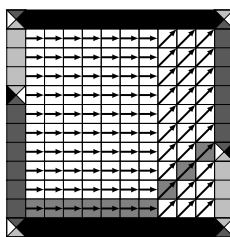


Figure 8: Tiles allowing to transmit the aperiodic background.

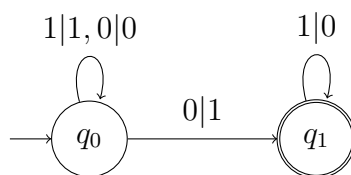


Figure 9: A transducer transforming n in binary into $n + 1$.

first column after a vertical breaking column since our initial aperiodic tiling system is East-deterministic.

In order to do that, we take these tiles $\{\boxminus, \boxplus, \boxtimes, \boxdot\}$, with the following rules :

- on the right, above and below a \boxplus there can only be a \boxplus or a \boxtimes .
- on the left of a \boxtimes we necessarily have a \boxtimes and the south western neighbor of a \boxtimes , if the tile is a white, is a \boxtimes or a \boxdot ,
- the lower left white tile of a square is necessarily a \boxdot . The rules on \boxdot is that there can only be a \boxdot or a \boxplus on a white tile to its right,
- the vertical/horizontal breaking tiles have necessarily a \boxminus on them.

The tiling obtained inside a square is shown on figure 8. We add a sublayer that is a copy of the white tiles with the rules that the tiles of this component on the right of this column are identical to the white ones on component C and that this copy is transmitted to the tile pointed by the arrow. Then with the property that the black tiles continue the rules on the whites, the whole aperiodic background between two vertical breaking columns is exactly the same but shifted by the offset.

Component P : Now each square contains two data: its size (p) and the offset to the next square q , both in unary. We will pass them as input to the Turing machine after some transformation.

The idea is to transform the unary input (p, q) into a smaller binary one (p', q') where $\text{gcd}(p', q')$ is not a multiple of two. Doing that is fairly easy : we first need to convert the input in binary; this can be done by the iteration of the transducer of figure 9: starting from $000\dots 00$ we obtain the binary representation of p (least significant bit on the rightmost part) in p iterations of the transducer. Then we strip the binary representation of p and q of their common last zeroes.

Component T_M : This layer implements the Turing machine M , the input has been computed by layer P . Note that the Turing machine has to halt for the tiling to be valid.

Component A: This layer is made of only two tiles, a yellow and a blue one. It will be superimposed to white tiles and to the ■ of the vertical breaking tiles of component C only. The rules are that two neighboring tiles (horizontally and vertically) have the same color. It is easy to see that the color is uniform inside a square and that it spreads to the upper right and lower left neighboring squares. Thus the squares along the direction of periodicity have the same color.

We now prove that the preceding construction works.

(1) Any slope is an accepted input of M : Let $\theta = q/p \in S_\tau$ be a slope of periodicity of τ , with $p > q > 0$ relatively prime.

By construction, the tiling has to be formed of squares of identical size with constant offset (components C , R , W). Their aperiodic background has to be the same on each column (component S), so that in fact the tiling is periodic along direction (m, n) where m and n denote respectively the width and offset of the tiling. As a consequence, the tiling is of slope $\theta = m/n = q/p \in]0; 1[$ and we have $(n, m) = 2^k k'(p, q)$ for some k, k' with k' odd.

Now the Turing Machine on each square has $(k'q, k'p)$ as an input and halts. Hence the slope $k'q/k'p$ is accepted by the machine, so $q/p \in R$, which proves $S_\tau \subseteq R \cap]0; 1[$.

(2) Any accepted input of M is a slope of some tiling: Let $\theta \in R$ be an accepted input of M with $\theta = q/p$, $p > q > 0$ and p, q relatively prime.

There exists a time t and a space s such that M accepts (p, q) in time t and space s and $s \leq t$. Take $(m, n) = 2^{\lceil \log t \rceil} (p, q) \geq (t, s)$. Now the $m \times m$ square is big enough for the computation on input (p, q) to succeed. Hence there is a tiling of period (m, n) and component A allows us to make the direction of periodicity unique by dividing the plane into two colors, half a plane yellow and half a plane blue. Hence $R \cap]0; 1[\subseteq S_\tau$.

This finishes the proof for the case $0 < \theta < 1$, i.e. $p > q > 0$.

The cases where $q > p > 0$, $-p > q > 0$, or $q > -p > 0$ are treated in a very similar way: rotating the tiling system we just constructed and changing the way the input is written on the tape (to invert the inputs, or add a minus sign) is enough. However the remaining cases ($p = \pm q, p = 0, q = 0$) need special treatment¹.

For these cases, the construction above does not work, by that we mean that just rotating it and modifying slightly the Turing machine of component T_M won't do the trick. However it is actually simpler. We now make squares facing one another, obtaining a regular grid. This requires less tiles for component C and no component W . Then according to the case, components C, S and A are modified as follows:

- for $p = q$ ($\theta = 1$), S just transmits diagonally the tiles. In component A , the color is synchronized from the top right corner to the next square at the north east. The case $p = -q$ is similar.
- for $q = 0$ ($\theta = 0$), S transmits horizontally, and the colors of component A are synchronized with the square on the right. The tiling can only be horizontally periodic if the Turing machine accepts it, this is the only way it can be periodic.

¹As this corresponds to four specific different θ s, note that we could treat them nonconstructively, adding if necessary four new tiling systems having prescribed slopes.

- for $p = 0$ ($\theta = \infty$), C has, instead of an east deterministic tiling, a north deterministic one. Components S and A are modified accordingly. The tiling can only be vertically periodic if the Turing machine accepts it and this is the only way it can be periodic. ■

5. Concluding remarks

We have shown that the sets of slopes of periodicity of tilings correspond exactly to the recursively enumerable (Σ_1^0) sets of rationals for tilings in dimension 2. Our intuition for analogous results in higher dimensions would be that the slopes of periodicity would then be characterized by Σ_2^0 sets [11], since knowing whether a tiling is periodic of vector v in dimension 3 is not decidable anymore but only Π_1^0 . Hence the following conjecture:

Conjecture 5.1. The sets of slopes of tilings in dimension $d \geq 3$ are exactly the Σ_2^0 subsets of $(\mathbb{Q} \cup \{\infty\})^{d-1}$.

An analogous construction to the one detailed here should work at least for dimension 3, it would however be tedious.

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