Decomposition Complexity

Alexander Shen

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DECOMPOSITION COMPLEXITY

ALEXANDER SHEN

LIF Marseille, CNRS & University Aix–Marseille; on leave from IITP RAS, Moscow
E-mail address: Alexander.Shen@lif.univ-mrs.fr
URL: http://www.lif.univ-mrs.fr/~ashen

Abstract. We consider a problem of decomposition of a ternary function into a composition of binary ones from the viewpoint of communication complexity and algorithmic information theory as well as some applications to cellular automata.

1. Introduction

The 13th Hilbert problem asks whether all functions can be represented as compositions of binary functions. This question can be understood in different ways. Initially Hilbert was interested in a specific function (roots of a polynomial as function of its coefficients). Kolmogorov and Arnold (see [5]) gave kind of a positive answer for continuous functions proving that any continuous function of several real arguments can be represented as a composition of continuous unary functions and addition (a binary function). On the other hand, for differentiable functions negative answer was obtained by Vituschkin. Later Kolmogorov interpreted this result in terms of information theory (see [4]): the decomposition is impossible since we have "much more" ternary functions than compositions of binary ones. In a discrete setting this information-theoretic argument was used by Hansen, Lachish and Miltersen ([3]. We consider similar questions in a (slightly) different setting.

Let us start with a simple decomposition problem. An input (say, a binary string) is divided into three parts x, y and z. We want to represent $T(x, y, z)$ (for some function T) as a composition of three binary functions:

$$T(x, y, z) = t(a(x, y), b(y, z)).$$

In other words, we want to compute $T(x, y, z)$ under the following restrictions: node A gets x and y and computes some function $a(x, y)$; node B gets y and z and computes some function $b(y, z)$; finally, the output node T gets $a(x, y)$ and $b(y, z)$ and should compute $T(x, y, z)$.

The two upper channels have limited capacity; the question is how much capacity is needed to make such a decomposition possible. If a- and b-channels are wide enough, we may transmit all the available information, i.e., let $a(x, y) = \langle x, y\rangle$ and

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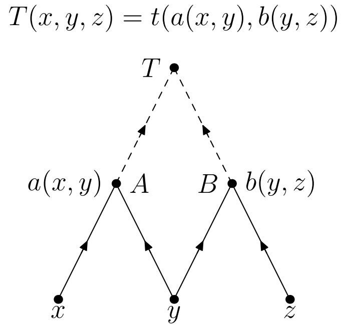


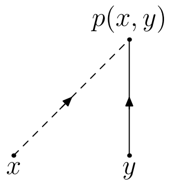
Figure 1: Information transmission for the decomposition.

$b(y,z)=\langle y,z\rangle$. Even better, we can split y in an arbitrary proportion and send one part with x and the other one with z.

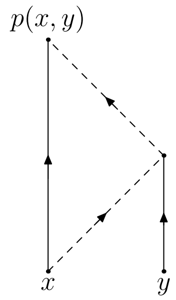
Is it possible to use less capacity? The answer evidently depends on the function T. If, say, $T(x,y,z)$ is `xor` of all bits in x, y and z, one bit for a- and b-values is enough. However, for other functions T it is not the case, as we see below.

In the sequel we prove different lower bounds for the necessary capacity of two upper channels in different settings; then we consider related questions in the framework of multi-source algorithmic information theory [7]).

Before going into details, let us note that the definition of communication complexity can be reformulated in similar terms: one-round communication complexity corresponds to the network



(dotted line indicates channel of limited capacity) while two-rounds communication complexity corresponds to the network



etc. Another related setting that appears in communication complexity theory: three inputs x,y,z are distributed between three participants; one knows x and y, the other knows y and z, the third one knows x and z; all three participants send their messages to the fourth one who should compute $T(x,y,z)$ based on their messages (see [6]).

One can naturally define communication complexity for other networks (we select some channels and count the bits that go through these channels).

2. Decomposition complexity

Now let us give formal definitions. Let $T = T(x, y, z)$ be a function defined on $\mathbb{B}^p \times \mathbb{B}^q \times \mathbb{B}^r$ (here $\mathbb{B}^k$ is the set of k-bit binary strings) whose values belong to some set M. We say that *decomposition complexity* of T does not exceed n if there exist $u + v \leqslant n$ and functions $a\colon \mathbb{B}^p \times \mathbb{B}^q \to \mathbb{B}^u$, $b\colon \mathbb{B}^q \times \mathbb{B}^r \to \mathbb{B}^v$ and $t\colon \mathbb{B}^u \times \mathbb{B}^v \to M$ such that

$$T(x, y, z) = t(a(x, y), b(y, z))$$

for all $x \in \mathbb{B}^p$, $y \in \mathbb{B}^q$, $z \in \mathbb{B}^r$. (As in communication complexity, we take into account the total number of bits transmitted via both restricted links. More detailed analysis could consider u and v separately.)

2.1. General upper and lower bounds

Since the logarithm of the image cardinality is an evident lower bound for decomposition complexity, it is natural to consider *predicates* T (so this lower bound is trivial). This makes our setting different from [3] where all the arguments and values have the same size. However, the same simple counting argument can be used to provide worst-case lower bounds for arbitrary functions.

Theorem 2.1. (Upper bounds) *Complexity of any function does not exceed $n = p + q + r$; complexity of any predicate does not exceed $2^r + r$ as well as $2^p + p$.*

(Lower bound) *If p and r are not too small (at least $\log n + O(1)$), then there exists a predicate with decomposition complexity $n - O(1)$.*

The second statement shows that the upper bounds provided by the first one are rather tight.

Proof. (Upper bounds) For the first bound one can let, say, $a(x, y) = \langle x, y\rangle$ and $b(y, z) = z$. (One can also split y between a and b in an arbitrary proportion.)

For the second bound: for each x, y the predicate $T_{x,y}$

$$z \mapsto T_{x,y}(z) = T(x, y, z)$$

can be encoded by 2^r bits, so we let $a(x, y) = T_{x,y}$ and $b(z) = z$ and get decomposition complexity at most $2^r + r$. The bound $2^p + p$ is obtained in a symmetric way.

(Lower bound) We can use a standard counting argument (in the same way as in [3]; they consider functions, not predicates, but this does not matter much.) Let us count how many possibilities we have for a predicate with decomposition complexity m or less. Choosing such a predicate, we first have to choose numbers u and v such that $u + v \leqslant m$. Without loss of generality we may assume that $u + v = m$ (adding dummy bits). First, let us count (for fixed u and v) all the decompositions where a has u-bit values and b has v-bit values. We have $(2^u)^{2^{p+q}}$ possible a's, $(2^v)^{2^{q+r}}$ possible b's and $2^{2^{u+v}}$ possible t's, i.e.,

$$2^{u2^{p+q}} \cdot 2^{v2^{q+r}} \cdot 2^{2^{u+v}} = 2^{u2^{p+q}+v2^{q+r}+2^{u+v}} \leqslant 2^{(u+v)2^{p+q}+(u+v)2^{q+r}+2^{u+v}}$$

possibilities (for fixed u, v). In total we get at most

$$m2^{m2^{p+q}+m2^{q+r}+2^m}$$

predicates of decomposition complexity m or less (the factor m appears since there are at most m decompositions of m into a sum of positive integers u and v). Therefore, if all 2^{2^n} predicates $\mathbb{B}^p \times \mathbb{B}^q \times \mathbb{B}^r \to \mathbb{B}$ have decomposition complexity at most m, then

$$m2^{m2^{p+q}+m2^{q+r}+2^m} \geqslant 2^{2^n}$$

or

$$\log m + m2^{p+q} + m2^{q+r} + 2^m \geqslant 2^n$$

At least one of the terms in the left-hand side should be $\Omega(2^n)$, therefore either $m \geqslant n - O(1)$ [if $2^m = \Omega(2^n)$], or $\log m \geqslant r - O(1)$ [if $m2^{p+q} \geqslant \Omega(2^n) = \Omega(2^{p+q+r})$], or $\log m \geqslant p - O(1)$ [if $m2^{q+r} \geqslant \Omega(2^n) = \Omega(2^{p+q+r})$]. ■

2.2. Bounds for explicit predicates

As with circuit complexity, an interesting question is to provide a lower bound for an explicit function; it is usually much harder than proving the existence results. The following statement provides a lower bound for a simple function.

Consider the predicate $T \colon \mathbb{B}^k \times \mathbb{B}^{2^{2k}} \times \mathbb{B}^k \to \mathbb{B}$ defined as follows:

$$T(x, y, z) = y(x, z)$$

where $y \in \mathbb{B}^{2^{2k}}$ is treated as a function $\mathbb{B}^k \times \mathbb{B}^k \to \mathbb{B}$.

Theorem 2.2. *The decomposition complexity of T is at least 2^k.*

(Note that this lower bound almost matches the second upper bound of Theorem 2.1, which is $k + 2^k$.)

Proof. Assume that some decomposition of T is given:

$$T(x, y, z) = t(a(x, y), b(y, z)),$$

where $a(x, y)$ and $b(y, z)$ consist of u and v bits respectively. Then every $y : \mathbb{B}^k \times \mathbb{B}^k \to \mathbb{B}$ determines two functions $a_y \colon \mathbb{B}^k \to \mathbb{B}^u$ and $b_y \colon \mathbb{B}^k \to \mathbb{B}^v$ obtained from a and b by fixing y. Knowing these two functions (and t) one should be able to reconstruct $T(x, y, z)$ for all x and z, since

$$T(x, y, z) = t(a_y(x), b_y(z)),$$

i.e., to reconstruct y. Therefore, the number of possible pairs $\langle a_y, b_y \rangle$, which is at most

$$2^{u2^k} \cdot 2^{v2^k},$$

is at least the number of all y's, i.e. $2^{2^{2k}}$. So we get

$$(u + v)2^k \geqslant 2^{2k},$$

or $u + v \geqslant 2^k$, therefore the decomposition complexity of T is at least 2^k. ■

Remarks.

1. In this way we get a lower bound $\Omega(\sqrt{n})$ (where n is the total input size) for the case when x and z are of size about $\frac{1}{2}\log n$. In this case this lower bound matches the upper bound of Theorem 2.1, as we have noted.

2. Here is another example where upper and lower bounds match. If the predicate $t(x,y,z)$ is defined as $x = z$, we need to transmit x and z completely (see [6] or use the pigeon-hole principle). So there is a trivial (and tight) linear lower bound if we let x and z be long (of $\Theta(n)$) size.

3. It would be interesting to get a linear bound for an explicit function in an intermediate case when x and z are short compared to y (preferable even of logarithmic size) but not as short as in Theorem 2.2 (so a non-constructive lower bound applies). Such a lower bound would mean that $a(x,y)$ or $b(y,z)$ has to retain a significant part of information in y. Intuitive explanation for this necessity could be: "since we do not know z when computing $a(x,y)$, we do not know which part of y-information is relevant and need to retain a significant fraction of y". Note that for the function T defined above this is not the case: not knowing z, we still know x so only one row (xth row) in the matrix y is relevant.

The natural candidate is the function $T'\colon \mathbb{B}^k \times \mathbb{B}^{2^k} \times \mathbb{B}^k \to \mathbb{B}$ defined by $T'(x,y,z) = y(x \oplus z)$. Here y is considered as a vector $\mathbb{B}^k \to \mathbb{B}$, not matrix, and $x \oplus z$ denotes bitwise XOR of two k-bit strings x and z. The size of x and z is about $\log n$ (where n is the total input size), and for these input sizes the worst-case lower bound is indeed linear. One could think that this lower bound could be obtained for T': "when computing $a(x,y)$ we do not know z, and $x \oplus z$ could be any bit string of length k, so all the information in y is relevant". However, this intuition is false, and there exists a sublinear upper bound $O(n^{0.92})$, see [1] or [6], p. 95.[1] (This upper bound should be compared to the $\Omega(\sqrt{n})$ lower bound obtained by reduction to T: in the special case when the left half of x and the right half of z contain only zeros, we get T out of T'.)

Question: what happens if we replace $x \oplus z$ by $x + z \bmod 2^k$ in the definition of T'? It seems that the upper bound argument does not work any more.

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[1]This upper bound is obtained as follows. Let us consider y as a Boolean function of k Boolean variables; $y\colon (u_1,\ldots,u_k) \mapsto y(u_1,\ldots,u_k)$. Such a Boolean function can be represented as a multilinear polynomial of degree k over the 2-element field $\mathbb{F}_2$. This polynomial $y(u_1,\ldots,u_k)$ has 2^k bit coefficients and is known when $a(x,y)$ or $b(y,z)$ are computed. Let us separate terms of "high" and "low" degree in this polynomial:

$$y(u_1,\ldots) = y_{\text{low}}(u_1,\ldots) + y_{\text{high}}(u_1,\ldots),$$

taking $\frac{2}{3}k$ as the threshold between "low" and "high". The polynomial y_{high} is included in a (or b) as is, just by listing all its coefficients. (We have about $2^{H(\frac{2}{3})k} \approx n^{0.92}$ of them, where H is Shannon entropy function.) For y_{low} we use the following trick. Consider $y(X_1 \oplus Z_1,\ldots,X_k \oplus Z_k)$ as a polynomial $\tilde{y}$ of $2k$ variables $X_1,\ldots,X_k,Z_1,\ldots,Z_k \in \mathbb{F}_2$. Its degree is at most $\frac{2}{3}k$, and each monomial includes at most $\frac{2}{3}k$ variables. So we can split $\tilde{y}$ again:

$$\tilde{y}(X_1,\ldots,Z_1,\ldots) = \tilde{y}_{x\text{-low}}(X_1,\ldots,Z_1,\ldots) + \tilde{y}_{z\text{-low}}(X_1,\ldots,Z_1,\ldots);$$

here the first term has small X-degree (Z-variables are treated as constants), and the second term has small Z-degree. Here "small" means "at most $\frac{1}{3}k$". All this could be done in both nodes (while computing a and b), since y is known there; X_i and Z_i are just variables. Now we include in $a(x,y)$ the coefficients of the polynomial $(Z_1,\ldots,Z_k) \mapsto \tilde{y}_{z\text{-low}}(x_1,\ldots,x_k,Z_1,\ldots,Z_k)$, and do the symmetric thing for $b(y,z)$. Both polynomial have degree at most $\frac{1}{3}k$, so we again need only $O(n^{0.92})$ bits to specify them.

3. Probabilistic decomposition

As in communication complexity theory, we may consider also probabilistic and distributional versions of decomposition complexity. In the probabilistic version we consider random variables instead of binary functions a, b, t (with shared random bits or independent random bits). In the distributional version we look for a decomposition that is Hamming-close to a given function.

It turns out that the lower bounds mentioned above are robust in that sense and remain valid for distributional (and therefore probabilistic) decomposition complexity almost unchanged.

Let ε be a positive number less than 1/2. We are interested in a minimum decomposition complexity of a function that ε-approximates a given one (coincides with it with probability at least $1-\varepsilon$ with respect to uniform distribution on inputs). For $\varepsilon \geqslant \frac{1}{2}$ this question is trivial (either 0 or 1 constant provide the required approximation). So we assume that some $\varepsilon < \frac{1}{2}$ is fixed (the $O()$-constants in the statements will depend on it).

A standard argument shows that lower bounds established for distributional decomposition complexity remain true for probabilistic complexity (where a, b, t use random bits and for every input x, y, z the random variable $t(a(x,y), b(y,z))$ should coincide with a given function with probability at least $1-\varepsilon$). So we may consider only the distributional complexity.

Theorem 3.1. (1) *Let $n = p+q+r$ and $p, r \geqslant \log n + O(1)$. Then there exists a predicate $T\colon \mathbb{B}^p \times \mathbb{B}^q \times \mathbb{B}^r \to \mathbb{B}$ such that decomposition complexity of any its ε-approximation is at least $n - O(1)$.*

(2) *For the predicate T used in Theorem 2.2 we get the lower bound $\Omega(2^k)$ (in the same setting).*

Proof. **1**. Assume this is not the case. We repeat the same counting argument as in Theorem 2.1. Now we have to count not only the predicates that have decomposition complexity at most m, but also their ε-approximations. The volume of an ε-ball in $\mathbb{B}^{2^n}$ is about $2^{H(\varepsilon)2^n}$, so the number of the centers of the balls that cover the entire space is at least $2^{(1-H(\varepsilon))2^n}$. So after taking the logarithms we get a constant factor $(1-H(\varepsilon))$, and the lower bound for m remains $n - O(1)$.

2. If the computation is correct for $1-\varepsilon$ fraction of all triples (x, y, z), then there exist $\varepsilon' < \frac{1}{2}$ and $\varepsilon'' > 0$ such that for at least ε''-fraction of all y the computation is correct with probability at least $1-\varepsilon'$ (with respect to uniform distribution on x and z). This means that ε'-balls around functions $(x,z) \mapsto t(a_y(x), b_y(z))$ cover at least ε''-fraction of all functions y. (See the proof of Theorem 2.2.) Again this gives us a constant factor before 2^{2k}, but here we do not take the logarithm second time, so we get $u + v \geqslant \Omega(2^k)$, not $2^k - O(1)$. ∎

4. Applications to cellular automata

An (one-dimensional) cellular automata is a linear array of cells. Each of the cells can be in some state from a finite set S of states (the same for all cells). At each step all the cells update their state; new state of a cell is some fixed function of its old state and the states of its two neighbors. All the updates are made synchronously.

Using a cellular automaton to compute a predicate, we assume that there are two special states 0 and 1 and a neutral state that is stable (if a cell and both its neighbors are in the neutral state, then the cell remains neutral). To compute $P(x)$ for a n-bit string x, we assemble n cells and put them into states that correspond to x; the rest of the (biinfinite) cell array is in a neutral state.

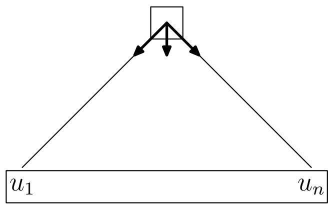
Then we start the computation; the answer should appear in some predefined cell (see below about the choice of this cell).

There is a natural non-uniform version of cellular automata: we assume that in each vertex of the time-space diagram an arbitrary ternary transition function (different for different vertices) is used. Then the only restriction is caused by the limited capacity of links: we require that inputs/outputs of all functions (in all vertices) belong to some fixed set S.

In this non-uniform setting a predicate P on binary strings is considered as a family of Boolean functions P_n (where P_n is a restriction of P onto n-bit strings) and for each P_n we measure the minimal size of a set S needed to compute P_n in a non-uniform way described above. If this size is an unbounded function of n, we conclude that predicate P is not computable by a cellular automaton. (In classical complexity theory we use the same approach when we try to prove that some predicate is not in P since it needs superpolynomial circuits in a non-uniform setting.)

As usual, getting lower bounds for nonuniform models is difficult, but it turns out that decomposition complexity can be used if the cellular automaton is required to produce the answer as soon as possible.

Since each cell gets information only from itself and its two neighbors, the first occasion to use all n input bits happens around time $n/2$ in the middle of the string:



Now we assume that the output of a cellular automaton is produced at this place (both in uniform and non-uniform model). (This is a very strong version of real-time computation by cellular automata; we could call it "as soon as possible"-computation.)

The next theorem observes that non-uniformly computable family of predicates is transformed into a function with small decomposition complexity if we split the input string in three parts.

Theorem 4.1. *Let* $T_k\colon \mathbb{B}^{k+f(k)+k} = \mathbb{B}^k \times \mathbb{B}^{f(k)} \times \mathbb{B}^k \to \mathbb{B}$ *be a family of predicates that is non-uniformly computable in this sense. Then the decomposition complexity of* T_k *is* $O(k)$*, and the constant in* O*-notation is the logarithm of the number of states.*

Proof. Consider Figure 2 where the (nonuniform) computation is presented (we use bigger units for time direction to make the picture more clear).

Let us look at the contents of the line of length $2k$ located k steps before the end of the computation. The left half is $a(x, y)$, the right half is $b(y, z)$ and the function t is computed by the upper part of the circuit. It is easy to see that $a(x, y)$ indeed depends only on x and y since information about z has not arrived yet; for

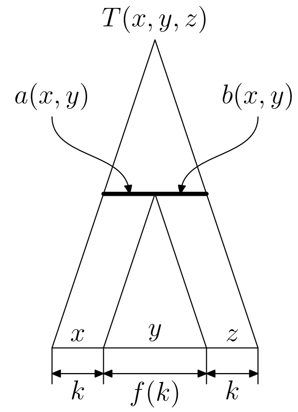


Figure 2: Automaton run and its decomposition.

the same reason $b(y, z)$ depends only on y and z. The bit size of $a(x, y)$ and $b(y, z)$ is $k \log \#S$. ■

Corollary 4.2. *The predicate T from Theorem 2.2 cannot be computed in this model.*

This predicate splits a string of length $k + 2^{2k} + k$ into three pieces x, y, z of length k, 2^{2k} and k respectively, and then computes $y(x, z)$. Note that this can be done by a cellular automaton in linear time. Indeed, we combine the string x and z into a $2k$-binary string; then we move this string across the middle part of input subtracting one at each step and waiting until our counter decreases to zero; then we know where the output bit should be read. So we get the following result:

Theorem 4.3. *There exists a linear-time computable predicate that is not computable "as soon as possible" even in a non-uniform model.*

Remark. This result and the intuition behind the proof are not new (see the paper of V. Terrier [8]; see also [2]). However, the explicit use of decomposition complexity helps to formalize the intuition behind the proof. It also allows us to show (in a similar way) that this predicate cannot be computed not only "as soon as possible", but even after $o(\sqrt{n})$ steps after this moment (which seems to be an improvement).

Another improvement that we get for free is that we cannot even ε-approximate this predicate in the "as soon as possible" model.

Question: There could be other ways to get lower bounds for non-uniform automata (=triangle circuits). Of course, there is a counting lower bound, but this does not give any explicit function. Are there some other tools?

5. Algorithmic Information Theory

Now we can consider the Kolmogorov complexity version of the same decomposition problem. Let us start with some informal comments. Assume that we have four binary strings x, y, z, t such that $\mathrm{K}(t|x, y, z)$ is small (we write $\mathrm{K}(t|x, y, z) \approx 0$, not specifying exactly how small should it be). Here $\mathrm{K}(\alpha|\beta)$ stands for conditional complexity of α when β is known, i.e., for the minimal length of a program that transforms β to α. (Hence our requirement says that there is a short program that produces t given x, y, z.)

We are looking for strings a and b such that $\mathrm{K}(a|x,y) \approx 0$, $\mathrm{K}(b|y,z) \approx 0$, and $\mathrm{K}(t|a,b) \approx 0$. Such a and b always exist, since we may let $a = \langle x, y\rangle$ and $b = \langle y, z\rangle$ (again, y can also be split between a and b). However, the situation changes if we restrict the complexities of a and b (or their lengths, this does not matter, since each string can be replaced by its shortest description). As we shall see, sometimes we need a and b of total complexity close to $\mathrm{K}(x)+\mathrm{K}(y)+\mathrm{K}(z)$ even if t has much smaller complexity. (Note that now we cannot restrict ourselves to one-bit strings t for evident reasons.)

To be specific, let us agree that all the strings x, y, z, t have the same length n; we look for strings a and b of length m, and "small" conditional complexity means that complexity is less than some c.

Theorem 5.1. *If $3c < n - O(1)$ and $2m+c < 3n-O(1)$, there exist strings x, y, z, t of length n such that $K(t|x,y,z) = O(\log n)$, but there are no strings a, b of length m such that*

$$K(a|x,y) < c, \qquad K(b|y,z) < c, \qquad K(t|a,b) < c.$$

For example, this is true if $c = O(\log n)$ and m is $1.5n - O(\log n)$ (note that for $m = 1.5n$ we can split y into two halves and combine the first half with x, and the second half with y).

Proof. Consider the following algorithm. Given n, we generate (in parallel for all $x, y \in \mathbb{B}^n$) the lists of those m-bit strings who have conditional complexity (with respect to x and y) less than c (one list for each pair x, y). Also we generate (in parallel for all strings a and b of length m) the lists of those strings t who have complexity less than c given a and b (one list for each pair a, b). At every step of enumeration we imagine that these lists are final and construct a quadruple x, y, z, t that satisfies the statement of the theorem. It is done as follows: we take a "fresh" triple x, y, z (that was not used on the previous steps of the construction), take all strings a that are in the list for x, y, take all strings b that are in the list for y, z, and take all strings t that are in the lists for those as and bs. Then we choose some t that does not appear in all these lists.

Such a t exists since we have at most 2^c strings a (for given x and y), and at most 2^c strings b (for given y and z). For every of 2^{2c} pairs (a, b) there are at most 2^c strings t, so in total at most 2^{3c} values of t are unsuitable, and we can choose a suitable one.

We also need to ensure that there are enough "fresh" pairs for all the steps of the construction. The new elements in the first series of lists may appear at most $2^n \times 2^n \times 2^c$ times (we have at most $2^n \times 2^n$ pairs (x, y) and at most 2^c values of a for each pair). Then we have $2^m \times 2^m \times 2^c$ events for the second series of lists. On the other hand, we have 2^{3n} triples (x, y, z), so we need the inequality

$$2^{2n+c} + 2^{2m+c} < 2^{3n},$$

which is guaranteed by our assumptions.

To run this process, it is enough to know n, so for every x, y, z, t generated by this algorithm we have $K(t|x,y,z) = O(\log n)$. (For given x, y, z only one t may appear since we take a fresh triple each time.) ∎

This result can be improved:

Theorem 5.2. *Assume that $3c < n - O(1)$ and $m \leqslant 1.5n - O(\log n)$. We can effectively construct for every n a total function $T : \mathbb{B}^n \times \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}^n$ such that for random (= incompressible) triple x, y, z and $t = T(x, y, z)$ the strings a and b of length m that provide a decomposition (as defined above) do not exist.*

The improvement is two-fold: first, we have a total function T (instead of a partial one provided by the previous construction); second, we claim that all random triples have the required property (instead of mere existence of such a triple).

Proof. Let us first deal with the first improvement. Consider multi-valued functions $A, B\colon \mathbb{B}^n \times \mathbb{B}^n \to \mathcal{P}(\mathbb{B}^m)$ that map every pair of n-bit strings into a 2^c-element set of m-bit strings. Consider also multi-valued function $F\colon \mathbb{B}^m \times \mathbb{B}^m \to \mathcal{P}(\mathbb{B}^n)$ whose values are 2^c-element sets of n-bit strings. We say that A, B, F *cover* a total function $T : \mathbb{B}^n \times \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}^n$ if for every $x, y, z \in \mathbb{B}^n$ there exist strings $a, b \in \mathbb{B}^m$ such that $a \in A(x, y)$, $b \in B(y, z)$, and $T(x, y, z) \in F(a, b)$.

Let us prove first the following combinatorial statement: *there exists a function T that is not covered by any triple of functions A, B, F*. This can be shown by a counting argument similar to the proof of Theorem 2.1. Indeed, let us compute the probability of the event "random function T is covered by some fixed A, B, F". This event is the intersection of independent events (for each triple x, y, z). For given x, y, z there are 2^c possible as, 2^c possible bs, and 2^c possible elements in $F(a, b)$ for each a and b, i.e., 2^{3c} possibilities altogether. Since $3c < n - O(1)$, each of the independent events has probability less than $\frac{1}{2}$, and their intersection has probability less than $2^{-2^{3n}}$.

This probability then should be multiplied by the number of triples A, B, F. For A and B we have at most $(2^m)^{2^n \times 2^n \times 2^c}$ possibilities, for F we have at most $(2^n)^{2^m \times 2^m \times 2^c}$ possibilities. So the existence of a function T not covered by any triple is guaranteed if

$$2^{m2^{2n+c}} \times 2^{m2^{2n+c}} \times 2^{n2^{2m+c}} \times 2^{-2^{3n}} < 1,$$

i.e.,

$$m2^{2n+c} + m2^{2n+c} + n2^{2m+c} < 2^{3n},$$

and this inequality follows from the assumptions.

The property "T can be covered by some triple A, B, F" can be computably tested by an exhaustive search over all triples A, B, F. So we can (for every n) computably find the first (in some order) function T that does not have this property. For these T there are some x, y, z that do not allow decomposition. Indeed, we can choose A so that $A(x, y)$ contains all strings a of length m such that $K(a|x, y) < c$, etc.

However, we promised more: we need to show not only the existence of x, y, z but that all incompressible triples (this means that $K(x, y, z) \geqslant 3n - O(1)$) have the required property. This is done in two steps. First, we show than (for some F that computably depends on n) most triples do not allow decomposition. Then we note that one can enumerate triples that allow decomposition, so they can be encoded by their ordinal number in the enumeration and therefore are compressible.

To make this plan work, we need to consider other property of function T. Now we say that T is covered by A, B, F if at least $2^{-O(1)}$-fraction of all triples (x, y, z) admit a and b. The probability of this event should now be estimated by Chernoff inequality (we guarantee first that the probability of each individual event is, say, twice smaller than the threshold), and we get a bound of the same type, with $\Omega(2^{3n})$ instead of 2^{3n}, which is enough. ■

In fact, this argument provides a decomposition complexity bound similar to Theorem 2.1, but now the functions a, b and t are multi-valued and we can choose any of their values to obtain $t(x, y, z)$.

Remarks and questions

1. Similar results can be obtained for more binary operations in the decomposition. Imagine that we have some strings x, y, z, t of length n such that $K(t|x, y, z)$ is small and want to construct some "intermediate" strings $u_1, \ldots, u_s$ such that in the sequence

$$x, y, z, u_1, u_2, \ldots, u_s, t$$

every string, starting from u_1, is conditionally simple with respect to some *pair* of its predecessors. We can use our technique to show that this is not possible if all u_i have length close to n and the number s is not large.

2. As before, it would be nice to get lower bounds for some explicit function $T(x, y, z)$ (even a non-optimal lower bound, like in Theorem 2.2) for the algorithmic information theory version of decomposition problem.

3. Many results of multi-source algorithmic information theory have some counterparts in classical information theory. Can we find some statement that corresponds to the lower bound for decomposition complexity?

4. Is it possible to use the techniques of [3] to get some bounds for explicit functions in algorithmic information theory setting?

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