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CHERN CLASSES OF TENSOR PRODUCTS

LAURENT MANIVEL

Abstract. We prove explicit formulas for Chern classes of tensor products of vector bundles, with coefficients given by certain universal polynomials in the ranks of the two bundles.

1. Introduction

Chern classes are ubiquitous in algebraic topology, differential geometry [Ch] or algebraic geometry [Gr, Fu]. They have nice formal properties like the Whitney sum formula, expressing the total Chern class of the direct sum of two complex vector bundles as the product of the total Chern classes of the two bundles. The situation is much more complicated for the other universal operation on vector bundles given by the tensor product: the Chern character is of course well behaved with respect to products, but computing the Chern classes of the tensor product of two vector bundles is often a painful task.

In this note we express the total Chern class of a tensor product in terms of the Schur classes of the two bundles. Recall that the Schur classes are certain universal polynomials in the Chern classes. They are indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$, and the Giambelli formula expresses them as determinants in the usual Chern classes:

$$s_\lambda(E) = \det \left(c_{\lambda_i^* - i + j}(E)\right)_{1 \leq i,j \leq s},$$

with the convention that $c_k(E) = 0$ for $k < 0$. Here λ^* denotes the conjugate partition of λ, defined by $\lambda_i^* = \#\{k, \lambda_k \geq j\}$, and s can be any integer greater or equal to λ_1. The Schur classes form an integral additive basis of the universal algebra generated by Chern classes, in particular there must be a universal formula of type

$$c(E \otimes F) = \sum_{\lambda,\mu} P_{\lambda,\mu}(e,f) s_\lambda(E) s_\mu(F)$$

for vector bundles E, F of respective ranks e, f, the coefficients $P_{\lambda,\mu}(e,f)$ being integers. In fact, the splitting principle allows to translate this identity into an identity of symmetric functions in two sets of variables, of size e and f respectively.

An expression of this type has already been given by A. Lascoux in [La] (see also [Mc, Ex.5 p.67]), the coefficients $P_{\lambda,\mu}(e,f)$ being expressed as determinants of binomial coefficients. Explicitly:

$$P_{\lambda,\mu}(e,f) = \det \left(\binom{f - \mu^*_{e+1-i} + e - i}{\lambda_j + e - j} \right)_{1 \leq i,j \leq e}.$$

Unfortunately, these determinants seem quite difficult to evaluate in practice. Moreover, their dependence in e and f appears quite unclear, while one can easily convince oneself that this dependence must be polynomial. Building on the work of Okounkov and Olchanski on shifted Schur functions [OO], we obtain a polynomial formula for $P_{\lambda,\mu}(e,f)$. This formula is very explicit, except maybe that it involves Littlewood-Richardson coefficients. Fortunately, our understanding of these fundamental coefficients has greatly improved

in the recent years. In particular, many very nice algorithms are known that allow to compute them quite efficiently.

2. The main result

For any two partitions λ and μ, consider the polynomial

$$P_{\lambda,\mu}(e,f) = \sum_{\nu} c^{\nu^*}_{\lambda^*,\mu}(e|\nu-\lambda)(f|\nu^*-\mu)/h(\nu). \tag{1}$$

The notation is the following. The partition λ^* is the *conjugate partition* of λ: when partitions are represented as Young diagrams, the lengths of the lines of λ^* are the lengths of the columns of λ. The coefficient $c^{\nu^*}_{\lambda^*,\mu}$ is a *Littlewood-Richardson coefficient* [Mc]. It can be non-zero only when $\nu^* \supset \lambda^*$, or equivalently $\nu \supset \lambda$, and $\nu^* \supset \mu$. The integer $h(\nu)$ is the product of the *hook-lengths* of the partition ν, where the hook-length of a box $\alpha = (i,j)$ in ν is $h(\alpha) = \nu_i + \nu^*_j - i - j + 1$. Finally, for a partition ρ, we let

$$(e|\rho) = \prod_{\alpha\in\rho}(e + c(\alpha)), \tag{2}$$

where $c(\alpha) = j - i$ is the *content* of the box $\alpha = (i,j)$. This is the *content polynomial* of [Mc, Ex.11 p.15]. In particular $(e|k) = e(e-1)\cdots(e-k+1)$. This definition extends to skew-partitions: if $\rho \supset \sigma$, we simply let $(e|\rho-\sigma) = (e|\rho)/(e|\sigma) = \prod_{\alpha\in\rho/\sigma}(e + c(\alpha))$.

Examples. Suppose that $\lambda = (\ell)$ and $\mu = (m)$ have only one non-zero part. Then $\lambda^* = (1^\ell)$ has all its non-zero parts equal to one. The Littlewood-Richardson coefficient $c^{\nu}_{\lambda^*,\mu}$ is non-zero only if $\nu = (m, 1^\ell)$ or $\nu = (m+1, 1^{\ell-1})$, and in both cases it is equal to one. We thus get

$$P_{(\ell),(m)}(e,f) = \frac{(f-1)\cdots(f-\ell)(e+\ell)(e-1)\cdots(e-m+1)}{\ell!(m-1)!(\ell+m)} + \frac{(f+m)(f-1)\cdots(f-\ell+1)(e-1)\cdots(e-m)}{(\ell-1)!m!(\ell+m)},$$

$$P_{(\ell),(m)}(e,f) = \binom{e-1}{m-1}\binom{f-1}{\ell-1}\frac{ef-\ell m}{\ell m}.$$

Suppose now that $\lambda = (1^\ell)$ and $\mu = (1^m)$ have no part bigger than one. By the previous computation and the symmetry properties stated in Proposition 1, we get that

$$P_{(1^\ell),(1^m)}(e,f) = \binom{e+m-1}{m-1}\binom{f+\ell-1}{\ell-1}\frac{ef-\ell m}{\ell m}.$$

The mixed case is more complicated. Suppose that $\lambda = (\ell)$ and $\mu = (1^m)$. Using the symmetry properties of our polynomials we may suppose that $\ell \geq m$. Then the Littlewood-Richardson coefficient $c^{\nu^*}_{\lambda^*,\mu}$ is non-zero only if $\nu = (\ell+m-n, n)$ for some n such that $0 \leq n \leq m$, in which case it is equal to one. We deduce the following formula:

$$P_{(\ell),(1^m)}(e,f) = \sum_{n=0}^{m} \frac{\binom{e+n-2}{n}\binom{e+\ell+m-n-1}{m-n}\binom{f+1}{n}\binom{f-m}{\ell-n}}{\binom{\ell+m-n+1}{n}\binom{\ell+m-2n}{m-n}}.$$

For a last example, suppose that $\lambda = \mu = (2,1)$. Then ν is one of the partitions $(4,2), (4,1,1), (3,3), (3,2,1), (3,1,1,1), (2,2,2), (2,2,1,1)$. The corresponding Littlewood-Richardson coefficients are one, except for $\nu = (3,2,1)$, for which it is two. We get

$$\begin{aligned} P_{(2,1),(2,1)}(e,f) &= \tfrac{e(e-2)(e-3)f(f+2)(f+3)+e(e+2)(e+3)f(f-2)(f-3)}{80} \\ &+ \tfrac{(e-3)(e-2)(e+2)(f-2)(f+2)(f+3)+(e-2)(e+2)(e+3)(f-3)(f-2)(f+2)}{72} \\ &+ \tfrac{e(e-1)(e-2)f(f+1)(f+2)+e(e+1)(e+2)f(f-1)(f-2)}{144} + 2\tfrac{e(e-2)(e+2)f(f-2)(f+2)}{45}, \\ P_{(2,1),(2,1)}(e,f) &= \tfrac{e(e^2-1)f(f^2-1)}{9} - e^2f^2 + e^2 + 2ef + f^2 - 4. \end{aligned}$$

Theorem 1. *Let E, F be two vector bundles of respective ranks e, f. The total Chern class of their tensor product is*

$$c(E \otimes F) = \sum_{\lambda,\mu} P_{\lambda,\mu}(e,f) s_\lambda(E) s_\mu(F).$$

Proof. By the splitting principle (see e.g. [Fu, Remark 3.2.3]), we are reduced to proving an identity between symmetric functions in two sets of variables $x_1, \dots, x_e$ and $y_1, \dots, y_f$. In the right hand side of the main formula, $s_\lambda(E)$ and $s_\mu(F)$ must then be replaced by the Schur functions $s_\lambda(x_1, \dots, x_e)$ and $s_\mu(y_1, \dots, y_f)$ in these variables.

In order to compute the right hand side, we start as in [La] with the Cauchy formula:

$$\prod_{\substack{1\le i\le e,\\ 1\le j\le f}} (1 + x_i y_j) = \sum_{\lambda \subset e\times f} s_\lambda(x_1, \dots, x_e) s_{\lambda^*}(y_1, \dots, y_f), \tag{3}$$

Replacing formally each x_i by x_i^{-1} and multiplying by $(x_1 \dots x_e)^f$ yields

$$\prod_{\substack{1\le i\le e,\\ 1\le j\le f}} (x_i + y_j) = \sum_{\lambda \subset e\times f} s_\lambda(x_1, \dots, x_e) s_{e\times f - \tilde{\lambda}}(y_1, \dots, y_f). \tag{4}$$

Here the notation is the following: the sum is over all partitions λ whose Young diagram fits into the rectangle $e \times f$, which means that $\lambda_1 \le f$ and $\lambda_1^* \le e$. Moreover we have denoted by $e \times f - \tilde{\lambda}$ the partition $(e - \lambda_f^*, \dots, e - \lambda_1^*)$. We deduce a first formula for the total Chern class of a tensor product:

$$\prod_{\substack{1\le i\le e,\\ 1\le j\le f}} (1 + x_i + y_j) = \sum_{\lambda \subset e\times f} s_\lambda(x_1, \dots, x_e) s_{e\times f - \tilde{\lambda}}(1 + y_1, \dots, 1 + y_f).$$

Now we use the *binomial theorem* [OO, Theorem 5.1] to obtain

$$s_{e\times f - \tilde{\lambda}}(1 + y_1, \dots, 1 + y_f) = \dim_{GL(f)}(e \times f - \tilde{\lambda}) \sum_\mu \frac{s_\mu^*(e \times f - \tilde{\lambda})}{(f|\mu)} s_\mu(y_1, \dots, y_f).$$

Hence the following expression for the coefficient $P_{\lambda,\mu}(e,f)$ of $s_\lambda(E)s_\mu(F)$ in $c(E \otimes F)$:

$$P_{\lambda,\mu}(e,f) = \dim_{GL(f)}(e \times f - \tilde{\lambda}) \frac{s_\mu^*(e \times f - \tilde{\lambda})}{(f|\mu)}.$$

As in [OO], we have denoted by $\dim_{GL(f)}(e \times f - \tilde{\lambda})$ the dimension of the Schur module $S_{e\times f - \tilde{\lambda}}\mathbb{C}^f$. It is given by the formula [Mc, Ex.4 p.45]:

$$\dim_{GL(f)}(e \times f - \tilde{\lambda}) = \frac{(f|e \times f - \tilde{\lambda})}{h(e \times f - \tilde{\lambda})}. \tag{5}$$

On the other hand s_μ^* denotes the *shifted Schur function* introduced in [OO]. Its evaluation on a partition can be expressed in terms of representations of symmetric groups. Indeed, [OO, Theorem 8.1] yields

$$s_\mu^*(e \times f - \tilde{\lambda}) = \frac{\dim[(e \times f - \tilde{\lambda})/\mu]}{\dim[e \times f - \tilde{\lambda}]} (|e \times f - \tilde{\lambda}| \mid |\mu|).$$

Here $[\rho]$ denotes the irreducible representation of the symmetric group $\mathcal{S}_{|\rho|}$ associated to the partition ρ. Its dimension is given by the celebrated hook-length formula [Mc, Ex.2

p.74]: letting $|\rho| = \rho_1 + \cdots + \rho_r$,

$$\dim[\rho] = \frac{|\rho|!}{h(\rho)}. \tag{6}$$

On the other hand $(e \times f - \tilde{\lambda})/\mu$ is not a partition but only a skew-partition, therefore the corresponding representation of the symmetric group is not irreducible and there is no generalization of the hook-length formula that would give its dimension. Nevertheless, its decomposition into irreducible representations is known to be given by Littlewood-Richardson coefficients [Mc, Ex.7 p.117]:

$$[(e \times f - \tilde{\lambda})/\mu] = \bigoplus_{\rho} c_{\rho,\mu}^{e\times f-\tilde{\lambda}}[\rho]. \tag{7}$$

For this Littlewood-Richardson coefficient to be non-zero, we need that ρ be contained in $e \times f - \tilde{\lambda}$. We can therefore write it as $\rho = e \times f - \tilde{\nu}$ for some partition ν containing λ. The coefficient $c_{\rho,\mu}^{e\times f-\tilde{\lambda}}$ is, by definition, equal to the multiplicity of the Schur module $S_{e\times f-\tilde{\lambda}}\mathbb{C}^f$ inside the tensor product $S_\rho\mathbb{C}^f \otimes S_\mu\mathbb{C}^f$. By [Ma, Lemma 1], it is also equal to the multiplicity of $S_{f\times e}\mathbb{C}^f = (\det \mathbb{C}^f)^e$ inside the triple tensor product $S_\rho\mathbb{C}^f \otimes S_\mu\mathbb{C}^f \otimes S_{\lambda^*}\mathbb{C}^f$. But then for the same reason, it is also equal to the multiplicity of $S_{\nu^*}\mathbb{C}^f$ inside $S_\mu\mathbb{C}^f \otimes S_{\lambda^*}\mathbb{C}^f$. In other words, we have proved that

$$c_{\rho,\mu}^{e\times f-\tilde{\lambda}} = c_{\lambda^*,\mu}^{\nu^*}.$$

Therefore we get from (7) the identity

$$\frac{\dim[(e \times f - \tilde{\lambda})/\mu]}{\dim[e \times f - \tilde{\lambda}]} = \sum_{\nu\subset e\times f} c_{\lambda^*,\mu}^{\nu^*} \frac{\dim[e \times f - \tilde{\nu}]}{\dim[e \times f - \tilde{\lambda}]}.$$

Using the hook-length formula (6) for $\dim[e \times f - \tilde{\nu}]$ and $\dim[e \times f - \tilde{\lambda}]$, we deduce that

$$P_{\lambda,\mu}(e, f) = \frac{(f|e \times f - \tilde{\lambda})}{(f|\mu)} \sum_{\nu\subset e\times f} \frac{c_{\lambda^*,\mu}^{\nu^*}}{h(e \times f - \tilde{\nu})}. \tag{8}$$

Lemma 1. $(f|e \times f - \tilde{\lambda})(e|\lambda) = (f|e \times f) = h(e \times f)$.

Proof. The quotient $(f|e \times f)/(f|e \times f - \tilde{\lambda})$ is the product of the $f + c(\alpha)$ for α a box in $e \times f$ but not in $e \times f - \tilde{\lambda}$. Such a box has coordinates $\alpha = (f - j + 1, e - i + 1)$ with $1 \leq j \leq \lambda_i$, and $f + c(\alpha) = f + (e - i + 1) - (f - j + 1) = e + j - i = e + c(\beta)$, where β is a box in λ. Hence $(f|e \times f)/(f|e \times f - \tilde{\lambda}) = (e|\lambda)$. The next identity is clear. □

This leads for our coefficient $P_{\lambda,\mu}(e, f)$ to the following expression:

$$P_{\lambda,\mu}(e, f) = \frac{1}{(e|\lambda)(f|\mu)} \sum_{\nu\subset e\times f} c_{\lambda^*,\mu}^{\nu^*} \frac{h(e \times f)}{h(e \times f - \tilde{\nu})}. \tag{9}$$

Our next task will be to evaluate the quotient $h(e \times f)/h(e \times f - \tilde{\nu})$. In order to do this we will divide the rectangle $e \times f$ into four sub-rectangles NO, NE, SO, SE, in such a way that NO ∪ NE is the set of boxes $\alpha = (i, j)$ with $i \leq f - \nu_1$, while NO ∪ SO is the set of boxes $\alpha = (i, j)$ with $j \leq e - \nu_1^*$. We will denote by $h_{NO}(e \times f - \tilde{\nu})$, and so on, the product of the hook-lengths of the boxes of $e \times f - \tilde{\nu}$ belonging to the rectangle NO.

Lemma 2. *The quotient $h(e\times f)/h(e\times f-\tilde{\nu})$ is the product of the following four partial quotients:*

$$\begin{aligned} h_{NO}(e\times f)/h_{NO}(e\times f-\tilde{\nu}) &= 1,\\ h_{NE}(e\times f)/h_{NE}(e\times f-\tilde{\nu}) &= (e|\nu)/(\nu_1^*|\nu),\\ h_{SO}(e\times f)/h_{SO}(e\times f-\tilde{\nu}) &= (f|\nu^*)/(\nu_1|\nu^*),\\ h_{NO}(e\times f)/h_{NO}(e\times f-\tilde{\nu}) &= h(\nu_1\times\nu_1^*)/h(\bar{\nu}), \end{aligned}$$

where $\bar{\nu}$ denotes the partition $\nu_1\times\nu_1^*-\tilde{\nu}$.

Proof. Straightforward. □

We deduce a polynomial expression for our coefficient $P_{\lambda,\mu}(e,f)$:

$$P_{\lambda,\mu}(e,f)=\sum_{\nu\subset e\times f} c^{\nu^*}_{\lambda^*,\mu}(e|\nu-\lambda)(f|\nu^*-\mu)\frac{h(\nu_1\times\nu_1^*)}{(\nu_1^*|\nu)(\nu_1|\nu^*)h(\bar{\nu})}. \tag{10}$$

Indeed, this expression is really polynomial in e and f since we can omit the condition that ν be contained inside the rectangle $e\times f$. If it is not, that is for example, if ν_1^* is bigger than e, then the box $\alpha=(e+1,1)$ is contained in ν and has content $c(\alpha)=-e$, which implies that $(e|\nu-\lambda)=0$.

In order to complete the proof of Theorem 1, there just remains to establish the following combinatorial lemma:

Lemma 3. *For any partition ν,*

$$(\nu_1^*|\nu)(\nu_1|\nu^*)h(\bar{\nu})=h(\nu_1\times\nu_1^*)h(\nu).$$

Proof. As $SL(\nu_1^*)$-modules, the Schur modules $S_\nu\mathbb{C}^{\nu_1^*}$ and $S_{\bar{\nu}}\mathbb{C}^{\nu_1^*}$ are dual one to each other. In particular they have the same dimension, which means that

$$\frac{(\nu_1^*|\nu)}{h(\nu)}=\frac{(\nu_1^*|\bar{\nu})}{h(\bar{\nu})}.$$

What remains to notice is the identity $(\nu_1^*|\bar{\nu})=h(\nu_1\times\nu_1^*)/(\nu_1|\nu^*)$, which is equivalent to Lemma 1. □

Remark. Each term in Lemma 3 is defined as a certain product of integers, and it seems that each integer p appears the same number of times in the left and right hand sides of the identity. What is the combinatorial explanation?

There is also a dual version of Theorem 1. Recall that total Segre class of a vector bundle E is defined as the formal inverse to the Segre class. More precisely, if we define the polynomial total Chern class of E as

$$c_t(E)=\sum_{k\geq 0}t^k c_k(E)=\prod_{i=1}^{e}(1+tx_i),$$

where $x_1,\ldots,x_e$ are the formal Chern roots, then the polynomial total Segre class of E is

$$h_t(E)=\sum_{k\geq 0}t^k h_k(E)=\prod_{i=1}^{e}(1-tx_i)^{-1}.$$

The total Segre class is $h(E)=h_1(E)$.

Theorem 2. *Let E, F be two vector bundles of respective ranks e, f. The total Segre class of their tensor product is*

$$h(E \otimes F) = \sum_{\lambda,\mu} (-1)^{|\lambda|} P_{\lambda,\mu^*}(e, -f) s_\lambda(E) s_\mu(F).$$

The coefficient $Q_{\lambda,\mu}(e, f) = (-1)^{|\lambda|} P_{\lambda,\mu^*}(e, -f)$ of $s_\lambda(E)s_\mu(F)$ in this formula is

$$Q_{\lambda,\mu}(e, f) = \sum_\nu c^\nu_{\lambda,\mu}(e|\nu - \lambda)(f|\nu - \mu)/h(\nu), \tag{11}$$

and is clearly symmetric.

Proof. A completely formal argument shows that Theorem 1 is also valid for formal bundles. Indeed, first observe that the identity $c(E \otimes (G \oplus H)) = c(E \otimes G)/c(E \otimes H)$ implies that the polynomials $P_{\lambda,\mu}(e, f)$ verify the relations

$$\sum_\mu c^\mu_{\varphi\psi} P_{\lambda,\mu}(e, g + h) = \sum_{\alpha,\beta} c^\lambda_{\alpha\beta} P_{\alpha,\varphi}(e, g) P_{\beta,\psi}(e, h). \tag{12}$$

This is a straightforward consequence of the fact that Littlewood-Richardson coefficients also govern the decomposition of Schur classes of direct sums [Mc, I, (5.9)]:

$$s_\mu(G \oplus H) = \sum_{\varphi,\psi} c^\mu_{\varphi\psi} s_\varphi(G) s_\psi(H). \tag{13}$$

Now suppose that the formal bundle $F = G - H$, of rank $f = g - h$, is the formal difference of two vector bundles G, H of ranks g, h. Here $f = g - h$ can be negative. Then $E \otimes F = E \otimes G - E \otimes H$, hence $c(E \otimes F) = c(E \otimes G)/c(E \otimes H)$. Theorem 1 for $F = G - H$ is thus equivalent to the identity

$$\begin{array}{rcl} \sum P_{\lambda,\mu}(e, f) s_\lambda(E) s_\mu(F) & = & \sum P_{\alpha,\beta}(e, f - g) s_\alpha(E) s_\beta(F - G) P_{\gamma,\delta}(e, g) s_\gamma(E) s_\delta(G) \\ & = & \sum P_{\alpha,\beta}(e, -h) P_{\gamma,\delta}(e, g) c^\theta_{\alpha,\gamma} s_\theta(E) s_\beta(F - G) s_\delta(G). \end{array}$$

But (12) being a polynomial identity, remains valid if we replace h by $-h$, and therefore the previous identity can be rewritten as

$$\sum P_{\lambda,\mu}(e, f) s_\lambda(E) s_\mu(F) = \sum P_{\epsilon,\eta}(e, g - h) c^\eta_{\beta,\delta} s_\epsilon(E) s_\beta(F - G) s_\delta(G),$$

which clearly holds true since (13) is also valid for formal bundles, meaning that

$$\sum_{\beta,\delta} c^\eta_{\beta,\delta} s_\beta(F - G) s_\delta(G) = s_\eta(F).$$

There just remains to apply Theorem 1, instead of F, to the formal bundle $-F$, of rank $-f$. We have $c_t(-F) = h_{-t}(F)$, and more generally $s_\mu(-F) = (-1)^{|\mu|} s_{\mu^*}(F)$. Therefore $h(E \otimes F) = c_{-1}(E \otimes (-F))$ is given by

$$\begin{array}{rcl} h(E \otimes F) & = & \sum_{\lambda,\mu} (-1)^{|\lambda|+|\mu|} P_{\lambda,\mu}(e, -f) s_\lambda(E) s_\mu(-F) \\ & = & \sum_{\lambda,\mu} (-1)^{|\lambda|} P_{\lambda,\mu}(e, -f) s_\lambda(E) s_{\mu^*}(F). \end{array}$$

This conclude the proof. □

3. Properties

3.1. Symmetries.

Proposition 1. *$P_{\lambda,\mu}(e,f)$ is an integer valued polynomial of degree $|\mu|$ in e and degree $|\lambda|$ in f, with the following symmetries:*

$$P_{\lambda,\mu}(e,f) = P_{\mu,\lambda}(f,e) = (-1)^{|\lambda|+|\mu|} P_{\lambda^*,\mu^*}(-e,-f).$$

Proof. The first assertion is obvious. To prove the first symmetry property we just need to notice that $c^{\nu^*}_{\lambda^*,\mu} = c^{\nu}_{\lambda,\mu^*}$ and $h(\nu) = h(\nu^*)$. To prove the second one we observe that if α is a box of $\nu - \lambda$, then the corresponding box α^* in the conjugate skew-partition $\nu^* - \lambda^*$ has opposite content. This implies that $(e|\nu - \lambda) = (-1)^{|\nu|-|\lambda|}(-e|\nu^* - \lambda^*)$, and the conclusion easily follows. □

3.2. Vanishing.

Proposition 2. *One has $P_{\lambda,\mu}(e,f) = 0$ whenever $\lambda_1^* \leq e < \mu_1$ or $\mu_1^* \leq f < \lambda_1$.*

Proof. If $c^{\nu^*}_{\lambda^*,\mu} \neq 0$, the Littlewood-Richardson rule implies that the first column of ν has length at least equal to μ_1. If $\lambda_1^* < \mu_1$, this implies that the intersection of $\nu - \lambda$ with the first column contains the boxes which belong to the lines numbered from $\lambda_1^* + 1$ to μ_1. These boxes have content $-\lambda_1^*, \ldots, -\mu_1 + 1$, hence $(e|\nu - \lambda)$ is divisible by $(e - \lambda_1^*) \cdots (e - \mu_1 + 1)$. Hence the first half of the claim, the second one following by symmetry. □

3.3. **Recursion.** Consider two complex vector bundles E, F of respective rank e, f and apply Theorem 1 to $E' = E \oplus \mathcal{O}$ and F, where $\mathcal{O}$ denotes the trivial line bundle. Then E' and E have the same Chern and Schur classes. Since $E' \otimes F = E \otimes F \oplus F$, the Whitney sum formula gives $c(E' \otimes F) = c(E \otimes F)c(F)$. Hence the relation

$$P_{\lambda,\mu}(e+1,f) = \sum_{\mu \to \theta} P_{\lambda,\theta}(e,f),$$

where $\mu \to \theta$ means that θ can be obtained from μ by suppressing some vertical strip. We can rewrite this as

Proposition 3. *The polynomials $P_{\lambda,\mu}(e,f)$ obey the following recursion rule:*

$$P_{\lambda,\mu}(e+1,f) - P_{\lambda,\mu}(e,f) = \sum_{\substack{\mu \to \theta, \\ \mu \neq \theta}} P_{\lambda,\theta}(e,f).$$

We can use the same idea to obtain more recursion formulas. Indeed, suppose that $E = M \oplus \mathcal{O}^{e-m}$ and $F = P \oplus \mathcal{O}^{f-p}$ for some vector bundles M, P or rank $m \leq e$ and $p \leq f$, respectively. Then E and M have the same Chern and Schur classes, as well as F and P. The relation

$$c(E \otimes F) = c(M \otimes P)c(M)^{f-p}c(P)^{e-m}$$

implies the following recursion formula, which is explicitly polynomial in e and f:

$$P_{\lambda,\mu}(e,f) = \sum_{\alpha,\beta} P_{\alpha,\beta}(m,p) \sum_{\sigma,\tau} d^{\lambda}_{\alpha,\sigma} d^{\mu}_{\beta,\tau} \frac{(e-m|\tau_1 + \cdots + \tau_p)}{\tau_1! \cdots \tau_p!} \frac{(f-p|\sigma_1 + \cdots + \sigma_m)}{\sigma_1! \cdots \sigma_m!}.$$

Here we have denoted by $d^{\lambda}_{\alpha,\sigma}$ the generalized Kostka coefficient defined as the multiplicity of s_λ inside the product $s_\alpha e_1^{\sigma_1} \ldots e_m^{\sigma_m}$.

3.4. **Leading term.**

Proposition 4. *The leading term of* $P_{\lambda,\mu}(e,f)$ *is* $e^{|\mu|}f^{|\lambda|}/h(\lambda)h(\mu)$.

Proof. Consider the previous formula for $P_{\lambda,\mu}(e,f)$. The term corresponding to the quadruple α,β,σ,τ has degree $|\tau| = \tau_1 + \cdots + \tau_p$ in e and $|\sigma| = \sigma_1 + \cdots + \sigma_m$ in f. But for $d^{\lambda}_{\alpha,\sigma}$ and $d^{\mu}_{\beta,\tau}$ to be non-zero we must have the relations $|\lambda| = |\alpha| + \sigma_1 + \cdots + m\sigma_m$ and $|\mu| = |\beta| + \tau_1 + \cdots + p\tau_p$. Hence $|\tau|$ and $|\sigma|$ will be maximal when α,β are empty and $\tau_u, \sigma_v = 0$ for $u, v > 1$. But then the coefficient $d^{\lambda}_{\alpha,\sigma}$ is just the Kostka number K_λ, the number of standard tableaux of shape λ. This is also the dimension of $[\lambda]$, and we can conclude the proof by applying the hook-length formula (6) once again. □

Comparing with the definition of $P_{\lambda,\mu}$ we deduce the following intriguing formula.

Corollary 1. *For any three partitions* λ,μ,ν, *let* $h^{\lambda,\mu}_{\nu} = h(\lambda)h(\mu)/h(\nu)$. *Then*

$$\sum_{\nu} h^{\lambda,\mu}_{\nu} c^{\nu}_{\lambda,\mu} = 1.$$

Is there any combinatorial interpretation ?

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