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Essential self-adjointness for combinatorial Schrödinger operators III- Magnetic fields

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Abstract

We define the magnetic Schrödinger operator on an infinite graph by the data of a magnetic field, some weights on vertices and some weights on edges. We discuss essential self-adjointness of this operator for graphs of bounded degree. The main result is a discrete version of a result of two authors of the present paper.

On définit l'opérateur de Schrödinger avec champ magnétique sur un graphe infini par la donnée d'un champ magnétique, de poids sur les sommets et de poids sur les arêtes. Lorsque le graphe est de degré borné, on étudie le caractère essentiellement auto-adjoint d'un tel opérateur. Le résultat principal est une version discrète d'un résultat de deux des auteurs du présent article.

Keywords: Magnetic field, Magnetic Schrödinger operator, Weighted graph Laplacian, Essential self-adjointness.

Math Subject Classification (2000): 05C63, 05C50, 05C12, 05C05, 35J10, 47B25.

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1 Introduction

In this work, we investigate essential self-adjointness for magnetic Schrödinger operators on infinite weighted graphs $G = (V, E)$ of bounded degree. It is a continuation of [To] and [CdV-To-Tr], where the same problem was studied in the non magnetic case, for metrically complete graphs ([To]) as well as non complete ones ([CdV-To-Tr]). The main result is a discrete version of the result in [CdV-Tr].

In the former paper ([CdV-To-Tr]), we proved that provided a growth condition on the potential W, namely $W \geq N/(2D^2)$ where D is the distance to infinity and N the maximal degree, the Schrödinger operator $\Delta_{\omega,c} + W$ is essentially self-adjoint. The operator $\Delta_{\omega,c}$ is defined, for any weights $\omega : V \longrightarrow \mathbb{R}_+^\star$ and $c : E \longrightarrow \mathbb{R}_+^\star$, by:

$$(\Delta_{\omega,c} f)(x) = \frac{1}{\omega_x^2} \sum_{y \sim x} c_{\{x,y\}} (f(x) - f(y)),$$

for any $f \in C_0(V)$ (finite supported function) and any vertex $x \in V$.

We will extend this result to the case of magnetic graph Laplacians. To do this, we establish a lower bound for the magnetic Dirichlet integral, in terms of an effective potential depending on the magnetic field (and not depending on the magnetic potential) and follow the method described in [CdV-To-Tr]. The precise setup of the result is described in the next section.

2 Magnetic fields on graphs

Discrete magnetic Schrödinger operators were already introduced by several authors, see [Li-Lo, CdV, CdV3, Tor].

2.1 Magnetic Schrödinger operators

Let $G = (V, E)$ be a *locally finite connected graph.* We will denote by $\{x, y\} \in E$ an edge and by $[x, y]$ and $[y, x]$ the two orientations of this edge.
We equip G with

(i) a set of non zero *complex* weights on oriented edges: $C_{xy} \in \mathbb{C}\backslash 0$ for $\{x, y\} \in E$ with $C_{yx} = \overline{C_{xy}}$. We write $C_{xy} = c_{xy} e^{i\alpha_{xy}}$ with $c_{xy} = |C_{xy}|$. We have $c_{xy} = c_{yx}$ and $\alpha_{xy} = -\alpha_{yx}$. The set $A = (\alpha_{xy})$ is called a *magnetic potential* on G.

(ii) a set of *strictly positive* weights on the vertices: $\omega_x,\ x \in V$.

The space of complex valued functions on the graph G is denoted here by

$$C(V) = \{f : V \longrightarrow \mathbb{C}\}$$

and $C_0(V)$ is the subspace of $C(V)$ of functions with finite support.

We consider the Hilbert space

$$l^2_\omega(V) = \{f \in C(V);\ \sum_{x\in V} \omega_x^2\ |f(x)|^2 < \infty\}$$

equipped with the Hermitian inner product given by

$$\langle f, g\rangle_{l^2_\omega} = \sum_{x\in V} \omega_x^2 f(x)\overline{g(x)}.$$

Let us consider the Hermitian form

$$Q_{c,A}(f) = \sum_{\{x,y\}\in E} c_{xy}|f(x) - e^{i\alpha_{xy}} f(y)|^2,$$

where we take only one term for each (unoriented) edge (the contribution is the same for both choices of orientations $[x, y]$ and $[y, x]$.)

The associated *magnetic Schrödinger operator* $H_{\omega,c,A}$ is given formally by

$$\langle H_{\omega,c,A} f|f\rangle_{l^2_\omega} = Q_{c,A}(f).$$

We get easily

$$H_{\omega,c,A} f(x) = \frac{1}{\omega_x^2} \sum_{y\sim x} c_{xy}[f(x) - e^{i\alpha_{xy}} f(y)].$$

This operator $H_{\omega,c,A}$ is Hermitian symmetric on $C_0(V)$ with the Hermitian product induced by $l^2_\omega(V)$.

2.2 Gauge transforms

Definition 2.1 *Let us consider a sequence of complex numbers $(u_x)_{x\in V}$ with $|u_x| \equiv 1$ and write $u_x = e^{i\sigma_x}$. The associated* gauge transform *U is the unitary map on $l^2_\omega(V)$ defined by*

$$(Uf)(x) = u_x f(x).$$

The map U acts on the quadratic forms $Q_{c,A}$ by

$$U^\star(Q_{c,A})(f) = Q_{c,A}(Uf).$$

Let us define the magnetic potential $U^\star(A)$ by $U^\star(Q_{c,A}) = Q_{c,U^\star(A)}$. The associated magnetic Schrödinger operator is $H_{\omega,c,U^\star(A)}$.

More explicitly, we get:

$$U^\star(A)_{xy} = \alpha_{xy} + \sigma_y - \sigma_x.$$

Let us denote by $C_1(G)$ the $\mathbb{Z}$-module generated by the oriented edges with the relation $[x, y] = -[y, x]$, and by ∂ the boundary operator

$$C_1(G) \xrightarrow{\partial} C(V, \mathbb{Z})$$

so that $\partial([x, y]) = \delta_y - \delta_x$ where, for $z \in V$, $\delta_z \in C(V, \mathbb{Z})$ is defined by $\delta_z(z) = 1$ and $\delta_z(z') = 0$ if $z' \neq z$. We will denote, for any $\gamma \in C_1(G)$, $\gamma = \sum_{e \in E} \gamma(e)e$.

The space of cycles, denoted by $Z_1(G)$, is the kernel of the boundary operator. If G is connected and finite, it is a known fact (see [Bi], part 1, chapters 4, 5) that $Z_1(G)$ is a free $\mathbb{Z}-$module, of rank $\#V - \#E + 1$, with a basis of geometric cycles $\gamma = [x_0, x_1] + [x_1, x_2] + \cdots + [x_{n-1}, x_n]$ with, for $i = 0, \cdots, n-1$, $\{x_i, x_{i+1}\} \in E$, and $x_n = x_0$.

We will construct a basis of cycles for any graph G.

Zorn's Lemma allows to show the existence of a maximal tree T of G. We have $V(T) = V(G)$ and we denote E' the set of all edges of G which are not edges of T. We choose an orientation for each edge of E'. For any (oriented) edge $[x, y] \in E'$, there exists a unique simple path β_{yx} in the tree T linking y to x. So $\gamma_{xy} = [x, y] + \beta_{yx}$ is a geometric cycle of G.
Let $\gamma \in Z_1(G)$, we set: $\gamma' = \gamma - \sum_{e \in \gamma \cap E'} \gamma(e)\gamma_e$, where we denote $\gamma_e = \gamma_{xy}$, if $e = [x, y]$ is an oriented edge which is in E' and included in the cycle γ. Then γ' is a cycle of G with support in T. So it vanishes and we have: $\gamma = \sum_{e \in \gamma \cap E'} \gamma(e)\gamma_e$.
We have proved the following Lemma:

Lemma 2.1 *Let T a maximal tree of G. To each oriented edge $[x, y]$ of G, if $\{x, y\}$ is not an edge of T, we have associated a unique cycle γ_{xy} of the graph G including $[x, y]$. The set of all such cycles is a basis of $Z_1(G)$.*

Definition 2.2 *Let us define the* holonomy map*:*

$$\mathrm{Hol}_A : Z_1(G) \to \mathbb{R}/2\pi\mathbb{Z}$$

by

$$\mathrm{Hol}_A([x_0, x_1] + [x_1, x_2] + \cdots + [x_{n-1}, x_0]) = \alpha_{x_0x_1} + \cdots + \alpha_{x_{n-1}x_0}.$$

Proposition 2.1 *With the notations above we have:*

(i) *The map $A \to \mathrm{Hol}_A$ is surjective onto $\mathrm{Hom}_{\mathbb{Z}}(Z_1(G), \mathbb{R}/2\pi\mathbb{Z})$.*

(ii) $\mathrm{Hol}_{A_1} = \mathrm{Hol}_{A_2}$ *if and only if there exists a gauge transform* U *so that* $U^\star(A_2) = A_1$.

Proof.–

For (i), let $hol \in \mathrm{Hom}_{\mathbb{Z}}(Z_1(G), \mathbb{R}/2\pi\mathbb{Z})$ and let T a maximal tree. We choose $A = (\alpha_{xy})$ such that $\alpha_{xy} = hol(\gamma_{xy})$ if $\{x, y\} \in E'$ and $\alpha_{xy} = 0$ if $\{x, y\} \in E(T)$, see Lemma 2.1.
Then we have $hol = Hol_A$.
For (ii), it suffices to prove that, if $Hol_A = 0$, then there exists a gauge transform U so that $U^\star(A) = 0$.
We must find a sequence (σ_x) satisfying the equality $\sigma_x = \alpha_{xy} + \sigma_y$, for any edge $\{x, y\}$.
We fix $x_0 \in V$, set $\sigma_{x_0} = 0$. If $x \in V \setminus \{x_0\}$, there exists a path $x_0, x_1, ..., x_{n-1}, x_n = x$ connecting x_0 to x, and we set $\sigma_x = \alpha_{xx_{n-1}} + ... + \alpha_{x_2x_1} + \alpha_{x_1x_0}$. This doesn't depend on the path from x_0 to x, since $Hol_A(\gamma) = 0$ for any cycle γ.

□

In the case of finite planar graphs, the assertions (i) and (ii) are similar to respectively Lemma 2.2 and Lemma 2.1 in [Li-Lo].

Definition 2.3 *A* magnetic field B *on the graph* G *is given by an holonomy map*

$$\mathrm{hol} \in \mathrm{Hom}_{\mathbb{Z}}(Z_1(G), \mathbb{R}/2\pi\mathbb{Z}).$$

If B *is associated to the* magnetic potential A*, we write* $B = dA$ *and we have* $\mathrm{hol} = \mathrm{Hol}_A$.

Remark 2.1 *The magnetic Schrödinger operator* $H_{\omega,c,A}$ *is uniquely defined, up to unitary conjugation, by the data of the magnetic field* B*, the weights* $(c_{xy})_{\{x,y\}\in E}$ *and the weights* $\omega = (\omega_x)_{x\in V}$.

2.3 Norms of magnetic fields

Definition 2.4 *If* $G = (V, E)$ *is a finite connected graph with a magnetic field* B*, we define the norm* $|B|$ *of* B *as the lowest eigenvalue of* $H_{1,1,A}$ *on* $l^2_\omega(V)$ *with* $\omega \equiv 1$*, for any* A *with* $dA = B$.

Lemma 2.2 *We have* $|B| = 0$ *if and only if* $\mathrm{Hol}_A = 0$.

Proof.–

If $\mathrm{Hol}_A = 0$, $H_{1,1,A}$ is unitarily equivalent to $H_{1,1,0}$ whose lowest eigenvalue is 0 with constant eigenfunctions.
Conversely, let $f \neq 0$ with $H_{1,1,A} f = 0$ and hence $Q_{1,A}(f) = 0$. This implies that all terms in the expression of $Q_{1,A}(f)$ vanish : for any edge $\{x, y\}$ we have $f(x) = e^{i\alpha_{xy}} f(y)$. If $\gamma = [x_0, x_1] + [x_1, x_2] + \cdots + [x_{n-1}, x_0]$ is a cycle, we have in particular

$$f(x_n) = e^{i\alpha_{x_n x_{n-1}}} f(x_{n-1}) = \cdots = e^{-i\mathrm{Hol}_A(\gamma)} f(x_0) \ .$$

Hence

$$e^{-i\mathrm{Hol}_A(\gamma)} = 1.$$

□

Lemma 2.3 *Let $G = \mathbb{Z}/N\mathbb{Z}$ be the cyclic graph with N vertices, Ω be the holonomy of $\gamma = [0, 1] + [1, 2] + \cdots [N - 1, 0]$ and $\delta = \inf_{k\in\mathbb{Z}} |\Omega - 2\pi k|$.*
If B denotes the magnetic field such that $B(\gamma) = \Omega$ we have

$$|B| = |1 - e^{i\delta/N}|^2.$$

In particular, the maximal value

$$|B| = |1 - e^{i\pi/N}|^2$$

is obtained for $B = \pi$.

Proof.–

We can choose A so that

$$Q_{1,A}(f) = \sum_{x=0}^{N-1} |f(x) - e^{i\Omega/N} f(x + 1)|^2.$$

The eigenvectors are the N complex functions on V:

$$f_\xi : x \to \xi^x$$

where $\xi^N = 1$. We have $\|f_\xi\|^2_{l^2} = N$ and

$$Q_{1,A}(f_\xi) = N|1 - \xi e^{i\Omega/N}|^2.$$

□

2.4 Lower bounds using an effective potential

Definition 2.5 *Let $m \in \mathbb{N}$. A* good covering of degree m *of $G = (V, E)$ is a family $G_l = (V_l, E_l)$ with $l \in L$ of finite connected sub-graphs of G so that*

(i) $V = \cup_{l \in L} V_l$

(ii) *for any $\{x, y\} \in E$,*

$$0 < \#\{l \in L \mid \{x, y\} \in E_l\} \leq m.$$

Example 2.1 *Let G the 1-skeleton of a triangulation of the plane $\mathbb{R}^2$. Then the set of all the triangles of this triangulation is a good covering of degree 2.*

Remark 2.2 *A graph G of bounded degree admits good coverings given by the:*

Proposition 2.2 *Let $G = (V, E)$ be a graph of bounded degree N. For $k \geq 1$ and $x \in V$, let*

$$G_x^k = \{y \in V,\ d(x, y) \leq k\}.$$

The family $(G_x^k)_{x \in V}$ is a good covering of G of degree $m = \frac{N(N-1)^k - 2}{N-2}$ of the graph G.

The main estimate is given by the following Theorem:

Theorem 2.1 *Let $(G_l)_{l \in L}$ a good covering of degree m of G. Then for any $f \in C_0(V)$,*

$$Q_{c,A}(f) \geq \sum_{x \in V} W(x) \omega_x^2 |f(x)|^2$$

with the effective potential

$$W(x) = \frac{1}{m} \sum_{\{l \in L | x \in V_l\}} |B_l| \inf_{\{y,z\} \in E_l} c_{yz} \tag{1}$$

where $|B_l|$ is the norm of the restriction of B to G_l.

Proof.–

From the definition of $Q_{c,A}$ and m, we have

$$Q_{c,A}(f) \geq \frac{1}{m} \sum_{l \in L} \sum_{\{x,y\} \in E_l} c_{xy} |f(x) - e^{i\alpha_{xy}} f(y)|^2.$$

Using the definition of $|B|_l$, we get

$$Q_{c,A}(f) \geq \frac{1}{m} \sum_{l} \left(\inf_{\{y,z\} \in E_l} c_{yz} \right) |B|_l \left(\sum_{x \in V_l} \omega_x^2 |f(x)|^2 \right)$$

which gives the lower bound.

□

3 Magnetic confinement

We want to find a criterion similar to the main result of [CdV-Tr] which says that if $(c, |B|)$ grows fast enough near infinity, then $H_{\omega,c,B}$ is essentially self-adjoint on $C_0(V)$. We will use Theorem 4.3 of [CdV-To-Tr] which gives the case where $B = 0$.

Let us define the distance d_p given in terms of the weights ω and c as follows:

$$p_{xy} = \frac{\min(\omega_x, \omega_y)}{\sqrt{c_{xy}}}$$

for any vertices $x, y \in V$.

Define also $D(x)$ $(\leq \infty)$ (see [CdV-To-Tr]) as the distance from a vertex x to the boundary V_∞.

Theorem 3.1 *Let G a graph with maximal degree N and let $(G_l)_{l\in L}$ a good covering of degree m of G. If there exists M so that*

$$W(x) \geq N/2D(x)^2 - M,$$

where W is the effective potential defined in (1), then $H_{\omega,c,B}$ is essentially self-adjoint.

Remark 3.1 *Theorem 3.1 holds in particular if (G, d_p) is a complete metric space.*

Proof.–

The proof follows the steps of the proof of Theorem 4.3. in [CdV-To-Tr]. In particular we use the following Agmon estimate.

Lemma 3.1 *Let v be a weak solution of $Hv = 0$, and let $f = \overline{f} \in C_0(V)$ a real function with finite support in V. Then*

$$\langle fv\ , H(fv)\rangle \ = \ \frac{1}{2}\sum_{x\in V}\sum_{y\sim x} \Re[v(x)\overline{v}(y)C_{yx}](f(x) - f(y))^2 \qquad (2)$$

Proof.–

We denote here $H \equiv H_{\omega,c,B}$.

The proof is a simple calculation:

$$\begin{aligned}\langle fv\ , H(fv)\rangle &= \sum_{x\in V} f(x)v(x)\left(\sum_{y\sim x} c_{xy}[f(x)\overline{v}(x) - e^{-i\alpha_{xy}} f(y)\overline{v}(y)]\right.\\ &\quad \left. + \sum_{y\sim x} W(x)f(x)\overline{v}(x)\right)\\ &= \sum_{x\in V} f(x)v(x)\left(\sum_{y\sim x} \overline{C_{xy}}(f(x) - f(y))\overline{v}(y)\right)\end{aligned}$$

where we used the fact that $\overline{H}\overline{v} = 0$.
An edge $\{x, y\}$ contributes to the first sum twice. So the total contribution is

$$f(x)v(x)\,\overline{C_{xy}}(f(x)-f(y))\overline{v}(y)-f(y)v(y)C_{xy}(f(x)-f(y))\overline{v}(x)$$

so

$$\langle fv\,, H(fv)\rangle = \sum_{\{x,y\}\in E} [f(x)-f(y)]\,[f(x)v(x)C_{yx}\overline{v}(y) - f(y)v(y)C_{xy}\overline{v}(x)]$$

Noticing that the quantity is real, we take the mean value of the expression and of its conjugate then we get the result.

□

From Lemma 3.1 we derive the following Theorem.

Theorem 3.2 *Let v be a solution of $(H-\lambda)v = 0$. Assume that v belongs to $l^2_\omega(V)$ and that there exists a constant $c > 0$ such that, for all $u \in C_0(V)$,*

$$\langle u|(H-\lambda)u\rangle_{l^2_\omega} \geq \frac{N}{2}\sum_{x\in V}\max\left(\frac{1}{D(x)^2}, 1\right)\omega_x^2|u(x)|^2 + c\|u\|^2_{l^2_\omega}, \quad (3)$$

then $v \equiv 0$.

Proof.–

We refer to [CdV-To-Tr] for the proof, since the fact that we use complex functions does not make any change in it.

□

Then Theorem 3.1 follows from Theorems 2.1 and 3.2 since we have for any $u \in C_0(V)$:

$$\langle u|Hu\rangle_{l^2_\omega} \geq \sum_{x\in V} W(x)\omega_x^2|u(x)|^2,$$

so

$$\langle u|(H-\lambda)u\rangle_{l^2_\omega} - \frac{N}{2}\sum_{x\in V}\frac{1}{D(x)^2}\omega_x^2|u(x)|^2 \geq \sum_{x\in V} -(M+\lambda)\|u\|^2_{l^2_\omega}.$$

□

4 Examples

The simplest example where we can make estimates is the "infinite ladder", see [Li-Lo]. That is the graph $G = (V, E)$ where the set of vertices V is the Cartesian product $V = \mathbb{N} \times \{-1, 1\}$ equipped with the "horizontal" edges $\{(l, \varepsilon), (l+1, \varepsilon)\}$ with $l = 0, 1, \cdots$ and $\varepsilon = \pm 1$ and the "vertical" edges $\{(l, -1), (l, +1)\}$ with $l = 0, 1, \cdots$. We will use the "square" cycles

$$\gamma_l = [(l,1),(l+1,1)] + [(l+1,1),(l+1,-1)] + [(l+1,-1),(l,-1)] + [(l,-1),(l,1)],$$

for $l = 0, 1, \cdots$, as a basis of the space of cycles. Let b_l be the holonomy of B in the cycle γ_l. We will take B so that the value of $|B_{\gamma_l}|$ is $2 - \sqrt{2}$, which is the maximal one by Lemma 2.3.

Using the good covering of G by the cycles γ_l, we get $m = 2$ and the effective potential

$$W((l, \varepsilon)) = \left(1 - \frac{\sqrt{2}}{2}\right) \inf_{\{x,y\} \in E_l} c_{xy}.$$

We will take

$$c_{(l,\varepsilon)(l+1,\varepsilon)} = c_{(l,-1)(l,+1)} = C_l \text{ and } \omega_{(l,\varepsilon)} = w_l \ .$$

If C_l is increasing, we get $W((l, \varepsilon)) = \left(1 - \frac{\sqrt{2}}{2}\right) C_l$. Let us assume that w_l is decreasing, we get

$$p_{(l,\varepsilon)(l+1,\varepsilon)} = \frac{\omega_{l+1}}{\sqrt{C_l}} \text{ and } D((l, \varepsilon)) = \sum_{m=l}^{\infty} \frac{w_{m+1}}{\sqrt{C_m}}.$$

We take now $C_l = l^a$ with $a > 0$ and $w_l = l^{-b}$ with $b > 0$. The graph is not complete for the distance d_p if $a + b/2 > 1$. In this case, we have

$$D((l, \varepsilon)) \sim c_1 l^{(1-a-b/2)} \text{ and } W((l, \varepsilon)) \sim c_2 l^a.$$

The assumption of Theorem 2.1 is satisfied if $b < 1$. So we get that, if $0 < b < 1$ and $a + b/2 > 1$, the operator $H_{\omega,c,B}$ is essentially self-adjoint. If $a > 2$, the operator $H_{\omega,c,0}$ is not essentially self-adjoint by Theorem 4.1 in [CdV-To-Tr].

5 Questions

The following questions are unsolved at the moment:

1. If $H_{\omega,c,0}$ is essentially self-adjoint, does it imply that $H_{\omega,c,B}$ is essentially self-adjoint for any B? Does it hold in the continuous case?

2. What would be a correct statement for a locally finite graph with unbounded degree (even if $B = 0$)?

3. In the case where the completion of (G, d_p) is compact and under the assumption of Theorem 2.1, $H_{\omega,c,B}$ has a compact resolvent. What is the asymptotic behavior of the eigenvalues? The continuous case is worked out in [CdV2].

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