Heisenberg algebras and rational double affine Hecke algebras

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HEISENBERG ALGEBRAS AND RATIONAL DOUBLE AFFINE HECKE ALGEBRAS

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Abstract. In this paper we categorify the Heisenberg action on the Fock space via the category $\mathcal{O}$ of cyclotomic rational double affine Hecke algebras. This permits us to relate the filtration by the support on the Grothendieck group of $\mathcal{O}$ to a representation theoretic grading defined using the Heisenberg action. This implies a recent conjecture of Etingof.

Contents

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1. Introduction and notation

1.1. **Introduction.** In this paper we study a relationship between the representation theory of certain rational double affine Hecke algebras (=RDAHA) and the representation theory of affine Kac-Moody algebras. Such connection is not new and appears already at several places in the literature. A first occurrence is Suzuki's functor [29] which maps the Kazhdan-Lusztig category of modules over the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_n$ at a negative level to the representation category of the RDAHA of $\mathfrak{sl}_m$. A second one is a cyclotomic version of Suzuki's functor [31] which maps a more general version of the parabolic category $\mathcal{O}$ of $\widehat{\mathfrak{sl}}_n$ at a negative level to the representation category of the cyclotomic RDAHA. A third one comes from the relationship between the cyclotomic RDAHA and quiver varieties, see e.g., [13], and from the relationship between quiver varieties and affine Kac-Moody algebras. Finally, a fourth one, which is closer to our study, comes from the relationship in [28] between the Grothendieck ring of cyclotomic RDAHA and the level ℓ Fock space $\mathcal{F}_{m,\ell}$ of $\widehat{\mathfrak{sl}}_m$. In this paper we focus on a recent conjecture of Etingof [9] which relates the support of the objects of the category $\mathcal{O}$ of $H(\Gamma_n)$, the RDAHA associated with the complex reflection group $\Gamma_n = \mathfrak{S}_n \ltimes (\mathbb{Z}_\ell)^n$, to a representation theoretic grading of the Fock space $\mathcal{F}_\ell = \mathcal{F}_{\ell,1}$. These conjectures yield in particular an explicit formula for the number of finite dimensional $H(\Gamma_n)$-modules. This was not known so far. The appearance of the Fock space $\mathcal{F}_\ell$ is not a hazard. It is due to the following two facts, already noticed in [9]. First, by level-rank duality, the $\widehat{\mathfrak{sl}}_m$-module $\mathcal{F}_{m,\ell}$ carries a level m action on $\widehat{\mathfrak{gl}}_\ell$. It carries also a level 1 action on $\widehat{\mathfrak{gl}}_\ell$, under which it is identified with $\mathcal{F}_\ell$. Next, the category $\mathcal{O}$ of the algebras $H(\Gamma_n)$ with $n \geqslant 1$ categorifies $\mathcal{F}_{m,\ell}$ by [28]. Our proof consists precisely to interpret the support of the $H(\Gamma_n)$-modules in terms of the $\widehat{\mathfrak{sl}}_m$-action on $\mathcal{F}_{m,\ell}$, and then to interpret this in terms of the $\widehat{\mathfrak{sl}}_\ell$ action on $\mathcal{F}_\ell$. An important ingredient is

a categorification (in a weak sense) of the action of the Heisenberg algebra on $\mathcal{F}_\ell$ and $\mathcal{F}_{m,\ell}$. The categorification of the Heisenberg algebra has recently been studied by several authors. We'll come back to this in another publication.

1.2. **Organisation.** The organisation of the paper is the following.

Section 2 is a reminder on rational DAHA. We recall some basic facts concerning parabolic induction/restriction functors. In particular we describe their behavior on the support of the modules.

Section 3 contains basic notations for complex reflection groups, for the cyclotomic rational DAHA $H(\Gamma_n)$ and for affine Lie algebras. In particular we introduce the category $\mathcal{O}(\Gamma_n)$ of $H(\Gamma_n)$-modules, the functor KZ, Rouquier's equivalence from $\mathcal{O}(\mathfrak{S}_n)$ to the module category of the ζ-Schur algebra. Next we recall the categorification of the Fock space representation of $\widehat{\mathfrak{sl}}_m$ in [28], and we describe the filtration by the support on $\mathcal{O}(\Gamma_n)$.

Section 4 is more combinatorial. We recall several constructions related to Fock spaces and symmetric polynomials. In particular we give a relation between symmetric polynomials and the representation ring of the group Γ_n, and we describe several representations on the level ℓ Fock space (of Heisenberg algebras and of affine Kac-Moody algebras).

Section 5 is devoted to the categorification of the Heisenberg action on the Fock space, using $\mathcal{O}(\Gamma_n)$. Then we introduce a particular class of simple objects in $\mathcal{O}(\Gamma_n)$, called the primitive modules, and we compute the endomorphism algebra of some modules induced from primitive modules. Finally we introduce the operators $\tilde{a}_\lambda$ which are analogues for the Heisenberg algebra of the Kashiwara's operators $\tilde{e}_q$, $\tilde{f}_q$ associated with Kac-Moody algebras.

Section 6 contains the main results of the paper. Using our previous constructions we compare the filtration by the support on $\mathcal{O}(\Gamma_n)$ with a representation-theoretic grading on the Fock space. This confirms a conjecture of Etingof, yielding in particular the number of finite dimensional simple objects in $\mathcal{O}(\Gamma_n)$ for integral ℓ-charge (this corresponds to some rational values of the parameters of $H(\Gamma_n)$).

Finally there are two appendices containing basic facts on Hecke algebras, Schur algebras, quantum groups, quantum Frobenius homomorphism and on the universal R-matrix.

1.3. **Notation.** Now we introduce some general notation. Let $\mathcal{A}$ be a $\mathbb{C}$-category, i.e., a $\mathbb{C}$-linear additive category. We'll write $Z(\mathcal{A})$ for the center of $\mathcal{A}$, a $\mathbb{C}$-algebra. Let $\mathrm{Irr}(\mathcal{A})$ be the set of isomorphism classes of simple objects of $\mathcal{A}$. If $\mathcal{A} = \mathrm{Rep}(\mathbf{A})$, the category of all finite-dimensional representations of a $\mathbb{C}$-algebra $\mathbf{A}$, we abbreviate

$$\mathrm{Irr}(\mathbf{A}) = \mathrm{Irr}(\mathrm{Rep}(\mathbf{A})).$$

For an Abelian or triangulated category let $K(\mathcal{A})$ denote its Grothendieck group. We abbreviate $K(\mathbf{A}) = K(\mathrm{Rep}(\mathbf{A}))$. We set

$$[\mathcal{A}] = K(\mathcal{A}) \otimes \mathbb{C}.$$

For an object M of $\mathcal{A}$ we write $[M]$ for the class of M in $[\mathcal{A}]$. For an Abelian category $\mathcal{A}$ let $D^b(\mathcal{A})$ denote its bounded derived category. We abbreviate $D^b(\mathbf{A}) = D^b(\mathrm{Rep}(\mathbf{A}))$. The symbol $H^*(\mathbb{P}^{m-1})$ will denote both the complex

$$\langle m \rangle = \bigoplus_{i=0}^{m-1} \mathbb{C}[-2i] \in D^b(\mathbb{C})$$

and the integer m in $K(\mathbb{C}) = \mathbb{Z}$. Given two Abelian $\mathbb{C}$-categories $\mathcal{A}$, $\mathcal{B}$ which are Artinian (i.e., objects are of finite length and Hom's are finite dimensional) we

define the tensor product (over $\mathbb{C}$)

$$\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$$

as in [4, sec. 5.1, prop. 5.13]. Recall that for $\mathcal{A} = \mathrm{Rep}(\mathbf{A})$ and $\mathcal{B} = \mathrm{Rep}(\mathbf{B})$ we have $\mathcal{A} \otimes \mathcal{B} = \mathrm{Rep}(\mathbf{A} \otimes \mathbf{B})$. Given a category $\mathcal{A}$ and objects $A, A' \in \mathcal{A}$, we write $\mathrm{Hom}_{\mathcal{A}}(A, A')$ for the collection of morphisms $A \to A'$. Given categories $\mathcal{A}$, $\mathcal{B}$ and functors $F, F' : \mathcal{A} \to \mathcal{B}$ we write $\mathrm{Hom}(F, F')$ for the collection of morphisms $F \to F'$. We denote the identity morphism $A \to A$ by $\mathbf{1}_A$ and the identity morphism $F \to F$ by $\mathbf{1}_F$. Given a category $\mathcal{C}$ and a functor $G : \mathcal{B} \to \mathcal{C}$ let $G \circ F$ be the composed functor $\mathcal{A} \to \mathcal{C}$. For a functor $G' : \mathcal{B} \to \mathcal{C}$ and morphisms of functors $\phi \in \mathrm{Hom}(F, F')$, $\psi \in \mathrm{Hom}(G, G')$ we write $\psi\phi$ for the morphism of functors $G \circ F \to G' \circ F'$ given by

$$(\psi\phi)(A) = \psi(F'(A)) \circ G(\phi(A)) \in \mathrm{Hom}_{\mathcal{C}}\big(G(F(A)), G'(F'(A))\big), \quad A \in \mathcal{A}.$$

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2. Reminder on rational DAHA's

2.1. **The category $\mathcal{O}(W)$.** Let W be any complex reflection group. Let $\mathfrak{h}$ be the reflection representation of W. Let S be the set of pseudo-reflections in W. Let $c : S \to \mathbb{C}$ be a map that is constant on the W-conjugacy classes. The rational DAHA attached to W with parameter c is the quotient $H(W)$ of the smash product of $\mathbb{C}W$ and the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = \langle x, y \rangle - \sum_{s \in S} c_s \langle \alpha_s, y \rangle \langle x, \check{\alpha}_s \rangle s,$$

for all $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$. Here $\langle \bullet, \bullet \rangle$ is the canonical pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$, the element α_s is a generator of $\mathrm{Im}(s|_{\mathfrak{h}^*} - 1)$ and $\check{\alpha}_s$ is the generator of $\mathrm{Im}(s|_{\mathfrak{h}} - 1)$ such that $\langle \alpha_s, \check{\alpha}_s \rangle = 2$. Let R_x, R_y be the subalgebras generated by $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. We may abbreviate

$$\mathbb{C}[\mathfrak{h}] = R_x, \quad \mathbb{C}[\mathfrak{h}^*] = R_y.$$

The category $\mathcal{O}$ of $H(W)$ is the full subcategory $\mathcal{O}(W)$ of the category of $H(W)$-modules consisting of objects that are finitely generated as $\mathbb{C}[\mathfrak{h}]$-modules and $\mathfrak{h}$-locally nilpotent. We recall from [12, sec. 3] the following properties of $\mathcal{O}(W)$. It is a quasi-hereditary category. The standard modules are labeled by the set $\mathrm{Irr}(\mathbb{C}W)$ of isomorphism classes of irreducible W-modules. Let Δ_χ be the standard module associated with the module $\chi \in \mathrm{Irr}(\mathbb{C}W)$. It is the induced module

$$\Delta_\chi = \mathrm{Ind}_{W \ltimes R_y}^{H(W)}(\chi).$$

Here χ is regarded as a $W \ltimes R_y$-module such that $\mathfrak{h}^* \subset R_y$ acts by zero. Let L_χ, P_χ denote the top and the projective cover of Δ_χ.

Remark 2.1. The definitions above still make sense if $\mathfrak{h}$ is any faithful finite dimensional $\mathbb{C}W$-module. To avoid any confusion we may write

$$\mathcal{O}(W, \mathfrak{h}) = \mathcal{O}(W), \quad H(W, \mathfrak{h}) = H(W).$$

2.2. **The stratification of $\mathfrak{h}$.** Let W be a complex reflection group. Let $\mathfrak{h}$ be the reflection representation of W. For a parabolic subgroup $W' \subset W$ let $X^\circ_{W'}$ be the set of points of $\mathfrak{h}$ whose stabilizer in W is conjugate (in W) to W'. By a theorem of Steinberg, the sets $X^\circ_{W',\mathfrak{h}}$, when W' runs over a set of representatives of the W-conjugacy classes of parabolic subgroups of W, form a stratification of $\mathfrak{h}$ by smooth locally closed subsets, see also [11, sec. 6] and the references there. Let $X_{W'}$ be the closure of $X^\circ_{W'}$ in $\mathfrak{h}$. To avoid any confusion we may write $X^\circ_{W',\mathfrak{h}} = X^\circ_{W'}$ and $X_{W',\mathfrak{h}} = X_{W'}$. The set $X_{W',\mathfrak{h}}$ consists of points of $\mathfrak{h}$ whose stabilizer is W-conjugate to W'. We have

$$X_{W',\mathfrak{h}} = \bigsqcup X^\circ_{W'',\mathfrak{h}},$$

where the union is over a set of representatives of the W-conjugacy classes of the parabolic subgroups W'' of W which contain W'. Further, the quotient $X_{W',\mathfrak{h}}/W$ is an irreducible closed subset of $\mathfrak{h}/W$.

2.3. **Induction and restriction functors on $\mathcal{O}(W)$.** Fix an element $b \in \mathfrak{h}$. Let $W_b \subset W$ be the stabilizer of b, and

$$\pi_b : \mathfrak{h} \to \mathfrak{h}/\mathfrak{h}^{W_b}$$

be the obvious projection onto the reflection representation of W_b. The parabolic induction/restriction functor relatively to the point b is a functor [1]

$$\mathrm{Ind}_b : \mathcal{O}(W_b, \mathfrak{h}/\mathfrak{h}^{W_b}) \to \mathcal{O}(W, \mathfrak{h}), \quad \mathrm{Res}_b : \mathcal{O}(W, \mathfrak{h}) \to \mathcal{O}(W_b, \mathfrak{h}/\mathfrak{h}^{W_b}).$$

Since the functors Ind_b, Res_b do not depend on b up to isomorphism, see [1, sec. 3.7], we may write

$${}^{\mathcal{O}}\mathrm{Ind}^W_{W_b} = \mathrm{Ind}_b, \quad {}^{\mathcal{O}}\mathrm{Res}^W_{W_b} = \mathrm{Res}_b$$

if it does not create any confusion. The *support* of a module M in $\mathcal{O}(W, \mathfrak{h})$ is the support of M regarded as a $\mathbb{C}[\mathfrak{h}]$-module. It is a closed subset $\mathrm{Supp}(M) \subset \mathfrak{h}$. For any simple module L in $\mathcal{O}(W, \mathfrak{h})$ we have $\mathrm{Supp}(L) = X_{W',\mathfrak{h}}$ for some parabolic subgroup $W' \subset W$. For $b \in X^\circ_{W',\mathfrak{h}}$ the module $\mathrm{Res}_b(L)$ is a nonzero finite dimensional module. See [1, sec. 3.8]. The support of a module is the union of the supports of all its constituents. So the support of any module in $\mathcal{O}(W, \mathfrak{h})$ is a union of $X_{W',\mathfrak{h}}$'s. Let us consider the behavior of the support under restriction.

Proposition 2.2. *Let $W' \subset W$ be a parabolic subgroup. Let $\mathfrak{h}'$ be the reflection representation of W'. Let $X \subset \mathfrak{h}$ be the support of a module M in $\mathcal{O}(W, \mathfrak{h})$. Let $X' \subset \mathfrak{h}'$ be the support of the module $M' = {}^{\mathcal{O}}\mathrm{Res}^W_{W'}(M)$.*

(a) We have $M' \neq 0$ if and only if $X_{W',\mathfrak{h}} \subset X$.

(b) Assume that $X = X_{W'',\mathfrak{h}}$ with $W'' \subset W$ a parabolic subgroup. If $M' \neq 0$ then W'' is W-conjugate to a subgroup of W' and we have

$$X' = \bigcup_{W_1} X_{W_1,\mathfrak{h}'} = \bigsqcup_{W_1} X^\circ_{W_1,\mathfrak{h}'},$$

where W_1 runs over a set of representatives of the W'-conjugacy classes of parabolic subgroups of W' containing a subgroup W-conjugated to W''.

Proof. Part (a) is immediate from the definition of the restriction, because for $b \in \mathfrak{h}$ it implies that $\mathrm{Res}_b(M) \neq 0$ if and only if $b \in X$. Now we prove (b). For a parabolic subgroup $W_1 \subset W'$ we have

$$\begin{aligned} X_{W_1,\mathfrak{h}'} \subset X' &\iff {}^{\mathcal{O}}\mathrm{Res}^{W'}_{W_1}(M') \neq 0 \\ &\iff {}^{\mathcal{O}}\mathrm{Res}^{W}_{W_1}(M) \neq 0 \\ &\iff X_{W_1,\mathfrak{h}} \subset X_{W'',\mathfrak{h}}. \end{aligned}$$

Here the first and third equivalence follow from (a), while the second one follows from the transitivity of the restriction functor [28, cor. 2.5]. Therefore $X^\circ_{W_1,\mathfrak{h}'} \subset X'$

if and only if $X_{W_1,\mathfrak{h}'} \subset X'$ if and only if W_1 contains a subgroup W-conjugate to W''.

□

Remark 2.3. For any closed point b of a scheme X we denote by $X_b^\wedge$ the completion of X at b (a formal scheme). Assume that $M' = {}^{\mathcal{O}}\mathrm{Res}^W_{W'}(M)$ is non zero. Let π be the canonical projection $\mathfrak{h} \to \mathfrak{h}' = \mathfrak{h}/\mathfrak{h}^{W'}$. For $b \in X^\circ_{W',\mathfrak{h}}$ the definition of the restriction functor yields the following formula

$$0 \in \pi^{-1}(X'), \quad X_b^\wedge = b + \pi^{-1}(X')_0^\wedge.$$

Next, we consider the behavior of the support under induction. Before this we need the following two lemmas. The $\mathbb{C}$-vector space $[\mathcal{O}(W)]$ is spanned by the set $\{[\Delta_\chi]; \chi \in \mathrm{Irr}(\mathbb{C}W)\}$. Thus there is a unique $\mathbb{C}$-linear isomorphism

$$\mathrm{spe} : [\mathrm{Rep}(\mathbb{C}W)] \to [\mathcal{O}(W)], \quad [\chi] \mapsto [\Delta_\chi]. \tag{2.1}$$

The parabolic induction/restriction functor is exact. We'll need the following lemma [1].

Lemma 2.4. *Let $W' \subset W$ be a parabolic subgroup. Let $\mathfrak{h}'$ be the reflection representations of W'. Under the isomorphism (2.1) the maps*

$${}^{\mathcal{O}}\mathrm{Ind}^W_{W'} : [\mathcal{O}(W', \mathfrak{h}')] \to [\mathcal{O}(W, \mathfrak{h})], \quad {}^{\mathcal{O}}\mathrm{Res}^W_{W'} : [\mathcal{O}(W, \mathfrak{h})] \to [\mathcal{O}(W', \mathfrak{h}')]$$

coincide with the induction and restriction

$$\mathrm{Ind}^W_{W'} : [\mathrm{Rep}(\mathbb{C}W')] \to [\mathrm{Rep}(\mathbb{C}W)], \quad \mathrm{Res}^W_{W'} : [\mathrm{Rep}(\mathbb{C}W)] \to [\mathrm{Rep}(\mathbb{C}W')].$$

We'll also need the following version of the Mackey induction/restriction theorem. First, observe that for any parabolic subgroup $W' \subset W$ and any $x \in W$ there is a canonical $\mathbb{C}$-algebra isomorphism

$$\varphi_x : \ H(W') \to H(x^{-1}W'x), \ w \mapsto x^{-1}wx, \ f \mapsto x^{-1}fx, \ f' \mapsto x^{-1}f'x,$$

for $w \in W'$, $f \in R_x$, $f' \in R_y$. It yields an exact functor

$$\mathcal{O}(W') \to \mathcal{O}(x^{-1}W'x), \quad M \mapsto {}^xM,$$

where ${}^xM$ is the $H(x^{-1}W'x)$-module obtained by twisting the $H(W')$-action on M by φ_x.

Lemma 2.5. *Let $W', W'' \subset W$ be parabolic subgroups. Let $\mathfrak{h}'$, $\mathfrak{h}''$ be the reflection representations of W', W''. For $M \in \mathcal{O}(W', \mathfrak{h}')$ we have the following formula in $[\mathcal{O}(W'', \mathfrak{h}'')]$*

$${}^{\mathcal{O}}\mathrm{Res}^W_{W''} \circ {}^{\mathcal{O}}\mathrm{Ind}^W_{W'}([M]) = \sum_x {}^{\mathcal{O}}\mathrm{Ind}^{W''}_{W'' \cap x^{-1}W'x} \circ {}^x\big({}^{\mathcal{O}}\mathrm{Res}^{W'}_{xW''x^{-1} \cap W'}([M])\big), \tag{2.2}$$

where x runs over a set of representatives of the cosets in $W' \setminus W / W''$.

Proof. Use Lemma 2.4 and the usual Mackey induction/restriction theorem associated with the triplet of groups W, W', W''. □

Remark 2.6. For a future use, note that the left hand side of (2.2) is zero if and only if each term in the sum of the right hand side is zero, because each of these terms is the class of a module in $\mathcal{O}(W'', \mathfrak{h}'')$.

Now, we can prove the following proposition.

Proposition 2.7. *Let $W'' \subset W' \subset W$ be a parabolic subgroups. Let $\mathfrak{h}'$ be the reflection representation of W'. For a simple module $L \in \mathcal{O}(W', \mathfrak{h}')$ with $\mathrm{Supp}(L) = X_{W'',\mathfrak{h}'}$, we have*

$${}^{\mathcal{O}}\mathrm{Ind}^W_{W'}(L) \neq 0, \quad \mathrm{Supp}\big({}^{\mathcal{O}}\mathrm{Ind}^W_{W'}(L)\big) = X_{W'',\mathfrak{h}}.$$

Proof. First, note that ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W}(L) \neq 0$ by Lemma 2.5, because

$$ {}^{\mathcal{O}}\mathrm{Res}_{W'}^{W} \circ {}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W}([L]) = [L] + [M] $$

for some $M \in \mathcal{O}(W', \mathfrak{h}')$ and $[L] \neq 0$. Therefore ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W}(L) \neq 0$. We abbreviate $M = {}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W}(L)$. To compute the support of M we first check that

$$ X_{W'',\mathfrak{h}} \subset \mathrm{Supp}(M). $$

By Proposition 2.2 we have

$$ \begin{aligned} X_{W'',\mathfrak{h}} \subset \mathrm{Supp}(M) &\iff X^{\circ}_{W'',\mathfrak{h}} \subset \mathrm{Supp}(M) \\ &\iff {}^{\mathcal{O}}\mathrm{Res}_{W''}^{W}(M) \neq 0. \end{aligned} $$

By Remark 2.6 the last equality holds if and only if

$$ {}^{\mathcal{O}}\mathrm{Res}_{xW''x^{-1}\cap W'}^{W'}(L) \neq 0 $$

for some $x \in W$. This identity is indeed true for $x = 1$ because $W'' \subset W'$ and

$$ X_{W'',\mathfrak{h}'} = \mathrm{Supp}(L) \Rightarrow {}^{\mathcal{O}}\mathrm{Res}_{W''}^{W'}(L) \neq 0. $$

Next we prove the inclusion

$$ \mathrm{Supp}(M) \subset X_{W'',\mathfrak{h}}. $$

Any point b of $\mathfrak{h} \setminus X_{W'',\mathfrak{h}}$ is contained in the set $X^{\circ}_{W''',\mathfrak{h}}$ for some parabolic subgroup $W''' \subset W$ such that W'' is not conjugate to a subgroup of W''' : it suffices to set $W''' = W_b$. We must check that for such a subgroup $W''' \subset W$ we have

$$ X^{\circ}_{W''',\mathfrak{h}} \not\subset \mathrm{Supp}(M). $$

By Proposition 2.2 it is enough to check that

$$ {}^{\mathcal{O}}\mathrm{Res}_{W'''}^{W}(M) = 0. $$

Now, by Lemma 2.5 we have the following formula in $[\mathcal{O}(W''', \mathfrak{h})]$

$$ {}^{\mathcal{O}}\mathrm{Res}_{W'''}^{W}([M]) = \sum_{x} {}^{\mathcal{O}}\mathrm{Ind}_{W'''\cap x^{-1}W'x}^{W'''} \circ {}^{x}\big({}^{\mathcal{O}}\mathrm{Res}_{xW'''x^{-1}\cap W'}^{W'}([L])\big). $$

Here x runs over a set of representatives of the cosets in $W' \setminus W / W'''$. Since W'' is not conjugate to a subgroup of W''' it is a fortiori not conjugate to a subgroup of $xW'''x^{-1} \cap W'$, i.e., we have

$$ X^{\circ}_{xW'''x^{-1}\cap W',\mathfrak{h}'} \cap X_{W'',\mathfrak{h}'} = \emptyset. $$

Therefore Proposition 2.2 yields

$$ {}^{\mathcal{O}}\mathrm{Res}_{xW'''x^{-1}\cap W'}^{W'}(L) = 0, $$

because $\mathrm{Supp}(L) = X_{W'',\mathfrak{h}'}$. This implies that

$$ {}^{\mathcal{O}}\mathrm{Res}_{W'''}^{W}([M]) = 0. $$

Hence we have also

$$ {}^{\mathcal{O}}\mathrm{Res}_{W'''}^{W}(M) = 0. $$

We are done. □

3. The cyclotomic rational DAHA

3.1. **Combinatorics.** For a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of integers $\geqslant 0$ we set $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. Let

$$\Lambda(\ell, n) = \{\lambda = (\lambda_1, \lambda_2, \dots \lambda_\ell) \in \mathbb{N}^\ell \,;\, |\lambda| = n\}.$$

It is the set of *compositions* of n with ℓ parts. Let $\mathcal{P}_n$ be the set of *partitions* of n, i.e., the set of non-increasing sequences λ of integers > 0 with sum $|\lambda| = n$. We write λ' for the transposed partition and $l(\lambda)$ for its length, i.e., for the number of parts in λ. We write also

$$z_\lambda = \prod_{i \geqslant 1} i^{m_i}\, m_i!, \tag{3.1}$$

where m_i is the number of parts of λ equal to i. Given a positive integer m and a partition λ we write also

$$m\lambda = (m\lambda_1, m\lambda_2, \dots).$$

To any partition we associate a *Young diagram*, which is a collection of rows of square boxes with λ_i boxes in the i-th row, $i = 1, \dots, l(\lambda)$. A box in a Young diagram is called a *node*. The coordinate of the j-th box in the i-th row is the pair of integers (i, j). The *content* of the node of coordinate (i, j) is the integer $j - i$. Let the set $\mathcal{P}_0$ consist of a single element, the unique partition of zero, which we denote by 0. Let $\mathcal{P} = \bigsqcup_{n \geqslant 0} \mathcal{P}_n$ be the set of all partitions. We'll abbreviate $\mathbb{Z}_\ell = \mathbb{Z}/\ell\mathbb{Z}$. Let $\mathcal{P}^\ell$ be the set of *ℓ-partitions*, i.e., the set of all partition valued functions on $\mathbb{Z}_\ell$. Let $\mathcal{P}^\ell_n$ be the subset of ℓ-tuples $\lambda = (\lambda(p))$ of partitions with $|\lambda| = \sum_p |\lambda(p)| = n$. Let Γ be the group of the ℓ-th roots of 1 in $\mathbb{C}^\times$. We define the sets $\mathcal{P}^\Gamma$, $\mathcal{P}^\Gamma_n$ of partition valued functions on Γ in the same way.

3.2. **The complex reflection group Γ_n.** Fix non negative integers ℓ, n. Unless specified otherwise we'll always assume that $\ell, n \neq 0$. Let $\mathfrak{S}_n$ be the symmetric group on n letters and Γ_n be the semi-direct product $\mathfrak{S}_n \ltimes \Gamma^n$, where Γ^n is the Cartesian product of n copies of Γ. We write also $\mathfrak{S}_0 = \Gamma^0 = \Gamma_0 = \{1\}$. For $\gamma \in \Gamma$ let $\gamma_i \in \Gamma^n$ be the element with γ at the i-th place and with 1 at the other ones. Let s_{ij} be the transposition (i, j) in $\mathfrak{S}_n$. We'll abbreviate $s_i = s_{i,i+1}$. Write $s^\gamma_{ij} = s_{ij}\gamma_i\gamma_j^{-1}$ for $\gamma \in \Gamma$, $i \neq j$. For $p \in \mathbb{Z}_\ell$ let $\chi_p : \Gamma \to \mathbb{C}^\times$ be the character $\gamma \mapsto \gamma^p$. The assignment $p \mapsto \chi_p$ identifies $\mathbb{Z}_\ell$ with the group of characters of Γ. The group Γ_n is a complex reflection group. For $\ell > 1$ it acts on the vector space $\mathfrak{h} = \mathbb{C}^n$ via the reflection representation. For $\ell = 1$ the reflection representation is given by the permutation of coordinates on the hyperplane

$$\mathbb{C}^n_0 = \{x_1 + \cdots + x_n = 0\} \subset \mathbb{C}^n.$$

We'll be interested in the following subgroups of Γ_n.

- To a composition ν of n we associate the set

$$I = \{1, 2, \dots, n-1\} \setminus \{\nu_1, \nu_1 + \nu_2, \dots\}.$$

  Let $\Gamma_\nu = \mathfrak{S}_\nu \ltimes \Gamma^n$, where $\mathfrak{S}_\nu = \mathfrak{S}_I$ is the subgroup of $\mathfrak{S}_n$ generated by the simple reflections $s_{i,i+1}$ with $i \in I$.
- For integers $m, n \geqslant 0$ and a composition ν we set $\Gamma_{n,\nu} = \Gamma_n \times \mathfrak{S}_\nu$. If $\nu = (m^j)$ for some integer $j \geqslant 0$ we abbreviate $\Gamma_{n,(m^j)} = \Gamma_{n,\nu}$. We write also $\Gamma_{n,m} = \Gamma_n \times \mathfrak{S}_m$. Any parabolic subgroup of Γ_n is conjugate to $\Gamma_{l,\nu}$ for some l, ν with $l + |\nu| \leqslant n$.

3.3. **Definition of the cyclotomic rational DAHA.** Fix a basis (x, y) of $\mathbb{C}^2$. Let x_i, y_i denote the elements x, y respectively in the i-th summand of $(\mathbb{C}^2)^{\oplus n}$. The group Γ_n acts on $(\mathbb{C}^2)^{\oplus n}$ such that for distinct i, j, k we have

$$\gamma_i(x_i) = \gamma^{-1}x_i, \quad \gamma_i(x_j) = x_j, \quad \gamma_i(y_i) = \gamma y_i, \quad \gamma_i(y_j) = y_j,$$

$$s_{ij}(x_i) = x_j, \quad s_{ij}(y_i) = y_j, \quad s_{ij}(x_k) = x_k, \quad s_{ij}(y_k) = y_k.$$

Fix $k \in \mathbb{C}$ and $c_\gamma \in \mathbb{C}$ for each $\gamma \in \Gamma$. Since Γ_n is a complex reflection group with the reflection representation $\mathfrak{h}$ equal to $(\mathbb{C}^2)^{\oplus n}$, see above, we can define the algebra $H(W) = H(W, \mathfrak{h})$ for $W = \Gamma_n$. We'll call $H(\Gamma_n)$ the *cyclotomic rational DAHA*. It is the quotient of the smash product of $\mathbb{C}\Gamma_n$ and the tensor algebra of $(\mathbb{C}^2)^{\oplus n}$ by the relations

$$[y_i, x_i] = -k\sum_{j\neq i}\sum_{\gamma\in\Gamma} s_{ij}^\gamma - \sum_{\gamma\in\Gamma} c_\gamma\gamma_i, \quad c_1 = -1,$$

$$[y_i, x_j] = k\sum_{\gamma\in\Gamma} s_{ij}^\gamma \quad \text{if } i \neq j,$$

$$[x_i, x_j] = [y_i, y_j] = 0.$$

Let R_x, R_y be the subalgebras generated by $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ respectively. We'll write $\mathfrak{h}$, $\mathfrak{h}^*$ for the maximal spectrum of R_x, R_y. The $\mathbb{C}$-vector space $\mathfrak{h}$ is identified with $\mathbb{C}^n$ in the obvious way. We'll use another presentation where the parameters are h, h_p with $p \in \mathbb{Z}_\ell$ where $k = -h$ and $-c_\gamma = \sum_{p\in\mathbb{Z}_\ell} \gamma^{-p}h_p$. Note that $1 = \sum_p h_p$.

3.4. **The Lie algebras $\widehat{\mathfrak{sl}}_\ell$ and $\widetilde{\mathfrak{sl}}_\ell$.** Given complex numbers h_p, $p \in \mathbb{Z}_\ell$, with $\sum_p h_p = 1$, it is convenient to consider the following level 1 weight

$$\Lambda = \sum_p h_p\,\omega_p. \tag{3.2}$$

Here the ω_p's are the fundamental weights of the affine Lie algebra

$$\widehat{\mathfrak{sl}}_\ell = (\mathfrak{sl}_\ell \otimes \mathbb{C}[\varpi, \varpi^{-1}]) \oplus \mathbb{C}\mathbf{1},$$

where $\mathbf{1}$ is a central element and the Lie bracket is given by

$$[x \otimes \varpi^r, y \otimes \varpi^s] = [x, y] \otimes \varpi^{r+s} + r(x, y)\delta_{r,-s}\mathbf{1}, \quad (x, y) = \tau(xy^t), \tag{3.3}$$

where $y \mapsto y^t$ is the transposition and τ is the trace. The affine Lie algebra $\widehat{\mathfrak{sl}}_\ell$ is generated by the symbols e_p, f_p, $p = 0, \ldots, \ell-1$, satisfying the Serre relations. For $p \neq 0$ we have

$$e_p = e_{p,p+1} \otimes 1, \quad e_0 = e_{\ell,1} \otimes \varpi, \quad f_p = e_{p+1,p} \otimes 1, \quad f_0 = e_{1,\ell} \otimes \varpi^{-1},$$

where $e_{p,q}$ is the usual elementary matrix in $\mathfrak{sl}_\ell$. We'll also use the extended affine Lie algebras $\widetilde{\mathfrak{sl}}_\ell$, obtained by adding to $\widehat{\mathfrak{sl}}_\ell$ the 1-dimensional vector space spanned by the scaling element D such that $[D, x \otimes \varpi^r] = r\,x \otimes \varpi^r$ and $[D, \mathbf{1}] = 0$. Let δ denote the dual of D, i.e., the smallest positive imaginary root. We equip the space of linear forms on the Cartan subalgebra of $\widetilde{\mathfrak{sl}}_\ell$ with the pairing such that

$$\langle\omega_p, \omega_q\rangle = \min(p, q) - pq/\ell, \quad \langle\omega_p, \delta\rangle = 1, \quad \langle\delta, \delta\rangle = 0.$$

Let $U(\widehat{\mathfrak{sl}}_\ell)$ be the enveloping algebra of $\widehat{\mathfrak{sl}}_\ell$, and let $U^-(\widehat{\mathfrak{sl}}_\ell)$ be the subalgebra generated by the elements f_p with $p = 0, \ldots, \ell-1$. For $r \geqslant 0$ we write $U^-(\widehat{\mathfrak{sl}}_\ell)_r$ for the subspace of $U^-(\widehat{\mathfrak{sl}}_\ell)$ spanned by the monomials whose weight is the sum of r negative simple roots.

3.5. **Representations of $\mathfrak{S}_n$, Γ_n.** The set of isomorphism classes of irreducible $\mathfrak{S}_n$-modules is

$$\mathrm{Irr}(\mathbb{C}\mathfrak{S}_n) = \{\bar{L}_\lambda; \lambda \in \mathcal{P}_n\},$$

see [24, sec. I.9]. The set of isomorphism classes of irreducible Γ_n-modules is

$$\mathrm{Irr}(\mathbb{C}\Gamma_n) = \{\bar{L}_\lambda; \lambda \in \mathcal{P}_n^\ell\},$$

where $\bar{L}_\lambda$ is defined as follows. Write $\lambda = (\lambda(p))$. The tuple of positive integers $\nu_\lambda = (|\lambda(p)|)$ is a composition in $\Lambda(\ell, n)$. Let

$$\bar{L}_{\lambda(p)}(\chi_{p-1})^{\otimes|\lambda(p)|} \in \mathrm{Irr}(\mathbb{C}\Gamma_{|\lambda(p)|})$$

be the tensor product of the $\mathfrak{S}_{|\lambda(p)|}$-module $\bar{L}_{\lambda(p)}$ and the one-dimensional $\Gamma^{|\lambda(p)|}$-module $(\chi_{p-1})^{\otimes|\lambda(p)|}$. The Γ_n-module $\bar{L}_\lambda$ is given by

$$\bar{L}_\lambda = \mathrm{Ind}_{\Gamma_{\nu_\lambda}}^{\Gamma_n}\big(\bar{L}_{\lambda(1)}\chi_\ell^{\otimes|\lambda(1)|} \otimes \bar{L}_{\lambda(2)}\chi_1^{\otimes|\lambda(2)|} \otimes \cdots \otimes \bar{L}_{\lambda(\ell)}\chi_{\ell-1}^{\otimes|\lambda(\ell)|}\big). \tag{3.4}$$

3.6. **The category $\mathcal{O}(\Gamma_n)$.** Consider the $\mathbb{C}$-algebra $H(\Gamma_n)$ with the parameter Λ in (3.2). The category $\mathcal{O}$ of $H(\Gamma_n)$ is the quasi-hereditary category $\mathcal{O}(\Gamma_n)$. The standard modules are the induced modules

$$\Delta_\lambda = \mathrm{Ind}_{\Gamma_n \ltimes R_y}^{H(\Gamma_n)}(\bar{L}_\lambda), \quad \lambda \in \mathcal{P}_n^\ell.$$

Here $\bar{L}_\lambda$ is viewed as a $\Gamma_n \ltimes R_y$-module such that $y_1, \dots y_n$ act trivially. Let L_λ, P_λ denote the top and the projective cover of Δ_λ. Recall the $\mathbb{C}$-linear isomorphism

$$\mathrm{spe} : [\mathrm{Rep}(\mathbb{C}\Gamma_n)] \to [\mathcal{O}(\Gamma_n)], \quad [\bar{L}_\lambda] \mapsto [\Delta_\lambda]. \tag{3.5}$$

To avoid cumbersome notation for induction/restriction functors in

$$\mathcal{O}(\Gamma) = \bigoplus_{n \geqslant 0} \mathcal{O}(\Gamma_n)$$

we'll abbreviate

$$\begin{aligned}
&{}^{\mathcal{O}}\mathrm{Ind}_n = {}^{\mathcal{O}}\mathrm{Ind}_{\Gamma_{n-1}}^{\Gamma_n}, && {}^{\mathcal{O}}\mathrm{Res}_n = {}^{\mathcal{O}}\mathrm{Res}_{\Gamma_{n-1}}^{\Gamma_n},\\
&{}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)} = {}^{\mathcal{O}}\mathrm{Ind}_{\Gamma_{n,(m^r)}}^{\Gamma_{n+mr}}, && {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)} = {}^{\mathcal{O}}\mathrm{Res}_{\Gamma_{n,(m^r)}}^{\Gamma_{n+mr}},\\
&{}^{\mathcal{O}}\mathrm{Ind}_{n,mr} = {}^{\mathcal{O}}\mathrm{Ind}_{\Gamma_{n,mr}}^{\Gamma_{n+mr}}, && {}^{\mathcal{O}}\mathrm{Res}_{n,mr} = {}^{\mathcal{O}}\mathrm{Res}_{\Gamma_{n,mr}}^{\Gamma_{n+mr}}.
\end{aligned} \tag{3.6}$$

We write also

$$\begin{aligned}
{}^{\mathcal{O}}\mathrm{Ind}_{(m^r)} &= {}^{\mathcal{O}}\mathrm{Ind}_{\mathfrak{S}_m^r}^{\mathfrak{S}_{mr}} : \mathcal{O}(\mathfrak{S}_m^r) \to \mathcal{O}(\mathfrak{S}_{mr}),\\
{}^{\mathcal{O}}\mathrm{Res}_{(m^r)} &= {}^{\mathcal{O}}\mathrm{Res}_{\mathfrak{S}_m^r}^{\mathfrak{S}_{mr}} : \mathcal{O}(\mathfrak{S}_{mr}) \to \mathcal{O}(\mathfrak{S}_m^r).
\end{aligned}$$

3.7. **The functor KZ.** For $\zeta \in \mathbb{C}^\times$ and $v_1, v_2, \dots, v_\ell \in \mathbb{C}^\times$ let $\mathbf{H}_\zeta(n, \ell)$ be the cyclotomic Hecke algebra associated with Γ_n and the parameters $\zeta, v_1, \dots, v_\ell$, see Section A.2. We'll abbreviate $\mathbf{H}(\Gamma_n) = \mathbf{H}_\zeta(n, \ell)$. Assume that

$$\zeta = \exp(2i\pi h), \quad v_p = v_1 \exp\big(-2i\pi(h_1 + h_2 + \cdots + h_{p-1})\big).$$

Then the KZ-functor [12] is a quotient functor

$$\mathrm{KZ} : \mathcal{O}(\Gamma_n) \to \mathrm{Rep}(\mathbf{H}(\Gamma_n)).$$

Since KZ is a quotient functor, it admits a right adjoint functor

$$S : \mathrm{Rep}(\mathbf{H}(\Gamma_n)) \to \mathcal{O}(\Gamma_n)$$

such that $\mathrm{KZ} \circ S = \mathbf{1}$. By [12, thm. 5.3], for each projective module $Q \in \mathcal{O}(\Gamma_n)$ the canonical adjunction morphism $\mathbf{1} \to S \circ \mathrm{KZ}$ yields an isomorphism

$$Q \to S(\mathrm{KZ}(Q)). \tag{3.7}$$

3.8. **The functor R.** Let $\mathbf{H}_\zeta(m)$ be the Hecke $\mathbb{C}$-algebra of GL_m, see Section A.2. Let $\mathbf{S}_\zeta(m)$ be the ζ-Schur $\mathbb{C}$-algebra, see Appendix B. The module categories of $\mathbf{S}_\zeta(m)$, $\mathbf{H}_\zeta(m)$ are related through the Schur functor

$$\Phi^* : \mathrm{Rep}(\mathbf{S}_\zeta(m)) \to \mathrm{Rep}(\mathbf{H}_\zeta(m)).$$

Set

$$\Lambda(m)_+ = \Lambda(m,m) \cap \mathbb{Z}_+^m, \quad \mathbb{Z}_+^m = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{Z}^m \,;\, \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_m\}.$$

The category $\mathrm{Rep}(\mathbf{S}_\zeta(m))$ is quasi-hereditary with respect to the dominance order, the standard objects being the modules Δ_λ^S with $\lambda \in \Lambda(m)_+$. The comultiplication Δ yields a bifunctor (B.6)

$$\dot\otimes : \mathrm{Rep}(\mathbf{S}_\zeta(m)) \otimes \mathrm{Rep}(\mathbf{S}_\zeta(m')) \to \mathrm{Rep}(\mathbf{S}_\zeta(m+m')).$$

Now, assume that h is a negative rational number with denominator d and let $\zeta \in \mathbb{C}^\times$ be a primitive d-th root of 1. Recall that h is the parameter of the $\mathbb{C}$-algebra $H(\mathfrak{S}_m)$. If $h \notin 1/2 + \mathbb{Z}$ then Rouquier's functor [27] is an equivalence of quasi-hereditary categories

$$R : \mathcal{O}(\mathfrak{S}_m) \to \mathrm{Rep}(\mathbf{S}_\zeta(m)), \quad \Delta_\lambda \mapsto \Delta_\lambda^S,$$

such that $\mathrm{KZ} = \Phi^* \circ R$. For $m = m' + m''$ we have a canonical equivalence of categories $\mathcal{O}(\mathfrak{S}_{m'}) \otimes \mathcal{O}(\mathfrak{S}_{m''}) = \mathcal{O}(\mathfrak{S}_{m'} \times \mathfrak{S}_{m''})$ and the induction yields a bifunctor

$$^{\mathcal{O}}\mathrm{Ind}_{m',m''} : \mathcal{O}(\mathfrak{S}_{m'}) \otimes \mathcal{O}(\mathfrak{S}_{m''}) \to \mathcal{O}(\mathfrak{S}_m). \tag{3.8}$$

We'll abbreviate

$$\mathcal{O}(\mathfrak{S}) = \bigoplus_{n \geqslant 0} \mathcal{O}(\mathfrak{S}_n), \quad \mathrm{Rep}(\mathbf{S}_\zeta) = \bigoplus_{n \geqslant 0} \mathrm{Rep}(\mathbf{S}_\zeta(n)).$$

Proposition 3.1. *For $h \notin 1/2 + \mathbb{Z}$ the functor R is tensor equivalence $\mathcal{O}(\mathfrak{S}) \to \mathrm{Rep}(\mathbf{S}_\zeta)$.*

Proof. We must check that R identifies the tensor product $\dot\otimes$ with the induction (3.8). First, fix two projective objects $X \in \mathcal{O}(\mathfrak{S}_{m'})$ and $Y \in \mathcal{O}(\mathfrak{S}_{m''})$. We have

$$\begin{aligned}
\Phi^*\big(R(X) \dot\otimes R(Y)\big) &= {}^{\mathbf{H}}\mathrm{Ind}_{m',m''}\big(\Phi^* R(X) \otimes \Phi^* R(Y)\big) \\
&= {}^{\mathbf{H}}\mathrm{Ind}_{m',m''}\big(\mathrm{KZ}(X) \otimes \mathrm{KZ}(Y)\big) \\
&= \mathrm{KZ}\big({}^{\mathcal{O}}\mathrm{Ind}_{m',m''}(X \otimes Y)\big) \\
&= \Phi^* R\big({}^{\mathcal{O}}\mathrm{Ind}_{m',m''}(X \otimes Y)\big).
\end{aligned}$$

The first equality follows from Corollary B.4, the second one and the fourth one come from $\mathrm{KZ} = \Phi^* \circ R$, and the third one is the commutation of KZ and the induction functors, see [28]. Since the modules $R(X) \dot\otimes R(Y)$ and $R\big({}^{\mathcal{O}}\mathrm{Ind}_{m',m''}(X \otimes Y)\big)$ are projective, and since Φ^* is fully faithful on projectives we get that

$$R(X) \dot\otimes R(Y) = R\big({}^{\mathcal{O}}\mathrm{Ind}_{m',m''}(X \otimes Y)\big).$$

Now, since the functors (B.6), (3.8) are exact and coincide on projective objects, and since the category $\mathcal{O}(\mathfrak{S}_m)$ has enough projectives, the proposition is proved. □

3.9. **The categorification of $\widetilde{\mathfrak{sl}}_m$.** Recall that $Z(\mathcal{O}(\Gamma_n))$ is the center of the category $\mathcal{O}(\Gamma_n)$. Let $D_n(z)$ be the polynomial in $Z(\mathcal{O}(\Gamma_n))[z]$ defined in [28, sec. 4.2]. For any $a \in \mathbb{C}(z)$ the projection to the generalized eigenspace of $D_n(z)$ with the eigenvalue a yields an exact endofunctor $Q_{n,a}$ of $\mathcal{O}(\Gamma_n)$. Next, consider the point

$$b_n = (0, 0, \dots, 0, 1) \in \mathfrak{h}, \quad \mathfrak{h} = \mathbb{C}^n.$$

The induction and the restriction relatively to b_n yield functors

$$^{\mathcal{O}}\mathrm{Ind}_n : \mathcal{O}(\Gamma_{n-1}) \to \mathcal{O}(\Gamma_n), \quad {}^{\mathcal{O}}\mathrm{Res}_n : \mathcal{O}(\Gamma_n) \to \mathcal{O}(\Gamma_{n-1}).$$

Definition 3.2. [28, sec. 4.2] The *q-restiction* and the *q-induction* functors

$$e_q : \mathcal{O}(\Gamma_n) \to \mathcal{O}(\Gamma_{n-1}), \quad f_q : \mathcal{O}(\Gamma_{n-1}) \to \mathcal{O}(\Gamma_n), \quad q = 0, 1, \dots, m-1$$

are given by

$$e_q = \bigoplus_{a \in \mathbb{C}(z)} Q_{n-1,a/(z-\zeta^q)} \circ {}^{\mathcal{O}}\mathrm{Res}_n \circ Q_{n,a},$$

$$f_q = \bigoplus_{a \in \mathbb{C}(z)} Q_{n,a(z-\zeta^q)} \circ {}^{\mathcal{O}}\mathrm{Ind}_n \circ Q_{n-1,a}.$$

We'll abbreviate

$$E = e_0 \oplus e_1 \oplus \cdots \oplus e_{m-1}, \quad F = f_0 \oplus f_1 \oplus \cdots \oplus f_{m-1}.$$

Following [28, sec. 6.3], for $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))$ we set

$$\tilde{e}_q(L) = \mathrm{top}(e_q(L)), \quad \tilde{f}_q(L) = \mathrm{soc}(f_q(L)), \quad \tilde{e}_q(0) = \tilde{f}_q(0) = 0.$$

Now, for each n we choose the parameters of $H(\Gamma_n)$ in the following way

$$h = -1/m, \quad h_p = (s_{p+1} - s_p)/m, \quad s_p \in \mathbb{Z}, \quad p \neq 0. \tag{3.9}$$

The following hypothesis is important for the rest of the paper :

from now on we'll always assume that $m > 1$.

The $\mathbb{C}$-vector space $[\mathcal{O}(\Gamma)]$ is canonically isomorphic to the *level* ℓ *Fock space* $\mathcal{F}^{(s)}_{m,\ell}$ *associated with the* ℓ*-charge* $s = (s_p)$, see (5.20) below for details. The latter is equipped with an integrable representation of $\widehat{\mathfrak{sl}}_m$ of level ℓ, see Section 4.6 below.

Proposition 3.3. *(a) The functors* e_q, f_q *are exact and biadjoint.*

(b) We have $E = {}^{\mathcal{O}}\mathrm{Res}_n$ *and* $F = {}^{\mathcal{O}}\mathrm{Ind}_n$.

(c) For $M \in \mathcal{O}(\Gamma_n)$ *we have* $E(M) = 0$ *(resp.* $F(M) = 0$*) iff* $E(L) = 0$ *(resp.* $F(L) = 0$*) for any constituent* L *of* M.

(d) The operators e_q, f_q *equip* $[\mathcal{O}(\Gamma)]$ *with a representation of* $\widehat{\mathfrak{sl}}_m$ *which is isomorphic via the map (5.20) to* $\mathcal{F}^{(s)}_{m,\ell}$.

(e) The tuple $(\mathrm{Irr}(\mathcal{O}(\Gamma)), \tilde{e}_q, \tilde{f}_q)$ *has a crystal structure. In particular, for* $L, L' \in \mathrm{Irr}(\mathcal{O}(\Gamma))$ *we have* $\tilde{e}_q(L), \tilde{f}_q(L) \in \mathrm{Irr}(\mathcal{O}(\Gamma)) \sqcup \{0\}$*, and* $\tilde{e}_q(L) = L'$ *if and only if* $\tilde{f}_q(L') = L$.

Proof. Parts $(a), (b)$ follows from [28, prop. 4.4], part (e) is contained in [28, thm. 6.3], part (c) is obvious, and part (d) is [28, cor. 4.5]. □

3.10. **The filtration of** $[\mathcal{O}(\Gamma_n)]$ **by the support.** Fix a positive integer n. Assume that $\ell \geqslant 2$. In this section we consider the tautological action of Γ_n on $\mathbb{C}^n$. For an integer $l \geqslant 0$ and a composition ν such that $l + |\nu| \leqslant n$ we abbreviate $X^\circ_{l,\nu} = X^\circ_{W,\mathfrak{h}}$ and $X_{l,\nu} = X_{W,\mathfrak{h}}$ where $W = \Gamma_{l,\nu}$. If $\nu = (m^j)$ for some integer $j \geqslant 0$ such that $l + jm \leqslant n$ we write

$$X^\circ_{l,j} = X^\circ_{l,\nu}, \quad X_{l,j} = X_{l,\nu}.$$

Therefore $X_{l,j}$ is the set of the points in $\mathbb{C}^n$ with l coordinates equal to zero and j collections of m coordinates which differ from each other by ℓ-th roots of one. To avoid confusions we may write $X_{l,j,\mathbb{C}^n} = X_{l,j}$. Unless specified otherwise, for l, j, m, n as above we'll set

$$i = n - l - jm. \tag{3.10}$$

Definition 3.4. For $i, j \geqslant 0$ we set

$$\mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j} = \{L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n)) \,;\, \mathrm{Supp}(L) = X_{l,j}\}.$$

Definition 3.5. For $i, j \geqslant 0$ let $F_{i,j}(\Gamma_n)$ be the $\mathbb{C}$-vector subspace of $[\mathcal{O}(\Gamma_n)]$ spanned by the classes of the modules whose support is contained in $X_{l,j}$, with l as in (3.10). If $i < 0$ or $j < 0$ we write $F_{i,j}(\Gamma_n) = 0$.

Definition 3.6. We define a partial order on the set of pairs of nonnegative integers (i, j) such that $i + jm \leqslant n$ given by $(i', j') \leqslant (i, j)$ if and only if $X_{l',j'} \subset X_{l,j}$, where $l = n - i - jm$ and $l' = n - i' - j'm$.

Since the support of a module is the union of the supports of all its constituents, the $\mathbb{C}$-vector space $F_{i,j}(\Gamma_n)$ is spanned by the classes of the modules in $\mathrm{Irr}(\mathcal{O}(\Gamma_n))$ whose support is contained in $X_{l,j}$, or, equivalently $F_{i,j}(\Gamma_n)$ is spanned by the classes of the modules in

$$\bigcup_{(i',j')\leqslant(i,j)} \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i',j'}.$$

Remark 3.7. We have $\bigcup_{i,j} F_{i,j}(\Gamma_n) = [\mathcal{O}(\Gamma_n)]$. Indeed, for $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))$ we have $\mathrm{Supp}(L) = X_{l,\nu}$ for some l, ν, see Section 2.2. For $b \in X^\circ_{l,\nu}$ the $H(\Gamma_{l,\nu})$-module $\mathrm{Res}_b(L)$ is finite dimensional. Thus, since the parameter h of $H(\Gamma_{l,\nu})$ is equal to $-1/m$ the parts of ν are all equal to m. Hence we have $\mathrm{Supp}(L) = X_{l,j}$ for some l, j as above.

The subspaces $F_{i,j}(\Gamma_n)$ give a filtration of $[\mathcal{O}(\Gamma_n)]$. Consider the associated graded $\mathbb{C}$-vector space

$$\mathrm{gr}(\Gamma_n) = \bigoplus_{i,j} \mathrm{gr}_{i,j}(\Gamma_n).$$

Note that the $\mathbb{C}$-vector spaces $F_{i,j}(\Gamma_n)$ and $\mathrm{gr}_{i,j}(\Gamma_n)$ differ slightly from the corresponding objects, denoted by $\mathbf{F}_{i,j}K_0$ and $\mathrm{gr}^{\mathbf{F}}_{i,j}K_0$, in [9, sec. 6.5].The images by the canonical projection $F_{i,j}(\Gamma_n) \to \mathrm{gr}_{i,j}(\Gamma_n)$ of the classes of the modules in $\mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}$ form a basis of the $\mathbb{C}$-vector space $\mathrm{gr}_{i,j}(\Gamma_n)$. So we may regard $\mathrm{gr}_{i,j}(\Gamma_n)$ as the subspace of $[\mathcal{O}(\Gamma_n)]$ spanned by $\mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}$. We'll abbreviate

$$F_{i,\bullet}(\Gamma_n) = \sum_j F_{i,j}(\Gamma_n), \quad F_{\bullet,j}(\Gamma_n) = \sum_i F_{i,j}(\Gamma_n),$$
$$\mathrm{gr}_{i,\bullet}(\Gamma_n) = \bigoplus_j \mathrm{gr}_{i,j}(\Gamma_n), \quad \mathrm{gr}_{\bullet,j}(\Gamma_n) = \bigoplus_i \mathrm{gr}_{i,j}(\Gamma_n).$$

Now, let us study the filtration of $[\mathcal{O}(\Gamma_n)]$ in details. The subgroup $\Gamma_{l,(m^j)}$ of Γ_n is contained in the subgroups $\Gamma_{l+1,(m^j)}$, $\Gamma_{l,(m^{j+1})}$ and $\Gamma_{l+m,(m^{j-1})}$ (up to conjugation by an element of Γ_n) whenever such subgroups exist. Thus we have the inclusions

$$X_{l+1,j},\ X_{l,j+1},\ X_{l+m,j-1} \subset X_{l,j},$$
$$F_{i-1,j}(\Gamma_n),\ F_{i-m,j+1}(\Gamma_n),\ F_{i,j-1}(\Gamma_n) \subset F_{i,j}(\Gamma_n).$$

Proposition 3.8. *(a) We have*

$$X_{l',j'} \subsetneq X_{l,j} \iff X_{l',j'} \subset X_{l+1,j} \cup X_{l,j+1} \cup X_{l+m,j-1}.$$

(b) We have an isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{gr}_{i,j}(\Gamma_n) = F_{i,j}(\Gamma_n)/\big(F_{i-1,j}(\Gamma_n) + F_{i-m,j+1}(\Gamma_n) + F_{i,j-1}(\Gamma_n)\big).$$

Proof. First we prove (a). Recall that $X_{l,j}$ is the set of the points in $\mathbb{C}^n$ with l coordinates equal to zero and j collections of m coordinates which differ from each other by ℓ-th roots of one. Therefore we have

$$X_{l',j'} \subset X_{l,j} \iff i - i' \geqslant \max\big(0, (j' - j)m\big). \tag{3.11}$$

In particular this inclusion implies that $l' \geqslant l$. We must prove that

$$X_{l',j'} \subsetneq X_{l,j} \Rightarrow X_{l',j'} \subset X_{l+1,j} \cup X_{l,j+1} \cup X_{l+m,j-1}.$$

First, assume that $l' = l$. Since $X_{l',j'} \subsetneq X_{l,j}$ we have $i > i'$. Then (3.10) implies that $i-i' = (j'-j)m$, hence that $j' > j$ and $i-i' \geqslant m$. So $i-i' \geqslant \max(m, (j'-j)m)$, and (3.11) implies that $X_{l',j'} \subset X_{l,j+1}$.

Next, assume that $l+m > l' > l$. Since $X_{l',j'} \subset X_{l,j}$ we have $i \geqslant i'$. Further (3.10) implies that $i-i' > (j'-j)m$ and $i'-i > (j-j'-1)m$. Thus $i \geqslant i'$ implies indeed that $i > i'$ and $j' \geqslant j$. So $i-1-i' \geqslant \max(0, (j'-j)m)$, and (3.11) implies that $X_{l',j'} \subset X_{l+1,j}$.

Finally, assume that $l' \geqslant l+m$. Since $X_{l',j'} \subset X_{l,j}$ we have $i \geqslant i'$. Further (3.10) implies that $i-i' \geqslant (j'-j+1)m$. So $i-i' \geqslant \max(0, (j'-j+1)m)$, and (3.11) implies that $X_{l',j'} \subset X_{l+m,j-1}$.

Part (b) is a consequence of (a) and of the definition of the filtration on $[\mathcal{O}(\Gamma_n)]$. □

Remark 3.9. The sets $X_{l+1,j}$, $X_{l,j+1}$, $X_{l+m,j-1}$ do not contain each other. Indeed, the variety $X_{l,j}$ has the dimension $i+j$. Thus the codimension of $X_{l+1,j}$, $X_{l,j+1}$, $X_{l+m,j-1}$ in $X_{l,j}$ are $1, m-1, 1$ respectively. However, since a point in $X^\circ_{l,j+1}$ has only l coordinates equal to 0, we have $X_{l,j+1} \not\subset X_{l+1,j}$ and $X_{l,j+1} \not\subset X_{l+m,j-1}$.

Remark 3.10. We have $F_{\bullet,0}(\Gamma_n) = [\mathcal{O}(\Gamma_n)]$, because $(i,j) \leqslant (i+jm, 0)$.

Remark 3.11. We have $(i',j') \leqslant (0,j)$ if and only if $i' = 0$ and $j' \leqslant j$.

Remark 3.12. Consider the set

$$F_{i,j}(\Gamma_n)^\circ = F_{i,j}(\Gamma_n) \setminus \big(F_{i-1,j}(\Gamma_n) + F_{i-m,j+1}(\Gamma_n) + F_{i,j-1}(\Gamma_n)\big).$$

For $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))$, by Proposition 3.8 and Remark 3.7 we have

$$\begin{aligned} [L] \in F_{i,j}(\Gamma_n)^\circ &\iff \mathrm{Supp}(L) = X_{l,j} \\ &\iff L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}. \end{aligned}$$

Remark 3.13. A representation is finite dimensional if and only if its support is zero. Thus $\mathrm{Irr}(\mathcal{O}(\Gamma_n))_{0,0}$ is the set of isomorphism classes of finite dimensional modules in $\mathcal{O}(\Gamma_n)$. Note that $(0,0) \leqslant (i,j)$ for all (i,j).

Remark 3.14. If $\ell = 1$ then, by Remark 2.1 and Section 3.2 we have $\mathcal{O}(\mathfrak{S}_n) = \mathcal{O}(\mathfrak{S}_n, \mathbb{C}^n_0)$. For an integer $j \geqslant 0$ we set $X_j = X_{\mathfrak{S}^j_m, \mathbb{C}^n_0}$, i.e., X_j is the set of the points in $\mathbb{C}^n_0$ with j collections of m equal coordinates. Then, we set $i = n - jm$ and the results of this section extend in the obvious way. In particular, we have

$$X_{j'} \subset X_j \iff j' \geqslant j, \quad X_{j'} \subsetneq X_j \iff X_{j'} \subset X_{j+1}.$$

Remark 3.15. For $\lambda \in \mathcal{P}_r$, $r \geqslant 1$, the support of the module $L_{m\lambda} \in \mathrm{Irr}(\mathcal{O}(\mathfrak{S}_{mr}))$ is

$$\mathrm{Supp}(L_{m\lambda}) = X_{\mathfrak{S}^r_m, \mathbb{C}^{mr}_0}.$$

Indeed, formula (5.16) below and Proposition 2.7 imply that

$$\mathrm{Supp}(L_{m\lambda}) \subset \mathrm{Supp}\big({}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L^{\otimes r}_{(m)})\big) = X_{\mathfrak{S}^r_m, \mathbb{C}^{mr}_0}.$$

Next, by Remark 3.7 there is $j = 0, 1, \ldots, r$ such that

$$\mathrm{Supp}(L_{m\lambda}) = X_{\mathfrak{S}^j_m, \mathbb{C}^{mr}_0}.$$

Finally the inclusion $X_{\mathfrak{S}^j_m,\mathbb{C}^{mr}_0} \subset X_{\mathfrak{S}^r_m,\mathbb{C}^{mr}_0}$ implies that $j = r$ by Remark 3.14. Note that this equality follows also from the work of Wilcox [32].

3.11. **The action of E, F on the filtration.** Let E, F denote the $\mathbb{C}$-linear operators on $[\mathcal{O}(\Gamma)]$ induced by the exact functors E, F. Recall that the parameters of $H(\Gamma)$ are chosen as in (3.9).

Proposition 3.16. *Let $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}$ and $l = n - i - mj$.*
(a) We have $\mathrm{Supp}(F(L)) = X_{l,j,\mathbb{C}^{n+1}}$.
(b) We have $E(L) = 0$ iff $i = 0$. We have $\mathrm{Supp}(E(L)) = X_{l,j,\mathbb{C}^{n-1}}$ if $i > 0$.

Proof. Recall that

$$\mathrm{Supp}(L) = X_{l,j} = X_{l,j,\mathbb{C}^n}, \quad E(L) = {}^{\mathcal{O}}\mathrm{Res}_n(L), \quad F(L) = {}^{\mathcal{O}}\mathrm{Ind}_n(L).$$

Thus by Proposition 2.2 we have $E(L) = 0$ iff $b_n \notin X_{l,j}$. Since $m > 1$ the definition of the stratum $X_{l,j}$ in Section 3.10 shows that $b_n \notin X_{l,j}$ iff $i = 0$. Now, assume that $i > 0$. Then $l + mj \leqslant n - 1$, and Proposition 2.2 yields

$$\mathrm{Supp}(E(L)) = \bigcup_W X_{W,\mathbb{C}^{n-1}},$$

where W runs over the parabolic subgroups of Γ_{n-1} which are Γ_n-conjugate to $\Gamma_{l,(m^j)}$ (inside the group Γ_n). We claim that a subgroup $W \subset \Gamma_{n-1}$ as above is Γ_{n-1}-conjugate to $\Gamma_{l,(m^j)}$ (inside the group Γ_{n-1}). Therefore, we have

$$\mathrm{Supp}(E(L)) = X_{l,j,\mathbb{C}^{n-1}}.$$

Indeed, fix $b' \in \mathbb{C}^{n-1}$ such that $W = (\Gamma_{n-1})_{b'}$. For $b = (b', z)$ with $z \in \mathbb{C}$ generic we have $(\Gamma_n)_b = W$, where W is regarded as a subgroup of Γ_n via the obvious inclusion $\Gamma_{n-1} \subset \Gamma_n$. Since W is Γ_n-conjugate to $\Gamma_{l,(m^j)}$, there is an element $g \in \Gamma_n$ such that the first l coordinates of $g(b)$ are 0, the next mj ones consist of j collections of m coordinates which differ from each other by ℓ-th roots of one, and the last i coordinates of $g(b)$ are in generic position. We'll abbreviate

$$g(b) \in 0^l(m)^j *^i .$$

Since z is generic it is taken by g to one of the coordinates of $g(b)$ in the packet $*^i$. Composing g by an appropriate reflection in $\mathfrak{S}_n$ we get an element $g' \in \Gamma_{n-1}$ such that

$$g'(b) = (g'(b'), z) \in 0^l(m)^j *^i .$$

Thus we have also

$$g'(b') \in 0^l(m)^j *^{i-1} .$$

This implies the claim. Hence, we have

$$\mathrm{Supp}(E(L)) = X_{l,j,\mathbb{C}^{n-1}}.$$

Finally, since $\mathrm{Supp}(L) = X_{l,j,\mathbb{C}^n}$, Proposition 2.7 implies that

$$\mathrm{Supp}(F(L)) = X_{l,j,\mathbb{C}^{n+1}}.$$

□

Corollary 3.17. *(a) We have $E(F_{i,j}(\Gamma_n)) \subset F_{i-1,j}(\Gamma_{n-1})$. If $i \neq 0$ we have also $E(F_{i,j}(\Gamma_n)^\circ) \subset F_{i-1,j}(\Gamma_{n-1})^\circ$.*
(b) For $M \in \mathcal{O}(\Gamma_n)$ with $[M] \in F_{i,j}(\Gamma_n)^\circ$ we have $E([M]) = 0$ iff $i = 0$.
(c) We have $F(F_{i,j}(\Gamma_n)) \subset F_{i+1,j}(\Gamma_{n+1})$ and $F(F_{i,j}(\Gamma_n)^\circ) \subset F_{i+1,j}(\Gamma_{n+1})^\circ$.

Proof. First, let $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))$ with $[L] \in F_{i,j}(\Gamma_n)$. Thus $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{i',j'}$ with $(i',j') \leqslant (i,j)$. Proposition 3.16 yields

$$\mathrm{Supp}(F(L)) = X_{l',j',\mathbb{C}^{n+1}}, \quad \mathrm{Supp}\big(E(L)\big) = X_{l',j',\mathbb{C}^{n-1}} \text{ if } i' \neq 0.$$

Hence we have $F([L]) \in F_{i+1,j}(\Gamma_{n+1})$ and $E([L]) \in F_{i-1,j}(\Gamma_{n-1})$. Part (b) follows from Proposition 3.16 and Remarks 3.11, 3.12. Part (c) follows from Proposition 3.16 and Remark 3.12. The second part of (a) follows from Proposition 3.16 and Remark 3.12. □

Corollary 3.18. *Let $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}$.*
(a) If $\tilde{e}_q(L) \neq 0$ then $\tilde{e}_q(L) \in \mathrm{Irr}(\mathcal{O}(\Gamma_{n-1}))_{i-1,j}$.
(b) If $\tilde{f}_q(L) \neq 0$ then $\tilde{f}_q(L) \in \mathrm{Irr}(\mathcal{O}(\Gamma_{n+1}))_{i+1,j}$.

Proof. Set $L' = \tilde{e}_q(L)$. Assume that $L' \neq 0$. By Proposition 3.3 we have

$$L' \in \mathrm{Irr}(\mathcal{O}(\Gamma_{n-1})), \quad \tilde{f}_q(L') = L.$$

Next, since $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{i,j}$ and since $\tilde{e}_q(L)$ is a constituent of $E(L)$, we have $[L'] \in F_{i-1,j}(\Gamma_{n-1})$ by Corollary 3.17. We must prove that $[L'] \in F_{i-1,j}(\Gamma_{n-1})^\circ$. If this is false then we have $[L'] \in F_{i',j'}(\Gamma_{n-1})$ with

$$(i',j') = (i-2,j),\ (i-m-1,j+1),\ (i-1,j-1).$$

Thus, since $\tilde{f}_q(L')$ is a constituent of $F(L')$, by Corollary 3.17 we have

$$[L] \in \mathrm{gr}_{i,j}(\Gamma_n) \cap F_{i'+1,j'}(\Gamma_n). \tag{3.12}$$

Therefore (3.11) yields $i'+1 \geqslant i$, so $i' = i-1$ and $j' = j-1$. So, applying (3.11) once again we get a contradiction with (3.12). This proves (a). The proof of (b) is similar. □

Corollary 3.19. *(a) For $x \in [\mathcal{O}(\Gamma)]$ we have*

$$\big(e_q(x) = 0,\ \forall q = 0,1,\dots,m-1\big) \iff x \in F_{0,\bullet}(\Gamma).$$

(b) For $M \in \mathcal{O}(\Gamma)$ we have

$$E(M) = 0 \iff E([M]) = 0 \iff [M] \in F_{0,\bullet}(\Gamma).$$

(c) The space $F_{0,\bullet}(\Gamma)$ is spanned by the set

$$\begin{aligned}\{[L]\,;\, L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,\bullet}\} &= \{[L]\,;\, L \in \mathrm{Irr}(\mathcal{O}(\Gamma)),\ E(L) = 0\}\\ &= \{[L]\,;\, L \in \mathrm{Irr}(\mathcal{O}(\Gamma)),\ \tilde{e}_q(L) = 0,\ \forall q = 0,1,\dots,m-1\}.\end{aligned}$$

Proof. For $x \in [\mathcal{O}(\Gamma)]$ we write $x = \sum_L x_L[L]$ where L runs over the set $\mathrm{Irr}(\mathcal{O}(\Gamma))$. By [28, lem. 6.1, prop. 6.2], for each q we have

$$e_q(x) = 0 \iff x_L = 0 \text{ if } e_q([L]) \neq 0.$$

Thus the $\mathbb{C}$-vector space

$$\big\{x \in [\mathcal{O}(\Gamma)]\,;\, e_q(x) = 0,\ \forall q = 0,1,\dots,m-1\big\}$$

is spanned by the classes of the simples modules L such that $e_q([L]) = 0$ for all $q = 0,1,\dots,m-1$. Then, apply Corollary 3.17. This proves (a). Parts (b), (c) are obvious. Note that

$$\tilde{e}_q(L) = 0,\ \forall q \iff e_q(L) = 0,\ \forall q,$$

because a non zero finitely generated module has a non zero top. □

4. The Fock space

From now on we'll abbreviate

$$R(\mathfrak{S}) = \bigoplus_{n\geqslant 0}[\mathrm{Rep}(\mathbb{C}\mathfrak{S}_n)], \quad R(\Gamma) = \bigoplus_{n\geqslant 0}[\mathrm{Rep}(\mathbb{C}\Gamma_n)].$$

4.1. **The Hopf $\mathbb{C}$-algebra $\mathbf{\Lambda}$.** This section and the following one are reminders on symmetric functions and the Heisenberg algebra. First, recall that the $\mathbb{C}$-vector space $R(\mathfrak{S})$ is identified with the $\mathbb{C}$-vector space of symmetric functions

$$\mathbf{\Lambda} = \mathbb{C}[x_1, x_2, \dots]^{\mathfrak{S}_\infty}$$

via the *characteristic map* [24, chap. I]

$$\mathrm{ch} : R(\mathfrak{S}) \to \mathbf{\Lambda}.$$

The map ch intertwines the induction/restriction in $R(\mathfrak{S})$ with the multiplication/comultiplication in $\mathbf{\Lambda}$. It takes the class of the simple module $\bar{L}_\lambda$ to the Schur function S_λ for each $\lambda \in \mathcal{P}$. The power sum polynomials are given by

$$P_\lambda = P_{\lambda_1}P_{\lambda_2}\dots, \quad P_r = \sum_i x_i^r, \quad P_0 = 1, \quad \lambda \in \mathcal{P}, \quad r > 0.$$

We equip the $\mathbb{C}$-vector space $\mathbf{\Lambda}$ with the level 1 action of $\widehat{\mathfrak{sl}}_m$ given by

$$e_q(S_\lambda) = \sum_\nu S_\nu, \quad f_q(S_\lambda) = \sum_\mu S_\mu, \quad q = 0, \dots, m-1, \tag{4.1}$$

where ν (resp. μ) runs through all partitions obtained from $\lambda \in \mathcal{P}$ by removing (resp. adding) a node of content q mod m. We equip $\mathbf{\Lambda}$ with the symmetric bilinear form such that the Schur functions form an orthonormal basis. The operators e_q, f_q are adjoint to each other for this pairing.

4.2. **The Heisenberg algebra.** The *Heisenberg algebra* is the Lie algebra $\mathfrak{H}$ spanned by the elements $\mathbf{1}$ and b_r, b'_r, $r > 0$, satisfying the following relations

$$[b'_r, b'_s] = [b_r, b_s] = 0, \quad [b'_r, b_s] = r\mathbf{1}\delta_{r,s}, \quad r, s > 0.$$

Let $U(\mathfrak{H})$ be the enveloping algebra of $\mathfrak{H}$, and let $U^-(\mathfrak{H}) \subset U(\mathfrak{H})$ be the subalgebra generated by the elements b_r with $r > 0$. Write $U^-(\mathfrak{H})_r$ for the subspace of $U^-(\mathfrak{H})$ spanned by the monomials $b_{r_1}b_{r_2}\cdots$ with $\sum_i r_i = r$. For $\lambda \in \mathcal{P}$ and $f \in \mathbf{\Lambda}$ we consider the following elements in $U(\mathfrak{H})$

$$b_\lambda = b_{\lambda_1}b_{\lambda_2}\cdots, \quad b'_\lambda = b'_{\lambda_1}b'_{\lambda_2}\cdots,$$
$$b_f = \sum_{\lambda\in\mathcal{P}} z_\lambda^{-1}\langle P_\lambda, f\rangle\, b_\lambda, \quad b'_f = \sum_{\lambda\in\mathcal{P}} z_\lambda^{-1}\langle P_\lambda, f\rangle\, b'_\lambda,$$

where z_λ is as in (3.1). For any integer ℓ we can equip $\mathbf{\Lambda}$ with the level ℓ action of $\mathfrak{H}$ such that b_r acts by multiplication by ℓP_r and b'_r acts by $r\partial/\partial_{P_r}$ for $r > 0$. The operators b_r, b'_r are adjoint to each other for the pairing on $\mathbf{\Lambda}$ introduced in Section 4.1. Further, they commute with the $\widehat{\mathfrak{sl}}_m$-action in (4.1), see e.g., [30]. We write $V_\ell^{\mathfrak{H}} = \mathbf{\Lambda}$ regarded as a level ℓ module of $\mathfrak{H}$. Consider the Casimir operator

$$\partial = \frac{1}{\ell}\sum_{r\geqslant 1} b_r b'_r. \tag{4.2}$$

To avoid any confusion we may call it also the *level ℓ Casimir operator*. This formal sum defines a diagonalisable $\mathbb{C}$-linear operator on $V_\ell^{\mathfrak{H}}$ such that

$$[\partial, b_r] = rb_r, \quad [\partial, b'_r] = -rb'_r.$$

Below, we'll equip $\mathbf{\Lambda}$ with the $\mathfrak{H}$-action of level 1, i.e., we'll identify $\mathbf{\Lambda} = V_1^{\mathfrak{H}}$, unless mentioning explicitly the contrary.

4.3. **The Lie algebras $\widehat{\mathfrak{gl}}_m$ and $\widetilde{\mathfrak{gl}}_m$.** We define the Lie algebra $\widehat{\mathfrak{gl}}_m$ in the same way as $\widehat{\mathfrak{sl}}_m$, with $\mathfrak{gl}_m$ instead of $\mathfrak{sl}_m$. We'll also use the extended affine Lie algebra $\widetilde{\mathfrak{gl}}_m$, obtained by adding to $\widehat{\mathfrak{gl}}_m$ the 1-dimensional vector space spanned by the scaling element D such that $[D, x \otimes \varpi^r] = r\, x \otimes \varpi^r$ and $[D, \mathbf{1}] = 0$. The Lie algebra

$$\big(\widehat{\mathfrak{sl}}_m \times \mathfrak{H}\big)/\big(m(\mathbf{1}, 0) - (0, \mathbf{1})\big). \tag{4.3}$$

is isomorphic to $\widehat{\mathfrak{gl}}_m$ via the obvious map, which takes the element b'_r to $\sum_{p=1}^m e_{pp} \otimes \varpi^r$ and the element b_r to $\sum_{p=1}^m e_{pp} \otimes \varpi^{-r}$ for each $r > 0$. Unless specified otherwise, by a $\widehat{\mathfrak{gl}}_m$-module we'll always mean a module over the Lie algebra (4.3), i.e., a $\widehat{\mathfrak{sl}}_m$-module with a compatible $\mathfrak{H}$-action. Similarly, a $\widetilde{\mathfrak{gl}}_m$-module we'll always mean a $\widehat{\mathfrak{gl}}_m$-module with a scaling operator D such that

$$[D, x \otimes \varpi^r] = r\, x \otimes \varpi^r, \quad [D, b_r] = -r b_r, \quad [D, b'_r] = r b'_r.$$

By a dominant integral weight of $\widehat{\mathfrak{gl}}_m$, $\widetilde{\mathfrak{gl}}_m$ we'll always mean a dominant integral weight of $\widehat{\mathfrak{sl}}_m$, $\widetilde{\mathfrak{sl}}_m$. We denote the sets of such weights by $P_+^{\widehat{\mathfrak{gl}}_m}$, $P_+^{\widetilde{\mathfrak{gl}}_m}$ or by $P_+^{\widehat{\mathfrak{sl}}_m}$, $P_+^{\widetilde{\mathfrak{sl}}_m}$. For $\lambda \in P_+^{\widetilde{\mathfrak{sl}}_m}$ let $V_\lambda^{\widetilde{\mathfrak{sl}}_m}$ and $V_\lambda^{\widetilde{\mathfrak{gl}}_m}$ be the irreducible integrable modules over $\widetilde{\mathfrak{sl}}_m$, $\widetilde{\mathfrak{gl}}_m$ with the highest weight λ. As a $\widetilde{\mathfrak{gl}}_m$-module we have

$$V_{\omega_0}^{\widetilde{\mathfrak{gl}}_m} = V_{\omega_0}^{\widetilde{\mathfrak{sl}}_m} \otimes V_m^{\mathfrak{H}}.$$

Let $Q^{\mathfrak{sl}_m}$, $P^{\mathfrak{sl}_m}$ be the root lattice and weight lattice of $\mathfrak{sl}_m$. The weights of the module $V_{\omega_0}^{\widetilde{\mathfrak{sl}}_m}$ are all the weights of the form

$$\mu = \omega_0 + \beta - \frac{1}{2}\langle \beta, \beta \rangle \delta - i\delta, \quad \beta \in Q^{\mathfrak{sl}_m}, \quad i \geqslant 0.$$

Among those, the *extremal weights* are the weights for which $i = 0$. The set of the extremal weights coincide with the set of the *maximal weights*, i.e., with the set of the weights μ such that $\mu + \delta$ is not a weight of $V_{\omega_0}^{\widetilde{\mathfrak{sl}}_m}$. A weight μ of $V_{\omega_0}^{\widetilde{\mathfrak{sl}}_m}$ is extremal if and only if

$$\langle \mu, \mu \rangle = 0.$$

Note also that we have $\langle \mu, \mu \rangle = -2i$ if and only if $\mu + i\delta$ is an extremal weight. See e.g., [3, sec. 20.3, 20.5] for details. Now, let T_m be the standard maximal torus in SL_m, and let $\mathfrak{t}_m$ be its Lie algebra. Let $\widehat{\mathfrak{S}}_m$ be the affine symmetric group. It is the semidirect product $\mathfrak{S}_m \ltimes Q^{\mathfrak{sl}_m}$. Note that $Q^{\mathfrak{sl}_m}$ is the group of cocharacters of T_m. We'll regard it as a lattice in $\mathfrak{t}_m$ in the usual way, and we'll identify $\mathfrak{t}_m$ with $\mathfrak{t}_m^*$ via the standard invariant pairing on $\mathfrak{t}_m$. The $\widehat{\mathfrak{S}}_m$-action on $\mathfrak{t}_m^* \oplus \mathbb{C}\omega_0 \oplus \mathbb{C}\delta$, see e.g., [20, sec. 13.1], is such that the element β in $Q^{\mathfrak{sl}_m}$ acts via the operator

$$\xi_\beta :\ \mu \mapsto \mu + \mu(\mathbf{1})\beta - \big(\langle \mu, \beta \rangle + \frac{1}{2}\langle \beta, \beta \rangle \mu(\mathbf{1})\big)\delta. \tag{4.4}$$

In particular, we have

$$\xi_\beta(\omega_0) = \omega_0 + \beta - \frac{1}{2}\langle \beta, \beta \rangle \delta.$$

We'll use the same notation for the $\widehat{\mathfrak{S}}_m$-action on $\mathfrak{t}_m^* \oplus \mathbb{C}\omega_0 \oplus \mathbb{C}\delta$ and on $\mathfrak{t}_m^* \oplus \mathbb{C}\omega_0$, hoping it will not create any confusion. Therefore, for $\lambda \in \mathfrak{t}_m^* \oplus \mathbb{C}\omega_0$ the symbol $\xi_\beta(\lambda)$ will denote both the weight (4.4) and the weight $\mu + \mu(\mathbf{1})\beta$. We can view the cocharacter $\beta \in Q^{\mathfrak{sl}_m}$ as a group-scheme homomorphism $\mathbb{G}_m \to T_m$. Thus the image $\beta(\varpi)$ of the element $\varpi \in K$ lies in $T_m(K)$. For any $\widetilde{\mathfrak{sl}}_m$-module V let $V[\mu]$,

$\mu \in P^{\widetilde{\mathfrak{sl}}_m}$, be the corresponding weight subspace in V. Since the coadjoint action of $\beta(\varpi)$ on $\mathfrak{t}_m^* \oplus \mathbb{C}\omega_0 \oplus \mathbb{C}\delta$ is given by ξ_β^{-1}, see e.g., [26], we have also

$$\beta(V[\mu]) = V[\xi_\beta^{-1}(\mu)] \tag{4.5}$$

if V is integrable.

4.4. **The Hopf $\mathbb{C}$-algebra $\mathbf{\Lambda}_\Gamma$.** Now, let us consider the Hopf $\mathbb{C}$-algebras $R(\Gamma)$. Once again, the multiplication/comultiplication on $R(\Gamma)$ is given by the induction/restriction. We equip $R(\Gamma)$ with the symmetric $\mathbb{C}$-bilinear form given by

$$\langle f, g\rangle = |\Gamma_n|^{-1} \sum_{x\in\Gamma_n} f(x)g(x^{-1}), \quad f, g \in [\mathrm{Rep}(\mathbb{C}\Gamma_n)].$$

Here we regard f, g as characters of $\mathbb{C}\Gamma_n$. This bilinear form is a Hopf pairing. Next, we consider the Hopf $\mathbb{C}$-algebra $\mathbf{\Lambda}_\Gamma = \mathbf{\Lambda}^{\otimes\Gamma}$. We'll use the following elements in $\mathbf{\Lambda}_\Gamma$

$$f^\gamma = 1 \otimes \cdots \otimes 1 \otimes f \otimes 1 \otimes \cdots \otimes 1, \quad f \in \mathbf{\Lambda}, \quad \gamma \in \Gamma,$$

with f at the γ-th place. We abbreviate

$$P_\mu^\gamma = (P_\mu)^\gamma, \quad P_\lambda = \prod_{\gamma\in\Gamma} P^\gamma_{\lambda(\gamma)}, \quad \mu \in \mathcal{P}, \quad \lambda \in \mathcal{P}^\Gamma.$$

The comultiplication in $\mathbf{\Lambda}_\Gamma$ is characterized by

$$\Delta(P_r^\gamma) = P_r^\gamma \otimes 1 + 1 \otimes P_r^\gamma, \quad r > 0, \quad \gamma \in \Gamma.$$

Following [24, chap. I, app. B, (7.1)] we write

$$P_{r,p} = \ell^{-1} \sum_{\gamma\in\Gamma} \gamma^p P_r^\gamma, \quad r \geqslant 0, \quad p \in \mathbb{Z}_\ell.$$

We equip $\mathbf{\Lambda}_\Gamma$ with the Hopf pairing such that

$$\langle P_{r,p}, P_{s,q}\rangle = r\delta_{p,q}\delta_{r,s}, \quad r, s > 0, \quad p, q \in \mathbb{Z}_\ell.$$

We may regard $P_{r,p}$, $r > 0$, as the r-th power sum of a new sequence of variables $x_{i,p}$, $i > 0$. We define the following elements in $\mathbf{\Lambda}_\Gamma$

$$S_{\mu,p} = S_\mu(x_{i,p}), \quad S_\lambda = \prod_{p\in\mathbb{Z}_\ell} S_{\lambda(p),p}, \quad \mu \in \mathcal{P}, \quad \lambda \in \mathcal{P}^\ell. \tag{4.6}$$

The Hopf $\mathbb{C}$-algebras $R(\Gamma)$ and $\mathbf{\Lambda}_\Gamma$ are identified via the characteristic map [24, chap. I, app. B, (6.2)]

$$\mathrm{ch} : R(\Gamma) \to \mathbf{\Lambda}_\Gamma.$$

This map intertwines the induction in $R(\Gamma)$ with the multiplication in $\mathbf{\Lambda}_\Gamma$ by [24, chap. I, app. B, (6.3)]. By [24, chap. I, app. B, (9.4)] and (3.4) we have

$$\mathrm{ch}(\bar{L}_\lambda) = S_{\tau\lambda}, \quad \lambda \in \mathcal{P}^\ell, \tag{4.7}$$

where τ is the permutation of $\mathcal{P}^\ell$ such that $(\tau\lambda)(p) = \lambda(p+1)$ for each $p \in \mathbb{Z}_\ell$. For $\lambda \in \mathcal{P}^\Gamma$ we write

$$z_\lambda = \prod_{\gamma\in\Gamma} z_{\lambda(\gamma)} \ell^{\,l(\lambda(\gamma))},$$

where $z_{\lambda(\gamma)}$ is as in (3.1), and we define $\bar{\lambda} \in \mathcal{P}^\Gamma$ by $\bar{\lambda}(\gamma) = \lambda(\gamma^{-1})$. Then we have

$$\begin{aligned} \langle S_\lambda, S_\mu\rangle &= \delta_{\lambda,\mu}, \quad \lambda, \mu \in \mathcal{P}^\ell, \\ \langle P_\lambda, P_{\bar{\mu}}\rangle &= \delta_{\lambda,\mu} z_\lambda, \quad \lambda, \mu \in \mathcal{P}^\Gamma. \end{aligned} \tag{4.8}$$

The first equality is proved as in [24, chap. I, app. B, (7.4)], while the second one is [24, chap. I, app. B, (5.3')]. By (4.7), (4.8) the map ch is an isometry. Thus it intertwines the restriction in $R(\Gamma)$ with the comultiplication in $\mathbf{\Lambda}_\Gamma$.

Proposition 4.1. *(a) The restriction* $\mathrm{Rep}(\mathbb{C}\Gamma_n) \to \mathrm{Rep}(\mathbb{C}\mathfrak{S}_n)$ *yields the* $\mathbb{C}$*-algebra homomorphism* $\mathrm{Res}^{\Gamma}_{\mathfrak{S}} : \mathbf{\Lambda}_\Gamma \to \mathbf{\Lambda}$ *such that* $S_\lambda \mapsto \prod_p S_{\lambda(p)}$*,* $P_{r,p} \mapsto P_r$*.*

(b) The induction $\mathrm{Rep}(\mathbb{C}\mathfrak{S}_n) \to \mathrm{Rep}(\mathbb{C}\Gamma_n)$ *yields the* $\mathbb{C}$*-algebra homomorphism* $\mathrm{Ind}^{\Gamma}_{\mathfrak{S}} : \mathbf{\Lambda} \to \mathbf{\Lambda}_\Gamma$ *such that* $P_r \mapsto P^1_r = \sum_{p\in\mathbb{Z}_\ell} P_{r,p}$*.*

Proof. The first part of (a) is easy by Section 3.5, and it is left to the reader. For the second one, observe that

$$\mathrm{ch}(\sigma_{r,p}) = P_{r,p}, \quad r > 0,$$

where $\sigma_{r,p}$ is the class function on Γ_r wich takes the value $r(\gamma_1\gamma_2\cdots\gamma_r)^p$ on pairs $(w, (\gamma_1, \gamma_2, \ldots, \gamma_r))$ such that w is a r-cycle, and 0 elsewhere, see [10, lem. 5.1]. Now we concentrate on (b). Note that

$$\mathrm{Res}^{\Gamma}_{\mathfrak{S}}(P^\gamma_0) = 1, \quad \mathrm{Res}^{\Gamma}_{\mathfrak{S}}(P^\gamma_r) = \ell\delta_{\gamma,1}P_r, \quad r > 0.$$

Therefore, for $\lambda \in \mathcal{P}^\Gamma$ we have

$$\mathrm{Res}^{\Gamma}_{\mathfrak{S}}(P_\lambda) = \prod_{\gamma\in\Gamma} \mathrm{Res}^{\Gamma}_{\mathfrak{S}}(P^\gamma_{\lambda(\gamma)}) = \begin{cases} \ell^{\,l(\lambda(1))}P_{\lambda(1)} & \text{if } \lambda(\gamma) = \emptyset \text{ for } \gamma \neq 1, \\ 0 & \text{else.} \end{cases}$$

If $f, g \in [\mathrm{Rep}(\mathbb{C}\Gamma_n)]$ are the characters of finite dimensional Γ_n-modules V, W, then $\langle f, g\rangle$ is the dimension of the space of $\mathbb{C}\Gamma_n$-linear maps $V \to W$. Hence, by Frobenius reciprocity the operator $\mathrm{Ind}^{\Gamma}_{\mathfrak{S}}$ is adjoint to the operator $\mathrm{Res}^{\Gamma}_{\mathfrak{S}}$. Thus,

$$\langle \mathrm{Ind}^{\Gamma}_{\mathfrak{S}}(P_r), P_\lambda\rangle = \begin{cases} r\ell^{\,l(\lambda(1))}\delta_{\lambda(1),(r)} & \text{if } \lambda(\gamma) = \emptyset \text{ for } \gamma \neq 1, \\ 0 & \text{else.} \end{cases}$$

This implies that $\mathrm{Ind}^{\Gamma}_{\mathfrak{S}}(P_r) = aP^1_r$ for some a. To determine a let λ be such that $\lambda(\gamma) = \emptyset$ if $\gamma \neq 1$ and $\lambda(1) = (r)$. Then we have

$$P_\lambda = P^1_r, \quad \langle P_\lambda, P_\lambda\rangle = r\ell.$$

This implies that $a = 1$. □

Remark 4.2. Let $f \mapsto \bar{f}$ be the $\mathbb{C}$-antilinear involution of $\mathbf{\Lambda}_\Gamma$ which fixes the P_λ's with $\lambda \in \mathcal{P}^\Gamma$, see [24, chap. I, app. B, (5.2)]. For $\lambda \in \mathcal{P}^\ell$ let $\bar{\lambda}$ be the ℓ-partition given by $\bar{\lambda}(p) = \lambda(-p)$. We have

$$\bar{P}_{r,p} = P_{r,-p}, \quad \bar{S}_\lambda = S_{\bar{\lambda}}, \quad r > 0, \quad p \in \mathbb{Z}_\ell, \quad \lambda \in \mathcal{P}^\ell.$$

Remark 4.3. Setting $\ell = 1$ in $\mathbf{\Lambda}_\Gamma$ we get the standard Hopf algebra structure and the Hopf pairing of $\mathbf{\Lambda}$.

Remark 4.4. We have [24, chap. I, app. B, (7.1')]

$$P^\gamma_r = \sum_{p\in\mathbb{Z}_\ell} \gamma^{-p}P_{r,p}, \quad r \geqslant 0, \quad P^\gamma_0 = 1, \quad P_{0,p} = \delta_{0,p}.$$

4.5. **The level 1 Fock space.** Fix once for all a basis $(\epsilon_1, \ldots, \epsilon_m)$ of $\mathbb{C}^m$. The *level 1 Fock space of* $\widehat{\mathfrak{sl}}_m$ is the space $\mathcal{F}_m$ of semi-infinite wedges of the $\mathbb{C}$-vector space $V_m = \mathbb{C}^m \otimes \mathbb{C}[t, t^{-1}]$. More precisely, we have

$$\mathcal{F}_m = \bigoplus_{d\in\mathbb{Z}} \mathcal{F}^{(d)}_m,$$

where $\mathcal{F}^{(d)}_m$ is the subspace spanned by the semi-infinite wedges of *charge* d, i.e., the semi-infinite wedges of the form

$$u_{i_1} \wedge u_{i_2} \wedge \cdots, \quad i_1 > i_2 > \ldots, \quad u_{i-jm} = \epsilon_i \otimes t^j, \tag{4.9}$$

where $i_k = d - k + 1$ if $k \gg 0$. We write

$$|\lambda, d\rangle = u_{i_1} \wedge u_{i_2} \wedge \cdots, \quad \lambda \in \mathcal{P}, \quad i_k = \lambda_k + d - k + 1, \quad k > 0. \tag{4.10}$$

The elements $|\lambda, d\rangle$ with $\lambda \in \mathcal{P}$ form a basis of $\mathcal{F}_m^{(d)}$. We equip $\mathcal{F}_m^{(d)}$ with the $\mathbb{C}$-bilinear symmetric form such that this basis is orthonormal.

The Fock space $\mathcal{F}_m^{(d)}$ is equipped with a level one representation of $\widehat{\mathfrak{sl}}_m$ in the following way. First, the $\mathbb{C}$-vector space V_m is given the level 0 action of $\widehat{\mathfrak{sl}}_m$ induced by the homomorphism

$$\widehat{\mathfrak{sl}}_m \to \mathfrak{sl}_m \otimes \mathbb{C}[t, t^{-1}], \quad \mathbf{1} \mapsto 0, \quad x \otimes \varpi \mapsto x \otimes t \tag{4.11}$$

and the obvious actions of $\mathfrak{sl}_m$ and $\mathbb{C}[t, t^{-1}]$ on V_m. Then, taking semi-infinite wedges, this action yields a level 1 action of $\widehat{\mathfrak{sl}}_m$ on $\mathcal{F}_m^{(d)}$, see e.g., [30].

Next, observe that the multiplication by t^r, $r > 0$, yields an endomorphism of V_m. Taking semi-infinite wedges it yields a linear operator b_r on $\mathcal{F}_m^{(d)}$. Let b'_r be the adjoint of b_r. Then b'_r, b_r define a level m action of $\mathfrak{H}$ on $\mathcal{F}_m^{(d)}$. The $\widehat{\mathfrak{sl}}_m$-action and the $\mathfrak{H}$-action on $\mathcal{F}_m^{(d)}$ glue together, yielding a level 1 representation of $\widehat{\mathfrak{gl}}_m$ on $\mathcal{F}_m^{(d)}$, see [30] again.

As a $\widehat{\mathfrak{gl}}_m$-module we have a canonical isomorphism

$$\mathcal{F}_m^{(d)} = V^{\widehat{\mathfrak{gl}}_m}_{\omega_{d \bmod m}}.$$

It identifies the symmetric bilinear form of $\mathcal{F}_m^{(d)}$ with the Shapovalov form on $V^{\widehat{\mathfrak{gl}}_m}_{\omega_{d \bmod m}}$, i.e., with the unique (up to a scalar) symmetric bilinear form such that the adjoint of b_r, e_q are b'_r, f_q respectively.

Remark 4.5. The $\mathbb{C}$-linear isomorphism

$$\mathcal{F}_m^{(d)} \to \mathbf{\Lambda}, \quad |\lambda, d\rangle \mapsto S_\lambda, \quad \lambda \in \mathcal{P} \tag{4.12}$$

takes the operators b'_r, b_r, e_q, f_q on the left hand side to the operators b'_{mr}, b_{mr}, e_{q-d}, f_{q-d} on the right hand side.

4.6. **The level ℓ Fock space.** Fix a basis $(\epsilon_1, \ldots, \epsilon_m)$ of $\mathbb{C}^m$ and a basis $(\dot\epsilon_1, \ldots, \dot\epsilon_\ell)$ of $\mathbb{C}^\ell$. The *level ℓ Fock space of* $\widehat{\mathfrak{sl}}_m$ is the $\mathbb{C}$-vector space

$$\mathcal{F}_{m,\ell} = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}_{m,\ell}^{(d)}$$

of semi-infinite wedges of the $\mathbb{C}$-vector space $V_{m,\ell} = \mathbb{C}^m \otimes \mathbb{C}^\ell \otimes \mathbb{C}[z, z^{-1}]$. The latter are defined as in (4.9) with

$$u_{i+(j-1)m-km\ell} = \epsilon_i \otimes \dot\epsilon_j \otimes z^k. \tag{4.13}$$

Here $i = 1, \ldots, m$, $j = 1, \ldots, \ell$, and $k \in \mathbb{Z}$. We define basis elements $|\lambda, d\rangle$, with $\lambda \in \mathcal{P}$, of $\mathcal{F}_{m,\ell}^{(d)}$ as in (4.10), using the semi-infinite wedges above. We equip $\mathcal{F}_{m,\ell}^{(d)}$ with the $\mathbb{C}$-bilinear symmetric form such that the basis elements $|\lambda, d\rangle$ are orthonormal. This yields a $\mathbb{C}$-linear isomorphism

$$\mathcal{F}_{m,\ell}^{(d)} \to \mathbf{\Lambda}, \quad |\lambda, d\rangle \mapsto S_\lambda, \quad \lambda \in \mathcal{P}. \tag{4.14}$$

We equip the $\mathbb{C}$-vector space $\mathcal{F}_{m,\ell}^{(d)}$ with the following actions, see [30] for details :

- The level $m\ell$ action of $\mathfrak{H}$ such that b'_r, b_r is taken to the operator $b'_{m\ell r}$, $b_{m\ell r}$ on $\mathbf{\Lambda}$ under the isomorphism (4.14) for $r > 0$.
- The level ℓ action of $\widehat{\mathfrak{sl}}_m$ defined as follows : equip the $\mathbb{C}[z, z^{-1}]$-module $V_{m,\ell}$ with the level 0 action of $\widehat{\mathfrak{sl}}_m$ given by the evaluation homomorphism (4.11) and the obvious actions of $\mathfrak{sl}_m$ and $\mathbb{C}[z, z^{-1}]$ on $V_{m,\ell}$. Taking semi-infinite wedges we get a level ℓ action of $\widehat{\mathfrak{sl}}_m$ on $\mathcal{F}_{m,\ell}^{(d)}$.

- The level m action of $\widehat{\mathfrak{sl}}_\ell$ which is defined as above by exchanging the role of m and ℓ.

The actions of $\mathfrak{H}$, $\widehat{\mathfrak{sl}}_m$ and $\widehat{\mathfrak{sl}}_\ell$ commute with each other. We call *ℓ-charge of weight d* an ℓ-tuple of integers $s = (s_p)$ such that $d = \sum_p s_p$. Set

$$\hat{\gamma}(s,m) = (m - s_1 + s_\ell)\,\omega_0 + \sum_{p=1}^{\ell-1}(s_p - s_{p+1})\omega_p. \tag{4.15}$$

The Fock space associated with the ℓ-charge s is the subspace

$$\mathcal{F}^{(s)}_{m,\ell} = \mathcal{F}^{(d)}_{m,\ell}[\hat{\gamma}(s,m)] \tag{4.16}$$

consisting of the elements of weight $\hat{\gamma}(s,m)$ with respect to the $\widehat{\mathfrak{sl}}_\ell$-action. It is an $\widehat{\mathfrak{sl}}_m \times \mathfrak{H}$-submodule of $\mathcal{F}^{(d)}_{m,\ell}$. Consider the basis elements $|\lambda, s\rangle$, $\lambda \in \mathcal{P}^\ell$, of $\mathcal{F}^{(s)}_{m,\ell}$ defined in [30, sec. 4.1]. The representation of $\widehat{\mathfrak{sl}}_m$ on $\mathcal{F}^{(s)}_{m,\ell}$ can be characterized in the following way, see e.g., [17], [30],

$$e_q|\lambda, s\rangle = \sum_\nu |\nu, s\rangle, \quad f_q|\lambda, s\rangle = \sum_\mu |\mu, s\rangle, \tag{4.17}$$

where ν (resp. μ) runs through all ℓ-partitions obtained by removing (resp. adding) a node of coordinate (i,j) in the p-th partition of λ such that $q = s_p + j - i$ modulo m. Consider the $\mathbb{C}$-vector space isomorphism

$$\mathbf{\Lambda}_\Gamma \to \mathcal{F}^{(s)}_{m,\ell}, \quad S_{\tau\lambda} \mapsto |\lambda, s\rangle, \quad \lambda \in \mathcal{P}^\ell. \tag{4.18}$$

By [30, sec. 4.1] we have an equality of sets

$$\{|\lambda, s\rangle\,;\, \lambda \in \mathcal{P}^\ell,\, s = (s_p) \in \mathbb{Z}^\ell,\, \sum_p s_p = d\} = \{|\lambda, d\rangle\,;\, \lambda \in \mathcal{P}\}. \tag{4.19}$$

Thus the elements $|\lambda, s\rangle$ form an orthonormal basis of $\mathcal{F}^{(d)}_{m,\ell}$ and the map (4.18) preserves the pairings by (4.8). The representation of $\mathfrak{H}$ on $\mathcal{F}^{(s)}_{m,\ell}$ can be characterized in the following way.

Proposition 4.6. *The operators b'_r, b_r, $r > 0$, on $\mathcal{F}^{(s)}_{m,\ell}$ are adjoint to each other. Further b_r acts as the multiplication by the element $P^1_{mr} = \sum_p P_{mr,p}$ of $\mathbf{\Lambda}_\Gamma$ under the isomorphism (4.18).*

Proof. The first claim is [30, prop. 5.8]. To prove the second one, observe that the formulas in [30, sec. 4.1, 4.3 and (25)] imply that the $\mathbb{C}$-linear map

$$\mathcal{F}^{(s)}_{m,\ell} \to \bigotimes_{p\in\mathbb{Z}_\ell} \mathcal{F}^{(s_p)}_m, \quad |\lambda, s\rangle \mapsto \bigotimes_{p\in\mathbb{Z}_\ell} |\lambda(p), s_p\rangle,$$

intertwines the operator b_r on the left hand side and the operator

$$b_r \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes b_r \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes b_r$$

on the right hand side. Thus the proposition follows from the definition of the $\mathfrak{H}$-action on $\mathcal{F}_m$ in Section 4.5 and from the definition of the $\mathfrak{H}$-action on $\mathbf{\Lambda}$ in Section 4.2. □

Remark 4.7. The $\widehat{\mathfrak{sl}}_m$-action on $\mathcal{F}^{(s)}_{m,\ell}$ can be extended to an $\widetilde{\mathfrak{sl}}_m$-action such that the weight of $|\lambda, s\rangle$ is

$$-\Delta(s,m)\delta + \sum_{p=1}^{\ell} \omega_{s_p} - \sum_{q=0}^{m-1} n_q(\lambda)\alpha_q,$$

see [30, sec. 4.2]. Here $n_q(\lambda)$ is the number of q-nodes in λ, i.e., it is the sum over all p's of the number of nodes of coordinate (i,j) in the p-th partition of λ such that $s_p + j - i = q \bmod m$. We have also used the notation

$$\Delta(s,m) = \frac{1}{2}\sum_{p=1}^{\ell}\langle \omega_{s_p \bmod m},\, \omega_{s_p \bmod m}\rangle + \frac{1}{2}\sum_{p=1}^{\ell} s_p(s_p/m-1).$$

In particular, we have

$$D(|\lambda, s\rangle) = -(\Delta(s,m) + n_0(\lambda))\,|\lambda, s\rangle.$$

4.7. **Comparison of the $\widetilde{\mathfrak{gl}}_\ell$-modules $\mathcal{F}^{(0)}_{m,\ell}$ and $V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}$.** The Fock space $\mathcal{F}_{m,\ell}$ can be equipped with a level 1 representation of $\widetilde{\mathfrak{gl}}_\ell$ in the following way. The assignment

$$z \mapsto t^m, \quad \epsilon_i \mapsto t^{1-i}, \quad i = 1, 2, \dots, m,$$

yields a $\mathbb{C}$-linear isomorphism

$$\begin{gathered} V_{m,\ell} = \mathbb{C}^m \otimes \mathbb{C}^\ell \otimes \mathbb{C}[z, z^{-1}] \to \mathbb{C}^\ell \otimes \mathbb{C}[t, t^{-1}] = V_\ell, \\ u_{i+(j-1)m-km\ell} \mapsto u_{j+(i-1)\ell-km\ell}, \end{gathered} \tag{4.20}$$

see (4.9), (4.13). Taking semi-infinite wedges, it yields a $\mathbb{C}$-linear isomorphism

$$\mathcal{F}_{m,\ell} \to \mathcal{F}_\ell. \tag{4.21}$$

Pulling back the representation of $\widetilde{\mathfrak{gl}}_\ell$ on $\mathcal{F}_\ell$ in Section 4.5 and Remark 4.7 by (4.21) we get a level 1 action of $\widetilde{\mathfrak{gl}}_\ell$ on $\mathcal{F}_{m,\ell}$ such that :

- For $d \in \mathbb{Z}$ the level 1 representation of $\widetilde{\mathfrak{gl}}_\ell$ on $\mathcal{F}_{m,\ell}$ yields an isomorphism
$$\mathcal{F}^{(d)}_{m,\ell} = V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_{d \bmod \ell}}. \tag{4.22}$$
- The level m-action of $\widehat{\mathfrak{gl}}_\ell$ in $\mathcal{F}_{m,\ell}$ given in Section 4.6 can be recovered from the level 1 action by composing it with the Lie algebra homomorphism
$$\widehat{\mathfrak{gl}}_\ell \to \widehat{\mathfrak{gl}}_\ell, \quad x \otimes \varpi^r \mapsto x \otimes \varpi^{mr}, \quad \mathbf{1} \mapsto m\mathbf{1}. \tag{4.23}$$
- Pulling back the level ℓ representation of $\mathfrak{H}$ on $\mathcal{F}_\ell$ in Section 4.5 by (4.21) we get a level ℓ action of $\mathfrak{H}$ on $\mathcal{F}_{m,\ell}$. The level $m\ell$-action of $\mathfrak{H}$ in $\mathcal{F}_{m,\ell}$ given in Section 4.6 can be recovered from the latter by composing it with the Lie algebra homomorphism
$$b_r \mapsto b_{mr}, \quad b'_r \mapsto b'_{mr}, \quad \mathbf{1} \mapsto m\mathbf{1}. \tag{4.24}$$
Hence, the action of the level $m\ell$ Casimir operator, i.e., the operator obtained by replacing ℓ by $m\ell$ in (4.2), associated with the representation of $\mathfrak{H}$ on $\mathcal{F}_{m,\ell}$ is the same as the action of the m-th Casimir operator
$$\partial_m = \frac{1}{m\ell}\sum_{r \geqslant 1} b_{mr} b'_{mr} \tag{4.25}$$
associated with the level ℓ representation of $\mathfrak{H}$ on $\mathcal{F}_{m,\ell}$.
- To a partition λ we associate an *ℓ-quotient* λ^*, an *ℓ-core* λ^c and a *content polynomial* $c_\lambda(X)$ as in [24, chap. I]. In [22, sec. 2.1] a bijection τ is given from the set of ℓ-cores to the set of ℓ-charges of weight 0. By [30, rem. 4.2(i)] the inverse of the map (4.21) is such that
$$\mathcal{F}^{(0)}_\ell \to \mathcal{F}^{(0)}_{m,\ell}, \quad |\lambda, 0\rangle \mapsto |\lambda^*, \tau(\lambda^c)\rangle.$$

Now, the same argument as in [24, ex. I.11] shows that

$$c_\lambda(X) = c_{\lambda^c}(X) \prod_{p=0}^{\ell-1} (X+p)^{|\lambda^*|} \bmod \ell. \tag{4.26}$$

Further, by Remark 4.7 the scaling element D of the level 1 representation of $\widetilde{\mathfrak{gl}}_\ell$ on $\mathcal{F}^{(0)}_{m,\ell}$ is given by

$$D(|\lambda, 0\rangle) = -n_0(\lambda)\, |\lambda, 0\rangle, \tag{4.27}$$

where $n_0(\lambda)$ is the number of 0-nodes in λ. Thus we have the following relation

$$[D, f_p] = -f_p, \quad \forall f_p \in \widehat{\mathfrak{sl}}_m. \tag{4.28}$$

Further we have

$$D(|\lambda, 0\rangle) = -\big(n_0(\lambda^c) + |\lambda^*|\big)\, |\lambda, 0\rangle. \tag{4.29}$$

Remark 4.8. We finish this section by several remarks concerning the Fock space that we'll not use in the rest of the paper. First, there is a tautological $\mathbb{C}$-linear isomorphism $\mathbb{C}^m \otimes \mathbb{C}^\ell = \mathbb{C}^{m\ell}$. It yields $\mathbb{C}$-linear isomorphisms $V_{m,\ell} \to V_{m\ell}$ and $\mathcal{F}_{m,\ell} \to \mathcal{F}_{m\ell}$. Recall that $\mathcal{F}_{m\ell}$ is equipped with a level 1 action of $\widehat{\mathfrak{sl}}_{m\ell}$, and that $\mathcal{F}_{m,\ell}$ is equipped with a level (ℓ, m)-action of $\widehat{\mathfrak{sl}}_m \times \widehat{\mathfrak{sl}}_\ell$. Now, there is a well-known Lie algebra inclusion

$$(\widehat{\mathfrak{sl}}_m \times \widehat{\mathfrak{sl}}_\ell)/(m(\mathbf{1}, 0) - \ell(0, \mathbf{1})) \subset \widehat{\mathfrak{sl}}_{m\ell}, \quad (\mathbf{1}, 0) \mapsto \ell\mathbf{1}, \quad (0, \mathbf{1}) \mapsto m\mathbf{1}.$$

This inclusion intertwines the $\widehat{\mathfrak{sl}}_m \times \widehat{\mathfrak{sl}}_\ell$-action on $\mathcal{F}_{m,\ell}$ and the $\widehat{\mathfrak{sl}}_{m\ell}$-action on $\mathcal{F}_{m,\ell} = \mathcal{F}_{m\ell}$. Further, we want to compare the $\widehat{\mathfrak{sl}}_{m\ell}$-action on $\mathcal{F}_{m,\ell}$ with the level one $\widehat{\mathfrak{sl}}_\ell$-action on $\mathcal{F}_{m,\ell}$ given in the begining of this section. The $\mathbb{C}$-linear isomorphisms (4.20) and (4.21) yield a $\mathbb{C}$-linear isomorphism

$$\mathcal{F}_\ell \to \mathcal{F}_{m,\ell} = \mathcal{F}_{m\ell}. \tag{4.30}$$

The right hand side is equipped with a level 1 action of $\widehat{\mathfrak{sl}}_{m\ell}$, and the left hand side with a level 1 action of $\widehat{\mathfrak{sl}}_\ell$. Consider the following elements in $\mathfrak{sl}_m \otimes \mathbb{C}[\varpi, \varpi^{-1}]$

$$x(i+km) = \sum_{j=1}^{m-i} e_{j,i+j} \otimes \varpi^k + \sum_{j=m-i+1}^{m} e_{j,i+j-m} \otimes \varpi^{k+1},$$
$$1 \leqslant i \leqslant m, \quad k \in \mathbb{Z}.$$

For $x \in \mathfrak{sl}_m \otimes \mathbb{C}[\varpi, \varpi^{-1}]$ and $p, q = 1, 2, \dots, \ell$ we define the element $x^{(p,q)} \in \mathfrak{sl}_{m\ell} \otimes \mathbb{C}[\varpi, \varpi^{-1}]$ by

$$x^{(p,q)} = \sum_{i,j=1}^{m} e_{i+(p-1)m,\, j+(q-1)m} \otimes a_{i,j} \quad \text{for} \quad x = \sum_{i,j=1}^{m} e_{i,j} \otimes a_{i,j}.$$

The following claim is proved by a direct computation which is left to the reader.

Proposition 4.9. *(a) There is a Lie algebra inclusion $\widehat{\mathfrak{sl}}_\ell \subset \widehat{\mathfrak{sl}}_{m\ell}$ given by*

$$\mathbf{1} \mapsto \mathbf{1}, \quad e_{p,q} \otimes \varpi^r \mapsto x(r)^{(p,q)}, \quad p, q = 1, 2, \dots, \ell, \quad r \in \mathbb{Z}.$$

(b) The map (4.30) intertwines the $\widehat{\mathfrak{sl}}_\ell$-action on $\mathcal{F}_\ell$ and the $\widehat{\mathfrak{sl}}_{m\ell}$-action on $\mathcal{F}_{m\ell}$.

5. The categorification of the Heisenberg algebra

We'll abbreviate

$$[\mathcal{O}(\Gamma)] = \bigoplus_{n\geqslant 0}[\mathcal{O}(\Gamma_n)].$$

Assume that h, h_p are rational numbers as in (3.9). Thus Λ is a rational weight of $\widehat{\mathfrak{sl}}_\ell$ of level 1. Let m be the denominator of h. We'll assume that $m > 2$.

5.1. **The functors $A_{\lambda,!}, A^*_\lambda, A_{\lambda,*}$ on $D^b(\mathcal{O}(\Gamma))$.** To simplify the exposition, from now on we'll assume that $\ell > 1$. All the statements below have an analoguous version for $\ell = 1$, by replacing everywhere $\mathbb{C}^n$ by $\mathbb{C}^n_0$. Let n, r be non-negative integers. Consider the point

$$b_{n,r} = (0, \dots, 0, 1, \dots, 1) \in \mathfrak{h} = \mathbb{C}^{n+r},$$

with $x_i = 0$ for $1 \leqslant i \leqslant n$, and $x_i = 1$ for $n < i \leqslant n + r$. The centralizer of $b_{n,r}$ in Γ_{n+r} is the parabolic subgroup $\Gamma_{n,r}$. We have

$$\mathfrak{h}/\mathfrak{h}^{\Gamma_{n,r}} = \mathbb{C}^n \times \mathbb{C}^r_0.$$

Here $\mathbb{C}^n$ is the reflection representation of Γ_n and $\mathbb{C}^r_0$ is the reflection representation of $\mathfrak{S}_r$. Note that

$$\mathcal{O}(\Gamma_{n,r}) = \mathcal{O}(\Gamma_{n,r}, \mathbb{C}^n \times \mathbb{C}^r_0), \quad \mathcal{O}(\mathfrak{S}_r) = \mathcal{O}(\mathfrak{S}_r, \mathbb{C}^r_0).$$

In particular we have a canonical equivalence of categories

$$\mathcal{O}(\Gamma_{n,r}) = \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathfrak{S}_r).$$

Thus the induction and restriction relative to $b_{n,r}$ yield functors

$$^{\mathcal{O}}\mathrm{Ind}_{n,r} : \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathfrak{S}_r) \to \mathcal{O}(\Gamma_{n+r}),$$
$$^{\mathcal{O}}\mathrm{Res}_{n,r} : \mathcal{O}(\Gamma_{n+r}) \to \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathfrak{S}_r).$$

Now consider the functors $^{\mathcal{O}}\mathrm{Ind}_{n,mr}$, $^{\mathcal{O}}\mathrm{Res}_{n,mr}$. The parameters of $H(\Gamma_{n+mr})$ and $H(\Gamma_n)$ are h, Λ. The parameter of $H(\mathfrak{S}_{mr})$ is h. Fix a partition $\lambda \in \mathcal{P}_r$. We define the functors

$$\mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathfrak{S}_{mr}) \to \mathcal{O}(\Gamma_n),$$
$$M \mapsto \mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_{mr})}(M, L_{m\lambda})^*, \quad M \mapsto \mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_{mr})}(L_{m\lambda}, M),$$

as the tensor product of the identity of $\mathcal{O}(\Gamma_n)$ and of the functors

$$\mathcal{O}(\mathfrak{S}_{mr}) \to \mathrm{Rep}(\mathbb{C}),$$
$$M \mapsto \mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_{mr})}(M, L_{m\lambda})^*, \quad M \mapsto \mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_{mr})}(L_{m\lambda}, M).$$

Here the upperscript $*$ denotes the dual $\mathbb{C}$-vector space. We denote the corresponding derived functors in the following way

$$M \mapsto \mathrm{RHom}_{\mathcal{O}(\mathfrak{S}_{mr})}(M, L_{m\lambda})^*, \quad M \mapsto \mathrm{RHom}_{\mathcal{O}(\mathfrak{S}_{mr})}(L_{m\lambda}, M).$$

Definition 5.1. For $\lambda \in \mathcal{P}_r$ with $r \geqslant 0$ we define the functors

$$A_{\lambda,!} : D^b(\mathcal{O}(\Gamma_{n+mr})) \to D^b(\mathcal{O}(\Gamma_n)), \quad M \mapsto \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{mr}))}(^{\mathcal{O}}\mathrm{Res}_{n,mr}(M), L_{m\lambda})^*,$$
$$A^*_\lambda : D^b(\mathcal{O}(\Gamma_n)) \to D^b(\mathcal{O}(\Gamma_{n+mr})), \quad M \mapsto {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(M \otimes L_{m\lambda}),$$
$$A_{\lambda,*} : D^b(\mathcal{O}(\Gamma_{n+mr})) \to D^b(\mathcal{O}(\Gamma_n)), \quad M \mapsto \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{mr}))}(L_{m\lambda}, {}^{\mathcal{O}}\mathrm{Res}_{n,mr}(M)).$$

Proposition 5.2. *We have a triple of exact adjoint endofunctors $(A_{\lambda,!}, A^*_\lambda, A_{\lambda,*})$ of the triangulated category $D^b(\mathcal{O}(\Gamma))$. For $M, N \in D^b(\mathcal{O}(\Gamma))$ we have*

$$\mathrm{RHom}_{D^b(\mathcal{O}(\Gamma))}(A^*_\lambda(M), N) = \mathrm{RHom}_{D^b(\mathcal{O}(\Gamma))}(M, A_{\lambda,*}(N)),$$
$$\mathrm{RHom}_{D^b(\mathcal{O}(\Gamma))}(A_{\lambda,!}(M), N) = \mathrm{RHom}_{D^b(\mathcal{O}(\Gamma))}(M, A^*_\lambda(N)).$$

Proof. Obvious because the functors ${}^{\mathcal{O}}\mathrm{Ind}_{n,mr}$ and ${}^{\mathcal{O}}\mathrm{Res}_{n,mr}$ are exact and biadjoint, see [1], [28]. □

5.2. **The $\mathfrak{S}_r$-action on $(A_!)^r$, $(A^*)^r$ and $(A_*)^r$.** For $\flat =!, *$ we write $A^* = A^*_{(1)}$ and $A_\flat = A_{(1),\flat}$. For $r \geqslant 1$, the transitivity of the induction and restriction functors [28, cor. 2.5] yield functor isomorphisms

$$\begin{aligned}(A_!)^r &= \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}\big({}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(\bullet), L\big)^*,\\ (A^*)^r &= {}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}(\bullet \otimes L) = {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}\big(\bullet \otimes {}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)\big),\\ (A_*)^r &= \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}\big(L, {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(\bullet)\big).\end{aligned} \tag{5.1}$$

Here, to unburden the notation we abbreviate $L = L_{(m)}^{\otimes r}$. The goal of this section is to construct a $\mathfrak{S}_r$-action on $(A_!)^r$, $(A^*)^r$ and $(A_*)^r$, and to decompose these functors using this action. To do this, let $\mathbf{H}(\Gamma_{n,(m^r)})$, $\mathbf{H}(\Gamma_n)$, $\mathbf{H}(\mathfrak{S}_m)$ be as in Appendix A, with the parameters ζ and v_p as in Section 3.7. There is an obvious isomorphism

$$\mathbf{H}(\Gamma_{n,(m^r)}) = \mathbf{H}(\Gamma_n) \otimes \mathbf{H}(\mathfrak{S}_m)^{\otimes r}.$$

Let $\tau_i \in \mathfrak{S}_{n+mr}$ be the unique permutation such that

- τ_i is minimal in the coset $\mathfrak{S}_{(n,m^r)}\tau_i\mathfrak{S}_{(n,m^r)}$,
- $\tau_i(vw_1w_2\dots w_r)\tau_i^{-1} = vw_1\dots w_{i+1}w_i\dots w_r$ for $v \in \mathfrak{S}_n$, $w_1, \dots, w_r \in \mathfrak{S}_m$.

Let τ_i denote also the algebra isomorphism $\mathbf{H}(\Gamma_{n,(m^r)}) \to \mathbf{H}(\Gamma_{n,(m^r)})$ given by

$$x \otimes y_1 \otimes \dots \otimes y_r \to x \otimes y_1 \otimes \dots \otimes y_{i+1} \otimes y_i \otimes \dots \otimes y_r.$$

We have the following relation in $\mathbf{H}(\Gamma_{n+mr})$

$$T_{\tau_i} z = \tau_i(z) T_{\tau_i}, \quad z \in \mathbf{H}(\Gamma_{n,(m^r)}). \tag{5.2}$$

Therefore, the element T_{τ_i} belongs to the normalizer of $\mathbf{H}(\Gamma_{n,(m^r)})$ in $\mathbf{H}(\Gamma_{n+mr})$. The twist of a module by τ_i yields the functor

$$\begin{gathered}\tau_i : \mathrm{Rep}(\mathbf{H}(\Gamma_{n,(m^r)})) \to \mathrm{Rep}(\mathbf{H}(\Gamma_{n,(m^r)})),\\ M \otimes N_1 \otimes \dots \otimes N_r \to M \otimes N_1 \otimes \dots \otimes N_{i+1} \otimes N_i \otimes \dots \otimes N_r.\end{gathered}$$

We define the morphism of functors

$$\begin{gathered}{}^{\mathbf{H}}\tau_i : {}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)} \to {}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)} \circ \tau_i, \quad {}^{\mathbf{H}}\tau_i(M)(h \otimes v) = hT_{\tau_i} \otimes \tau_i(v),\\ h \in \mathbf{H}(\Gamma_{n+mr}), \quad v \in M, \quad M \in \mathrm{Rep}(\mathbf{H}(\Gamma_{n,(m^r)})).\end{gathered}$$

It is well-defined by (5.2). Next, the permutation τ_i yields also a functor

$$\begin{gathered}\tau_i : \mathcal{O}(\Gamma_{n,(m^r)}) \to \mathcal{O}(\Gamma_{n,(m^r)}),\\ M \otimes N_1 \otimes \dots \otimes N_r \to M \otimes N_1 \otimes \dots \otimes N_{i+1} \otimes N_i \otimes \dots \otimes N_r.\end{gathered}$$

The functor KZ yields a $\mathbb{C}$-algebra isomorphism [28, lem. 2.4]

$$\mathrm{KZ} : \mathrm{End}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}\big) \to \mathrm{End}\big(\mathrm{KZ} \circ {}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}\big) = \mathrm{End}\big({}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)} \circ \mathrm{KZ}\big). \tag{5.3}$$

For the same reason we have also an isomorphism

$$\mathrm{KZ} : \mathrm{Hom}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}, {}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)} \circ \tau_i\big) \to \mathrm{Hom}\big({}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)} \circ \mathrm{KZ}, {}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)} \circ \tau_i \circ \mathrm{KZ}\big).$$

So there is a unique morphism of functors

$${}^{\mathcal{O}}\tau_i : {}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)} \to {}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)} \circ \tau_i$$

which satisfies the following identity

$$\mathrm{KZ}({}^{\mathcal{O}}\tau_i(M)) = {}^{\mathbf{H}}\tau_i(\mathrm{KZ}(M)), \quad M \in \mathcal{O}(\Gamma_{n,(m^r)}). \tag{5.4}$$

The functor $\bullet \otimes L$ yields a map

$$\mathrm{Hom}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}, {}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)} \circ \tau_i\big) \to \mathrm{End}((A^*)^r). \tag{5.5}$$

Let $\bar\tau_i$ denote the image of ${}^{\mathcal{O}}\tau_i$ by this map.

Lemma 5.3. *The following relations hold in* $\mathrm{End}\big((A^*)^r\big)$

- $\bar\tau_i^2 = 1$,
- $\bar\tau_i\bar\tau_j = \bar\tau_j\bar\tau_i$ *if* $j \neq i-1, i+1$,
- $\bar\tau_i\bar\tau_{i+1}\bar\tau_i = \bar\tau_{i+1}\bar\tau_i\bar\tau_{i+1}$.

Proof. We'll write $L^S = (L^S_{(m)})^{\dot\otimes r}$. Consider the morphism of functors

$$\begin{gathered}{}^{\mathbf{H}}\tau_i^0 : {}^{\mathbf{H}}\mathrm{Ind}_{(m^r)} \to {}^{\mathbf{H}}\mathrm{Ind}_{(m^r)} \circ \tau_i, \quad {}^{\mathbf{H}}\tau_i^0(M)(h\otimes v) = hT_{\tau_i}\otimes \tau_i(v), \\ h \in \mathbf{H}(\mathfrak{S}_{mr}), \quad v \in M, \quad M \in \mathrm{Rep}(\mathbf{H}(\mathfrak{S}_m)^{\otimes r}).\end{gathered} \tag{5.6}$$

It is well-defined by (5.2). By (5.3) there is a unique morphism of functors

$${}^{\mathcal{O}}\tau_i^0 : {}^{\mathcal{O}}\mathrm{Ind}_{(m^r)} \to {}^{\mathcal{O}}\mathrm{Ind}_{(m^r)} \circ \tau_i$$

such that

$$\mathrm{KZ}({}^{\mathcal{O}}\tau_i^0(M)) = {}^{\mathbf{H}}\tau_i^0(\mathrm{KZ}(M)). \tag{5.7}$$

We define the endomorphism $\bar\tau_i^0$ of the module ${}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)$ by

$$\bar\tau_i^0 = {}^{\mathcal{O}}\tau_i^0(L). \tag{5.8}$$

The transitivity of the induction functor [28, cor. 2.5] yields

$$\begin{gathered}(A^*)^r(M) = {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}\big(M \otimes {}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)\big), \\ \bar\tau_i(M) = {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(\mathbf{1}\otimes\bar\tau_i^0).\end{gathered} \tag{5.9}$$

Therefore, we are reduced to check the following relations

- $(\bar\tau_i^0)^2 = 1$,
- $\bar\tau_i^0\bar\tau_j^0 = \bar\tau_j^0\bar\tau_i^0$ if $j \neq i-1, i+1$,
- $\bar\tau_i^0\bar\tau_{i+1}^0\bar\tau_i^0 = \bar\tau_{i+1}^0\bar\tau_i^0\bar\tau_{i+1}^0$.

To prove this, recall that Rouquier's functor R yields an equivalence

$$\mathcal{O}(\mathfrak{S}_{mr}) \to \mathrm{Rep}(\mathbf{S}_\zeta(mr)). \tag{5.10}$$

Here ζ is a primitive m-th root of 1. We have

$$R(L_{m\lambda}) = L^S_{m\lambda}. \tag{5.11}$$

By Proposition 3.1 we have also

$$R\big({}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)\big) = L^S.$$

Thus the functor R yields a $\mathbb{C}$-algebra isomorphism

$$\mathrm{End}_{\mathcal{O}(\mathfrak{S}_{mr})}\big({}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)\big) = \mathrm{End}_{\mathbf{S}_\zeta(mr)}(L^S).$$

Therefore, we are reduced to check the following relations in $\mathrm{End}_{\mathbf{S}_\zeta(mr)}(L^S)$

- $R(\bar\tau_i^0)^2 = 1$,
- $R(\bar\tau_i^0)R(\bar\tau_j^0) = R(\bar\tau_j^0)R(\bar\tau_i^0)$ if $j \neq i-1, i+1$,
- $R(\bar\tau_i^0)R(\bar\tau_{i+1}^0)R(\bar\tau_i^0) = R(\bar\tau_{i+1}^0)R(\bar\tau_i^0)R(\bar\tau_{i+1}^0)$.

By Proposition 3.1 there is an isomorphism of functors $\mathcal{O}(\mathfrak{S}_{(m^r)}) \to \mathrm{Rep}(\mathbf{S}_\zeta(mr))$

$$(\bullet)^{\dot\otimes r} \circ R = R \circ {}^{\mathcal{O}}\mathrm{Ind}_{(m^r)} \circ (\bullet)^{\otimes r}.$$

Since $\mathbf{1}_R\, {}^{\mathcal{O}}\tau_i^0\, \mathbf{1}_{(\bullet)^{\otimes r}}$ is an endomorphism of the right hand side and since R is an equivalence, there is an unique endomorphism ${}^{\mathbf{S}}\tau_i^0$ of the functor

$$(\bullet)^{\dot\otimes r} : \mathrm{Rep}(\mathbf{S}_\zeta(m)) \to \mathrm{Rep}(\mathbf{S}_\zeta(mr))$$

such that

$${}^{\mathbf{S}}\tau_i^0\, \mathbf{1}_R = \mathbf{1}_R\, {}^{\mathcal{O}}\tau_i^0\, \mathbf{1}_{(\bullet)^{\otimes r}}. \tag{5.12}$$

Consider the diagram

$$\begin{array}{ccc} \mathrm{End}\big({}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}\circ(\bullet)^{\otimes r}\big) & \xrightarrow{\mathrm{KZ}} & \mathrm{End}\big({}^{\mathbf{H}}\mathrm{Ind}_{(m^r)}\circ\mathrm{KZ}\circ(\bullet)^{\otimes r}\big) \\ \Big\downarrow{\scriptstyle R} & \nearrow{\scriptstyle \Phi^*} & \\ \mathrm{End}\big((\bullet)^{\dot\otimes r}\circ R\big). & & \end{array}$$

The upper map is invertible by (5.3), the vertical one by Proposition 3.1, and the lower one by Corollary B.4. The diagram is commutative because $\Phi^*\circ R=\mathrm{KZ}$. By (5.7) and (5.12) the image of ${}^{\mathcal{O}}\tau_i^0\,\mathbf{1}_{(\bullet)^{\otimes r}}$ is given by

$$\begin{array}{ccc} {}^{\mathcal{O}}\tau_i^0\mathbf{1}_{(\bullet)^{\otimes r}} & \longmapsto & {}^{\mathbf{H}}\tau_i^0\,\mathbf{1}_{\mathrm{KZ}\circ(\bullet)^{\otimes r}} \\ \Big\downarrow & \nearrow & \\ {}^{\mathbf{S}}\tau_i^0\,\mathbf{1}_{R(\bullet)}. & & \end{array} \tag{5.13}$$

Now, recall the endomorphisms of functors $\mathcal{R}_{\bullet,i}$, $\mathcal{S}_{\bullet,i}$ defined in (B.10), (B.9). By Corollary B.7 the functor Φ^* yields a map

$$\mathrm{End}\big((\bullet)^{\dot\otimes r}\big)\to\mathrm{End}\big({}^{\mathbf{H}}\mathrm{Ind}_{(m^r)}\circ(\bullet)^{\otimes r}\circ\Phi^*\big),\quad \mathcal{R}_{\bullet,i}\mapsto\mathcal{S}_{\Phi^*(\bullet),i}.$$

By (5.6) we have

$$\mathcal{S}_{M,i}={}^{\mathbf{H}}\tau_i^0(M^{\otimes r}),\quad M\in\mathrm{Rep}(\mathbf{H}(\mathfrak{S}_m)).$$

Therefore, by (5.13) we have also

$$\mathcal{R}_{M,i}={}^{\mathbf{S}}\tau_i^0(M),\quad M\in\mathrm{Rep}(\mathbf{S}_\zeta(m)). \tag{5.14}$$

Now, by (5.8), (5.11) and (5.12) we have

$$R(\bar\tau_i^0)={}^{\mathbf{S}}\tau_i^0\big(L^S_{(m)}\big).$$

Thus, by (5.14) we must check that the operators $\mathcal{R}_{L^S_{(m)},i}$ satisfies the same relations as above. The quantum Frobenius homomorphism yields a functor

$$\mathrm{Fr}^*:\mathrm{Rep}(\mathbf{S}_1(r))\to\mathrm{Rep}(\mathbf{S}_\zeta(mr))$$

such that $L^S_{(m)}=\mathrm{Fr}^*(\bar L^S_{(1)})$, see Section B.7. It is a braided tensor functor by Proposition B.9. Thus the claim follows from Proposition B.8. □

We can now prove the following, which is the main result of this section.

Proposition 5.4. *Let $r\geqslant 1$.*

(a) The group $\mathfrak{S}_r$ acts on the functors $(A^*)^r$, $(A_*)^r$.

(b) We have the following $\mathfrak{S}_r$-equivariant isomorphisms of functors

$$(A^*)^r=\bigoplus_{\lambda\in\mathcal{P}_r}\bar L_\lambda\otimes A^*_\lambda,\quad (A_*)^r=\bigoplus_{\lambda\in\mathcal{P}_r}\bar L_\lambda\otimes A_{\lambda,*}.$$

Proof. First, we concentrate on part (a). To unburden the notation we abbreviate

$$L=L^{\otimes r}_{(m)},\quad L^S=(L^S_{(m)})^{\dot\otimes r}.$$

By Lemma 5.3 the assignment $s_i\mapsto\bar\tau_i$ yields a $\mathfrak{S}_r$-action on $(A^*)^r$. Under the adjunction $\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)},{}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}\big)$ the isomorphism ${}^{\mathcal{O}}\tau_i$ yields a (right transposed)

isomorphism of ${}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}$. We'll denote it by ${}^{\mathcal{O}}\tau_i$ again. By definition of the right transposition, the following square is commutative for $M \in \mathcal{O}(\Gamma_{n+mr})$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}(\Gamma_{n+mr})}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}(L), M\big) & \xrightarrow{\circ\,{}^{\mathcal{O}}\tau_i(L)} & \mathrm{Hom}_{\mathcal{O}(\Gamma_{n+mr})}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}(L), M\big) \\ \| & & \| \\ \mathrm{Hom}_{\mathcal{O}(\Gamma_{n,(m^r)})}\big(L, {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(M)\big) & \xrightarrow{{}^{\mathcal{O}}\tau_i(M)\,\circ} & \mathrm{Hom}_{\mathcal{O}(\Gamma_{n,(m^r)})}\big(L, {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(M)\big). \end{array}$$

Here and in the rest of the proof, we use the canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}(\Gamma_{n+mr})}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}(L), M\big) &= \mathrm{Hom}_{\mathcal{O}(\Gamma_{n+mr})}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}(\tau_i(L)), M\big), \\ \mathrm{Hom}_{\mathcal{O}(\Gamma_{n,(m^r)})}\big(L, {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(M)\big) &= \mathrm{Hom}_{\mathcal{O}(\Gamma_{n,(m^r)})}\big(L, \tau_i({}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(M))\big) \end{aligned}$$

given by $\tau_i(L) = L$ without mentionning them explicitly. Let $\langle \bullet, \bullet \rangle$ denote the canonical pairing

$$\mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}(\bullet, L)^* \times \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}(\bullet, L) \to \mathbb{C}.$$

We define the $\mathfrak{S}_r$-action on $(A_*)^r$ by

$$\begin{aligned} s_i(f) &= {}^{\mathcal{O}}\tau_i(M) \circ f, \\ f \in (A_*)^r(M) &= \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}\big(L, {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(M)\big). \end{aligned} \tag{5.15}$$

Note that the formulas (5.15) do define an action of the group $\mathfrak{S}_r$ by Lemma 5.3, because the square above is commutative.

Now, we prove part (b). It is convenient to rewrite the $\mathfrak{S}_r$-action on $(A^*)^r$ in a slightly different way. Setting $n = 0$ in the construction above we get a $\mathfrak{S}_r$-action on ${}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)$ such that s_i acts through the operator $\bar{\tau}_i^0$ in (5.8), and by (5.9) the reflection s_i acts on $(A^*)^r$ through the automorphism

$${}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(\mathbf{1} \otimes {}^{\mathcal{O}}\bar{\tau}_i^0).$$

We claim that the following identity holds in $\mathrm{Rep}(\mathbb{C}\mathfrak{S}_r) \otimes \mathcal{O}(\mathfrak{S}_{mr})$

$${}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L) = \bigoplus_{\lambda \in \mathcal{P}_r} \bar{L}_\lambda \otimes L_{m\lambda}. \tag{5.16}$$

To prove (5.16) we use Rouquier's functor R as in the proof of Lemma 5.3. It is enough to check the following identity in $\mathrm{Rep}(\mathbb{C}\mathfrak{S}_r) \otimes \mathrm{Rep}(\mathbf{S}_\zeta(mr))$

$$L^S = \bigoplus_{\lambda \in \mathcal{P}_r} \bar{L}_\lambda \otimes L^S_{m\lambda}.$$

To do that, note that by Proposition B.9 the functor in Section B.7

$$\mathrm{Fr}^* : \mathrm{Rep}(\mathbf{S}_1(r)) = \mathrm{Rep}(\mathbf{S}_{(-1)^m}(r)) \to \mathrm{Rep}(\mathbf{S}_\zeta(mr))$$

given by the quantum Frobenius homomorphism is a braided tensor functor. Further we have

$$\mathrm{Fr}^*(\bar{L}^S_\lambda) = L^S_{m\lambda}, \quad \mathrm{Fr}^*((\bar{L}^S_{(1)})^{\dot{\otimes} r}) = L^S,$$

where $\bar{L}^S_\lambda$ is the simple $\mathbf{S}_1(r)$-module with the highest weight λ. Therefore, to prove (5.16) we are reduced to check the following identity in $\mathrm{Rep}(\mathbb{C}\mathfrak{S}_r) \otimes \mathrm{Rep}(\mathbf{S}_1(r))$

$$(\bar{L}^S_{(1)})^{\dot{\otimes} r} = \bigoplus_{\lambda \in \mathcal{P}_r} \bar{L}_\lambda \otimes \bar{L}^S_\lambda.$$

This is a trivial consequence of the Schur duality. The decomposition

$$(A^*)^r = \bigoplus_{\lambda \in \mathcal{P}_r} \bar{L}_\lambda \otimes A^*_\lambda \tag{5.17}$$

is a direct consequence of (5.16). The decomposition of the functor $(A_*)^r$ follows from (5.16) and the commutativity of the diagram above, because it implies that the canonical isomorphism

$$(A_*)^r(M) = \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}\big({}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L), {}^{\mathcal{O}}\mathrm{Res}_{n,mr}(M)\big)$$

is $\mathfrak{S}_r$-equivariant. □

Remark 5.5. Using an adjunction $\big({}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}, {}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}\big)$ for each r, we can construct in a similar way a $\mathfrak{S}_r$-action on the functor $(A_!)^r$ such that we have the decomposition

$$(A_!)^r = \bigoplus_{\lambda\in\mathcal{P}_r} \bar{L}_\lambda \otimes A_{\lambda,!}.$$

Then, by Propositions 5.2 and 5.4 we have the triple $\big((A_!)^r, (A^*)^r, (A_*)^r\big)$ of adjoint $\mathfrak{S}_r$-equivariant functors.

Remark 5.6. We have used the hypothesis $m > 2$ in the proof of Proposition 5.4 when using Rouquier's functor R. Probably this is not necessary.

Proposition 5.7. *For $r \geqslant 1$ we have an isomorphism of functors*

$$(A_!)^r[2r(1-m)] = (A_*)^r.$$

Proof. Once again we'll abbreviate $L = L_{(m)}^{\otimes r}$. Let $\mathcal{P}erv(\mathbb{P}^{m-1})$ be the category of perverse sheaves on $\mathbb{P}^{m-1}$ which are constructible with respect to the standard stratification $\mathbb{P}^{m-1} = \mathbb{C}^0 \cup \mathbb{C}^1 \cup \cdots \cup \mathbb{C}^{m-1}$. By [2, thm. 1.3] the category $\mathcal{O}(\mathfrak{S}_m)$ decomposes as the direct sum of $\mathcal{P}erv(\mathbb{P}^{m-1})$ and semisimple blocks. Under this equivalence the module $L_{(m)}$ is taken to the perverse sheaf $\mathbb{C}_{\mathbb{P}^{m-1}}[m-1]$. So, by Verdier duality [18, (3.1.8)] we have an isomorphism of functors $D^b(\mathcal{O}(\mathfrak{S}_m)) \to D^b(\mathbb{C})$

$$\mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_m))}(L_{(m)}, \bullet) \to \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_m))}(\bullet, L_{(m)})^*[2(1-m)]. \tag{5.18}$$

The tensor power of (5.18) is an isomorphism of functors $D^b(\mathcal{O}(\mathfrak{S}_{(m^r)})) \to D^b(\mathbb{C})$

$$\theta^0 : \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}(L, \bullet) \to \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}(\bullet, L)^*[2r(1-m)].$$

The group $\mathfrak{S}_r$ acts on $D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))$ in such a way that the simple reflection s_i acts via the permutation functor

$$\tau_i : \mathcal{O}(\mathfrak{S}_{(m^r)}) \to \mathcal{O}(\mathfrak{S}_{(m^r)}), \quad M_1 \otimes \cdots \otimes M_r \to M_1 \otimes \cdots \otimes M_{i+1} \otimes M_i \otimes \cdots \otimes M_r.$$

The isomorphism θ^0 is $\mathfrak{S}_r$-equivariant, i.e., we have

$$\theta^0(\tau_i(M))(\tau_i(f)) = \tau_i(\theta^0(M)(f)),$$
$$M \in \mathcal{O}(\mathfrak{S}_{(m^r)}), \quad f \in \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}(L, M).$$

It yields an isomorphism of functors $D^b(\mathcal{O}(\Gamma_{n,(m^r)})) \to D^b(\mathcal{O}(\Gamma_n))$

$$\theta : \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}(L, \bullet) \to \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}(\bullet, L)^*[2r(1-m)]$$

such that

$$\begin{gathered}\theta(\tau_i(M))(\tau_i(f)) = \tau_i(\theta(M)(f)),\\ M \in \mathcal{O}(\Gamma_{n,(m^r)}), \quad f \in \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}(L, M).\end{gathered} \tag{5.19}$$

We define an isomorphism of functors $D^b(\mathcal{O}(\Gamma_{n+mr})) \to D^b(\mathcal{O}(\Gamma_n))$ by

$$\theta' = \theta\, \mathbf{1}_{\mathcal{O}\mathrm{Res}_{n,(m^r)}}.$$

More precisely, we have

$$\theta' : \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}\big(L, {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(\bullet)\big) \to$$
$$\to \mathrm{RHom}_{D^b(\mathcal{O}(\mathfrak{S}_{(m^r)}))}\big({}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(\bullet), L\big)^*[2r(1-m)].$$

By (5.1) we may view θ' as an isomorphism $(A_*)^r \to (A_!)^r[2r(1-m)]$. □

Remark 5.8. Probably we can choose the $\mathfrak{S}_r$-action on $(A_!)^r$ in such a way that the isomorphism $(A_*)^r \to (A_!)^r[2r(1-m)]$ is $\mathfrak{S}_r$-equivariant. This would imply that for $\lambda \in \mathcal{P}_r$ we have $A_{\lambda,!}[2r(1-m)] = A_{\lambda,*}$. We'll not use this.

Remark 5.9. The transitivity of the induction functor [28, cor. 2.5] yields an isomorphism of functors $A^*_\lambda A^*_\mu = A^*_\mu A^*_\lambda$ for $\lambda, \mu \in \mathcal{P}$. Taking the adjoint functors we get also the isomorphisms $A_{\lambda,!} A_{\mu,!} = A_{\mu,!} A_{\lambda,!}$ and $A_{\lambda,*} A_{\mu,*} = A_{\mu,*} A_{\lambda,*}$.

Remark 5.10. The functors $A_{\lambda,!}$, A^*_λ, $A_{\lambda,*}$ yield linear endomorphisms of the $\mathbb{C}$-vector space $[\mathcal{O}(\Gamma)]$. Let us denote them $A_{\lambda,!}$, A^*_λ, $A_{\lambda,*}$ again.

Remark 5.11. Recall that $\langle m\rangle = \bigoplus_{i=0}^{m-1} \mathbb{C}[-2i]$. For any object M of $D^b(\mathcal{O}(\Gamma))$ there should be a distinguished triangle

$$\ell\langle m\rangle M \longrightarrow A_* A^*(M) \longrightarrow A^* A_*(M) \xrightarrow{+1} \cdot$$

5.3. **The functors a^*_λ, $a_{\lambda,*}$ on $\mathcal{O}(\Gamma)$ and the $\mathfrak{H}$-action on the Fock space.** For $i \in \mathbb{Z}$ and $\flat = !, *$ we consider the endofunctor $H^i(A_{\lambda,\flat})$ of $\mathcal{O}(\Gamma)$ given by

$$H^i(A_{\lambda,\flat})(M) = H^i(A_{\lambda,\flat}(M)), \quad M \in \mathcal{O}(\Gamma).$$

From now on we'll write $Ra_{\lambda,\flat} = A_{\lambda,\flat}$ and $R^i a_{\lambda,\flat} = H^i(A_{\lambda,\flat})$.

Definition 5.12. Let a^*_λ be the restriction of A^*_λ to the Abelian category $\mathcal{O}(\Gamma)$. Since a^*_λ is an exact endofunctor of $\mathcal{O}(\Gamma)$, we may write a^*_λ for A^*_λ if it does not create any confusion. We abbreviate $a_{\lambda,\flat} = R^0 a_{\lambda,\flat}$. The functor $a_{\lambda,*}$ is a left exact endofunctor of $\mathcal{O}(\Gamma)$, while $a_{\lambda,!}$ is right exact.

Consider the chain of $\mathbb{C}$-linear isomorphisms which is the composition of (3.5), of the characteristic map ch, and of (4.18),

$$\begin{array}{ccccccc} [\mathcal{O}(\Gamma)] & \to & R(\Gamma) & \to & \mathbf{\Lambda}_\Gamma & \to & \mathcal{F}^{(s)}_{m,\ell}, \\ \Delta_\lambda & \mapsto & \bar{L}_\lambda & \mapsto & S_{\tau\lambda} & \mapsto & |\lambda, s\rangle. \end{array} \tag{5.20}$$

Recall that symmetric bilinear form on $\mathcal{F}^{(s)}_{m,\ell}$ defined in Section 4.6.

Proposition 5.13. *(a) The map (5.20) identifies the symmetric $\mathbb{C}$-bilinear form on $\mathcal{F}^{(s)}_{m,\ell}$ with the $\mathbb{C}$-bilinear form*

$$[\mathcal{O}(\Gamma)] \times [\mathcal{O}(\Gamma)] \to \mathbb{C}, \quad (M, N) \mapsto \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{O}(\Gamma)}(M, N).$$

(b) The map (5.20) identifies the operators b_{S_λ}, b'_{S_λ} on $\mathcal{F}^{(s)}_{m,\ell}$ with the operators a^*_λ, $Ra_{\lambda,*}$ on $[\mathcal{O}(\Gamma)]$.

Proof. Part (a) is obvious because we have

$$\dim \mathrm{Ext}^i_{\mathcal{O}(\Gamma_n)}(\Delta_\lambda, \nabla_\mu) = \delta_{i,0}\delta_{\lambda,\mu}, \quad [\Delta_\mu] = [\nabla_\mu], \quad \forall \lambda, \mu \in \mathcal{P}^\ell_n,$$

because $\mathcal{O}(\Gamma_n)$ is a quasi-hereditary category, see e.g., [7, prop. A.2.2]. Now we concentrate on (b). By (a) and Proposition 5.2, the pairs $(b_{S_\lambda}, b'_{S_\lambda})$ and $(a^*_\lambda, Ra_{\lambda,*})$

consist of adjoint linear operators on $\mathcal{F}^{(s)}_{m,\ell}$. So it is enough to check that under (5.20) we have the following equality

$$b_{S_\lambda} = a^*_\lambda. \tag{5.21}$$

To do that, observe first that, by Proposition 4.1, for $r > 0$ the map $\mathrm{ch} : R(\Gamma) \to \mathbf{\Lambda}_\Gamma$ intertwines the operator

$$R(\Gamma) \to R(\Gamma), \quad M \mapsto \mathrm{Ind}^\Gamma_{\Gamma\times\mathfrak{S}}\big(M \otimes \mathrm{ch}^{-1}(P_{mr})\big)$$

and the multiplication by $\sum_{p\in\mathbb{Z}_\ell} P_{mr,p}$. Here we have abbreviated

$$\mathrm{Ind}^\Gamma_{\Gamma\times\mathfrak{S}} = \bigoplus_{n,r\geqslant 0} \mathrm{Ind}_{n,mr}\,.$$

Next, by Proposition 4.6, the map $\mathbf{\Lambda}_\Gamma \to \mathcal{F}^{(s)}_{m,\ell}$ above intertwines the multiplication by $\sum_{p\in\mathbb{Z}_\ell} P_{mr,p}$ and the operator b_r. By definition, the plethysm with the power sum P_m is the $\mathbb{C}$-algebra endomorphism

$$\psi^m :\ \mathbf{\Lambda} \to \mathbf{\Lambda}, \quad f \mapsto \sum_{\lambda\in\mathcal{P}} z_\lambda^{-1}\langle f, P_\lambda\rangle\, P_{m\lambda}.$$

The discussion above implies that the map $R(\Gamma) \to \mathcal{F}^{(s)}_{m,\ell}$ above identifies the action of b_{S_λ} on $\mathcal{F}^{(s)}_{m,\ell}$ with the operator

$$R(\Gamma) \to R(\Gamma), \quad M \mapsto \mathrm{Ind}^\Gamma_{\Gamma\times\mathfrak{S}}(M \otimes \mathrm{ch}^{-1}\,\psi^m(S_\lambda)).$$

Now, recall the maps

$$\mathrm{spe} : [\mathrm{Rep}(\mathbb{C}\Gamma_n)] \to [\mathcal{O}(\Gamma_n)], \quad \mathrm{spe} : [\mathrm{Rep}(\mathbb{C}\mathfrak{S}_{mr})] \to [\mathcal{O}(\mathfrak{S}_{mr})].$$

By Lemma 2.4, they commute with the induction and restriction. We claim that

$$\mathrm{spe}\circ\mathrm{ch}^{-1}\circ\psi^m(S_\lambda) = L_{m\lambda}.$$

Thus (5.21) follows from (5.20). To prove the claim, set ζ equal to a primitive m-th root of 1. Then Rouquier's functor yields an isomorphism, see (5.10),

$$[\mathcal{O}(\mathfrak{S}_{mr})] = [\mathrm{Rep}(\mathbf{S}_\zeta(mr))].$$

Next, the quantum Frobenius homomorphism yields a commutative diagram

$$\begin{array}{ccc} [\mathrm{Rep}(\mathbf{S}_1(r))] & \xrightarrow{\mathrm{Fr}^*} & [\mathrm{Rep}(\mathbf{S}_\zeta(mr))] \\ \chi\big\| & \overset{\psi}{\nearrow} & \big\|\chi \\ \mathbf{\Lambda} & \xrightarrow[\psi^m]{} & \mathbf{\Lambda} \end{array} \tag{5.22}$$

where χ is the formal character, see e.g., [15, sec. II.H.9]. Consider the chain of maps

$$\theta\ :\ \ [\mathrm{Rep}(\mathbb{C}\mathfrak{S}_{mr})] \overset{(3.5)}{=\!=} [\mathcal{O}(\mathfrak{S}_{mr})] \overset{(5.10)}{=\!=} [\mathrm{Rep}(\mathbf{S}_\zeta(mr))]\ .$$

We have

$$\psi(S_\lambda) = L^S_{m\lambda}, \quad \theta\ \mathrm{ch}^{-1}(S_\mu) = \Delta^S_\mu, \quad \lambda\in\mathcal{P}_r, \quad \mu\in\mathcal{P}_{mr}.$$

Thus we have

$$\chi(\theta\ \mathrm{ch}^{-1}(S_\mu)) = \chi(\Delta^S_\mu) = S_\mu, \quad \mu\in\mathcal{P}_{mr}.$$

Therefore we have also

$$\chi(\theta\circ\mathrm{ch}^{-1}\circ\psi^m(S_\lambda)) = \psi^m(S_\lambda) = \chi(\psi(S_\lambda)) = \chi(L^S_{m\lambda}).$$

This implies that $\theta\circ\mathrm{ch}^{-1}\circ\psi^m(S_\lambda) = L^S_{m\lambda}$, proving the claim and the proposition. □

Remark 5.14. It has been conjectured in [9, sec. 6.6] that the Shapovalov form on $V^{\widehat{\mathfrak{gl}}_\ell}_{\omega_{d \bmod \ell}}$ should be related to the bilinear form on $[\mathcal{O}(\Gamma)]$ in Proposition 5.13. Recall that

$$\mathcal{F}^{(d)}_\ell = V^{\widehat{\mathfrak{gl}}_\ell}_{\omega_{d \bmod \ell}}$$

and that the Shapovalov form on the right hand side is identified with the symmetric bilinear form on the left hand side considered in Section 4.5. By (4.19), under the isomorphism (4.21), the latter is identified with the bilinear form on $\mathcal{F}^{(d)}_{m,\ell}$ in Section 4.6. Thus Proposition 5.13 implies Etingof's conjecture.

Proposition 5.15. *Let $\lambda \in \mathcal{P}_r$ with $r \geqslant 0$.*

(a) We have a triple of adjoint functors $(a_{\lambda,!},\, a^*_\lambda,\, a_{\lambda,*})$.

(b) For $\flat = *, !$, $q = 0, 1, \dots, m-1$, and $i \geqslant 0$ there are isomorphisms of functors

$$e_q\, R^i a_{\lambda,\flat} = R^i a_{\lambda,\flat}\, e_q, \quad e_q\, a^*_\lambda = a^*_\lambda\, e_q, \quad f_q\, R^i a_{\lambda,\flat} = R^i a_{\lambda,\flat}\, f_q, \quad f_q\, a^*_\lambda = a^*_\lambda\, f_q.$$

Proof. By definition of the functors $A_{\lambda,*}$, $A_{\lambda,!}$ we have

$$A_{\lambda,*}(\mathcal{O}(\Gamma)) \subset D^{\geqslant 0}(\mathcal{O}(\Gamma)), \quad A_{\lambda,!}(\mathcal{O}(\Gamma)) \subset D^{\leqslant 0}(\mathcal{O}(\Gamma)).$$

Thus, by Proposition 5.2 we have the triple of adjoint endofunctors of $\mathcal{O}(\Gamma)$

$$(a_{\lambda,!},\, a^*_\lambda,\, a_{\lambda,*}) = (H^0(A_{\lambda,!}),\, A^*_\lambda,\, H^0(A_{\lambda,*})).$$

This proves (a). Next, let us prove part (b). It is enough to give isomorphisms of functors

$$e_q\, a^*_\lambda = a^*_\lambda\, e_q, \quad f_q\, a^*_\lambda = a^*_\lambda\, f_q. \tag{5.23}$$

First, observe that we have an isomorphism of functors

$$F\, a^*_\lambda = a^*_\lambda\, F. \tag{5.24}$$

Indeed, for $M \in \mathcal{O}(\Gamma_n)$ the transitivity of the induction functor [28, cor. 2.5] yields

$$\begin{aligned} F\, a^*_\lambda(M) &= {}^{\mathcal{O}}\mathrm{Ind}_{n+mr}\, {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(M \otimes L_{m\lambda}) \\ &= {}^{\mathcal{O}}\mathrm{Ind}^{\Gamma_{n+mr+1}}_{\Gamma_{n+mr}}\, {}^{\mathcal{O}}\mathrm{Ind}^{\Gamma_{n+mr}}_{\Gamma_{n,mr}}(M \otimes L_{m\lambda}) \\ &= {}^{\mathcal{O}}\mathrm{Ind}^{\Gamma_{n+mr+1}}_{\Gamma_{n,mr}}(M \otimes L_{m\lambda}), \end{aligned}$$

$$\begin{aligned} a^*_\lambda F(M) &= {}^{\mathcal{O}}\mathrm{Ind}_{n+1,mr}({}^{\mathcal{O}}\mathrm{Ind}_n(M) \otimes L_{m\lambda}) \\ &= {}^{\mathcal{O}}\mathrm{Ind}^{\Gamma_{n+mr+1}}_{\Gamma_{n+1,mr}}\big({}^{\mathcal{O}}\mathrm{Ind}^{\Gamma_{n+1}}_{\Gamma_n}(M) \otimes L_{m\lambda}\big) \\ &= {}^{\mathcal{O}}\mathrm{Ind}^{\Gamma_{n+mr+1}}_{\Gamma_{n+1,mr}}\, {}^{\mathcal{O}}\mathrm{Ind}^{\Gamma_{n+1,mr}}_{\Gamma_{n,mr}}(M \otimes L_{m\lambda}) \\ &= {}^{\mathcal{O}}\mathrm{Ind}^{\Gamma_{n+mr+1}}_{\Gamma_{n,mr}}(M \otimes L_{m\lambda}). \end{aligned}$$

By (5.24) for each $M \in \mathcal{O}(\Gamma_n)$ we have

$$\bigoplus_q f_q\, a^*_\lambda(M) = \bigoplus_q a^*_\lambda\, f_q(M). \tag{5.25}$$

We must prove that we have also an isomorphism $f_q\, a^*_\lambda(M) = a^*_\lambda\, f_q(M)$. Let $\mathcal{O}(\Gamma)_\nu \subset \mathcal{O}(\Gamma)$ be the full subcategory consisting of the modules whose class is a weight vector of weight ν of $[\mathcal{O}(\Gamma)]$. Here ν is any weight of the $\widetilde{\mathfrak{sl}}_m$-module $[\mathcal{O}(\Gamma)]$. Recall that

Lemma 5.16. *We have the block decomposition $\mathcal{O}(\Gamma) = \bigoplus_\nu \mathcal{O}(\Gamma)_\nu$, where ν runs over the set of all weights of the $\widetilde{\mathfrak{sl}}_m$-module $[\mathcal{O}(\Gamma)]$.*

Proof. By [28] the image by KZ : $[\mathcal{O}(\Gamma)] \to [\mathrm{Rep}(\mathbf{H}(\Gamma))]$ of the class of a standard module is the class of a Specht module. By [21, thm. 2.11] we have a block decomposition

$$\mathrm{Rep}(\mathbf{H}(\Gamma)) = \bigoplus_{\nu} \mathrm{Rep}(\mathbf{H}(\Gamma))_\nu,$$

where ν runs over a set of weights of $\widetilde{\mathfrak{sl}}_m$ and the block $\mathrm{Rep}(\mathbf{H}(\Gamma))_\nu$ is generated by the constituents of the Specht modules whose classes are the images by KZ of the class of a standard module in $\mathcal{O}(\Gamma)_\nu$. In particular, each Specht module belongs to a single block of $\mathrm{Rep}(\mathbf{H}(\Gamma))$. Now, since the standard modules in $\mathcal{O}(\Gamma)$ are indecomposable (they have a simple top), each of them belong to a single block and any block is generated by the constituents of the standard modules in this block. Finally, by [12] the functor KZ induces a bijection from the blocks of $\mathcal{O}(\Gamma)$ to the blocks of $\mathrm{Rep}(\mathbf{H}(\Gamma)$. Hence two standard modules belong to the same block of $\mathcal{O}(\Gamma)$ if and only if their images by KZ belong to the same block of $\mathrm{Rep}(\mathbf{H}(\Gamma))$. Therefore $\mathcal{O}(\Gamma)_\nu$ is a block of $\mathcal{O}(\Gamma)$. This proves the lemma. □

Therefore, to prove the isomorphism $f_q\, a_\lambda^*(M) = a_\lambda^*\, f_q(M)$ we may assume that M lies in $\mathcal{O}(\Gamma)_\nu$. Then $f_q\, a_\lambda^*(M)$ and $a_\lambda^*\, f_q(M)$ belong to $\mathcal{O}(\Gamma)_{\nu+\alpha_q}$ by Proposition 5.13. Thus the isomorphism above follows from (5.25). The second isomorphism in (5.23) is proved. Next, let us prove that we have an isomorphism of functors

$$E\, a_\lambda^* = a_\lambda^*\, E. \tag{5.26}$$

The first isomorphism in (5.23) follows from (5.26) by a similar argument to the one above. For $M \in \mathcal{O}(\Gamma_n)$ we have

$$E\, a_\lambda^*(M) = {}^{\mathcal{O}}\mathrm{Res}_{n+mr}\, {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(M \otimes L_{m\lambda}),$$
$$a_\lambda^*\, E(M) = {}^{\mathcal{O}}\mathrm{Ind}_{n-1,mr}({}^{\mathcal{O}}\mathrm{Res}_n(M) \otimes L_{m\lambda}),$$

As above, we abbreviate $L = L_{(m)}^{\otimes r}$. By Proposition 5.4 it is enough to prove that we have a natural isomorphism

$${}^{\mathcal{O}}\mathrm{Res}_{n+mr}\, {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(M \otimes {}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)) \to {}^{\mathcal{O}}\mathrm{Ind}_{n-1,mr}({}^{\mathcal{O}}\mathrm{Res}_n(M) \otimes {}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L))$$

that is equivariant with respect to the $\mathfrak{S}_r$-action induced by the $\mathfrak{S}_r$-action on ${}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)$ given in (5.16). To see this, note that Proposition A.2 yields the following decomposition of functors

$${}^{\mathbf{H}}\mathrm{Res}_{n+mr} \circ {}^{\mathbf{H}}\mathrm{Ind}_{n,mr} = \big({}^{\mathbf{H}}\mathrm{Ind}_{n-1,mr} \circ ({}^{\mathbf{H}}\mathrm{Res}_n \otimes \mathbf{1})\big) \oplus$$
$$\oplus \big({}^{\mathbf{H}}\mathrm{Ind}_{n,mr-1} \circ (\mathbf{1} \otimes {}^{\mathbf{H}}\mathrm{Res}_{mr})\big)^{\oplus \ell}.$$

Therefore we have also the following decomposition of functors

$$\mathrm{KZ} \circ {}^{\mathcal{O}}\mathrm{Res}_{n+mr} \circ {}^{\mathcal{O}}\mathrm{Ind}_{n,mr} =$$
$$= \big(\mathrm{KZ} \circ {}^{\mathcal{O}}\mathrm{Ind}_{n-1,mr} \circ ({}^{\mathcal{O}}\mathrm{Res}_n \otimes \mathbf{1})\big) \oplus \big(\mathrm{KZ} \circ {}^{\mathcal{O}}\mathrm{Ind}_{n,mr-1} \circ (\mathbf{1} \otimes {}^{\mathcal{O}}\mathrm{Res}_{mr})\big)^{\oplus \ell}.$$

The induction and restriction functors on $\mathcal{O}(\Gamma)$ take projective modules to projective ones, because they are exact and biadjoint. Thus, by (3.7) we have a natural isomorphism

$${}^{\mathcal{O}}\mathrm{Res}_{n+mr}\, {}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(P) =$$
$$= {}^{\mathcal{O}}\mathrm{Ind}_{n-1,mr}\,({}^{\mathcal{O}}\mathrm{Res}_n \otimes \mathbf{1})(P) \oplus \big({}^{\mathcal{O}}\mathrm{Ind}_{n,mr-1}\,(\mathbf{1} \otimes {}^{\mathcal{O}}\mathrm{Res}_{mr})(P)\big)^{\oplus \ell}$$

for any projective module $P \in \mathcal{O}(\Gamma)$. Since $\mathcal{O}(\Gamma)$ has enough projective objects, this yields an isomorphism of functors

$${}^{\mathcal{O}}\mathrm{Res}_{n+mr}\, {}^{\mathcal{O}}\mathrm{Ind}_{n,mr} = {}^{\mathcal{O}}\mathrm{Ind}_{n-1,mr}\,({}^{\mathcal{O}}\mathrm{Res}_n \otimes \mathbf{1}) \oplus \big({}^{\mathcal{O}}\mathrm{Ind}_{n,mr-1}\,(\mathbf{1} \otimes {}^{\mathcal{O}}\mathrm{Res}_{mr})\big)^{\oplus \ell}.$$

In particular, the projection yields a morphism of functors

$$^{\mathcal{O}}\mathrm{Res}_{n+mr}\ {}^{\mathcal{O}}\mathrm{Ind}_{n,mr} \to {}^{\mathcal{O}}\mathrm{Ind}_{n-1,mr}\,({}^{\mathcal{O}}\mathrm{Res}_n \otimes \mathbf{1}).$$

Applying this to the module $M \otimes {}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)$ yields an $\mathfrak{S}_r$-equivariant surjective morphism

$$\Psi(M) : {}^{\mathcal{O}}\mathrm{Res}_{n+mr}\,{}^{\mathcal{O}}\mathrm{Ind}_{n,mr}(M\otimes{}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)) \to {}^{\mathcal{O}}\mathrm{Ind}_{n-1,mr}({}^{\mathcal{O}}\mathrm{Res}_n(M)\otimes{}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L)).$$

Now, by (5.1) the left hand side is equal to $E \circ (a^*)^r(M)$ and the right hand side is equal to $(a^*)^r \circ E(M)$. So by Proposition 5.13 and the fact that the actions of $\mathfrak{H}$ and $\widehat{\mathfrak{sl}}_m$ on $\mathcal{F}^{(s)}_{m,\ell}$ commute with each other, we have

$$[E \circ (a^*)^r(M)] = [(a^*)^r \circ E(M)].$$

Thus $\Psi(M)$ is indeed an isomorphism. So (5.26) is proved. □

5.4. **Primitive modules.**

Definition 5.17. A module $M \in \mathcal{O}(\Gamma)$ is *primitive* if $Ra_*(M) = 0$ and $E(M) = 0$ (or, equivalently, if $R^i a_*(M) = e_q(M) = 0$ for all q, i). Let $\mathrm{PI}(\mathcal{O}(\Gamma))$ be the set of isomorphism classes of primitive simple modules.

Proposition 5.18. *For $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))$ the following are equivalent*
(a) $L \in \mathrm{PI}(\mathcal{O}(\Gamma_n))$,
(b) $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{0,0}$,
(c) $\dim(L) < \infty$.

Proof. Assume that $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))$. The equivalence of (b) and (c) is Remark 3.13. Let us prove that $(a) \Rightarrow (b)$. Fix $l, j \geqslant 0$ such that $\mathrm{Supp}(L) = X_{l,j}$. Set $i = n - l - mj$. We first prove that $j = 0$. Assume that $j > 0$. Then we have

$$\Gamma_{l,(m^j)} = \Gamma_{l,(m^{j-1})} \times \mathfrak{S}_m, \quad \Gamma_{l,(m^{j-1})} \subset \Gamma_{n-m}.$$

There are modules $M_\mu \in \mathcal{O}(\Gamma_{n-m})$, $\mu \in \mathcal{P}_m$, such that in $[\mathcal{O}(\Gamma_{n,m})]$ we have

$$[\mathrm{Res}_{n,m}(L)] = \sum_{\mu\in\mathcal{P}_m} [M_\mu \otimes L_\mu].$$

The transitivity of the restriction functor [28, cor. 2.5] yields the following formula

$$[\mathrm{Res}_1(L)] = \sum_\mu [\mathrm{Res}_2(M_\mu) \otimes L_\mu], \quad \mathrm{Res}_1 = {}^{\mathcal{O}}\mathrm{Res}^{\Gamma_n}_{\Gamma_{l,(m^j)}}, \quad \mathrm{Res}_2 = {}^{\mathcal{O}}\mathrm{Res}^{\Gamma_{n-m}}_{\Gamma_{l,(m^{j-1})}}.$$

The $H(\Gamma_{l,(m^j)})$-module $\mathrm{Res}_1(L)$ is finite dimensional, because $\mathrm{Supp}(L) = X_{l,j}$. Thus we have $\mathrm{Res}_2(M_\mu) = 0$ unless $\mu = (m)$, and

$$[\mathrm{Res}_1(L)] = [\mathrm{Res}_2(M_{(m)}) \otimes L_{(m)}]. \tag{5.27}$$

Next, since $Ra_*([L]) = 0$ we have

$$\begin{aligned} 0 &= [\mathrm{Res}_2\, Ra_*(L)] \\ &= \sum_{\mu\in\mathcal{P}_m} [\mathrm{Res}_2(M_\mu) \otimes \mathrm{RHom}_{\mathcal{O}(\mathfrak{S}_m)}(L_{(m)}, L_\mu)], \\ &= [\mathrm{Res}_2(M_{(m)}) \otimes \mathrm{REnd}_{\mathcal{O}(\mathfrak{S}_m)}(L_{(m)})]. \end{aligned}$$

Thus, using [2, thm. 1.3] we get $\mathrm{Res}_2(M_{(m)}) = 0$. This yields a contradiction with (5.27) because $\mathrm{Res}_1(L) \neq 0$. So we have $j = 0$. Next, since $E(L) = 0$, by Corollary 3.19 and Remark 3.11 we have $i = 0$.

Finally, we prove that $(c) \Rightarrow (a)$. We must prove that if L is finite dimensional then it is primitive. This is obvious, because ${}^{\mathcal{O}}\mathrm{Res}_{n,m}(L) = {}^{\mathcal{O}}\mathrm{Res}_n(L) = 0$. □

Remark 5.19. By Proposition 5.18 the elements of $\mathrm{PI}(\mathcal{O}(\Gamma_n))$ form a basis of $F_{0,0}(\Gamma_n)$.

5.5. **Endomorphisms of induced modules.** For $r \geqslant 1$ we consider the algebras

$$B_r = \mathfrak{S}_r \ltimes \mathbb{C}[x_1, x_2, \dots, x_r], \quad B_{r,\ell} = B_r/(x_1^\ell, x_2^\ell, \dots, x_r^\ell).$$

The following proposition is the main result of this subsection.

Proposition 5.20. *Let $r \geqslant 1$.*

(a) The $\mathbb{C}$-algebra homomorphism $\mathbb{C}\mathfrak{S}_r \to \mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r)$ in Proposition 5.4 extends to a $\mathbb{C}$-algebra homomorphism $B_r \to \mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r)$ such that $x_1, x_2, \dots, x_r$ map to nilpotent operators in $\mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L))$ for each $L \in \mathcal{O}(\Gamma)$.

(b) The $\mathbb{C}$-algebra homomorphism $B_r \to \mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r)$ factors to an isomorphism $B_{r,\ell} = \mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L))$ for $L \in \mathrm{PI}(\mathcal{O}(\Gamma))$.

Proof. The proof of this proposition is done in several steps. Let $\mathbf{H}(\Gamma_{n,(m^r)})$, $\mathbf{H}(\Gamma_n)$ and X_i be as in Appendix A. Consider the elements

$$\xi_i = X_{n+m(i-1)+1} X_{n+m(i-1)+2} \cdots X_{n+mi}, \quad i = 1, 2, \dots, r.$$

They belong to the centralizer of $\mathbf{H}(\Gamma_{n,(m^r)})$ in $\mathbf{H}(\Gamma_{n+mr})$. Thus the right multiplication by ξ_i, $i = 1, 2, \dots, r$, defines an automorphism ${}^{\mathbf{H}}\xi_i$ of the functor ${}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)}$. More precisely, for a $\mathbf{H}(\Gamma_{n,(m^r)})$-module M we set

$${}^{\mathbf{H}}\xi_i(h \otimes v) = h\xi_i \otimes v, \quad h \in \mathbf{H}(\Gamma_{n+mr}), \quad v \in M.$$

The functor KZ yields a $\mathbb{C}$-algebra isomorphism (5.3)

$$\mathrm{KZ} : \mathrm{End}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}\big) \to \mathrm{End}\big({}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)} \circ \mathrm{KZ}\big).$$

Thus there is a unique endomorphism ${}^{\mathcal{O}}\xi_i$ of the functor ${}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}$ such that

$$\mathrm{KZ}({}^{\mathcal{O}}\xi_i(M)) = {}^{\mathbf{H}}\xi_i(\mathrm{KZ}(M)), \quad \forall M \in \mathcal{O}(\Gamma_{n,(m^r)}). \tag{5.28}$$

The functor $\bullet \otimes L : \mathcal{O}(\Gamma_n) \to \mathcal{O}(\Gamma_{n,(m^r)})$ yields a $\mathbb{C}$-algebra homomorphism

$$\mathrm{End}\big({}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}\big) \to \mathrm{End}((a^*)^r). \tag{5.29}$$

Let $\bar{\xi}_i$ denote the image of ${}^{\mathcal{O}}\xi_i$ by the map (5.29). Next, recall the operators

$$\bar{\tau}_i \in \mathrm{End}((a^*)^r) = \mathrm{End}((A^*)^r), \quad i = 1, 2, \dots, r-1$$

defined in Proposition 5.4, see also the proof of Lemma 5.3.

Lemma 5.21. *The following relations hold in $\mathrm{End}((a^*)^r)$ for $j \neq i, i+1$*

$$\bar{\tau}_i \circ \bar{\xi}_i \circ \bar{\tau}_i = \bar{\xi}_{i+1}, \quad \bar{\tau}_i \circ \bar{\xi}_j \circ \bar{\tau}_i = \bar{\xi}_j.$$

Proof. By (5.28) and (5.4) it is enough to prove that

$$({}^{\mathbf{H}}\tau_i\, \mathbf{1}_{\tau_i}) \circ ({}^{\mathbf{H}}\xi_i\, \mathbf{1}_{\tau_i}) \circ {}^{\mathbf{H}}\tau_i = {}^{\mathbf{H}}\xi_{i+1}, \qquad ({}^{\mathbf{H}}\tau_i\, \mathbf{1}_{\tau_i}) \circ ({}^{\mathbf{H}}\xi_j\, \mathbf{1}_{\tau_i}) \circ {}^{\mathbf{H}}\tau_i = {}^{\mathbf{H}}\xi_j.$$

To do so, we are reduced to check the following relations in $\mathbf{H}(\Gamma_{n+mr})$

$$T_{\tau_i} \xi_i T_{\tau_i} = \xi_{i+1}, \quad T_{\tau_i} \xi_j T_{\tau_i} = \xi_j.$$

Recall that ζ is a m-th root of 1. Let $a_i = n + (i-1)m + 1$, $b_i = n + im$, and

$$K_l = T_{b_i - l} T_{b_i - l + 2} \cdots T_{b_i + l - 2} T_{b_i + l}.$$

A direct computation yields that

$$T_{\tau_i} = K_0 K_1 \cdots K_{m-2} K_{m-1} K_{m-2} \cdots K_1 K_0.$$

Further, for $0 \leqslant l \leqslant m-1$ we have

$$K_l X_{a_i} X_{a_i+1} \cdots X_{b_i-l-2} X_{b_i-l-1} (X_{b_i-l} X_{b_i-l+2} \cdots X_{b_i+l}) K_l =$$
$$= \zeta^{l+1} X_{a_i} X_{a_i+1} \cdots X_{b_i-l-2} (X_{b_i-l-1} X_{b_i-l+1} X_{b_i-l+3} \cdots X_{b_i+l+1}),$$

and for $0 \leqslant l \leqslant m-2$ we have

$$K_l(X_{b_i-l}X_{b_i-l+2}\cdots X_{b_i+l})X_{b_i+l+2}X_{b_i+l+3}\cdots X_{b_i+m}K_l =$$
$$= \zeta^{l+1}(X_{b_i-l+1}X_{b_i-l+3}\cdots X_{b_i+l-1})X_{b_i+l+1}X_{b_i+l+2}X_{b_i+l+3}\cdots X_{b_i+m}.$$

We deduce that

$$\begin{aligned} T_{\tau_i}\xi_i T_{\tau_i} &= T_{\tau_i}X_{a_i}X_{a_i+1}\cdots X_{b_i}T_{\tau_i} \\ &= \zeta^{1+2+\cdots+m}K_0\cdots K_{m-2}X_{a_i+1}X_{a_i+3}\cdots X_{b_i+m-2}X_{b_i+m}K_{m-2}\cdots K_0 \\ &= \zeta^{1+2+\cdots+m}\zeta^{1+2+\cdots+m-1}X_{a_i+m}X_{a_i+m+1}\cdots X_{b_i+m} \\ &= \zeta^{m^2}\xi_{i+1} \\ &= \xi_{i+1}. \end{aligned}$$

The relation $T_{\tau_i}\xi_j T_{\tau_i} = \xi_j$ for $j \neq i, i+1$ is obvious. □

For any element $w \in \mathfrak{S}_r$ we set

$$\bar{\tau}_w = \bar{\tau}_{i_1}\bar{\tau}_{i_2}\cdots\bar{\tau}_{i_k} \in \mathrm{End}((a^*)^r)$$

for $w = s_{i_1}s_{i_2}\cdots s_{i_k}$. This definition does not depend on the choice of the decomposition of w by Lemma 5.3. Next, for a tuple $p = (p_1, p_2, \ldots, p_r) \in \mathbb{Z}^r$ such that $0 \leqslant p_i < \ell$ we set

$$\xi^p = \xi_1^{p_1}\xi_2^{p_2}\cdots\xi_r^{p_r}, \quad \bar{\xi}^p = \bar{\xi}_1^{p_1}\bar{\xi}_2^{p_2}\cdots\bar{\xi}_r^{p_r}.$$

Lemma 5.22. *For any $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))$ the elements $\bar{\xi}^p\,\bar{\tau}_w(L)$ of $\mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L))$, with $w \in \mathfrak{S}_r$ and $p \in [0,\ell)^r$, are linearly independent.*

Proof. For $w, i_1, \ldots, i_k, p$ as above the expression $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}$ is reduced. Let us define the following elements in $\mathbf{H}(\Gamma_{n+mr})$

$$t_w = T_{\tau_{i_1}}T_{\tau_{i_2}}\cdots T_{\tau_{i_k}}, \quad \xi^p = \xi_1^{p_1}\xi_2^{p_2}\cdots\xi_r^{p_r}.$$

Recall that the elements

$$X_1^{p_1}X_2^{p_2}\cdots X_{n+mr}^{p_{n+mr}}T_w, \quad p_i \in [0,\ell), \quad w \in \mathfrak{S}_{n+mr},$$

form a $\mathbb{C}$-basis of $\mathbf{H}(\Gamma_{n+mr})$. Further ξ^p centralizes $\mathbf{H}(\Gamma_{n,(m^r)})$ and the element $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}$ above is minimal in its right $\mathfrak{S}_{(n,m^r)}$-coset. Therefore the left $\mathbf{H}(\Gamma_{n,(m^r)})$-submodule of $\mathbf{H}(\Gamma_{n+mr})$ spanned by

$$\{\xi^p t_w\,;\, w \in \mathfrak{S}_r,\, p \in [0,\ell)^r\},$$

is indeed the direct sum

$$\bigoplus_{p,w}\mathbf{H}(\Gamma_{n,(m^r)})\,\xi^p t_w,$$

where p runs over $[0,\ell)^r$ and w over $\mathfrak{S}_r$. In other words, there is an injective $\mathbf{H}(\Gamma_{n,(m^r)})$-module homomorphism

$${}^{\mathbf{H}}\psi : \mathbf{H}(\Gamma_{n,(m^r)})^{\oplus \ell^r r!} \to \mathbf{H}(\Gamma_{n+mr}), \quad (h_{p,w}) \mapsto \sum_{p,w} h_{p,w}\,\xi^p\,t_w, \tag{5.30}$$

where w, p run over $\mathfrak{S}_r$, $[0,\ell)^r$ respectively. Further, since ξ^p centralizes $\mathbf{H}(\Gamma_{n,(m^r)})$, the relation (5.2) yields

$$z\xi^p t_w = \xi^p z t_w = \xi^p t_w w^{-1}(z), \quad z \in \mathbf{H}(\Gamma_{n,(m^r)}),$$

where $w^{-1}(z) = \tau_{i_k}\cdots\tau_{i_2}\tau_{i_1}(z)$. Therefore ${}^{\mathbf{H}}\psi$ is a $(\mathbf{H}(\Gamma_{n,(m^r)}), \mathbf{H}(\Gamma_{n,(m^r)}))$-bimodule homomorphism, where the right $\mathbf{H}(\Gamma_{n,(m^r)})$-action on $\mathbf{H}(\Gamma_{n,(m^r)})^{\oplus \ell^r r!}$ is

twisted in the obvious way. Since ${}^{\mathbf{H}}\psi$ is injective, and both sides are free $\mathbf{H}(\Gamma_{n,(m^r)})$-modules, for each $M \in \mathcal{O}(\Gamma_{n,(m^r)})$ we have an injective homomorphism

$$ {}^{\mathbf{H}}\psi(\mathrm{KZ}(M)) : \bigoplus_{p,w} w\,\mathrm{KZ}(M) \to {}^{\mathbf{H}}\mathrm{Res}_{n,(m^r)} \circ {}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)}\,\mathrm{KZ}(M) = $$
$$ = \mathrm{KZ} \circ {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)} \circ {}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}(M), $$

where

$$ w = \tau_{i_1}\tau_{i_2}\dots\tau_{i_k} : \mathrm{Rep}(\mathbf{H}(\Gamma_{n,(m^r)})) \to \mathrm{Rep}(\mathbf{H}(\Gamma_{n,(m^r)})). $$

Further, we have

$$ w\,\mathrm{KZ}(M) = \mathrm{KZ}(wM), $$

where

$$ w : \mathcal{O}(\Gamma_{n,(m^r)}) \to \mathcal{O}(\Gamma_{n,(m^r)}) $$

is the twist by the permutation

$$ w : H(\Gamma_{n,(m^r)}) = H(\Gamma_n) \otimes H(\mathfrak{S}_m)^{\otimes r} \to H(\Gamma_n) \otimes H(\mathfrak{S}_m)^{\otimes r} = H(\Gamma_{n,(m^r)}). $$

The canonical adjunction morphism

$$ P \to S(\mathrm{KZ}(P)) $$

is an isomorphism for each projective module $P \in \mathcal{O}(\Gamma)$. Here $S : \mathrm{Rep}(\mathbf{H}(\Gamma)) \to \mathcal{O}(\Gamma)$ is the functor from Section 3.7. Further, the functors ${}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}$ and ${}^{\mathcal{O}}\mathrm{Ind}_{n,(m^r)}$ preserve the projective objects, because they are bi-adjoint and exact. Therefore, applying the left exact functor S to the map ${}^{\mathbf{H}}\psi(\mathrm{KZ}(P))$, with P projective in $\mathcal{O}(\Gamma_{n,(m^r)})$, we get an injection

$$ {}^{\mathcal{O}}\psi(P) : \bigoplus_{p,w} wP \to {}^{\mathcal{O}}\,\mathrm{Res}_{n,(m^r)} \circ {}^{\mathcal{O}}\,\mathrm{Ind}_{n,(m^r)}(P). $$

Since the category $\mathcal{O}(\Gamma_{n,(m^r)})$ has enough projective objects and since the functor ${}^{\mathcal{O}}\,\mathrm{Res}_{n,(m^r)} \circ {}^{\mathcal{O}}\,\mathrm{Ind}_{n,(m^r)}$ is exact, the five lemma implies that there is a functorial injective morphism

$$ {}^{\mathcal{O}}\psi(M) : \bigoplus_{p,w} wM \to {}^{\mathcal{O}}\,\mathrm{Res}_{n,(m^r)} \circ {}^{\mathcal{O}}\,\mathrm{Ind}_{n,(m^r)}(M), \quad M \in \mathcal{O}(\Gamma_{n,(m^r)}). $$

Now, set $M = L \otimes L_{(m)}^{\otimes r}$ with $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))$. Then we have $wM = M$ for all w as above. Therefore we get an injective linear map

$$ \mathbb{C}^{\ell^r r!} = \mathrm{Hom}_{\mathcal{O}(\Gamma)}(L \otimes L_{(m)}^{\otimes r}, L \otimes L_{(m)}^{\otimes r})^{\oplus \ell^r r!} \to $$
$$ \to \mathrm{Hom}_{\mathcal{O}(\Gamma)}(L \otimes L_{(m)}^{\otimes r}, {}^{\mathcal{O}}\,\mathrm{Res}_{n,(m^r)}\,{}^{\mathcal{O}}\,\mathrm{Ind}_{n,(m^r)}(L \otimes L_{(m)}^{\otimes r})) = \mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L)). $$

It maps the canonical basis elements to the elements $\bar{\xi}^p\,\bar{\tau}_w(L)$ with $w \in \mathfrak{S}_r$ and $p \in [0,\ell)^r$. □

Lemma 5.23. *For $L \in \mathrm{PI}(\mathcal{O}(\Gamma_n))$ the following identity holds in $[\mathcal{O}(\Gamma_{n,(m^r)})]$*

$$ [{}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(a^*)^r(L)] = \ell^r r!\,[L \otimes L_{(m)}^{\otimes r}]. $$

Proof. By Lemma 2.5 the left hand side is equal to

$$ \sum_x {}^{\mathcal{O}}\mathrm{Ind}_{x^{-1}W_x x}^{\Gamma_{n,(m^r)}} \circ {}^x\big({}^{\mathcal{O}}\mathrm{Res}_{W_x}^{\Gamma_{n,(m^r)}}([L \otimes L_{(m)}^{\otimes r}])\big), $$

where $W_x = x\Gamma_{n,(m^r)}x^{-1} \cap \Gamma_{n,(m^r)}$ and x runs over a set of representatives of the double cosets in $\Gamma_{n,(m^r)} \setminus \Gamma_{n+mr}/\Gamma_{n,(m^r)}$. Since W_x is a parabolic subgroup of $\Gamma_{n,(m^r)}$, it is generated by reflections. Hence we can decompose the group W_x in the following way

$$ W_x = W_x' \times W_x'', \quad W_x' \subset \Gamma_n, \quad W_x'' \subset \mathfrak{S}_m^r. \tag{5.31} $$

Here W'_x, W''_x are parabolic subgroups. We have

$$^{\mathcal{O}}\mathrm{Res}^{\Gamma_{n,(m^r)}}_{W_x}(L\otimes L^{\otimes r}_{(m)}) = {}^{\mathcal{O}}\mathrm{Res}^{\Gamma_n}_{W'_x}(L)\otimes {}^{\mathcal{O}}\mathrm{Res}^{\mathfrak{S}^r_m}_{W''_x}(L^{\otimes r}_{(m)}),$$

and a similar decomposition holds for the induction functor. Further, since $L\in \mathrm{PI}(\mathcal{O}(\Gamma_n))$ we have ${}^{\mathcal{O}}\mathrm{Res}^{\Gamma_n}_{W'_x}(L)=0$ if W'_x is proper by Proposition 5.18. Thus we can assume that $W'_x=\Gamma_n$, i.e., we can assume that x belongs to the subgroup $\{1\}\times\Gamma_{mr}\subset\Gamma_{n+mr}$. We'll abbreviate

$$\mathfrak{S}^r_m=\{1\}\times\mathfrak{S}^r_m,\quad \Gamma_{mr}=\{1\}\times\Gamma_{mr}.$$

Then we have $W''_x = x\mathfrak{S}^r_m x^{-1}\cap\mathfrak{S}^r_m$, and we are reduced to check that

$$\sum_x {}^{\mathcal{O}}\mathrm{Ind}^{\mathfrak{S}^r_m}_{x^{-1}W_x x}\circ {}^x\big({}^{\mathcal{O}}\mathrm{Res}^{\mathfrak{S}^r_m}_{W_x}([L^{\otimes r}_{(m)}])\big) = \ell^r r!\,[L^{\otimes r}_{(m)}],$$

where $W_x = x\mathfrak{S}^r_m x^{-1}\cap\mathfrak{S}^r_m$ and x runs over a set of representatives of the double cosets in $\mathfrak{S}^r_m\setminus\Gamma_{mr}/\mathfrak{S}^r_m$. Now, observe that

$$^{\mathcal{O}}\mathrm{Res}^{\mathfrak{S}^r_m}_{W_x}(L^{\otimes r}_{(m)})=0$$

unless $x\mathfrak{S}^r_m x^{-1}=\mathfrak{S}^r_m$, and that $x\mathfrak{S}^r_m x^{-1}=\mathfrak{S}^r_m$ if and only if x belongs to $N_{\Gamma_{mr}}(\mathfrak{S}^r_m)$, the normalizer of $\mathfrak{S}^r_m$ in Γ_{mr}. Further, we have a group isomorphism

$$N_{\Gamma_{mr}}(\mathfrak{S}^r_m)/\mathfrak{S}^r_m=\Gamma_r.$$

This proves the lemma. □

Lemma 5.24. *For $L\in\mathrm{PI}(\mathcal{O}(\Gamma))$ the elements $\bar\xi^p\,\bar\tau_w(L)$ with $w\in\mathfrak{S}_r$ and $p\in[0,\ell)^r$ form a basis of $\mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L))$.*

Proof. By Lemma 5.22 it is enough to check that

$$\dim\,\mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L))\leqslant \ell^r r!.$$

For $L\in\mathrm{PI}(\mathcal{O}(\Gamma_n))$ Lemma 5.23 yields

$$\dim\,\mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L)) = \dim\mathrm{Hom}_{\mathcal{O}(\Gamma)}(L\otimes L^{\otimes r}_{(m)},\,{}^{\mathcal{O}}\,\mathrm{Res}_{n,(m^r)}(a^*)^r(L))\leqslant \ell^r r!.$$

□

Lemma 5.25. *For $i=1,2,\dots,r$ and $L\in\mathcal{O}(\Gamma)$ the operator $\bar\xi_i(L)+1$ on $(a^*)^r(L)$ is nilpotent. Further, if $L\in\mathrm{PI}(\mathcal{O}(\Gamma))$ we have $(\bar\xi_i(L)+1)^\ell=0$.*

Proof. The $\mathbb{C}$-vector space $[\mathcal{O}(\Gamma)]$ is equipped with an $\widetilde{\mathfrak{sl}}_m$-action via the isomorphism (5.20), see also Remark 4.7. For a weight μ of $\widetilde{\mathfrak{sl}}_m$ let $\mathcal{O}(\Gamma)_\mu\subset\mathcal{O}(\Gamma)$ be the Serre subcategory generated by the simple modules L whose class in $[\mathcal{O}(\Gamma)]$ has the weight μ. Set $\mathcal{O}(\Gamma_n)_\mu=\mathcal{O}(\Gamma)_\mu\cap\mathcal{O}(\Gamma_n)$. Although we'll not need this formula, note that if $\Delta_\lambda\in\mathcal{O}(\Gamma_n)_\mu$ then we have

$$\mu=\mu_0-\sum_{q=0}^{m-1} n_q(\lambda)\alpha_q$$

where μ_0 is a weight which does not depend on n, λ, and $n_q(\lambda)$ is the number of q-nodes in the ℓ-partition λ. The element

$$z_n=X_1X_2\cdots X_n$$

belongs to the center of $\mathbf{H}(\Gamma_n)$. Thus it yields an element ${}^{\mathbf{H}}z_n$ in the center of $\mathrm{Rep}(\mathbf{H}(\Gamma_n))$. Since KZ identifies the centers of $\mathcal{O}(\Gamma_n)$ and $\mathrm{Rep}(\mathbf{H}(\Gamma_n))$, it yields also an element ${}^{\mathcal{O}}z_n$ in the center of $\mathcal{O}(\Gamma_n)$. Let $L\in\mathrm{Irr}(\mathcal{O}(\Gamma_n)_\mu)$. Then ${}^{\mathcal{O}}z_n$ acts on L by multiplication by the scalar $\zeta^{\nu(\mu)}$, where ν is a linear form such that $\nu(\alpha_i)=i$ for $i=0,1,\dots,m-1$, see e.g., [28, sec. 4.1]. Now the operator a^* maps $\mathcal{O}(\Gamma_n)_\mu$ to

$\mathcal{O}(\Gamma_{n+m})_{\mu+\delta}$ by Proposition 5.13. Thus ${}^{\mathcal{O}}z_{n+m}$ acts on $a^*(L)$ by multiplication by the scalar $\zeta^{\nu(\mu+\delta)}$. Therefore $\bar{\xi}_1$ acts on $a^*(L)$ by multiplication by the scalar

$$\zeta^{\nu(\delta)} = \zeta^{m(m-1)/2} = -1.$$

By Lemmas 5.21, 5.3 this implies that for any $L \in \mathcal{O}(\Gamma)$ we have $(\bar{\xi}_i(L)+1)^N = 0$ in $\mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L))$ for $i = 1, 2, \dots, r$ and N large enough.

Now, assume that $L \in \mathrm{PI}(\mathcal{O}(\Gamma))$. Let N_i be the minimal integer such that $(\bar{\xi}_i(L)+1)^{N_i} = 0$. By Lemmas 5.21, 5.3 we have $N_1 = N_2 = \cdots = N_r$. Hence, by Lemma 5.22 we have also $\ell = N_1 = N_2 = \cdots = N_r$. □

Now we complete the proof of Proposition 5.20. The previous lemmas imply that the assignment

$$x_i \mapsto \bar{\xi}_i + 1, \quad s_j \mapsto \bar{\tau}_j, \quad i = 1, 2, \dots, r, \quad j = 1, 2, \dots, r-1, \tag{5.32}$$

yields a $\mathbb{C}$-algebra morphism $B_r \to \mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r)$ such that x_i maps to a nilpotent operator in $\mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L))$ for each $L \in \mathcal{O}(\Gamma)$. The action of s_j on $(a^*)^r$ given above is the same as the action of s_j on $(A^*)^r$ in Proposition 5.4. This proves part (a). Part (b) follows from Lemmas 5.24, 5.25. □

For a module M in $\mathcal{O}(\Gamma)$ the adjunction yields a morphism

$$\eta(M) : M \otimes L_{(m)}^{\otimes r} \to {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(a^*)^r(M).$$

Corollary 5.26. *For $r \geqslant 1$ and $L \in \mathrm{PI}(\mathcal{O}(\Gamma_n))$ the $\mathbb{C}$-algebra isomorphism (5.32)*

$$B_{r,\ell} = \mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L))$$

yields an isomorphism of $B_{r,\ell} \times H(\Gamma_{n,(m^r)})$-modules

$$B_{r,\ell} \otimes (L \otimes L_{(m)}^{\otimes r}) \to {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(a^*)^r(L), \quad w \otimes v \mapsto {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(w) \cdot \eta(L)(v).$$

Proof. The corollary follows from Proposition 5.20 and Lemma 5.23, because

$$\mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L)) = \mathrm{Hom}_{\mathcal{O}(\Gamma)}(L \otimes L_{(m)}^{\otimes r}, {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(a^*)^r(L))$$

is a free $B_{r,\ell}$-module of rank one and, in $[\mathcal{O}(\Gamma_{n,(m^r)})]$, we have

$$[{}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(a^*)^r(L)] = \dim(B_{r,\ell})\,[L \otimes L_{(m)}^{\otimes r}].$$

□

Definition 5.27. For $\lambda \in \mathcal{P}_r$, $r \geqslant 1$, we can regard the $\mathfrak{S}_r$-module $\bar{L}_\lambda$ as a $B_{r,\ell}$-module such that $x_1, x_2, \dots, x_r$ act by zero. For $L \in \mathrm{PI}(\mathcal{O}(\Gamma_n))$ we define

$$\bar{a}_\lambda^*(L) = \bar{L}_\lambda \otimes_{B_{r,\ell}} (a^*)^r(L) \in \mathcal{O}(\Gamma_{n+mr}).$$

Definition 5.28. For $r \geqslant 1$ we define a functor $\mathcal{O}(\Gamma_{n+mr}) \to \mathrm{Rep}(\mathfrak{S}_r) \otimes \mathcal{O}(\Gamma_n)$ by

$$\begin{aligned}\Psi(M) &= \mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_m^r)}(L_{(m)}^{\otimes r}, {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(M)) \\ &= \mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_{mr})}({}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L_{(m)}^{\otimes r}), {}^{\mathcal{O}}\mathrm{Res}_{n,mr}(M)).\end{aligned}$$

The $\mathfrak{S}_r$-action on $\Psi(M)$ is the $\mathfrak{S}_r$-action on ${}^{\mathcal{O}}\mathrm{Ind}_{(m^r)}(L_{(m)}^{\otimes r})$ in the proof of Proposition 5.4. In other words, we have $\Psi = (a_*)^r$, viewed as a $\mathfrak{S}_r$-equivariant functor as in the proof of Proposition 5.4.

Corollary 5.29. *For $r \geqslant 1$ and $L \in \mathrm{PI}(\mathcal{O}(\Gamma_n))$ we have an isomorphism*

$$(L \otimes L_{(m)}^{\otimes r})^{\oplus \dim(\bar{L}_\lambda)} = {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(\bar{a}_\lambda^*(L))$$

as $H(\Gamma_{n,(m^r)})$-modules, and we have an isomorphism of $\mathfrak{S}_r \times H(\Gamma_n)$-modules

$$\bar{L}_\lambda \otimes L = \Psi(\bar{a}_\lambda^*(L)).$$

Proof. Corollary 5.26 yields an isomorphism

$$B_{r,\ell} \otimes (L \otimes L_{(m)}^{\otimes r}) = {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}\big((a^*)^r(L)\big)$$

which factors to an isomorphism

$$\mathbb{C}\mathfrak{S}_r \otimes (L \otimes L_{(m)}^{\otimes r}) = {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}\big(\overline{(a^*)^r}(L)\big), \tag{5.33}$$

with

$$\overline{(a^*)^r}(L) = (a^*)^r(L) \Big/ \sum_i x_i\,(a^*)^r(L).$$

Further, taking the isotypic components the isomorphism (5.33) factors to an isomorphism

$$(L \otimes L_{(m)}^{\otimes r})^{\oplus \dim(\bar{L}_\lambda)} = {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}\big(\bar{a}_\lambda^*(L)\big).$$

This proves the first claim. To prove the second claim, observe that Corollary 5.26 and (5.33) yield compatible $\mathfrak{S}_r \times \mathfrak{S}_r \times H(\Gamma_n)$-module isomorphism

$$B_{r,\ell} \otimes L = \Psi\big((a^*)^r(L)\big), \quad \mathbb{C}\mathfrak{S}_r \otimes L = \Psi\big(\overline{(a^*)^r}(L)\big). \tag{5.34}$$

The first $\mathfrak{S}_r$-action on $\Psi\big(\overline{(a^*)^r}(L)\big)$ is the $\mathfrak{S}_r$-action in the definition of Ψ, and the first $\mathfrak{S}_r$-action on $\mathbb{C}\mathfrak{S}_r \otimes L$ is the dual of the right $\mathfrak{S}_r$-action on $\mathbb{C}\mathfrak{S}_r$. The second $\mathfrak{S}_r$-action on $\Psi\big(\overline{(a^*)^r}(L)\big)$ is the $\mathfrak{S}_r$-action on $\overline{(a^*)^r}(L)$ in Corollary 5.26, and the second $\mathfrak{S}_r$-action on $\mathbb{C}\mathfrak{S}_r \otimes L$ is the left $\mathfrak{S}_r$-action on $\mathbb{C}\mathfrak{S}_r$. To identify the actions as above, it is enough to note that the isomorphism

$$\begin{aligned} B_{r,\ell} = \mathrm{Hom}_{\mathcal{O}(\Gamma_n)}(L, B_{r,\ell} \otimes L) = \mathrm{Hom}_{\mathcal{O}(\Gamma_n)}\big(L, \Psi(a^*)^r(L)\big) = \\ = \mathrm{End}_{\mathcal{O}(\Gamma)}((a^*)^r(L)) \end{aligned} \tag{5.35}$$

given by (5.34) is equal to the isomorphism (5.32), and that the $\mathfrak{S}_r$-actions on $(a^*)^r(L)$ are taken to the left and to the dual right $\mathfrak{S}_r$-action on $B_{r,\ell}$ by the map (5.35). Next, write

$$\mathbb{C}\mathfrak{S}_r = \bigoplus_\lambda \bar{L}_\lambda \otimes \bar{L}_\lambda$$

as an $\mathfrak{S}_r \times \mathfrak{S}_r$-module, and take the isotypic component. □

5.6. **Definition of the map $\tilde{a}_\lambda$.**

Proposition 5.30. *For $\lambda \in \mathcal{P}_r$ with $r \geqslant 1$ we have*

$$a_\lambda^*(F_{i,j}(\Gamma_n)) \subset F_{i,j+r}(\Gamma_{n+mr}), \quad a_\lambda^*(F_{i,j}(\Gamma_n)^\circ) \subset F_{i,j+r}(\Gamma_{n+mr})^\circ.$$

Proof. By Remark 3.15 we have

$$\mathrm{Supp}(L_{m\lambda}) = X_{\mathfrak{S}_m^r, \mathbb{C}_0^{mr}}.$$

Let $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))$. First, assume that $L \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}$, i.e., that

$$\mathrm{Supp}(L) = X_{l,j,\mathbb{C}^n}$$

by Remark 3.12. Hence the module $L \otimes L_{m\lambda}$ has the following support

$$\mathrm{Supp}(L \otimes L_{m\lambda}) = X_{l,j,\mathbb{C}^n} \times X_{\mathfrak{S}_m^r, \mathbb{C}_0^{mr}}.$$

So by Proposition 2.7 we have

$$\mathrm{Supp}(a_\lambda^*(L)) = X_{l,j+r,\mathbb{C}^{n+mr}}.$$

Thus the class of $a_\lambda^*(L)$ belongs to $F_{i,j+r}(\Gamma_{n+mr})^\circ$ by Remark 3.12. Next, assume that $[L] \in F_{i,j}(\Gamma_n)$, i.e.,

$$\mathrm{Supp}(L) = X_{l',j',\mathbb{C}^n}, \quad X_{l',j',\mathbb{C}^n} \subset X_{l,j,\mathbb{C}^n}.$$

Thus we have

$$\mathrm{Supp}(a_\lambda^*(L')) = X_{l',j'+r,\mathbb{C}^{n+mr}}.$$

So (3.11) yields

$$X_{l',j'+r,\mathbb{C}^{n+mr}} \subset X_{l,j+r,\mathbb{C}^{n+mr}},$$

i.e., the class of $a^*_\lambda(L)$ lies in $F_{i,j+r}(\Gamma_{n+mr})$. □

Proposition 5.31. *Let $\lambda \in \mathcal{P}_r$ with $r \geqslant 1$, and let $L \in \mathrm{PI}(\mathcal{O}(\Gamma_n))$. The module $\mathrm{top}(\bar{a}^*_\lambda(L))$ has a unique constituent in $\mathrm{Irr}(\mathcal{O}(\Gamma_{n+mr}))_{0,r}$.*

Proof. Since the module L is primitive, it belongs to $\mathrm{Irr}(\mathcal{O}(\Gamma_n))_{0,0}$ by Proposition 5.18. Thus $[a^*_\lambda(L)] \in F_{0,r}(\Gamma_{n+mr})$ by Proposition 5.30. Thus the constituents of $\bar{a}^*_\lambda(L)$ belong to the set

$$\bigcup_{j \leqslant r} \mathrm{Irr}(\mathcal{O}(\Gamma_{n+mr}))_{0,j}$$

by Remark 3.11. Now, for L' in $\mathrm{Irr}(\mathcal{O}(\Gamma_{n+mr}))_{0,j}$ we have ${}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(L') = 0$ if $j < r$, and $\dim {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(L') < \infty$ if $j = r$. Further, the constituents of a finite dimensional module in $\mathcal{O}(\mathfrak{S}^r_m)$ are all isomorphic to $L^{\otimes r}_{(m)}$, and, using [2, thm. 1.3] as in the proof of Proposition 5.7, we get

$$\mathrm{Ext}^1_{\mathcal{O}(\mathfrak{S}^r_m)}(L^{\otimes r}_{(m)}, L^{\otimes r}_{(m)}) = 0.$$

Thus if L' is a constituent of $\mathrm{top}(\bar{a}^*_\lambda(L))$ then we have a surjective map

$$\Psi(\bar{a}^*_\lambda(L)) \to \Psi(L'). \tag{5.36}$$

We have also

$$\Psi(L') = \bigoplus_{\mu \in \mathcal{P}_r} \bar{L}_\mu \otimes \mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_{mr})}(L_{m\mu}, {}^{\mathcal{O}}\mathrm{Res}_{n,mr}(L')).$$

Finally, Corollary 5.29 yields an isomorphism of $\mathfrak{S}_r \otimes H(\Gamma_n)$-modules

$$\bar{L}_\lambda \otimes L = \Psi(\bar{a}^*_\lambda(L)).$$

Thus the surjectivity of (5.36) implies that

$$\mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_{mr})}(L_{m\mu}, {}^{\mathcal{O}}\mathrm{Res}_{n,mr}(L')) = 0, \quad \forall \mu \neq \lambda. \tag{5.37}$$

Since the $\mathfrak{S}_r \otimes H(\Gamma_n)$-module $\bar{L}_\lambda \otimes L$ is simple, the map (5.36) is invertible if it is nonzero. Assume further that $L' \in \mathrm{Irr}(\mathcal{O}(\Gamma_{n+mr}))_{0,r}$. Then Proposition 2.2 yields

$${}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(L') \neq 0.$$

Since $\dim {}^{\mathcal{O}}\mathrm{Res}_{n,(m^r)}(L') < \infty$ and the constituents of a finite dimensional module in $\mathcal{O}(\mathfrak{S}^r_m)$ are all isomorphic to $L^{\otimes r}_{(m)}$, we have also $\Psi(L') \neq 0$. Therefore (5.36) is indeed invertible. This implies that $\mathrm{top}(\bar{a}^*_\lambda(L))$ has a unique constituent in $\mathrm{Irr}(\mathcal{O}(\Gamma_{n+mr}))_{0,r}$. Indeed, otherwise we would have a surjective map

$$\bar{a}^*_\lambda(L) \to L' \oplus L'', \quad L', L'' \in \mathrm{Irr}(\mathcal{O}(\Gamma_{n+mr}))_{0,r},$$

yielding a surjective map

$$\bar{L}_\lambda \otimes L = \Psi(\bar{a}^*_\lambda(L)) \to \Psi(L') \oplus \Psi(L'') = (\bar{L}_\lambda \otimes L)^{\oplus 2}.$$

This is absurd. □

Definition 5.32. For $\lambda \in \mathcal{P}_r$ and $L \in \mathrm{PI}(\mathcal{O}(\Gamma))$ we define $\tilde{a}_\lambda(L)$ to be the unique constituent of $\mathrm{top}(\bar{a}^*_\lambda(L))$ in $\mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r}$.

Proposition 5.33. *For $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r}$ there is $L' \in \mathrm{PI}(\mathcal{O}(\Gamma))$, $\lambda \in \mathcal{P}_r$ such that $\tilde{a}_\lambda(L') \simeq L$. In other words, there is a surjective map*

$$\mathrm{PI}(\mathcal{O}(\Gamma)) \times \mathcal{P}_r \to \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r}, \quad (L', \lambda) \mapsto \tilde{a}_\lambda(L'). \tag{5.38}$$

Proof. By Proposition 5.18 the module L is primitive if and only if $r = 0$. Thus we can assume that $r > 0$, i.e., that $a_*(L) \neq 0$ by Corollary 3.19, else the claim is obvious. Now, we first claim that there is a module $L_1 \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r-1}$ with a surjective morphism $\bar{a}^*(L_1) \to L$. Indeed, since $a_*(L) \neq 0$, the adjunction map $\epsilon : a^*(a_*(L)) \to L$ is non-zero, hence it is surjective. Hence, there is a constituent L_1 of $a_*(L)$ such that ϵ yields a surjective morphism $a^*(L_1) \to L$.

Lemma 5.34. *If $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r}$ and L_1 is a constituent of $a_*(L)$ such that $a^*(L_1)$ maps onto L then $L_1 \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r-1}$.*

Fix the integer n such that $L_1 \in \mathrm{Irr}(\mathcal{O}(\Gamma_n))$. Then $\bar{\xi}_1$ acts on $a^*(L_1)$ as the operator

$$^{\mathcal{O}}z_{n+m}(a^*(L_1)) \circ a^*(^{\mathcal{O}}z_n(L_1))^{-1}.$$

The second factor is a scalar because L_1 is a simple module. Hence x_1 acts on $a^*(L_1)$ as an element of the center of $\mathcal{O}(\Gamma_{n+m})$, see (5.32). Therefore, since L is simple and since the operator x_1 on $a^*(L_1)$ is nilpotent by Proposition 5.20, the operator x_1 is 0 on L. Thus the map $a^*(L_1) \to L$ factors to a surjective morphism

$$\epsilon_1 : \bar{a}^*(L_1) \to L.$$

This proves the claim.

Now, assume that for $0 < k < r$ there is a module $L_k \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r-k}$ with a surjective homomorphism

$$\epsilon_k : \overline{(a^*)^k}(L_k) \to L, \qquad \overline{(a^*)^k}(L_k) = (a^*)^k(L_k)/\sum_i x_i(a^*)^k(L_k).$$

By the claim above, there is a module $L_{k+1} \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r-k-1}$ with a surjective homomorphism

$$\bar{a}^*(L_{k+1}) \to L_k.$$

Applying the functor $(a^*)^k$, which is exact, we get a surjective map

$$(a^*)^k\bar{a}^*(L_{k+1}) \to (a^*)^k(L_k).$$

Taking the quotient by the action of $x_2, \dots, x_k, x_{k+1}$ it yields a surjective map

$$(a^*)^k\bar{a}^*(L_{k+1})/\sum_{i=2}^{k+1} x_i(a^*)^k\bar{a}^*(L_{k+1}) \to \overline{(a^*)^k}(L_k).$$

Now, since a^* is exact, we have

$$(a^*)^k\bar{a}^*(L_{k+1}) = (a^*)^{k+1}(L_{k+1})/x_1(a^*)^{k+1}(L_{k+1}).$$

Therefore we get a surjective map

$$\overline{(a^*)^{k+1}}(L_{k+1}) = (a^*)^{k+1}(L_{k+1})/\sum_{i=1}^{k+1} x_i(a^*)^k\bar{a}^*(L_{k+1}) \to \overline{(a^*)^k}(L_k).$$

Composing it with ϵ_k we get a surjective homomorpism

$$\epsilon_{k+1} : \overline{(a^*)^{k+1}}(L_{k+1}) \to L.$$

By induction, this yields a module $L_r \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,0}$ with a surjective homomorphism

$$\epsilon_r : \overline{(a^*)^r}(L_r) \to L.$$

Then we have $L_r \in \mathrm{PI}(\mathcal{O}(\Gamma))$ by Proposition 5.18, and there is $\lambda \in \mathcal{P}_r$ such that $\bar{a}^*_\lambda(L_r)$ maps onto L. The proposition follows from Proposition 5.31. □

Proof of Lemma 5.34. Fix $i, j \geqslant 0$ such that $L_1 \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{i,j}$. By Proposition 5.15, since $E(L) = 0$ we have $E\, a_*(L) = 0$. Hence $E(L_1) = 0$ by Proposition 3.3. Thus $i = 0$ by Corollary 3.19. So, by Proposition 5.30 we have $a^*(L_1) \in F_{0,j+1}(\Gamma)$. Since $a^*(L_1)$ maps onto L, we have also $[L] \in F_{0,j+1}(\Gamma)$. Since $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,r}$ this implies that $r \leqslant j+1$ by Remark 3.11.

Now, we prove that $j+1 \leqslant r$. Fix $n \geqslant 1$ such that $L \in \mathcal{O}(\Gamma_n)$. Recall that

$$a_*(L) = \mathrm{Hom}_{\mathcal{O}(\mathfrak{S}_m)}\big(L_{(m)}, {}^{\mathcal{O}}\mathrm{Res}_{n-m,m}(L)\big).$$

Thus there is an obvious inclusion

$$a_*(L) \otimes L_{(m)} \subset {}^{\mathcal{O}}\mathrm{Res}_{n-m,m}(L).$$

Hence, since L_1 is a constituent of $a_*(L)$, the module $L_1 \otimes L_{(m)}$ is a constituent of ${}^{\mathcal{O}}\mathrm{Res}_{n-m,m}(L)$. Let us abbreviate

$$W' = \Gamma_{l,(m^j)}, \quad l = n - (j+1)m,$$

regarded as a subgroup of Γ_{n-m}. Then $W' \times \mathfrak{S}_m \subset \Gamma_{n-m} \times \mathfrak{S}_m$ in the obvious way. Since $L_1 \in \mathrm{Irr}(\mathcal{O}(\Gamma_{n-m}))_{0,j}$, we have

$$\mathrm{Supp}(L_1 \otimes L_{(m)}) = X_{W' \times \mathfrak{S}_m,\, \mathbb{C}^{n-m} \times \mathbb{C}_0^m}.$$

By Proposition 2.2 applied to the module $M = L$, we have also

$$\mathrm{Supp}(L_1 \otimes L_{(m)}) = X_{W_1,\, \mathbb{C}^{n-m} \times \mathbb{C}_0^m},$$

where W_1 is a parabolic subgroup of $\Gamma_{n-m,m}$ containing a subgroup Γ_n-conjugate to $\Gamma_{n-mr,(m^r)}$. Hence we have $F_{0,j+1}(\Gamma_n) \subset F_{0,r}(\Gamma_n)$. Therefore we have $j+1 \leqslant r$ by Remark 3.11.

□

6. The filtration of the Fock space and Etingof's conjecture

Recall that $[\mathcal{O}(\Gamma)]$ is identified with the Fock space $\mathcal{F}^{(s)}_{m,\ell}$ via the map (5.20). The aim of this section is to identify the filtration on $[\mathcal{O}(\Gamma)]$ defined in Section 3.10 in terms of supports of irreducible modules, with a filtration on the Fock space given by representation theoretic tools. We'll use the following notation : n, m, j, i are integers with $n > 0$, $m > 2$, $i, j \geqslant 0$ and $i = n - l - jm$.

6.1. **The representation theoretic interpretation of $F_{0,0}(\Gamma)$.** The goal of this section is to give a representation theoretic interpretation of $F_{0,0}(\Gamma)$ using the actions of $\widehat{\mathfrak{sl}}_m$ and $\mathfrak{H}$ on $[\mathcal{O}(\Gamma)]$ defined in the previous sections. Note that the set $\mathrm{Irr}(\mathcal{O}(\Gamma))_{0,0}$ is a basis of the $\mathbb{C}$-vector space $F_{0,0}(\Gamma)$. Further, we have proved that $\mathrm{Irr}(\mathcal{O}(\Gamma))_{0,0} = \mathrm{PI}(\mathcal{O}(\Gamma))$. in Proposition 5.18. Recall that the operators b'_r, $r \geqslant 1$, on $\mathcal{F}^{(s)}_{m,\ell}$ given in Section 4.6 act on $[\mathcal{O}(\Gamma)]$ via the map (5.20).

Lemma 6.1. *For $L \in \mathrm{Irr}(\mathcal{O}(\Gamma))$ we have $L \in \mathrm{PI}(\mathcal{O}(\Gamma))$ if and only if $E([L]) = b'_r([L]) = 0$ in $[\mathcal{O}(\Gamma)]$ for all $r \geqslant 1$*

Proof. It is enough to prove that for $L \in \mathrm{PI}(\mathcal{O}(\Gamma))$ we have $b'_r(L) = 0$ for all $r \geqslant 1$. A direct summand of the zero object is zero in any additive category. Further, for $L \in \mathrm{PI}(\mathcal{O}(\Gamma))$ we have $(Ra_*)^r(L) = 0$ for $r \geqslant 1$. Thus we have also $Ra_{\lambda,*}(L) = 0$ for all $\lambda \in \mathcal{P}$ by Proposition 5.4. By Proposition 5.13 the map (5.20) identifies the $\mathbb{C}$-linear operator $Ra_{\lambda,*}$ on $[\mathcal{O}(\Gamma)]$ with the action of b'_{S_λ} on $\mathcal{F}^{(s)}_{m,\ell}$ given in Section 4.6. This proves the lemma. □

In particular the lemma yields an inclusion

$$F_{0,0}(\Gamma) \subset \{x \in [\mathcal{O}(\Gamma)]\,;\, e_q(x) = b'_r(x) = 0,\ \forall q, r\}.$$

However it is not obvious that the right hand side is spanned by classes of irreducible objects of $\mathcal{O}(\Gamma)$. This follows indeed from Proposition 6.2 below. Before to prove it we need the following lemma.

Proposition 6.2. *We have*

$$\{x \in [\mathcal{O}(\Gamma)]\,;\, e_q(x) = b'_r(x) = 0,\ \forall q, r\} = F_{0,0}(\Gamma).$$

Proof. Consider the set

$$F_{0,0}(\Gamma)' = \{x \in F_{0,\bullet}(\Gamma)\,;\, b'_r(x) = 0,\ \forall r \geqslant 1\}.$$

By Corollary 3.19 it is enough to prove that

$$F_{0,0}(\Gamma) = F_{0,0}(\Gamma)'.$$

We have

$$F_{0,0}(\Gamma)' = \bigoplus_{n\geqslant 0} F_{0,0}(\Gamma_n)', \quad F_{0,0}(\Gamma_n)' = F_{0,0}(\Gamma)' \cap F_{0,\bullet}(\Gamma_n).$$

The actions of $\widehat{\mathfrak{sl}}_m$ and $\mathfrak{H}$ on $\mathcal{F}^{(s)}_{m,\ell}$ commute to each other. Thus, by Corollary 3.19 the $\mathbb{C}$-vector space $F_{0,\bullet}(\Gamma)$ is identified with a $\mathfrak{H}$-submodule of $\mathcal{F}^{(s)}_{m,\ell}$ via the map (5.20), and we have

$$\sum_{n\geqslant 0} \dim(F_{0,\bullet}(\Gamma_n)) \cdot t^n = \sum_{n\geqslant 0} \sharp\,\mathrm{Irr}(\mathcal{O}(\Gamma_n))_{0,\bullet} \cdot t^n. \tag{6.1}$$

The representation theory of $\mathfrak{H}$ yields the following formula in $\mathbb{Z}[[t]]$

$$\big(\sum_{k\geqslant 0} \dim(F_{0,0}(\Gamma_k)') \cdot t^k\big)\big(\sum_{r\geqslant 0} \sharp\mathcal{P}_r \cdot t^{mr}\big) = \sum_{n\geqslant 0} \dim(F_{0,\bullet}(\Gamma_n)) \cdot t^n. \tag{6.2}$$

Finally, Proposition 5.33 yields a surjective map

$$\mathrm{PI}(\mathcal{O}(\Gamma_k)) \times \mathcal{P}_r \to \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{0,r}, \quad (L, \lambda) \mapsto \tilde{a}_\lambda(L) \tag{6.3}$$

for $k, r \geqslant 0$ such that $n = k + mr$. From (6.1) and (6.3) we get

$$\big(\sum_{k\geqslant 0} \sharp\,\mathrm{PI}(\mathcal{O}(\Gamma_k)) \cdot t^k\big)\big(\sum_{r\geqslant 0} \sharp\mathcal{P}_r \cdot t^{mr}\big) - \sum_{n\geqslant 0} \dim(F_{0,\bullet}(\Gamma_n)) \cdot t^n \in \mathbb{N}[[t]]. \tag{6.4}$$

By Corollary 3.19 and Lemma 6.1 we have $\mathrm{PI}(\mathcal{O}(\Gamma_k)) \subset F_{0,0}(\Gamma_k)'$, hence we have

$$\sharp\,\mathrm{PI}(\mathcal{O}(\Gamma_k)) \leqslant \dim(F_{0,0}(\Gamma_k)').$$

Therefore, comparing (6.2) and (6.4) we get the equality

$$\sharp\,\mathrm{PI}(\mathcal{O}(\Gamma_k)) = \dim(F_{0,0}(\Gamma_k)'). \tag{6.5}$$

In other words $\mathrm{PI}(\mathcal{O}(\Gamma_k))$ is a basis of $F_{0,0}(\Gamma_k)'$. Since $\mathrm{PI}(\mathcal{O}(\Gamma_k))$ is a basis of $F_{0,0}(\Gamma_k)$ by Proposition 5.18, we have also

$$F_{0,0}(\Gamma_k) = F_{0,0}(\Gamma_k)'.$$

□

Remark 6.3. The proof of Proposition 6.2 and Corollary 3.19 imply that the map (6.3) yields a bijection

$$\mathrm{PI}(\mathcal{O}(\Gamma_k)) \times \mathcal{P}_r \to \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{0,r}, \quad (L, \lambda) \mapsto \tilde{a}_\lambda(L)$$

for $k, r \geqslant 0$ such that $n = k + mr$. Note that Proposition 5.18 yields

$$\mathrm{PI}(\mathcal{O}(\Gamma_k)) = \mathrm{Irr}(\mathcal{O}(\Gamma_k))_{0,0}.$$

6.2. **The representation theoretic grading on** $[\mathcal{O}(\Gamma)]$**.** Using the actions of the Lie algebras $\mathfrak{H}$ and $\widehat{\mathfrak{sl}}_m$ we can now define a grading

$$[\mathcal{O}(\Gamma)] = \bigoplus_{i,j\geqslant 0} [\mathcal{O}(\Gamma)]_{i,j}.$$

Then, we'll compare it with the filtration by the support introduced in Section 3.10, i.e., we'll compare it with the grading

$$[\mathcal{O}(\Gamma)] = \bigoplus_{i,j\geqslant 0} \mathrm{gr}_{i,j}(\Gamma).$$

To do so, let us consider the level $m\ell$ Casimir operator

$$\partial = \frac{1}{m\ell} \sum_{r\geqslant 1} b_r b'_r,$$

see (4.2). Under the map (5.20) this formal sum defines a diagonalisable $\mathbb{C}$-linear operator on $[\mathcal{O}(\Gamma)]$. For any integer j let $[\mathcal{O}(\Gamma)]_{\bullet,j}$ be the eigenspace of ∂ with the eigenvalue j. Note that $[\mathcal{O}(\Gamma)]_{\bullet,j} = 0$ if $j < 0$. Next, given an integer $i \geqslant 0$ we define $[\mathcal{O}(\Gamma)]_{i,\bullet}$ to be the image of

$$\bigoplus_{\mu,\alpha} V_\mu^{\widehat{\mathfrak{sl}}_m}[\mu-\alpha] \otimes \mathrm{Hom}_{\widehat{\mathfrak{sl}}_m}(V_\mu^{\widehat{\mathfrak{sl}}_m}, [\mathcal{O}(\Gamma)])$$

under the canonical maps

$$V_\mu^{\widehat{\mathfrak{sl}}_m} \otimes \mathrm{Hom}_{\widehat{\mathfrak{sl}}_m}(V_\mu^{\widehat{\mathfrak{sl}}_m}, [\mathcal{O}(\Gamma)]) \to [\mathcal{O}(\Gamma)].$$

Here the sum runs over all sums α of i affine simple roots of $\widehat{\mathfrak{sl}}_m$, and over all dominant affine weight μ of $\widehat{\mathfrak{sl}}_m$. If $i < 0$ we set $[\mathcal{O}(\Gamma)]_{i,\bullet} = 0$.

Definition 6.4. We define a grading on $[\mathcal{O}(\Gamma)]$ by the following formula

$$[\mathcal{O}(\Gamma)]_{i,j} = [\mathcal{O}(\Gamma)]_{i,\bullet} \cap [\mathcal{O}(\Gamma)]_{\bullet,j}, \quad [\mathcal{O}(\Gamma_n)]_{i,j} = [\mathcal{O}(\Gamma)]_{i,j} \cap [\mathcal{O}(\Gamma_n)]$$

Proposition 6.5. *We have* $\dim[\mathcal{O}(\Gamma_n)]_{i,j} = \dim \mathrm{gr}_{i,j}(\Gamma_n)$ *for all* $n, i, j \geqslant 0$.

Proof. The vector space $[\mathcal{O}(\Gamma)]_{0,\bullet}$ is a $\mathfrak{H}$-submodule of $[\mathcal{O}(\Gamma)]$. Thus it is preserved by the linear operator ∂ and $[\mathcal{O}(\Gamma)]_{0,j}$ is the eigenspace with the eigenvalue j. Since the $\mathfrak{H}$-action on $[\mathcal{O}(\Gamma)]_{0,\bullet}$ has the level $m\ell$ we have $[\partial, b_j] = jb_j$ for all $j > 0$. Next, we have

$$[\mathcal{O}(\Gamma)]_{0,\bullet} = F_{0,\bullet}(\Gamma), \quad [\mathcal{O}(\Gamma)]_{0,0} = F_{0,0}(\Gamma)$$

by Corollary 3.19 and Proposition 6.2. Further, the $\mathfrak{H}$-action yields an isomorphism

$$U^-(\mathfrak{H})_j \otimes [\mathcal{O}(\Gamma)]_{0,0} = [\mathcal{O}(\Gamma)]_{0,j}. \tag{6.6}$$

By Remark 6.3, for $n = k + mj$ we have a bijection

$$\mathrm{Irr}(\mathcal{O}(\Gamma_k))_{0,0} \times \mathcal{P}_j \to \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{0,j}, \quad (L, \lambda) \mapsto \tilde{a}_\lambda(L). \tag{6.7}$$

Thus the isomorphism (6.6) yields the following equality

$$\dim [\mathcal{O}(\Gamma_n)]_{0,j} = \sharp\, \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{0,j}. \tag{6.8}$$

Now, to compare $\dim [\mathcal{O}(\Gamma_n)]_{i,j}$ and $\sharp\, \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}$ for any $i \geqslant 0$, we need some tools from canonical bases. Since the integrable $\widehat{\mathfrak{sl}}_m$-module $[\mathcal{O}(\Gamma)]$ is not simple, the choice of a canonical basis of this module depends on a choice of a basis of $[\mathcal{O}(\Gamma)]_{0,\bullet}$. The general theory of canonical bases yields a bijection G between the canonical basis of $[\mathcal{O}(\Gamma)]$ and its crystal basis, the latter being identified with $\mathrm{Irr}(\mathcal{O}(\Gamma))$ by Proposition 3.3. The bijection G is such that a basis of $[\mathcal{O}(\Gamma)]_{0,\bullet}$ is given by

$$\{\mathrm{G}(L)\,;\, \tilde{e}_q(L) = 0,\, \forall q\}.$$

By Corollary 3.19 we have

$$\{L \in \mathrm{Irr}(\mathcal{O}(\Gamma))\,;\, \tilde{e}_q(L) = 0,\, \forall q\} = \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,\bullet}$$
$$= \{\tilde{a}_\lambda(L)\,;\, \forall \lambda \in \mathcal{P},\, \forall L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,0}\}.$$

We'll choose the canonical basis of $[\mathcal{O}(\Gamma)]$ such that

$$\mathrm{G}(\tilde{a}_\lambda(L)) = a^*_\lambda(L), \quad \forall \lambda \in \mathcal{P}, \quad \forall L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,0}.$$

Then the set $\{\mathrm{G}(L)\,;\, L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,j}\}$ is a basis of $[\mathcal{O}(\Gamma)]_{0,j}$ by (6.6) and (6.7). The $\widehat{\mathfrak{sl}}_m$-action on $[\mathcal{O}(\Gamma)]$ commutes with the operator ∂. Thus $[\mathcal{O}(\Gamma)]_{\bullet,j}$ is an $\widehat{\mathfrak{sl}}_m$-module and the $\widehat{\mathfrak{sl}}_m$-action yields a surjective $\mathbb{C}$-linear map

$$U^-(\widehat{\mathfrak{sl}}_m)_i \otimes [\mathcal{O}(\Gamma)]_{0,j} \to [\mathcal{O}(\Gamma)]_{i,j}. \tag{6.9}$$

For weight reasons, the crystal of $[\mathcal{O}(\Gamma)]$ decomposes in the following way

$$\mathrm{Irr}(\mathcal{O}(\Gamma)) = \bigsqcup_{i,j\geqslant 0} \mathrm{Irr}(\mathcal{O}(\Gamma))'_{i,j}, \quad \mathrm{Irr}(\mathcal{O}(\Gamma))'_{i,j} = \{L \in \mathrm{Irr}(\mathcal{O}(\Gamma))\,;\, \mathrm{G}(L) \in [\mathcal{O}(\Gamma)]_{i,j}\}.$$

Since $\{\mathrm{G}(L)\,;\, L \in \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,j}\}$ is a basis of $[\mathcal{O}(\Gamma)]_{0,j}$, we have

$$\mathrm{Irr}(\mathcal{O}(\Gamma))'_{0,j} = \mathrm{Irr}(\mathcal{O}(\Gamma))_{0,j}.$$

Next $\mathrm{Irr}(\mathcal{O}(\Gamma))'_{\bullet,j}$ is the union of connected components of $\mathrm{Irr}(\mathcal{O}(\Gamma))$ whose highest weight vector is in $\mathrm{Irr}(\mathcal{O}(\Gamma))'_{0,j}$, and by Corollary 3.18, the set $\mathrm{Irr}(\mathcal{O}(\Gamma))_{\bullet,j}$ is the union of connected components of $\mathrm{Irr}(\mathcal{O}(\Gamma))$ whose highest weight vector is in $\mathrm{Irr}(\mathcal{O}(\Gamma))_{0,j}$. Thus, for all n we have

$$\mathrm{Irr}(\mathcal{O}(\Gamma_n))'_{\bullet,j} = \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{\bullet,j}.$$

By Corollary 3.18 and (6.9), for all i we have also the inclusion

$$\mathrm{Irr}(\mathcal{O}(\Gamma_n))'_{i,j} \subset \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}. \tag{6.10}$$

Thus (6.10) is indeed an equality. By definition, we have

$$\dim \mathrm{gr}_{i,j}(\Gamma_n) = \sharp\, \mathrm{Irr}(\mathcal{O}(\Gamma_n))_{i,j}, \quad \dim\, [\mathcal{O}(\Gamma_n)]_{i,j} = \sharp\, \mathrm{Irr}(\mathcal{O}(\Gamma_n))'_{i,j}.$$

Thus the corollary is proved. □

Remark 6.6. Recall that $\mathrm{gr}_{i,j}(\Gamma)$ is identified with the subspace of $[\mathcal{O}(\Gamma)]$ spanned by $\mathrm{Irr}(\mathcal{O}(\Gamma))_{i,j}$, see Section 3.10. Proposition 6.5 does not imply that $[\mathcal{O}(\Gamma)]_{i,j}$ is also spanned by $\mathrm{Irr}(\mathcal{O}(\Gamma))_{i,j}$. However, since

$$[\mathcal{O}(\Gamma)]_{0,0} = \{x \in [\mathcal{O}(\Gamma)]\,;\, e_q(x) = b'_r(x) = 0,\, \forall q, r\},$$

the subspace $[\mathcal{O}(\Gamma)]_{0,0}$ is indeed spanned by $\mathrm{Irr}(\mathcal{O}(\Gamma))_{0,0}$ by Proposition 6.2.

6.3. **Etingof's conjecture.** Let $\alpha_{p,q}$ be the root of the elementary matrix $e_{p,q}$. Recall that $\omega_0, \omega_1, \ldots, \omega_{\ell-1}$ are the affine fundamental weights. Fix a level 1 weight

$$\Lambda = \sum_p h_p\, \omega_p.$$

Definition 6.7. Let $\tilde{\mathfrak{a}}_\Lambda$ be the Lie subalgebra of $\widetilde{\mathfrak{gl}}_\ell$ spanned by $\mathbf{1}$, D and the elements $e_{p,q} \otimes \varpi^r$ with $p, q = 1, 2, \ldots, \ell$ and $r \in \mathbb{Z}$ such that $\langle \Lambda, \alpha_{p,q}\rangle - hr \in \mathbb{Z}$. We abbreviate $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_\Lambda$ and $\hat{\mathfrak{a}} = \tilde{\mathfrak{a}} \cap \widehat{\mathfrak{gl}}_\ell$.

We define the set of *positive real roots of* $\tilde{\mathfrak{a}}$ to be the set $\Delta^{\hat{\mathfrak{a}}}_+$ consisting of the real roots of $\widehat{\mathfrak{gl}}_\ell$ of the form $\alpha + (r - \langle \Lambda, \alpha\rangle/h)\delta$ where α is a root of $\mathfrak{gl}_\ell$ and $\alpha + r\delta$ is a positive real root of $\widehat{\mathfrak{gl}}_\ell$. Let $P^{\tilde{\mathfrak{a}}}_+$ be the set of *dominant integral weights for* $\tilde{\mathfrak{a}}$, i.e., the set of integral weights of $\widetilde{\mathfrak{gl}}_\ell$ which are $\geqslant 0$ on $\Delta^{\hat{\mathfrak{a}}}_+$. For $\mu \in P^{\tilde{\mathfrak{a}}}_+$ let $V^{\tilde{\mathfrak{a}}}_\mu$ be the

corresponding irreducible integrable $\tilde{\mathfrak{a}}$-module. We'll say that a non zero vector of an $\tilde{\mathfrak{a}}$-module is *primitive for* $\tilde{\mathfrak{a}}$ or $\tilde{\mathfrak{a}}$-*primitive* if it is a weight vector whose weight belongs to $P_+^{\tilde{\mathfrak{a}}}$, and if it is killed by the action of the weight vectors of $\tilde{\mathfrak{a}}$ whose weights are positive roots of $\tilde{\mathfrak{a}}$. Now, let h, h_p be the parameters of the $\mathbb{C}$-algebra $H(\Gamma_n)$ for each $n > 0$. Assume that h is a rational number with the denominator $m > 1$. The elements of $\mathfrak{H}$ can be regarded as elements of $\widetilde{\mathfrak{gl}}_\ell$ as in (4.3). We have $b_{mr}, b'_{mr} \in \tilde{\mathfrak{a}}$ for each $r > 0$. The formal sum

$$\partial_m = \frac{1}{m\ell} \sum_{r \geqslant 1} b_{mr} b'_{mr}$$

acts on every $\tilde{\mathfrak{a}}$-module $V_\mu^{\tilde{\mathfrak{a}}}$. It is the m-th Casimir operator of $\widetilde{\mathfrak{gl}}_\ell$ introduced in (4.25). For any weight λ and any integer j we denote by $V_\mu^{\tilde{\mathfrak{a}}}[\lambda, j]$ the subspace of weight λ and eigenvalue j of ∂_m. We are interested in the following conjecture [9, conj. 6.7].

Conjecture 6.8. *There exists an isomorphism of* $\mathbb{C}$*-vector spaces*

$$\mathrm{gr}_{i,j}(\Gamma_n) = \bigoplus_\mu V_\mu^{\tilde{\mathfrak{a}}}[\omega_0 - n\delta, j] \otimes \mathrm{Hom}_{\tilde{\mathfrak{a}}}(V_\mu^{\tilde{\mathfrak{a}}}, V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}), \tag{6.11}$$

where the sum is over all weights $\mu \in P_+^{\tilde{\mathfrak{a}}}$ *such that* $\langle \mu, \mu \rangle = -2i$.

Remark 6.9. If $\Lambda = \omega_0$ then we have

$$\tilde{\mathfrak{a}}_{\omega_0} = \big(\mathfrak{gl}_\ell \otimes \mathbb{C}[\varpi^m, \varpi^{-m}]\big) \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}D,$$

and the map (4.23) below yields a Lie algebra isomorphism $\tilde{\mathfrak{a}}_{\omega_0} = \widetilde{\mathfrak{gl}}_\ell$.

Remark 6.10. Assume that the h_p's are rational numbers. Let $\bar{K}$ be the algebraic closure of the field $K = \mathbb{C}((\varpi))$. Set

$$\gamma = -\sum_{p=1}^{\ell-1} \frac{h_p}{h} \, (\omega_p - \omega_0) \in P^{\mathfrak{sl}_\ell} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We have $\alpha(\gamma) = -\langle \Lambda, \alpha \rangle / h$ for each root α of $\mathfrak{sl}_\ell$. We may view γ as the element $\gamma(\varpi)$ in $T_\ell(\bar{K})$. We have $\tilde{\mathfrak{a}} = \mathrm{ad}(\gamma)^{-1}(\tilde{\mathfrak{a}}_{\omega_0})$. Now, assume that h, h_p are as in (3.9). Then we have $\gamma \in P^{\mathfrak{sl}_\ell}$, because

$$\gamma = \sum_{p=1}^{\ell-1} (s_{p+1} - s_p) \, (\omega_p - \omega_0).$$

A short computation using the standard identification of $\omega_p - \omega_0$ with the ℓ-tuple

$$(1^p 0^{\ell-p}) - (p/\ell) \, (1^\ell) \tag{6.12}$$

shows that γ belongs to $Q^{\mathfrak{sl}_\ell}$ if and only if the ℓ-charge s has weight 0. In this case $\gamma \in T_\ell(K)$, more precisely, γ is a cocharacter of T_ℓ. Thus the element ξ_γ of the affine symmetric group is well-defined. For a future use note that

$$\hat{\gamma}(s, m) = \xi_\gamma^{-1}(\omega_0)', \tag{6.13}$$

where $\hat{\gamma}(s, m) \in P^{\widehat{\mathfrak{sl}}_\ell}$ is as in (4.15), and that $\xi_\gamma(\mu) \in P_+^{\tilde{\mathfrak{a}}}$ if and only if $\mu' \in P_+^{\widehat{\mathfrak{gl}}_\ell}$. Here μ' is as in (6.16) below.

6.4. **Reminder on the level-rank duality.** For $\lambda \in \mathbb{Z}^\ell$ we consider the weights

$$\tilde{\gamma}(\lambda, m) = \hat{\gamma}(\lambda, m) - \Delta(\lambda, m)\delta \in P^{\widetilde{\mathfrak{sl}}_\ell},$$

with $\hat{\gamma}(\lambda, m) \in P^{\widehat{\mathfrak{sl}}_\ell}$, see Remark 4.7 and (4.15). Note that $\tilde{\gamma}(\lambda, m)$ is dominant if and only if

$$\lambda \in A(\ell, m) = \{(\lambda_1, \lambda_2, \dots, \lambda_\ell) \in \mathbb{Z}_+^\ell \,;\, \lambda_1 - \lambda_\ell \leqslant m\}.$$

For $d \in \mathbb{Z}$ we write

$$A(\ell, m)_d = \{\lambda \in A(\ell, m) \,;\, \sum_p \lambda_p = d\}.$$

The *level-rank duality* yields a bijection $A(\ell, m)_d \to A(m, \ell)_d$, $\lambda \mapsto \lambda^\dagger$ such that

- we have the equality of weights

$$\hat{\gamma}(\lambda, m) = \sum_{p=1}^{m} \omega_{\lambda_p^\dagger \bmod \ell},$$

- we have an $\widetilde{\mathfrak{sl}}_m \times \mathfrak{H} \times \widetilde{\mathfrak{sl}}_\ell$-module isomorphism

$$\mathcal{F}_{m,\ell}^{(d)} = \bigoplus_{\lambda \in A(\ell,m)_d} V^{\widetilde{\mathfrak{sl}}_m}_{\tilde{\gamma}(\lambda^\dagger,\ell)} \otimes V^{\mathfrak{H}}_{m\ell} \otimes V^{\widetilde{\mathfrak{sl}}_\ell}_{\tilde{\gamma}(\lambda,m)} \tag{6.14}$$

  and there are highest weight vectors $v_{\tilde{\gamma}(\lambda^\dagger,\ell)}$, $v_{m\ell}$, $v_{\tilde{\gamma}(\lambda,m)}$ of $V^{\widetilde{\mathfrak{sl}}_m}_{\tilde{\gamma}(\lambda^\dagger,\ell)}$, $V^{\mathfrak{H}}_{m\ell}$, $V^{\widetilde{\mathfrak{sl}}_\ell}_{\tilde{\gamma}(\lambda,m)}$ such that $|0, \lambda\rangle = v_{\tilde{\gamma}(\lambda^\dagger,\ell)} \otimes v_{m\ell} \otimes v_{\tilde{\gamma}(\lambda,m)}$ for $\lambda \in A(\ell, m)_d$.

See e.g., [25, (3.17)], [30, sec. 4.2, 4.3], for details. Let $s = (s_p)$ be an ℓ-charge of weight d. Setting $d = 0$, the formula (4.16) yields

$$\mathcal{F}_{m,\ell}^{(s)} = \bigoplus_{\lambda \in A(\ell,m)_0} V^{\widehat{\mathfrak{sl}}_m}_{\hat{\gamma}(\lambda^\dagger,\ell)} \otimes V^{\mathfrak{H}}_{m\ell} \otimes \big(V^{\widehat{\mathfrak{sl}}_\ell}_{\hat{\gamma}(\lambda,m)}[\hat{\gamma}(s, m)]\big). \tag{6.15}$$

Here the bracket indicates the weight for the $\widehat{\mathfrak{sl}}_\ell$-action of level m.

6.5. **Proof of Etingof's conjecture for an integral ℓ-charge.** In this subsection we prove Etingof's conjecture in the particular case where the parameters h, h_p are as in (3.9). Note that our terminology differs from [9] because this case corresponds indeed to *rational (possibly non integral)* values of the parameters. From now on, unless specified otherwise we'll assume that the parameters h, h_p are as in (3.9), and we'll also assume that the ℓ-charge s has weight zero. To any level one weight μ of $\widehat{\mathfrak{gl}}_\ell$ we associate the level m weight μ' given by

$$\mu' = m\,\omega_0 + \sum_{p=1}^{\ell-1} \mu_p(\omega_p - \omega_0) \quad \text{where} \quad \mu = \omega_0 + \sum_{p=1}^{\ell-1} \mu_p(\omega_p - \omega_0). \tag{6.16}$$

Note that $\gamma \in Q_\ell^{\mathfrak{sl}}$ and that $\hat{\gamma}(s, m) = \xi_\gamma^{-1}(\omega_0)'$ by (6.13). Using this and (6.15) we get a $\widehat{\mathfrak{sl}}_m \times \mathfrak{H}$-module isomorphism

$$\mathcal{F}_{m,\ell}^{(s)} = \bigoplus_{\lambda \in A(\ell,m)_0} V^{\widehat{\mathfrak{sl}}_m}_{\hat{\gamma}(\lambda^\dagger,\ell)} \otimes V^{\mathfrak{H}}_{m\ell} \otimes \big(V^{\widehat{\mathfrak{sl}}_\ell}_{\hat{\gamma}(\lambda,m)}[\xi_\gamma^{-1}(\omega_0)']\big).$$

Thus, by (4.16), (4.22) and (4.23) we have

$$\mathcal{F}_{m,\ell}^{(s)} = V^{\widehat{\mathfrak{gl}}_\ell}_{\omega_0}[\xi_\gamma^{-1}(\omega_0)],$$

where the bracket indicates the weight subspace for the $\widehat{\mathfrak{gl}}_\ell$-action of level 1. Since the map (5.20) yields an isomorphism $[\mathcal{O}(\Gamma)] = \mathcal{F}_{m,\ell}^{(s)}$, we get also an isomorphism

$$[\mathcal{O}(\Gamma)] = V^{\widehat{\mathfrak{gl}}_\ell}_{\omega_0}[\xi_\gamma^{-1}(\omega_0)]. \tag{6.17}$$

Under this isomorphism we have

$$[\mathcal{O}(\Gamma_n)] = V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}[\xi_\gamma^{-1}(\omega_0) - n\delta]$$

by (4.29) and the following lemma.

Lemma 6.11. *(a) If λ^c is an ℓ-core such that $\tau(\lambda^c) = s$ then $n_0(\lambda^c) = \frac{1}{2}\langle\gamma,\gamma\rangle$.*

(b) The element $|0,s\rangle$ is an extremal weight vector of the module $\mathcal{F}^{(0)}_{m,\ell} = V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}$ with the weight $\xi_\gamma^{-1}(\omega_0)$.

The formula (6.11) we want to prove is

$$\mathrm{dimgr}_{i,j}(\Gamma_n) = \sum_\mu \dim\big(V^{\tilde{\mathfrak{a}}}_\mu[\omega_0 - n\delta, j] \otimes \mathrm{Hom}_{\tilde{\mathfrak{a}}}(V^{\tilde{\mathfrak{a}}}_\mu, V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0})\big),$$

where the sum is over all weights $\mu \in P^{\tilde{\mathfrak{a}}}_+$ such that $\langle\mu,\mu\rangle = -2i$. The proof consists of three steps.

Case 1: First, let us consider the sum over all i's. We must prove that

$$\dim \mathrm{gr}_{\bullet,j}(\Gamma_n) = \dim\big(V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}[\omega_0 - n\delta, j]\big).$$

Note that

$$\dim\big(V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}[\omega_0 - n\delta, j]\big) = \dim\big(V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}[\xi_\gamma^{-1}(\omega_0) - n\delta, j]\big),$$

because the Casimir operator ∂_m commutes with the γ-action on $V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}$ by (4.24). Therefore, by Proposition 6.5 we are thus reduced to prove that under (6.17) we have

$$[\mathcal{O}(\Gamma_n)]_{\bullet,j} = V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}[\xi_\gamma^{-1}(\omega_0) - n\delta, j].$$

This follows from the equality of the Casimir operators (**??**) and (4.25), see (4.24).

Case 2 : Next, consider the case $i = 0$. Let $\Theta_{n,0}$ be the image of

$$\bigoplus_{\tilde\mu} V^{\tilde{\mathfrak{a}}}_{\tilde\mu}[\omega_0 - n\delta, j] \otimes \mathrm{Hom}_{\tilde{\mathfrak{a}}}(V^{\tilde{\mathfrak{a}}}_{\tilde\mu}, V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}) \tag{6.18}$$

by the canonical maps $V^{\tilde{\mathfrak{a}}}_{\tilde\mu} \otimes \mathrm{Hom}_{\tilde{\mathfrak{a}}}(V^{\tilde{\mathfrak{a}}}_{\tilde\mu}, V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}) \to V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}$. Here $\tilde\mu$ runs over the set of all weights in $P^{\tilde{\mathfrak{a}}}_+$ with $\langle\tilde\mu,\tilde\mu\rangle = 0$. By Proposition 6.5 and the discussion above we must prove that the image of $[\mathcal{O}(\Gamma_n)]_{0,j}$ by (6.17) is isomorphic to $\Theta_{n,0}$ as a vector space. To do that, observe first that by definition of $[\mathcal{O}(\Gamma_n)]_{0,j}$ the map (6.17) takes $[\mathcal{O}(\Gamma_n)]_{0,j}$ onto the subspace

$$V^{\widetilde{\mathfrak{gl}}_\ell}_{\omega_0}[\xi_\gamma^{-1}(\omega_0) - n\delta] \cap \bigoplus_{\lambda \in A(\ell,m)_0} v_{\hat\gamma(\lambda^\dagger,\ell)} \otimes V^{\mathfrak{H}}_{m\ell}[j] \otimes V^{\widehat{\mathfrak{sl}}_\ell}_{\hat\gamma(\lambda,m)}. \tag{6.19}$$

Note that $v_{\hat\gamma(\lambda^\dagger,\ell)} \otimes V^{\mathfrak{H}}_{m\ell}[j] \otimes V^{\widehat{\mathfrak{sl}}_\ell}_{\hat\gamma(\lambda,m)}$ is the submodule of $\mathcal{F}^{(0)}_{m,\ell} = V^{\widehat{\mathfrak{gl}}_\ell}_{\omega_0}$ generated by the vector $|0,\lambda\rangle$ for the level m action of $\widehat{\mathfrak{gl}}_\ell$. Note also that $\tilde{\mathfrak{a}}_{\omega_0} \simeq \widetilde{\mathfrak{gl}}_\ell$ by Remark 6.9. Finally, the set of weights of $V^{\widehat{\mathfrak{gl}}_\ell}_{\omega_0}$ is

$$\mathrm{Wt}(V^{\widehat{\mathfrak{gl}}_\ell}_{\omega_0}) = \{\omega_0 + \beta\,;\, \beta \in Q^{\mathfrak{sl}_\ell}\},$$

see Section 4.3, and we have the following lemma.

Lemma 6.12. *(a) We have $\nu \in P^{\hat{\mathfrak{a}}_{\omega_0}}_+$ if and only if $\nu' \in P^{\widehat{\mathfrak{gl}}_\ell}_+$.*

(b) We have $\{\nu'\,;\, \nu \in P^{\hat{\mathfrak{a}}_{\omega_0}}_+ \cap \mathrm{Wt}(V^{\widehat{\mathfrak{gl}}_\ell}_{\omega_0})\} = \{\hat\gamma(\lambda,m)\,;\, \lambda \in A(\ell,m)_0\}$.

Thus, by Lemmas 6.11, 6.12 the space (6.19) is indeed equal to

$$\bigoplus_{\tilde\nu} V^{\tilde{\mathfrak{a}}_{\omega_0}}_{\tilde\nu}[\xi_\gamma^{-1}(\omega_0) - n\delta, j], \tag{6.20}$$

where the sum is over all extremal weights $\tilde{\nu}$ in $P_+^{\tilde{\mathfrak{a}}_{\omega_0}} \cap \mathrm{Wt}(V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell})$ and $V_{\tilde{\nu}}^{\tilde{\mathfrak{a}}_{\omega_0}}$ is identified with the $\tilde{\mathfrak{a}}_{\omega_0}$-submodule of $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ generated by a non zero extremal weight vector of weight $\tilde{\nu}$. Now, let us consider the space $\Theta_{n,0}$. Recall that $\langle \tilde{\mu}, \tilde{\mu} \rangle = 0$ if and only if $\tilde{\mu}$ is an extremal weight of $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$. Further an extremal weight have a one-dimensional weight subspace, see Section 4.3. Thus $\Theta_{n,0}$ is equal to the sum

$$\bigoplus_{\tilde{\mu}} V_{\tilde{\mu}}^{\tilde{\mathfrak{a}}}[\omega_0 - n\delta, j], \tag{6.21}$$

where $\tilde{\mu}$ runs over the set of all extremal weights such that $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ contains an $\tilde{\mathfrak{a}}$-primitive vector of weight $\tilde{\mu}$, say $v_{\tilde{\mu}}$, and $V_{\tilde{\mu}}^{\tilde{\mathfrak{a}}}$ is identified with the $\tilde{\mathfrak{a}}$-submodule of $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ generated by $v_{\tilde{\mu}}$. Now, Remark 6.10 yields

$$\tilde{\mathfrak{a}} = \mathrm{ad}(\gamma)^{-1}(\tilde{\mathfrak{a}}_{\omega_0}), \quad \xi_\gamma(\tilde{\mu}) \in P_+^{\tilde{\mathfrak{a}}} \iff \tilde{\mu} \in P_+^{\tilde{\mathfrak{a}}_{\omega_0}}.$$

Thus the γ-action yields a linear automorphism of $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ such that

$$\gamma^{-1}(V_{\tilde{\mu}}^{\tilde{\mathfrak{a}}_{\omega_0}}[\xi_\gamma^{-1}(\omega_0) - n\delta, j]) = V_{\xi_\gamma(\tilde{\mu})}^{\tilde{\mathfrak{a}}}[\omega_0 - n\delta, j], \quad \forall \tilde{\mu} \in P_+^{\tilde{\mathfrak{a}}_{\omega_0}}.$$

Thus (6.20) is equal to $\Theta_{n,0}$ by the following lemma.

Lemma 6.13. *For all weight μ in $P_+^{\hat{\mathfrak{a}}_{\omega_0}} \cap \mathrm{Wt}(V_{\omega_0}^{\widehat{\mathfrak{gl}}_\ell})$ the module $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ contains a $\tilde{\mathfrak{a}}_{\omega_0}$-primitive vector of weight $\tilde{\mu}$.*

Case 3 : Finally, consider the general case. Fix the integers n, j. Let $\Theta_{n,i}$ be the image of

$$\bigoplus_{\tilde{\nu}} V_{\tilde{\nu}}^{\tilde{\mathfrak{a}}}[\omega_0 - n\delta, j] \otimes \mathrm{Hom}_{\tilde{\mathfrak{a}}}(V_{\tilde{\nu}}^{\tilde{\mathfrak{a}}}, V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}),$$

by the canonical maps $V_{\tilde{\nu}}^{\tilde{\mathfrak{a}}} \otimes \mathrm{Hom}_{\tilde{\mathfrak{a}}}(V_{\tilde{\nu}}^{\tilde{\mathfrak{a}}}, V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}) \to V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$. Here the sum is over all weights $\tilde{\nu} \in P_+^{\tilde{\mathfrak{a}}}$ such that $\langle \tilde{\nu}, \tilde{\nu} \rangle = -2i$. The same argument as for Case 2 implies that $\Theta_{n,i} = \gamma^{-1}(\Theta'_{n,i})$ where $\Theta'_{n,i}$ is the image of

$$\bigoplus_{\tilde{\mu}} V_{\tilde{\mu}}^{\tilde{\mathfrak{a}}_{\omega_0}}[\xi_\gamma^{-1}(\omega_0) - n\delta, j] \otimes \mathrm{Hom}_{\tilde{\mathfrak{a}}_{\omega_0}}(V_{\tilde{\mu}}^{\tilde{\mathfrak{a}}_{\omega_0}}, V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}),$$

by the canonical maps $V_{\tilde{\mu}}^{\tilde{\mathfrak{a}}_{\omega_0}} \otimes \mathrm{Hom}_{\tilde{\mathfrak{a}}_{\omega_0}}(V_{\tilde{\mu}}^{\tilde{\mathfrak{a}}_{\omega_0}}, V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}) \to V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$, because the composition by the automorphism γ^{-1} of $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ yields a linear isomorphism

$$\mathrm{Hom}_{\tilde{\mathfrak{a}}}(V_{\xi_\gamma(\tilde{\mu})}^{\tilde{\mathfrak{a}}}, V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}) = \mathrm{Hom}_{\tilde{\mathfrak{a}}_{\omega_0}}(V_{\tilde{\mu}}^{\tilde{\mathfrak{a}}_{\omega_0}}, V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}).$$

Here the sum is over all weights $\tilde{\mu} \in P_+^{\tilde{\mathfrak{a}}_{\omega_0}}$ such that $\langle \tilde{\mu}, \tilde{\mu} \rangle = -2i$. Let us prove that (6.17) maps $[\mathcal{O}(\Gamma_n)]_{i,j}$ onto $\Theta'_{n,i}$. The proof of Case 2 implies that (6.17) maps $[\mathcal{O}(\Gamma_n)]_{0,j}$ onto $\Theta'_{n,0}$. By (6.9) we have

$$U^-(\widehat{\mathfrak{gl}}_m)_i \left([\mathcal{O}(\Gamma_n)]_{0,j}\right) = [\mathcal{O}(\Gamma_n)]_{i,j}.$$

By (4.28) we have also

$$U^-(\widehat{\mathfrak{gl}}_m)_i \left(\Theta'_{n,0}\right) \subset \Theta'_{n,i},$$

because the actions of $\widehat{\mathfrak{gl}}_m$ and $\hat{\mathfrak{a}}_{\omega_0}$ commute with each other. Therefore, we have

$$[\mathcal{O}(\Gamma_n)]_{i,j} \subset \Theta'_{n,i}.$$

On the other hand, the proof of the first case implies that

$$[\mathcal{O}(\Gamma_n)]_{\bullet,j} = \bigoplus_{i \geqslant 0} \Theta'_{n,i}.$$

Thus we have the equality $[\mathcal{O}(\Gamma_n)]_{i,j} = \Theta'_{n,i}$.

Proof of Lemma 6.11. A direct computation shows that

$$\frac{1}{2}\langle\gamma,\gamma\rangle = \frac{1}{2}\sum_{p=1}^{\ell} s_p^2.$$

Now, consider the partition $\lambda^c = (\lambda_1, \dots, \lambda_{k\ell})$. We choose k to be large enough such that $\lambda_{k\ell} = 0$. Write

$$\lambda_i - i + 1 = (a_i - 1)\ell + b_i, \quad 1 \leqslant b_i \leqslant \ell,$$

$$i - 1 = a_i'\ell + b_i', \quad 0 \leqslant b_i' \leqslant \ell - 1.$$

The number of 0-nodes in the i-th row of the Young diagram associated with λ^c is equal to $a_i + a_i'$. So

$$n_0(\lambda^c) = \sum_{i=1}^{k\ell}(a_i + a_i').$$

We have

$$\sum_{i=1}^{k\ell} a_i' = \frac{-k(-k+1)\ell}{2}.$$

By the definition of the bijection τ,

$$\begin{aligned}\sum_{i=1}^{k\ell} a_i &= \sum_{p=1}^{\ell}\big((-k+1) + (-k+2) + \cdots + s_p\big)\\ &= \frac{1}{2}\sum_{p=1}^{\ell} s_p^2 - \frac{-k(-k+1)\ell}{2}.\end{aligned}$$

This proves part (a). For part (b), note that (a) and (4.27) yield

$$D(|0,s\rangle) = -\frac{1}{2}\langle\gamma,\gamma\rangle|0,s\rangle$$

Further $|0,s\rangle$ is a weight vector for the level one representation of $\widehat{\mathfrak{gl}}_\ell$ with the weight $\omega_0 - \gamma$, see [30, (28)]. Thus $|0,s\rangle$ is a weight vector for the level one representation of $\widetilde{\mathfrak{gl}}_\ell$ with the weight

$$\xi_\gamma^{-1}(\omega_0) = \omega_0 - \gamma - \frac{1}{2}\langle\gamma,\gamma\rangle\delta.$$

The latter is an extremal weight, see Section 4.3. □

Proof of Lemma 6.12. The set of all dominant integral weights of $\widehat{\mathfrak{sl}}_\ell$ is

$$\begin{aligned}\big\{\hat\gamma(\lambda,m)\,;\,\lambda\in A(\ell,m)\big\} &= \Big\{(m-\lambda_1+\lambda_\ell)\,\omega_0 + \sum_{p=1}^{\ell-1}(\lambda_p-\lambda_{p+1})\,\omega_p\,;\,\lambda\in A(\ell,m)\Big\}\\ &= \Big\{m\,\omega_0 + \sum_{p=1}^{\ell-1}(\lambda_p-\lambda_{p+1})\,(\omega_p-\omega_0)\,;\,\lambda\in A(\ell,m)\Big\}.\end{aligned}$$

Set $\beta = \sum_{p=1}^{\ell-1}(\lambda_p - \lambda_{p+1})\,(\omega_p - \omega_0)$ with $\lambda \in A(\ell,m)$. Identifying $\omega_p - \omega_0$ with the ℓ-tuple (6.12), a short computation shows that $\beta \in Q^{\mathfrak{sl}_\ell}$ if and only if $\lambda \in A(\ell,m)_0$. □

Proof of Lemma 6.13. Fix a weight μ in $P_+^{\hat{\mathfrak{a}}_{\omega_0}} \cap \mathrm{Wt}(V_{\omega_0}^{\widehat{\mathfrak{gl}}_\ell})$. Fix a non zero element $v \in V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ of weight $\tilde\mu = \mu - \frac{1}{2}\langle\mu,\mu\rangle\delta$. We must prove that v is $\tilde{\mathfrak{a}}_{\omega_0}$-primitive. The

argument is taken from [9, sec. 6.2]. By Remark 6.9 it is enough to prove that $\tilde{\mu}+\nu$ is not a weight of $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ for any element ν in the set

$$\{\alpha_{p,p+1},\ \tilde{\mu}-\alpha_{1,\ell}+m\delta\,;\, p=1,2,\dots,\ell-1\}.$$

In fact, since $\tilde{\mu}\in P_+^{\tilde{\mathfrak{a}}}$, for such a ν we have

$$\langle\tilde{\mu}+\nu,\tilde{\mu}+\nu\rangle=\langle\nu,\nu\rangle+2\langle\tilde{\mu},\nu\rangle=2+2\langle\tilde{\mu},\nu\rangle>0.$$

Therefore $\tilde{\mu}+\nu$ is not a weight of $V_{\omega_0}^{\widetilde{\mathfrak{gl}}_\ell}$ by Section 4.3. □

Remark 6.14. Assume that the parameters h, h_p are as in (3.9). Since γ belongs to $T_\ell(K)$, it acts on any integrable $\widehat{\mathfrak{sl}}_\ell$-module. Let 0^ℓ denote the trivial ℓ-charge. The γ-action on the representation of $\widehat{\mathfrak{sl}}_\ell$ on $\mathcal{F}_{m,\ell}$ of level 1 takes $\mathcal{F}_{m,\ell}^{(0^\ell)}$ onto $\mathcal{F}_{m,\ell}^{(s)}$. Indeed, since γ is a cocharacter of T_ℓ the formula (4.5) yields the following equality

$$\begin{aligned}\gamma(\mathcal{F}_{m,\ell}^{(0^\ell)})&=\gamma(\mathcal{F}_{m,\ell}^{(0)}[m\omega_0])\\&=\gamma(V_{\omega_0}^{\widehat{\mathfrak{gl}}_\ell}[\omega_0])\\&=V_{\omega_0}^{\widehat{\mathfrak{gl}}_\ell}[\xi_\gamma^{-1}(\omega_0)]\\&=\mathcal{F}_{m,\ell}^{(0)}[\xi_\gamma^{-1}(\omega_0)'].\end{aligned}$$

Here the upper script $'$ is as in (6.16). Therefore, by Section 4.6 we are reduced to check the following identity

$$\xi_\gamma^{-1}(\omega_0)=\omega_0+\frac{1}{h}\sum_{p=1}^{\ell-1}h_p\,(\omega_p-\omega_0).$$

Recall that $\gamma=-\frac{1}{h}\sum_{p=1}^{\ell-1}h_p\,(\omega_p-\omega_0)$. Thus the claim follows from the formula (4.4) for the $\widehat{\mathfrak{S}}_\ell$-action on $\mathfrak{t}_\ell^*\oplus\mathbb{C}\omega_0\oplus\mathbb{C}\delta$.

Appendix A. Reminder on Hecke algebras

A.1. **Affine Hecke algebras.** The affine Hecke algebra of type GL_n with parameter $\zeta\in\mathbb{C}^\times$ is the $\mathbb{C}$-algebra $\hat{\mathbf{H}}_\zeta(n)$ generated by the symbols $X_1,X_2,\dots,X_n$, $T_1,T_2,\dots,T_{n-1}$ modulo the defining relations

$$\begin{gathered}X_iX_j=X_jX_i,\quad 1\leqslant i,j\leqslant n,\\T_iX_j=X_jT_i,\quad j\neq i,i+1,\\T_iX_iT_i=\zeta X_{i+1},\quad 1\leqslant i\leqslant n-1,\\(T_i+1)(T_i-\zeta)=0,\quad 1\leqslant i\leqslant n-1,\\T_iT_j=T_jT_i,\quad |i-j|>2,\\T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\quad 1\leqslant i\leqslant n-2.\end{gathered}$$

For $I\subset\{1,2,\dots,n-1\}$ let $\hat{\mathbf{H}}_\zeta(I)\subset\hat{\mathbf{H}}_\zeta(n)$ be the corresponding parabolic subalgebra. It is generated by the elements T_i, X_j with $i\in I$, $j=1,2,\dots,n$. For a reduced expression $w=s_{i_1}s_{i_2}\cdots s_{i_k}$ of an element $w\in\mathfrak{S}_n$ we write $T_w=T_{i_1}T_{i_2}\cdots T_{i_k}$. We abbreviate $T_{ij}=T_{s_{ij}}$. Let D_I be the set of minimal length representatives of the left cosets in $\mathfrak{S}_n/\mathfrak{S}_I$. We'll abbreviate $D_{I,J}=D_I^{-1}\cap D_J$. For $x\in D_{I,J}$ the map

$$\mathfrak{S}_{I\cap xJ}\to\mathfrak{S}_{x^{-1}I\cap J},\quad w\mapsto x^{-1}wx$$

defines a length preserving homomorphism. Hence there is a $\mathbb{C}$-algebra isomorphism

$$\hat{\mathbf{H}}_\zeta(I \cap xJ) \to \hat{\mathbf{H}}_\zeta(x^{-1}I \cap J), \quad T_w \mapsto T_{x^{-1}wx}, \quad X_j \mapsto X_{x^{-1}(j)}.$$

Let

$$\mathrm{Rep}(\hat{\mathbf{H}}_\zeta(x^{-1}I \cap J)) \to \mathrm{Rep}(\hat{\mathbf{H}}_\zeta(I \cap xJ)), \quad M \mapsto {}^xM$$

be the corresponding twist functor. The following is well-known.

Lemma A.1 (Affine Mackey theorem)**.** *Let $M \in \mathrm{Rep}(\hat{\mathbf{H}}_\zeta(J))$. The module*

$$\mathrm{Res}^{\hat{\mathbf{H}}_\zeta(n)}_{\hat{\mathbf{H}}_\zeta(I)} \mathrm{Ind}^{\hat{\mathbf{H}}_\zeta(n)}_{\hat{\mathbf{H}}_\zeta(J)}(M)$$

admits a filtration with subquotients isomorphic to

$$\mathrm{Ind}^{\hat{\mathbf{H}}_\zeta(I)}_{\hat{\mathbf{H}}_\zeta(I\cap xJ)} {}^x\mathrm{Res}^{\hat{\mathbf{H}}_\zeta(J)}_{\hat{\mathbf{H}}_\zeta(x^{-1}I\cap J)}(M),$$

one for each $x \in D_{I,J}$. The subquotients are taken in any order refining the Bruhat order on $D_{I,J}$. In particular we have the inclusion

$$\mathrm{Ind}^{\hat{\mathbf{H}}_\zeta(I)}_{\hat{\mathbf{H}}_\zeta(I\cap J)} \mathrm{Res}^{\hat{\mathbf{H}}_\zeta(J)}_{\hat{\mathbf{H}}_\zeta(I\cap J)}(M) \subset \mathrm{Res}^{\hat{\mathbf{H}}_\zeta(n)}_{\hat{\mathbf{H}}_\zeta(I)} \mathrm{Ind}^{\hat{\mathbf{H}}_\zeta(n)}_{\hat{\mathbf{H}}_\zeta(J)}(M).$$

A.2. **Cyclotomic Hecke algebras.** The cyclotomic Hecke algebra $\mathbf{H}_\zeta(n,\ell)$ associated with Γ_n and the parameters $\zeta, v_1, v_2, \dots, v_\ell \in \mathbb{C}^\times$ is the quotient of $\hat{\mathbf{H}}_\zeta(n)$ by the two-sided ideal generated by the element

$$(X_1 - v_1)(X_1 - v_2)\dots(X_1 - v_\ell).$$

We'll denote the image of the generator X_1 in $\mathbf{H}_\zeta(n,\ell)$ by the symbol T_0. For a subset $I \subset \{0,1,\dots,n-1\}$ we define $\Gamma_I \subset \Gamma_n$ as the subgroup $\mathfrak{S}_I$ if $0 \notin I$, or as the subgroup generated by $\mathfrak{S}_{I\setminus\{0\}}$ and $\{\gamma_1; \gamma \in \Gamma\}$ else. This yields all parabolic subgroup of Γ_n. We consider also the parabolic subalgebra $\mathbf{H}_\zeta(I,\ell) \subset \mathbf{H}_\zeta(n,\ell)$ which is the subalgebra generated by the elements T_i with $i \in I$. To unburden the notation, we abbreviate

$$\mathbf{H}(\Gamma_n) = \mathbf{H}_\zeta(n,\ell), \quad \mathbf{H}(\mathfrak{S}_m) = \mathbf{H}_\zeta(m), \quad \mathbf{H}(\Gamma_I) = \mathbf{H}_\zeta(I,\ell).$$

For $r > 0$ and $I = \{0,1,\dots,n+mr-1\} \setminus \{n\}$ we write also

$$\mathbf{H}(\Gamma_{n,mr}) = \mathbf{H}(\Gamma_I).$$

A.3. **Induction/restriction for cyclotomic Hecke algebras.** We'll abbreviate

$$\begin{aligned}
{}^{\mathbf{H}}\mathrm{Ind}_n &= \mathrm{Ind}^{\mathbf{H}(\Gamma_n)}_{\mathbf{H}(\Gamma_{n-1})}, & {}^{\mathbf{H}}\mathrm{Res}_n &= \mathrm{Res}^{\mathbf{H}(\Gamma_n)}_{\mathbf{H}(\Gamma_{n-1})}, \\
{}^{\mathbf{H}}\mathrm{Ind}_{n,(m^r)} &= \mathrm{Ind}^{\mathbf{H}(\Gamma_{n+mr})}_{\mathbf{H}(\Gamma_{n,(m^r)})}, & {}^{\mathbf{H}}\mathrm{Res}_{n,(m^r)} &= \mathrm{Res}^{\mathbf{H}(\Gamma_{n+mr})}_{\mathbf{H}(\Gamma_{n,(m^r)})}, \\
{}^{\mathbf{H}}\mathrm{Ind}_{n,mr} &= \mathrm{Ind}^{\mathbf{H}(\Gamma_{n+mr})}_{\mathbf{H}(\Gamma_{n,mr})}, & {}^{\mathbf{H}}\mathrm{Res}_{n,mr} &= \mathrm{Res}^{\mathbf{H}(\Gamma_{n+mr})}_{\mathbf{H}(\Gamma_{n,mr})}.
\end{aligned} \tag{A.1}$$

We write also

$$\begin{aligned}
{}^{\mathbf{H}}\mathrm{Ind}_{(m^r)} &= {}^{\mathbf{H}}\mathrm{Ind}^{\mathfrak{S}_{mr}}_{\mathfrak{S}_m^r} : \mathrm{Rep}(\mathbf{H}(\mathfrak{S}_m^r)) \to \mathrm{Rep}(\mathbf{H}(\mathfrak{S}_{mr})), \\
{}^{\mathbf{H}}\mathrm{Res}_{(m^r)} &= {}^{\mathbf{H}}\mathrm{Res}^{\mathfrak{S}_{mr}}_{\mathfrak{S}_m^r} : \mathrm{Rep}(\mathbf{H}(\mathfrak{S}_{mr})) \to \mathrm{Rep}(\mathbf{H}(\mathfrak{S}_m^r)).
\end{aligned}$$

Now, we consider the Mackey decomposition of the functor

$${}^{\mathbf{H}}\mathrm{Res}_{n+m} \circ {}^{\mathbf{H}}\mathrm{Ind}_{n,m} : \mathrm{Rep}(\mathbf{H}(\Gamma_{n,m})) \to \mathrm{Rep}(\mathbf{H}(\Gamma_{n+m-1})).$$

A short computation shows that a set of representatives of the double cosets in

$$\Gamma_{n+m-1} \setminus \Gamma_{n+m} / \Gamma_{n,m}$$

is $\{\gamma_{n+m}, s_{n,n+m}\,;\, \gamma \in \Gamma\}$. For

$$I = \{0,\dots,n+m-1\} \setminus \{n-1,n\}, \quad J = \{0,\dots,n+m-2\} \setminus \{n-1\}$$

we have

$$\mathbf{H}(\Gamma_I) \subset \mathbf{H}(\Gamma_{n,m}), \quad \mathbf{H}(\Gamma_J) = \mathbf{H}(\Gamma_{n-1,m}) \subset \mathbf{H}(\Gamma_{n+m-1}).$$

Further, there is an algebra isomorphism

$$\varphi:\ \mathbf{H}(\Gamma_J) \to \mathbf{H}(\Gamma_I),\ T_w \mapsto T_{sws^{-1}},\ X_i \mapsto X_{si},$$

where $s = s_n s_{n+1} \cdots s_{n+m-1}$. For each i, p we write $X_i^p = (X_i)^p$. We have the following decomposition. It is well known in the case $m = 1$, see e.g., [19, lem. 5.6.1] in the degenerate case.

Proposition A.2. *(a) We have an isomorphism of $\mathbf{H}(\Gamma_{n+m-1})$-modules*

$$\mathbf{H}(\Gamma_{n+m}) = \bigoplus_{0\leqslant p<\ell} \bigoplus_{1\leqslant j\leqslant n+m} \mathbf{H}(\Gamma_{n+m-1})\, T_{j,n+m}\, X_j^p.$$

(b) We have an isomorphism of $\big(\mathbf{H}(\Gamma_{n+m-1}), \mathbf{H}(\Gamma_{n,m})\big)$-bimodules

$$\mathbf{H}(\Gamma_{n+m}) = \mathbf{H}(\Gamma_{n+m-1})\, T_{n,n+m}\, \mathbf{H}(\Gamma_{n,m}) \oplus \bigoplus_{0\leqslant p<\ell} \mathbf{H}(\Gamma_{n+m-1}) X_{n+m}^p \mathbf{H}(\Gamma_{n,m}).$$

(c) There are isomorphisms of $\big(\mathbf{H}(\Gamma_{n+m-1}), \mathbf{H}(\Gamma_{n,m})\big)$-bimodules

$$\mathbf{H}(\Gamma_{n+m-1})\, T_{n,n+m} \mathbf{H}(\Gamma_{n,m}) = \mathbf{H}(\Gamma_{n+m-1}) \otimes_{\mathbf{H}(\Gamma_{n-1,m})} \mathbf{H}(\Gamma_{n,m}),$$
$$\mathbf{H}(\Gamma_{n+m-1})\, X_{n+m}^p \mathbf{H}(\Gamma_{n,m}) = \mathbf{H}(\Gamma_{n+m-1}) \otimes_{\mathbf{H}(\Gamma_{n,m-1})} \mathbf{H}(\Gamma_{n,m}),$$

where the algebra homomorphism $\mathbf{H}(\Gamma_{n-1,m}) \to \mathbf{H}(\Gamma_{n,m})$ is given by φ.

Proof. Part (a) is standard, see e.g., [19, lem. 5.6.1] in the degenerate case. Let us concentrate on (b). Write $t_{j,i} = T_j T_{j-1} \cdots T_i$ for $1 \leqslant i \leqslant j$, and $t_{j,i} = 1$ for $i > j$. By (a) we are reduced to prove the following identities

$$\bigoplus_{0\leqslant p<\ell} \bigoplus_{1\leqslant j\leqslant n} \mathbf{H}(\Gamma_{n+m-1})\, t_{n+m-1,j} X_j^p = \mathbf{H}(\Gamma_{n+m-1})\, t_{n+m-1,n}\, \mathbf{H}(\Gamma_{n,m}), \tag{A.2}$$

$$\bigoplus_{0\leqslant p<\ell} \bigoplus_{n<j\leqslant n+m} \mathbf{H}(\Gamma_{n+m-1})\, t_{n+m-1,j} X_j^p = \bigoplus_{0\leqslant p<\ell} \mathbf{H}(\Gamma_{n+m-1})\, X_{n+m}^p\, \mathbf{H}(\Gamma_{n,m}). \tag{A.3}$$

We have

$$u\, t_{n+m-1,n} = t_{n+m-1,n}\, \varphi(u), \quad u \in \mathbf{H}(\Gamma_{n-1,m}), \tag{A.4}$$

because for $i = 1, 2, \ldots, n-1$ and $j \in J \setminus \{0\}$ we have

$$T_j\, t_{n+m-1,n} = t_{n+m-1,n} T_{s(j)} = t_{n+m-1,n} \varphi(T_j),$$
$$X_i\, t_{n+m-1,n} = t_{n+m-1,n} X_i = t_{n+m-1,n} \varphi(X_i).$$

Hence, by (a) the right hand side of (A.2) is

$$= \bigoplus_{0\leqslant p<\ell} \bigoplus_{1\leqslant j\leqslant n} \mathbf{H}(\Gamma_{n+m-1})\, t_{n+m-1,n}\, \mathbf{H}(\Gamma_I)\, t_{n-1,j}\, X_j^p,$$
$$= \bigoplus_{0\leqslant p<\ell} \bigoplus_{1\leqslant j\leqslant n} \mathbf{H}(\Gamma_{n+m-1})\, \mathbf{H}(\Gamma_{n-1,m})\, t_{n+m-1,n}\, t_{n-1,j}\, X_j^p,$$
$$= \bigoplus_{0\leqslant p<\ell} \bigoplus_{1\leqslant j\leqslant n} \mathbf{H}(\Gamma_{n+m-1})\, t_{n+m-1,j}\, X_j^p.$$

This proves the first identity. Next, a short calculation involving the relation

$$X_{j+1}^p T_j - T_j X_j^p \in \mathbb{C}[X_j, X_{j+1}]$$

proves that the sum

$$\sum_{0\leqslant p<\ell} \sum_{n<j\leqslant n+m} \mathbf{H}(\Gamma_{n+m-1})\, t_{n+m-1,j} X_j^p$$

is indeed a direct sum, i.e., it is equal to the left hand side of (A.3). Thus the identity (A.3) follows from the following equalities

$$\begin{aligned}\mathbf{H}(\Gamma_{n+m-1})\, X^p_{n+m}\, \mathbf{H}(\Gamma_{n,m}) &= \sum_{n<j\leqslant n+m} \mathbf{H}(\Gamma_{n+m-1})\, X^p_{n+m}\, T_{j,n+m} \\ &= \sum_{n<j\leqslant n+m} \mathbf{H}(\Gamma_{n+m-1})\, X^p_{n+m}\, t_{n+m-1,j} \\ &= \sum_{n<j\leqslant n+m} \mathbf{H}(\Gamma_{n+m-1})\, t_{n+m-1,j} X^p_j.\end{aligned}$$

Finally, let us prove (c). The second claim is obvious because

$$\mathbf{H}(\Gamma_{n+m-1})X^p_{n+m}\mathbf{H}(\Gamma_{n,m}) = X^p_{n+m}\mathbf{H}(\Gamma_{n+m-1})\mathbf{H}(\Gamma_{n,m}) = \mathbf{H}(\Gamma_{n+m-1})\mathbf{H}(\Gamma_{n,m})$$

as $\big(\mathbf{H}(\Gamma_{n+m-1}), \mathbf{H}(\Gamma_{n,m})\big)$-bimodules. For the first one we define a map

$$\begin{aligned}\mathbf{H}(\Gamma_{n+m-1}) \times \mathbf{H}(\Gamma_{n,m}) &\to \mathbf{H}(\Gamma_{n+m-1})\, T_{n,n+m}\, \mathbf{H}(\Gamma_{n,m}), \\ (u, v) &\mapsto u\, t_{n+m-1,n} v.\end{aligned}$$

By (A.4) it factors to a surjective homomorphism

$$\psi:\ \mathbf{H}(\Gamma_{n+m-1}) \otimes_{\mathbf{H}(\Gamma_{n-1,m})} \mathbf{H}(\Gamma_{n,m}) \to \mathbf{H}(\Gamma_{n+m-1})\, T_{n,n+m}\, \mathbf{H}(\Gamma_{n,m}).$$

By (a) the left hand side is a free $\mathbf{H}(\Gamma_{n+m-1})$-module on basis

$$1 \otimes t_{n-1,j} X^p_j, \quad 1 \leqslant j \leqslant n, \quad 0 \leqslant p < \ell.$$

But ψ maps these elements to

$$t_{n+m-1,j} X^p_j, \quad 1 \leqslant j \leqslant n, \quad 0 \leqslant p < \ell.$$

Further, the latter form a $\mathbf{H}(\Gamma_{n+m-1})$-basis of the right hand side by (a) again. We are done. □

Appendix B. Reminder on ζ-Schur algebras

B.1. **The quantized modified algebra.** Let v be a formal variable. The *quantized modified algebra* $\dot{\mathbf{U}}(n)$ *of* $\mathfrak{gl}_n$ is the associative $\mathbb{Q}(v)$-algebra with generators E_i, F_i where $i = 1, \dots, n-1$ and 1_λ where $\lambda \in \mathbb{Z}^n$, with the defining relations [23, sec. 23]

- $1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda$,
- $E_i F_j - F_j E_i = \delta_{ij} \sum_\lambda [\lambda_i - \lambda_{i+1}] 1_\lambda$,
- $E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i$,
- $1_\lambda F_i = F_i 1_{\lambda+\alpha_i}$,
- $E_i E_j = E_j E_i$ if $i \neq j \pm 1$, $E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ else,
- $F_i F_j = F_j F_i$ if $i \neq j \pm 1$, $F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ else,

where $[m]$ is the usual v-analogue of m for any $m \in \mathbb{N}$. The comultiplication of $\dot{\mathbf{U}}(n)$ is the $\mathbb{Q}(v)$-algebra homomorphism

$$\Delta : \dot{\mathbf{U}}(n) \to \prod_{\lambda,\lambda'} \big(\dot{\mathbf{U}}(n) 1_\lambda \otimes \dot{\mathbf{U}}(n) 1_{\lambda'}\big)$$

given by

- $\Delta(1_\lambda) = \prod_{\lambda=\lambda'+\lambda''} 1_{\lambda'} \otimes 1_{\lambda''}$,
- $\Delta(E_i 1_\lambda) = \prod_{\lambda=\lambda'+\lambda''} (E_i 1_{\lambda'} \otimes 1_{\lambda''} + v^{(\alpha_i, \lambda')} 1_{\lambda'} \otimes E_i 1_{\lambda''})$,
- $\Delta(F_i 1_\lambda) = \prod_{\lambda=\lambda'+\lambda''} (F_i 1_{\lambda'} \otimes v^{-(\alpha_i, \lambda'')} 1_{\lambda''} + 1_{\lambda'} \otimes F_i 1_{\lambda''})$.

Set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. The *integral quantized modified algebra* is the $\mathcal{A}$-subalgebra $\dot{\mathbf{U}}_{\mathcal{A}}(n) \subset \dot{\mathbf{U}}(n)$ generated by the 1_λ's and all quantum divided powers $E_i^{(d)}$, $F_i^{(d)}$. The comultiplication yieds an $\mathcal{A}$-algebra homomorphism $\dot{\mathbf{U}}_{\mathcal{A}} \to \dot{\mathbf{U}}_{\mathcal{A}} \otimes_{\mathcal{A}} \dot{\mathbf{U}}_{\mathcal{A}}$. For $\epsilon \in \mathbb{C}^\times$ we consider the $\mathbb{C}$-algebra

$$\dot{\mathbf{U}}_\epsilon(n) = \dot{\mathbf{U}}_{\mathcal{A}}(n) \otimes_{\mathcal{A}} \mathbb{C}[v, v^{-1}]/(v - \epsilon).$$

For V, $V' \in \mathrm{Rep}(\dot{\mathbf{U}}_\epsilon(n))$ let $\mathbf{s}_{V,V'} : V \otimes V' \to V' \otimes V$ be the permutation $v \otimes v' \mapsto v' \otimes v$. The *R-matrix* is a $\mathbb{C}$-linear endomorphism $R_{V,V'}$ of $V \otimes V'$ such that the composed map

$$\mathcal{R}_{V,V'} = \mathbf{s}_{V,V'} \circ R_{V,V'}$$

is an isomorphism of $\dot{\mathbf{U}}_\epsilon(n)$-modules $V \otimes V' \to V' \otimes V$. The map $R_{V,V'}$ decomposes in the following form

$$R_{V,V'}(v \otimes v') = R(v \otimes v'), \quad R = \bar{\Pi}\bar{\Theta},$$
$$\bar{\Pi} = \prod_{\lambda,\lambda'} v^{-(\lambda,\lambda')}\, 1_\lambda \otimes 1_{\lambda'}, \quad \bar{\Theta} \in \prod_{\lambda,\lambda'} \big(\dot{\mathbf{U}}_\epsilon(n) 1_\lambda \otimes \dot{\mathbf{U}}_\epsilon(n) 1_{\lambda'}\big).$$

The notation is chosen to agree with [23, sec. 32]. We call R the *universal R-matrix*. To avoid confusions we may write R_ϵ for R. We'll write $\mathcal{R}_{V,V'}$ again for the braiding of right $\dot{\mathbf{U}}_\epsilon(n)$-modules V, V'. If ϵ is a primitive $2d$-th root of 1 then we have $\epsilon^{d^2} = (-1)^d$. Hence the *quantum Frobenius homomorphism* [23, sec. 35.1] is the unique $\mathbb{C}$-algebra homomorphism

$$\mathrm{Fr} : \dot{\mathbf{U}}_\epsilon(n) \to \dot{\mathbf{U}}_{(-1)^d}(n)$$

such that

- $\mathrm{Fr}(E_i^{(m)} 1_\lambda) = E_i^{(m/d)} 1_{\lambda/d}$ if $m \in d\mathbb{Z}$ and $\lambda \in d\mathbb{Z}^n$, and 0 otherwise,
- $\mathrm{Fr}(F_i^{(m)} 1_\lambda) = F_i^{(m/d)} 1_{\lambda/d}$ if $m \in d\mathbb{Z}$ and $\lambda \in d\mathbb{Z}^n$, and 0 otherwise.

The formulas in [23, sec. 3.1.5] imply that

$$\Delta \circ \mathrm{Fr} = \mathrm{Fr} \circ \Delta.$$

Proposition B.1. *We have* $(\mathrm{Fr} \otimes \mathrm{Fr})(R_\epsilon) = R_{(-1)^d} = \prod_{\lambda,\lambda'} (-1)^{d(\lambda,\lambda')} (1_\lambda \otimes 1_{\lambda'})$.

Proof. To avoid confusions we'll write $\bar{\Theta}_\epsilon$, $\bar{\Pi}_\epsilon$ for $\bar{\Theta}$, $\bar{\Pi}$. If $n = 2$ the proposition follows from the formula [23, sec. 4.1.4]. More precisely, since

$$\bar{\Theta}_\epsilon = \prod_{\lambda,\lambda'} \sum_{k \geqslant 0} (-1)^k \epsilon^{-k(k-1)/2} \{k\}_\epsilon F^{(k)} 1_\lambda \otimes E^{(k)} 1_{\lambda'}, \quad \{k\}_\epsilon = \prod_{i=1}^{k} (\epsilon^i - \epsilon^{-i}),$$

we have the following formula

$$(\mathrm{Fr} \otimes \mathrm{Fr})(\bar{\Theta}_\epsilon) = \prod_{\lambda,\lambda'} (1_\lambda \otimes 1_{\lambda'}). \tag{B.1}$$

Further, in $\dot{\mathbf{U}}_{(-1)^d}(n) \otimes \dot{\mathbf{U}}_{(-1)^d}(n)$ we have also

$$(\mathrm{Fr} \otimes \mathrm{Fr})(\bar{\Pi}_\epsilon) = \prod_{\lambda,\lambda'} (-1)^{d(\lambda,\lambda')} (1_\lambda \otimes 1_{\lambda'}), \tag{B.2}$$

and

$$\bar{\Theta}_{(-1)^d} = \prod_{\lambda,\lambda'} (1_\lambda \otimes 1_{\lambda'}), \quad \bar{\Pi}_{(-1)^d} = \prod_{\lambda,\lambda'} (-1)^{d(\lambda,\lambda')} (1_\lambda \otimes 1_{\lambda'}). \tag{B.3}$$

This proves the formula for $n = 2$. Now, let n be any integer $\geqslant 2$. The braid group of $\mathfrak{S}_n$ acts on $\dot{\mathbf{U}}_\epsilon(n)$ via the operators $T''_{1,1}, T''_{2,1}, \ldots, T''_{n-1,1}$ in [23, sec. 41]. For $i = 1, 2, \ldots, n-1$ we set

$$S_i = T''_{i,1} \otimes T''_{i,1}, \quad \bar{\theta}_{i,\epsilon} = \sum_{k \geqslant 0} (-1)^k \epsilon^{-k(k-1)/2} \{k\}_\epsilon F_i^{(k)} \otimes E_i^{(k)}.$$

For a reduced decomposition $s_{i_1}s_{i_2}\cdots s_{i_r}$ of the longuest element in $\mathfrak{S}_n$, the universal R-matrix is given by the following formula, see [14, thm. 3],

$$\bar{\Theta}_\epsilon = \prod_{\lambda,\lambda'} \bar{\theta}_\epsilon(1_\lambda \otimes 1_{\lambda'}), \quad \bar{\theta}_\epsilon = S_{i_r}^{-1}\cdots S_{i_3}^{-1}S_{i_2}^{-1}(\bar{\theta}_{i_1,\epsilon})\cdots S_{i_r}^{-1}(\bar{\theta}_{i_{r-1},\epsilon})\bar{\theta}_{i_r,\epsilon}.$$

Thus (B.3) yields

$$\bar{\Theta}_{(-1)^d} = \prod_{\lambda,\lambda'}(1_\lambda \otimes 1_{\lambda'}),$$

Since the braid group action is compatible with the quantum Frobenius homomorphism, see [23, sec. 41.1.9], by (B.1) we have also

$$(\mathrm{Fr}\otimes\mathrm{Fr})(\bar{\Theta}_\epsilon) = \prod_{\lambda,\lambda'}(1_\lambda \otimes 1_{\lambda'}).$$

Finally, a direct computation yields

$$(\mathrm{Fr}\otimes\mathrm{Fr})(\bar{\Pi}_\epsilon) = \prod_{\lambda,\lambda'}(-1)^{d(\lambda,\lambda')}(1_\lambda \otimes 1_{\lambda'}) = \bar{\Pi}_{(-1)^d}.$$

This proves the proposition. □

Remark B.2. It is proved in [23, prop. 33.2.3] that the assignment

$$E_i 1_\lambda \mapsto (-1)^{id(\lambda_i-\lambda_{i+1})}E_i 1_\lambda, \quad F_i 1_\lambda \mapsto (-1)^{(i+1)d(\lambda_i-\lambda_{i+1})+d}E_i 1_\lambda, \quad 1_\lambda \mapsto 1_\lambda$$

yields a $\mathbb{C}$-algebra isomorphism $\dot{\mathbf{U}}_1(n) \to \dot{\mathbf{U}}_{(-1)^d}(n)$. Thus we can regard Fr as a map $\dot{\mathbf{U}}_\epsilon(n) \to \dot{\mathbf{U}}_1(n)$. Note that the isomorphism above does not commute with the comultiplication.

B.2. **The ζ-Schur algebra.** The v-Schur algebra $\mathbf{S}(n,m)$ is the associative $\mathbb{Q}(v)$-algebra with 1 generated by E_i, F_i where $i = 1,\dots,n-1$ and by 1_λ where $\lambda \in \Lambda(n,m)$, modulo the defining relations [8, thm. 2.4]

- $1_\lambda 1_\mu = \delta_{\lambda,\mu}1_\lambda$, $\sum_\lambda 1_\lambda = 1$,
- $E_iF_j - F_jE_i = \delta_{ij}\sum_\lambda[\lambda_i - \lambda_{i+1}]1_\lambda$,
- $E_i1_\lambda = 1_{\lambda+\alpha_i}E_i$ if $\lambda+\alpha_i \in \Lambda(n,m)$, 0 else,
- $1_\lambda E_i = E_i 1_{\lambda-\alpha_i}$ if $\lambda-\alpha_i \in \Lambda(n,m)$, 0 else,
- $F_i1_\lambda = 1_{\lambda-\alpha_i}F_i$ if $\lambda-\alpha_i \in \Lambda(n,m)$, 0 else,
- $1_\lambda F_i = F_i 1_{\lambda+\alpha_i}$ if $\lambda+\alpha_i \in \Lambda(n,m)$, 0 else,
- $E_iE_j = E_jE_i$ if $i \neq j \pm 1$, $E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ else,
- $F_iF_j = F_jF_i$ if $i \neq j \pm 1$, $F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0$ else.

The integral v-Schur algebra is the $\mathcal{A}$-subalgebra $\mathbf{S}_\mathcal{A}(n,m) \subset \mathbf{S}(n,m)$ generated by the 1_λ's and all quantum divided powers $E_i^{(d)}$, $F_i^{(d)}$. In other words, we have a canonical isomorphism

$$\mathbf{S}_\mathcal{A}(n,m) = 1_m\dot{\mathbf{U}}_\mathcal{A}(n)1_m, \quad 1_m = \sum_{\lambda\in\Lambda(n,m)} 1_\lambda.$$

The comultiplication of $\dot{\mathbf{U}}_\mathcal{A}(n)$ factors through an $\mathcal{A}$-algebra homomorphism

$$\Delta : \mathbf{S}_\mathcal{A}(n,m) \to \bigoplus_{m=m'+m''} \mathbf{S}_\mathcal{A}(n,m') \otimes \mathbf{S}_\mathcal{A}(n,m''). \tag{B.4}$$

For $\zeta, \epsilon \in \mathbb{C}^\times$ with $\zeta = \epsilon^2$ we consider the $\mathbb{C}$-algebra

$$\begin{aligned}\mathbf{S}_\zeta(n,m) &= \mathbf{S}_\mathcal{A}(n,m) \otimes_\mathcal{A} \mathbb{C}[v,v^{-1}]/(v-\epsilon)\\ &= 1_m\dot{\mathbf{U}}_\epsilon(n)1_m.\end{aligned}$$

Indeed $\mathbf{S}_\zeta(n,m)$ depends only on ζ and not on the choice of ϵ. If ζ is a primitive d-th root of 1, we choose ϵ to be a primitive $2d$-th root of 1. Then the *quantum Frobenius homomorphism* $\mathrm{Fr} : \dot{\mathbf{U}}_\epsilon(n) \to \dot{\mathbf{U}}_1(n)$ factors through a $\mathbb{C}$-algebra homomorphism

$$\mathrm{Fr} : \mathbf{S}_\zeta(n, dm) \to \mathbf{S}_1(n,m). \tag{B.5}$$

Note that we have used the identification $\dot{\mathbf{U}}_{(-1)^d}(n) = \dot{\mathbf{U}}_1(n)$ in Remark B.2.

B.3. **The module category of $\mathbf{S}_\zeta(n,m)$.** For $\lambda \in \mathbb{Z}_+^n$ let $\Delta_\lambda^U, L_\lambda^U \in \mathrm{Rep}(\dot{\mathbf{U}}_\epsilon(n))$ denote the Weyl module and the simple module with highest weight λ. Set

$$\Lambda(n,m)_+ = \Lambda(n,m) \cap \mathbb{Z}_+^n.$$

The category $\mathrm{Rep}(\mathbf{S}_\zeta(n,m))$ is equivalent to the full subcategory of $\mathrm{Rep}(\dot{\mathbf{U}}_\epsilon(n))$ consisting of the modules such that all constituents have a highest weight in the set $\Lambda(n,m)_+$. It is quasi-hereditary with respect to the dominance order, the standard objects being the modules Δ_λ^S with $\lambda \in \Lambda(n,m)_+$. Here, for $\lambda \in \Lambda(n,m)_+$, we write

$$\Delta_\lambda^S = \Delta_\lambda^U, \quad L_\lambda^S = L_\lambda^U,$$

regarded as objects in $\mathrm{Rep}(\mathbf{S}_\zeta(n,m))$.

B.4. **The Schur functor.** Assume that $n \geqslant m$. There is a $\mathbb{C}$-algebra isomorphism [8, sec. 11]

$$\mathbf{H}_\zeta(m) = f\,\mathbf{S}_\zeta(n,m)\,f, \quad f = 1_{(1^m 0^{n-m})}.$$

Thus the vector space $\mathbf{T}_\zeta(n,m) = \mathbf{S}_\zeta(n,m)f$ is a $(\mathbf{S}_\zeta(n,m), \mathbf{H}_\zeta(m))$-bimodule, and $\mathbf{V}_\zeta(n,m) = f\mathbf{S}_\zeta(n,m)$ is a $(\mathbf{H}_\zeta(m), \mathbf{S}_\zeta(n,m))$-bimodule. Consider the triple of adjoint functors $(\Phi_!, \Phi^*, \Phi_*)$

$$\begin{aligned}
&\Phi^* : \mathrm{Rep}(\mathbf{S}_\zeta(n,m)) \to \mathrm{Rep}(\mathbf{H}_\zeta(m)), \quad M \mapsto fM,\\
&\Phi_* : \mathrm{Rep}(\mathbf{H}_\zeta(m)) \to \mathrm{Rep}(\mathbf{S}_\zeta(n,m)), \quad N \mapsto \mathrm{Hom}_{\mathbf{H}_\zeta(m)}(\mathbf{V}_\zeta(n,m), N),\\
&\Phi_! : \mathrm{Rep}(\mathbf{H}_\zeta(m)) \to \mathrm{Rep}(\mathbf{S}_\zeta(n,m)), \quad M \mapsto \mathbf{T}_\zeta(n,m) \otimes_{\mathbf{H}_\zeta(m)} M.
\end{aligned}$$

We call Φ^* the *Schur functor*. It is a *quotient functor*, i.e., it is exact and the counit $\Phi^*\,\Phi_* \to 1$ is invertible. The *double centralizer property* holds, i.e., we have

$$\mathbf{S}_\zeta(n,m) = \mathrm{End}_{\mathbf{H}_\zeta(m)}(\mathbf{V}_\zeta(n,m)).$$

Equivalently, the functor Φ^* is fully faithful on projectives, or, equivalently again, the unit $P \to \Phi_*\,\Phi^*(P)$ is invertible whenever P is projective. See [27, prop. 4.33] for details. Since Φ^* is a quotient functor, the functor $\Phi_!$ takes projectives to projectives and the unit $1 \to \Phi^*\Phi_!$ is an isomorphism of functors. For $m = m' + m''$ the comultiplication (B.4) yields a functor

$$\dot{\otimes} : \mathrm{Rep}(\mathbf{S}_\zeta(n,m')) \otimes \mathrm{Rep}(\mathbf{S}_\zeta(n,m'')) \to \mathrm{Rep}(\mathbf{S}_\zeta(n,m)). \tag{B.6}$$

We'll abbreviate ${}^{\mathbf{H}}\mathrm{Ind}_{m',m''} = \mathrm{Ind}_{\mathbf{H}_\zeta(m')\otimes\mathbf{H}_\zeta(m'')}^{\mathbf{H}_\zeta(m)}$.

Proposition B.3. *(a) We have a $(\mathbf{S}_\zeta(n,m), \mathbf{H}_\zeta(m') \otimes \mathbf{H}_\zeta(m''))$-bimodules isomorphism* $\mathrm{can} : \mathbf{T}_\zeta(n,m')\dot{\otimes}\mathbf{T}_\zeta(n,m'') \to \mathbf{T}_\zeta(n,m)$. *For $M' \in \mathrm{Rep}(\mathbf{H}_\zeta(m'))$, $M'' \in \mathrm{Rep}(\mathbf{H}_\zeta(m''))$ the map* can *yields an isomorphism*

$$\mathrm{can} : \Phi_!\big({}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(M' \otimes M'')\big) \to \Phi_!(M')\dot{\otimes}\Phi_!(M'').$$

(b) We have an isomorphism of $(\mathbf{H}_\zeta(m') \otimes \mathbf{H}_\zeta(m''), \mathbf{S}_\zeta(n,m))$-bimodules $\mathrm{can} : \mathbf{V}_\zeta(n,m')\dot{\otimes}\mathbf{V}_\zeta(n,m'') \to \mathbf{V}_\zeta(n,m)$. *For $M' \in \mathrm{Rep}(\mathbf{H}_\zeta(m'))$, $M'' \in \mathrm{Rep}(\mathbf{H}_\zeta(m''))$ the map* can *yields an isomorphism*

$$\mathrm{can} : \Phi_*\big({}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(M' \otimes M'')\big) \to \Phi_*(M')\dot{\otimes}\Phi_*(M'').$$

Proof. By definition $\mathbf{T}_\zeta(n,m)$ is the *v-tensor space* in [6, def. 2.6]. According to [5, sec. 3.3, 4.4] it is identified with the m-th tensor power of the natural representation of the (modified) quantized enveloping algebra of $\mathfrak{gl}_n$, in such a way that the $\mathbf{H}_\zeta(m)$-action comes from the R-matrix, see also [16]. This proves part (a). Part (b) follows also by taking the dual spaces. □

Corollary B.4. *We have an isomorphism*

$$\mathrm{can} : {}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(\Phi^* M' \otimes \Phi^* M'') \to \Phi^*(M' \dot{\otimes} M'')$$

for $M' \in \mathrm{Rep}(\mathbf{S}_\zeta(n,m'))$ *and* $M'' \in \mathrm{Rep}(\mathbf{S}_\zeta(n,m''))$.

Proof. For $M' \in \mathrm{Rep}(\mathbf{S}_\zeta(n,m'))$ and $M'' \in \mathrm{Rep}(\mathbf{S}_\zeta(n,m''))$, Proposition B.3 yields an isomorphism

$$\Phi_* {}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(\Phi^* M' \otimes \Phi^* M'') = \Phi_*\Phi^* M' \dot{\otimes} \Phi_*\Phi^* M''.$$

Composing it with Φ^* we get an isomorphism

$${}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(\Phi^* M' \otimes \Phi^* M'') = \Phi^*\big(\Phi_*\Phi^* M' \dot{\otimes} \Phi_*\Phi^* M''\big).$$

Composing it with the unit $1 \to \Phi_*\Phi^*$ we get a functorial map

$$\Phi^*(M' \dot{\otimes} M'') \to {}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(\Phi^* M' \otimes \Phi^* M'')$$

which is invertible whenever M', M'' are projectives, because the unit is invertible on projective modules. Thus it is always invertible, because Φ^* and ${}^{\mathbf{H}}\mathrm{Ind}_{m',m''}$ are exact and because there are enough projectives in $\mathrm{Rep}(\mathbf{S}_\zeta(n,m))$. □

B.5. **The braiding and the Schur functor.** For $M' \in \mathrm{Rep}(\mathbf{H}_\zeta(m'))$ and $M'' \in \mathrm{Rep}(\mathbf{H}_\zeta(m''))$ the R-matrix yields an isomorphism of $\mathbf{S}_\zeta(n,m)$-modules

$$\mathcal{R}_{\Phi_* M', \Phi_* M''} : \Phi_* M' \dot{\otimes} \Phi_* M'' \to \Phi_* M'' \dot{\otimes} \Phi_* M'.$$

Let $\tau \in \mathfrak{S}_m$ be the unique element such that

- τ is minimal in the coset $(\mathfrak{S}_{m'} \times \mathfrak{S}_{m''})\tau(\mathfrak{S}_{m''} \times \mathfrak{S}_{m'})$,
- we have $\tau^{-1}(\mathfrak{S}_{m'} \times \mathfrak{S}_{m''})\tau = \mathfrak{S}_{m''} \times \mathfrak{S}_{m'}$.

We have the following formula in $\mathbf{H}_\zeta(m)$

$$T_\tau(h'' \otimes h') = (h' \otimes h'')\, T_\tau, \quad h' \in \mathbf{H}_\zeta(m'),\ h'' \in \mathbf{H}_\zeta(m''). \tag{B.7}$$

Thus there is a unique functorial $\mathbf{H}_\zeta(m)$-module isomorphism

$$\mathcal{S}_{M',M''} : {}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(M' \otimes M'') \to {}^{\mathbf{H}}\mathrm{Ind}_{m'',m'}(M'' \otimes M')$$

given by

$$\mathcal{S}_{M',M''}(h \otimes (v' \otimes v'')) = hT_\tau \otimes (v'' \otimes v'), \quad h \in \mathbf{H}_\zeta(m),\ v' \in M',\ v'' \in M''.$$

Proposition B.5. *For* $M' \in \mathrm{Rep}(\mathbf{H}_\zeta(m'))$, $M'' \in \mathrm{Rep}(\mathbf{H}_\zeta(m''))$ *the following square is commutative*

$$\begin{array}{ccc} \Phi_* {}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(M' \otimes M'') & \xrightarrow{\Phi_*(\mathcal{S}_{M',M''})} & \Phi_* {}^{\mathbf{H}}\mathrm{Ind}_{m'',m'}(M'' \otimes M') \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ \Phi_* M' \dot{\otimes} \Phi_* M'' & \xrightarrow{\mathcal{R}_{\Phi_* M', \Phi_* M''}} & \Phi_* M'' \dot{\otimes} \Phi_* M'. \end{array}$$

Proof. We abbreviate $\mathbf{H} = \mathbf{H}_\zeta(m)$, $\mathbf{H}' = \mathbf{H}_\zeta(m')$, $\mathbf{H}'' = \mathbf{H}_\zeta(m'')$, $\mathbf{V} = \mathbf{V}_\zeta(n,m)$, $\mathbf{V}' = \mathbf{V}_\zeta(n,m')$ and $\mathbf{V}'' = \mathbf{V}_\zeta(n,m'')$. First, we have a commutative square

$$\begin{array}{ccc} \mathbf{V}'' \dot{\otimes} \mathbf{V}' & \xrightarrow{\mathcal{R}_{\mathbf{V}'',\mathbf{V}'}} & \mathbf{V}' \dot{\otimes} \mathbf{V}'' \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathbf{V} & \xrightarrow{T_\tau} & \mathbf{V} \end{array} \tag{B.8}$$

where the lower map is the left multiplication with T_τ. See [16] and the discussion in the proof of Proposition B.3. In particular, we have

$$\mathcal{R}_{\mathbf{V}'',\mathbf{V}'}(h''v''\otimes h'v') = (h''\otimes h')\mathcal{R}_{\mathbf{V}'',\mathbf{V}'}(v'\otimes v''), \quad v'\in\mathbf{V}',\, v''\in\mathbf{V}'',\, h'\in\mathbf{H}',\, h''\in\mathbf{H}''.$$

Therefore, the composition by $\mathcal{R}_{\mathbf{V}'',\mathbf{V}'}$ yields a linear map

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{H}'\otimes\mathbf{H}''}(\mathbf{V}, M'\otimes M'') = {\Phi_*}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(M'\otimes M'') \to\\ &\to \mathrm{Hom}_{\mathbf{H}''\otimes\mathbf{H}'}(\mathbf{V}, M''\otimes M') = {\Phi_*}^{\mathbf{H}}\mathrm{Ind}_{m'',m'}(M''\otimes M').\end{aligned}$$

The commutativity of the square (B.8) implies that this map is equal to $\Phi_*(\mathcal{S}_{M',M''})$. It is easy to see that this map coincides also with $\mathcal{R}_{\Phi_*M',\Phi_*M''}$. □

Corollary B.6. *For $M'\in\mathrm{Rep}(\mathbf{S}_\zeta(n,m'))$, $M''\in\mathrm{Rep}(\mathbf{S}_\zeta(n,m''))$ the following square is commutative*

$$\begin{array}{ccc}{}^{\mathbf{H}}\mathrm{Ind}_{m',m''}(\Phi^*M'\otimes\Phi^*M'') & \xrightarrow{\mathcal{S}_{\Phi^*M',\Phi^*M''}} & {}^{\mathbf{H}}\mathrm{Ind}_{m'',m'}(\Phi^*M''\otimes\Phi^*M')\\ \downarrow{\scriptstyle\mathrm{can}} & & \downarrow{\scriptstyle\mathrm{can}}\\ \Phi^*(M'\dot\otimes M'') & \xrightarrow{\Phi^*(\mathcal{R}_{M',M''})} & \Phi^*(M''\dot\otimes M')\end{array}$$

Proof. Use the same argument as in the proof of Corollary B.4. □

Let $r\geqslant 1$ and $i=1,2,\dots,r-1$. For $M\in\mathrm{Rep}(\mathbf{H}_\zeta(m))$ we consider the automorphism of the $\mathbf{H}_\zeta(mr)$-module ${}^{\mathbf{H}}\mathrm{Ind}_{(m^r)}(M^{\otimes r})$ given by

$$\begin{aligned}\mathcal{S}_{M,i} &= {}^{\mathbf{H}}\mathrm{Ind}_{\mathbf{H}}^{\mathbf{H}_\zeta(mr)}(\mathbf{1}^{\otimes i-1}\otimes\mathcal{S}_{M,M}\otimes\mathbf{1}^{\otimes r-i-1}),\\ \mathbf{H} &= \mathbf{H}_\zeta(m)^{\otimes i-1}\otimes\mathbf{H}_\zeta(2m)\otimes\mathbf{H}_\zeta(m)^{\otimes r-i-1}.\end{aligned}\tag{B.9}$$

For $M\in\mathrm{Rep}(\mathbf{S}_\zeta(n,m))$ we consider the automorphism of the $\mathbf{S}_\zeta(n,mr)$-module $M^{\dot\otimes r}$ given by

$$\mathcal{R}_{M,i} = \mathbf{1}^{\dot\otimes i-1}\dot\otimes\mathcal{R}_{M,M}\dot\otimes\mathbf{1}^{\dot\otimes r-i-1}.\tag{B.10}$$

Corollary B.7. *For $M\in\mathrm{Rep}(\mathbf{S}_\zeta(n,m))$, $r\geqslant 1$ and $i=1,2,\dots,r-1$ we have a commutative square with invertible vertical maps*

$$\begin{array}{ccc}{}^{\mathbf{H}}\mathrm{Ind}_{(m^r)}\,\Phi^*(M)^{\otimes r} & \xrightarrow{\mathcal{S}_{\Phi^*(M),i}} & {}^{\mathbf{H}}\mathrm{Ind}_{(m^r)}\,\Phi^*(M)^{\otimes r}\\ \downarrow & & \downarrow\\ \Phi^*(M^{\dot\otimes r}) & \xrightarrow{\Phi^*(\mathcal{R}_{M,i})} & \Phi^*(M^{\dot\otimes r})\end{array}$$

B.6. **The braiding and the quantum Frobenius homomorphism.** Recall that if ζ is a primitive d-th root of 1 then the quantum Frobenius homomorphism (B.5) yields a functor

$$\mathrm{Fr}^*:\mathrm{Rep}(\mathbf{S}_1(n,m)) = \mathrm{Rep}(\mathbf{S}_{(-1)^d}(n,m))\to\mathrm{Rep}(\mathbf{S}_\zeta(n,dm)).$$

Here we have identified $\mathbf{S}_{(-1)^d}(n,m)$ and $\mathbf{S}_1(n,m)$ as in Remark B.2. Let $m',m''>0$ with $m=m'+m''$. By Proposition B.1, for $M\in\mathrm{Rep}(\mathbf{S}_{(-1)^d}(n,m))$, $M'\in\mathrm{Rep}(\mathbf{S}_{(-1)^d}(n,m'))$ the braiding operator

$$\mathcal{R}_{M,M'}: M\dot\otimes M'\to M'\dot\otimes M$$

is the composition of the permutation $\mathbf{s}_{M,M'}$ and of the operator

$$R_{M,M'} = \prod_{\lambda,\lambda'}(-1)^{d(\lambda,\lambda')}(1_\lambda\dot\otimes 1_{\lambda'}).$$

Proposition B.8. *For $r \geqslant 1$, $i, j = 1, 2, \dots r-1$, and $M \in \mathrm{Rep}(\mathbf{S}_{(-1)^d}(n,m))$ the following relations hold in* $\mathrm{End}_{\mathbf{S}_{(-1)^d}(n,mr)}(M^{\dot\otimes r})$

- $\mathcal{R}_{M,i}^2 = 1$,
- $\mathcal{R}_{M,i}\mathcal{R}_{M,j} = \mathcal{R}_{M,j}\mathcal{R}_{M,i}$ *if* $j \neq i-1, i+1$,
- $\mathcal{R}_{M,i}\mathcal{R}_{M,i+1}\mathcal{R}_{M,i} = \mathcal{R}_{M,i+1}\mathcal{R}_{M,i}\mathcal{R}_{M,i+1}$ *if* $i \neq r-1$.

Proof. The first relation is obvious by definition of the braiding operator, see above. The other relations are consequences of the general properties of a braiding. □

Further, the functor Fr^* is a braided tensor functor, i.e., we have the following.

Proposition B.9. *For $M \in \mathrm{Rep}(\mathbf{S}_{(-1)^d}(n,m'))$, $M' \in \mathrm{Rep}(\mathbf{S}_{(-1)^d}(n,m''))$ we have a functorial isomorphism $\mathrm{Fr}^*(M \dot\otimes M') = \mathrm{Fr}^*(M) \dot\otimes \mathrm{Fr}^*(M')$ in $\mathrm{Rep}(\mathbf{S}_\zeta(n,dm))$ such that $\mathrm{Fr}^*(\mathcal{R}_{M,M'}) = \mathcal{R}_{\mathrm{Fr}^*(M),\mathrm{Fr}^*(M')}$.*

Proof. Obvious by Proposition B.1. □

B.7. **The algebra $\mathbf{S}_\zeta(m)$.** We'll abbreviate $\mathbf{S}_\zeta(m) = \mathbf{S}_\zeta(m,m)$. If $n \geqslant m$ the algebra $\mathbf{S}_\zeta(n,m)$ is Morita equivalent to $\mathbf{S}_\zeta(m)$, see e.g., [6, lem. 1.3]. Thus $\dot\otimes$ can be viewed as a functor (choosing $n \geqslant m = m' + m''$)

$$\dot\otimes : \mathrm{Rep}(\mathbf{S}_\zeta(m')) \otimes \mathrm{Rep}(\mathbf{S}_\zeta(m'')) \to \mathrm{Rep}(\mathbf{S}_\zeta(m)).$$

If ζ is a primitive d-th root of 1 then the quantum Frobenius homomorphism can be viewed as a functor (choosing $n \geqslant dm$)

$$\mathrm{Fr}^* : \mathrm{Rep}(\mathbf{S}_1(m)) = \mathrm{Rep}(\mathbf{S}_{(-1)^d}(m)) \to \mathrm{Rep}(\mathbf{S}_\zeta(dm)).$$

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