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# Boundary asymptotic analysis for an incompressible viscous flow: Navier wall laws

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## Abstract

We consider a new way of establishing Navier wall laws. Considering a bounded domain  $\Omega$  of  $\mathbf{R}^N$ ,  $N = 2, 3$ , surrounded by a thin layer  $\Sigma_\varepsilon$ , along a part  $\Gamma_2$  of its boundary  $\partial\Omega$ , we consider a Navier-Stokes flow in  $\Omega \cup \partial\Omega \cup \Sigma_\varepsilon$  with Reynolds' number of order  $1/\varepsilon$  in  $\Sigma_\varepsilon$ . Using  $\Gamma$ -convergence arguments, we describe the asymptotic behaviour of the solution of this problem and get a general Navier law involving a matrix of Borel measures having the same support contained in the interface  $\Gamma_2$ . We then consider two special cases where we characterize this matrix of measures. As a further application, we consider an optimal control problem within this context.

Navier law, Navier-Stokes flow,  $\Gamma$ -convergence, asymptotic behaviour, optimal control problem.

**AMS Classification.** 76D05, 76D10, 76M45, 35Q30.

## 1 Introduction

A common hypothesis used in fluid mechanics is that, at the interface between a solid and a fluid, the velocity  $u$  of the fluid is equal to that of the solid. If the solid is at rest, the velocity of the fluid must thus vanish:  $u = 0$ , on the boundary of the solid. These are the so-called rigid boundary conditions. When writing this condition, one assumes that the fluid perfectly adheres to the solid.

This hypothesis has not always been accepted for a viscous fluid, although some verifications have been made through experiments. G. Taylor indeed verified in 1923 the correctness of this hypothesis, when studying the stability of the motion of a fluid flowing between two cylinders in rotation (Taylor-Couette's problem).

Another approach has then been suggested. A thin layer adhering to the solid exists with a tangential velocity different from 0 on the surface of the solid. Navier suggested that this tangential velocity is proportional to the shearing strains and thus is given through

$$\left\{ \begin{array}{lcl} (Id - n \otimes n) \nu \frac{\partial u}{\partial n} & = & \kappa u, \\ u \cdot n & = & 0, \end{array} \right.$$

where  $Id$  is the identity matrix,  $n$  is the unit outer normal vector to the surface of the solid,  $\nu$  is the viscosity of the fluid and  $\kappa$  is a proportionality coefficient.

Many works have already been devoted to the derivation of Navier boundary conditions, see for example [2], [3], [13] and [14]. In [2] and [3], the authors considered a viscous and incompressible fluid, whose Reynolds number is of order  $1/\varepsilon$ , flowing in a domain with rugosities of thinness  $\varepsilon$  and  $\varepsilon$ -periodically

distributed on its boundary surface, and assuming an homogeneous Dirichlet boundary condition on the boundary of these rugosities. Using the asymptotic expansion method, they deduced, at the first-order level, a kind of Navier wall law

$$\begin{cases} \varepsilon (Id - n \otimes n) \nu \frac{\partial u}{\partial n} &= \kappa u, \\ u \cdot n &= 0. \end{cases}$$

In [13], the authors considered the laminar flow in a pipe with rough pieces  $\varepsilon$ -periodically distributed on the surface of the pipe, and imposing an homogeneous Dirichlet boundary condition on the boundary of these rough pieces. They used an homogenization process and obtained a Navier wall law, computing a corrector term. In [14], the author considered an  $\varepsilon$ -periodic geometry built with rough pieces of thinness  $\varepsilon^m$  and imposed there a boundary condition of the type

$$\begin{cases} (Id - n \otimes n) \nu \frac{\partial u^\varepsilon}{\partial n} &= \varepsilon^k (g^\varepsilon - \kappa u^\varepsilon), \\ u^\varepsilon \cdot n &= 0. \end{cases}$$

The following limit law was obtained, depending on  $k$  and  $m$

$$\begin{cases} (Id - n \otimes n) \nu \frac{\partial u}{\partial n} &= \lambda (g - \kappa u), \\ u \cdot n &= 0. \end{cases}$$

Throughout the present work, we consider a bounded domain  $\Omega \subset \mathbf{R}^N$ ,  $N = 2, 3$ , whose boundary  $\partial\Omega$  is Lipschitz continuous. We suppose that  $\partial\Omega = \Gamma_1 \cup \Gamma_2$ , with  $|\Gamma_1|, |\Gamma_2| > 0$ , where  $|\Gamma_i|$  denotes the Lebesgue measure of  $\Gamma_i$ . We suppose that near  $\Gamma_2$  there exists a thin layer  $\Sigma_\varepsilon$  of thinness  $\varepsilon > 0$ , which extends  $\Omega$  into  $\Omega_\varepsilon = \Omega \cup \Gamma_2 \cup \Sigma_\varepsilon$ .

Figure 1: The domain under consideration.

We consider the steady-state, viscous and incompressible Navier-Stokes flow in  $\Omega_\varepsilon$

$$\begin{cases} -\nu \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon &= f & \text{in } \Omega, \\ -\nu \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon &= f & \text{in } \Sigma_\varepsilon, \\ \operatorname{div}(u^\varepsilon) &= 0 & \text{in } \Omega_\varepsilon, \\ (u^\varepsilon)^+ &= (u^\varepsilon)^- & \text{on } \Gamma_2, \\ \nu \left( \frac{\partial u^\varepsilon}{\partial n} \right)^+ &= \nu \varepsilon \left( \frac{\partial u^\varepsilon}{\partial n} \right)^- & \text{on } \Gamma_2, \\ u^\varepsilon &= 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \quad (1)$$

where the superscript  $+$  (resp.  $-$ ) denotes the trace seen from  $\Omega$  (resp. from  $\Sigma_\varepsilon$ ) on  $\Gamma_2$ . The thin layer  $\Sigma_\varepsilon$  is here considered as an unstable thin boundary layer whose Reynolds' number  $R_\varepsilon$  is of order  $1/\varepsilon$  (see [12, pages 239-240], where Reynolds' number is allowed to depend on the thinness of the layer). In the problem (1), we suppose that the density  $f$  of volumic forces belongs to  $\mathbf{L}^\infty(\mathbf{R}^N, \mathbf{R}^N)$ .

Our purpose is to describe the asymptotic behavior of the solution  $u^\varepsilon$  of (1) when  $\varepsilon$  goes to 0, in order to derive the Navier wall law. We use  $\Gamma$ -convergence arguments (see [5] for the definition and the

properties of the  $\Gamma$ -convergence) in order to characterize the limit problem. Our approach is based on the tools developed in [1], [4], [7], [8] and [9]. On  $\Gamma_2$ , we will get a general Navier law of the kind

$$\begin{cases} (Id - n \otimes n) \nu \frac{\partial u}{\partial n} + \mu^\bullet u &= 0, \\ u \cdot n &= 0, \end{cases}$$

where  $\mu^\bullet$  is a symmetric matrix  $(\mu_{ij})_{i,j=1,\dots,N}$  of Borel measures having their support contained in  $\Gamma_2$ , which do not charge the polar subsets of  $\mathbf{R}^N$  and which satisfy  $\mu_{ij}(B) \zeta_i \zeta_j \geq 0, \forall \zeta \in \mathbf{R}^N, \forall B \in \mathcal{B}(\mathbf{R}^N)$ , where  $\mathcal{B}(\mathbf{R}^N)$  denotes the set of all Borel subsets of  $\mathbf{R}^N$  and where we have used the summation convention with respect to repeated indices.

As a first special case, we prove that when  $\Omega \subset \{x_3 > 0\}$ ,  $\Gamma_2 = \partial\Omega \cap \{x_3 = 0\}$  and

$$\Sigma_\varepsilon = \left\{ x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon h\left(\frac{x'}{\varepsilon}\right) < x_3 < 0 \right\},$$

where  $h$  is a periodic function, we get on  $\Gamma_2$  the Robin type boundary conditions

$$\begin{cases} \frac{\partial u_1}{\partial x_3}(x', 0) &= -c_1 u_1(x', 0), \\ \frac{\partial u_2}{\partial x_3}(x', 0) &= -c_2 u_2(x', 0), \\ u_3(x', 0) &= 0, \end{cases}$$

where  $c_m, m = 1, 2$ , are constants which will be computed in terms of the solution of appropriate local thin layer problems (21). This situation can be generalized to the case of a general open and bounded set  $\Omega$ , surrounded on a part of its boundary by such a rough thin layer.

As a second example, we will consider the case where

$$\Sigma_\varepsilon = \{s + tn(s) \mid s \in \Gamma_2, -\varepsilon h(s) < x_3 < 0\},$$

where  $h$  is a Lipschitz continuous and positive function on  $\Gamma_2$ . We here prove that Navier's law takes the following expression on  $\Gamma_2$

$$\begin{cases} (Id - n \otimes n) \frac{\partial u}{\partial n} + \frac{1}{h} u &= 0, \\ u \cdot n &= 0. \end{cases}$$

In the last part of this work, we consider an optimal control problem. Choosing  $m > 0$ , we consider the set  $\Xi_m$  of all the matrices  $\mathbf{h} = \text{Diag}(h_i)_{i=1,\dots,N}$  of functions  $h_i : \Gamma_2 \rightarrow [0, +\infty]$ , which are  $d\Gamma_2$ -measurable and satisfy  $\int_{\Gamma_2} h_i d\Gamma_2 = m, \forall i = 1, \dots, N$ . We suppose that  $\Omega$  is smooth enough and consider the following problem with Navier conditions on  $\Gamma_2$

$$\begin{cases} -\nu \Delta u^h + (u^h \cdot \nabla) u^h + \nabla p^h &= f & \text{in } \Omega, \\ \text{div}(u^h) &= 0 & \text{in } \Omega, \\ \mathbf{h}(Id - n \otimes n) \frac{\partial u^h}{\partial n} + u^h &= 0 & \text{on } \Gamma_2, \\ u^h \cdot n &= 0 & \text{on } \Gamma_2. \end{cases} \quad (2)$$

Let  $(u^h, p^h)$  be the solution of (2) and define the functional  $\mathbf{F}$  through

$$\mathbf{F}(\mathbf{h}, u) = \begin{cases} \frac{\nu}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \sum_{i=1}^N \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2 \\ \quad + \int_{\Omega} (u^h \cdot \nabla) u^h \cdot v dx - \int_{\Omega} f \cdot u dx & \text{if } u \in \mathbf{V}_{0,\Gamma_1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where  $\mathbf{V}_{0,\Gamma_1}(\Omega)$  is the functional space defined in (7). We consider the optimal control problem

$$\min_{\mathbf{h} \in \Xi_m} \min_{u \in \mathbf{V}_{0,\Gamma_1}(\Omega)} \mathbf{F}(\mathbf{h}, u). \quad (3)$$

In the last section of this work, we describe the asymptotic behavior of the solution of (3), when  $m$  goes to 0, and characterize the zones where some thin boundary layer appears. A problem of this kind has been considered in [11], but for a linear diffusion problem.

## 2 Functional framework

We define the  $(H^1(\mathbf{R}^N))$  capacity of any compact subset  $K$  of  $\mathbf{R}^N$  as

$$\begin{aligned} \text{Cap}(K) \\ = \inf \left\{ \int_{\mathbf{R}^N} |\nabla \varphi|^2 dx + \int_{\mathbf{R}^N} |\varphi|^2 dx \mid \varphi \in \mathbf{C}_c^\infty(\mathbf{R}^N), \varphi \geq 1 \text{ on } K \right\}. \end{aligned}$$

If  $U$  is an open subset of  $\mathbf{R}^N$ , then we define

$$\text{Cap}(U) = \sup \{ \text{Cap}(K) \mid K \subset U, K \text{ compact} \}.$$

If  $B \subset \mathbf{R}^N$  is a Borel subset of  $\mathbf{R}^N$ , then we define

$$\text{Cap}(B) = \inf \{ \text{Cap}(U) \mid B \subset U, U \text{ open} \}.$$

**Definition 1** Let  $\mathcal{B}(\mathbf{R}^N)$  be the  $\sigma$ -algebra of all Borel subsets of  $\mathbf{R}^N$ .

1. A property is said to be true quasi-everywhere (q.e.) on  $B \in \mathcal{B}(\mathbf{R}^N)$  if it is true except on a subset of  $B$  of capacity  $\text{Cap}$  equal to 0.
2. A function  $u : B \rightarrow \overline{\mathbf{R}}$ , with  $B \in \mathcal{B}(\mathbf{R}^N)$ , is quasi-continuous on  $B$  if, for every  $\varepsilon > 0$ , there exists an open subset  $U \subset B$  with  $\text{Cap}(U) < \varepsilon$  and such that the restriction of  $u$  on  $B \setminus U$  is continuous.
3. Every function  $u \in \mathbf{H}^1(\mathbf{R}^N)$  has a quasi-continuous representative  $\tilde{u}$ , which is unique for the equality quasi-everywhere in  $\mathbf{R}^N$ , (see [17], for example).  $\tilde{u}$  is given through

$$\tilde{u}(x) = \lim_{r \rightarrow 0^+} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy,$$

for q.e.  $x \in \mathbf{R}^N$ , where  $|B(x, r)|$  is the Lebesgue measure of the ball  $B(x, r)$  of  $\mathbf{R}^N$  of radius  $r > 0$  and centered at  $x$ .

We define some notions concerning families of subsets of  $\mathbf{R}^N$ .

**Definition 2** 1. A subset  $\mathcal{D} \subset \mathcal{B}(\mathbf{R}^N)$  is a dense family in  $\mathcal{B}(\mathbf{R}^N)$  if, for every  $A, B \in \mathcal{B}(\mathbf{R}^N)$  with  $\overline{A} \subset \overset{\circ}{B}$ , there exists  $D \in \mathcal{D}$  such that:  $\overline{A} \subset \overset{\circ}{D} \subset \overline{D} \subset \overset{\circ}{B}$ , where  $\overset{\circ}{A}$  (resp.  $\overline{A}$ ) denotes the interior (resp. the closure) of  $A$ .

2. A subset  $\mathcal{R} \subset \mathcal{B}(\mathbf{R}^N)$  is a rich family in  $\mathcal{B}(\mathbf{R}^N)$  if, for every family  $(A_t)_{t \in ]0, 1[} \subset \mathcal{B}(\mathbf{R}^N)$  such that  $\overline{A}_s \subset \overset{\circ}{A}_t$ , for every  $s < t$ , the set  $\{t \in ]0, 1[ \mid A_t \notin \mathcal{R}\}$  is at most countable.

Let  $\mathcal{O}(\mathbf{R}^N)$  be the set of all open subsets of  $\mathbf{R}^N$ . We consider the class  $\mathbb{F}$  of functionals  $F$  from  $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{O}(\mathbf{R}^N)$  to  $[0, +\infty]$  satisfying:

- i) (*Lower semi-continuity*): for every open subset  $\omega \in \mathcal{O}(\mathbf{R}^N)$ , the functional  $u \mapsto F(u, \omega)$  is lower semi-continuous with respect to the strong topology of  $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ ;
- ii) (*Measure property*): for every  $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ ,  $\omega \mapsto F(u, \omega)$  is the restriction to  $\mathcal{O}(\mathbf{R}^N)$  of some Borel measure still denoted  $F(u, \omega)$ ;
- iii) (*Localization*): for every  $\omega \in \mathcal{O}(\mathbf{R}^N)$  and every  $u, v \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ :

$$u|_{\omega} = v|_{\omega} \Rightarrow F(u, \omega) = F(v, \omega);$$

- iv) ( $\mathbf{C}^1$ -*convexity*): for every  $\omega \in \mathcal{O}(\mathbf{R}^N)$ , the functional  $u \mapsto F(u, \omega)$  is convex on  $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$  and moreover

$$\forall \varphi \in \mathbf{C}^1(\mathbf{R}^N), 0 \leq \varphi \leq 1 : F(\varphi u + (1 - \varphi)v, \omega) \leq F(u, \omega) + F(v, \omega).$$

**Example 3** Let us define  $\Gamma_{2,\varepsilon} = \partial\Omega_{\varepsilon} \cap \overline{\Sigma_{\varepsilon}}$ , for some thin layer  $\Sigma_{\varepsilon}$ , as defined above. We consider the functional  $F^{\varepsilon}$  defined on the space  $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{O}(\mathbf{R}^N)$  through

$$F^{\varepsilon}(u, \omega) = \begin{cases} 0 & \text{if } \tilde{u} = 0, \text{ q.e. on } \Gamma_{2,\varepsilon} \cap \omega, \\ +\infty & \text{otherwise.} \end{cases} \quad (4)$$

One can prove that  $F^{\varepsilon}$  belongs to  $\mathbb{F}$ , for every  $\varepsilon > 0$ .

Let us set the following definitions.

**Definition 4** Let  $Cap$  be the above-defined capacity.

1. A Borel measure  $\lambda$  is absolutely continuous with respect to the capacity  $Cap$  if

$$\forall B \in \mathcal{B}(\mathbf{R}^N) : Cap(B) = 0 \Rightarrow \lambda(B) = 0.$$

2.  $\mathcal{M}_0$  is the set of nonnegative Borel measures  $\mathbf{R}^N$  which are absolutely continuous with respect to the capacity  $Cap$ .

We have the following example of measure in  $\mathcal{M}_0$ .

**Example 5** For every  $E \subset \mathbf{R}^N$  such that  $Cap(E) > 0$ , we define the measure  $\infty_E$  through

$$\infty_E(B) = \begin{cases} 0 & \text{if } Cap(B \cap E) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then  $\infty_E \in \mathcal{M}_0$ .

Notice that, for every  $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$  and every  $\omega \in \mathcal{O}(\mathbf{R}^N)$ , the functional  $F^{\varepsilon}$  defined in (4) can be written as

$$F^{\varepsilon}(u, \omega) = \int_{\omega} |\tilde{u}|^2 d\infty_{\Gamma_{2,\varepsilon}} = \int_{\omega} |u|^2 d\infty_{\Gamma_{2,\varepsilon}}.$$

One has the following representation theorem for the functionals of  $\mathbb{F}$ .

**Theorem 6** (see [9]) For every  $F \in \mathbb{F}$ , there exist a finite measure  $\lambda \in \mathcal{M}_0$ , a nonnegative Borel measure  $\nu$  and a Borel function  $g : \mathbf{R}^N \times \mathbf{R}^N \rightarrow [0, +\infty]$ , with  $\zeta \mapsto g(x, \zeta)$  convex and lower semi-continuous on  $\mathbf{R}^N$ , such that

$$\forall u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N), \forall \omega \in \mathcal{O}(\mathbf{R}^N) : F(u, \omega) = \int_{\omega} g(x, \tilde{u}(x)) d\lambda + \nu(\omega).$$

Throughout the paper, we will need the following Corollary (see [9, Corollary 8.4]).

**Corollary 7** *Let  $F \in \mathbb{F}$ . If  $F(\cdot, \omega)$  is quadratic for every  $\omega \in \mathcal{O}(\mathbf{R}^N)$ , there exist  $\lambda \in \mathcal{M}_0$  finite, a symmetric matrix  $(a_{ij})_{i,j=1,\dots,N}$ , of Borel functions from  $\mathbf{R}^N$  to  $\mathbf{R}$  satisfying  $a_{ij}(x)\zeta_i\zeta_j \geq 0$ ,  $\forall \zeta \in \mathbf{R}^N$  and for q.e.  $x \in \mathbf{R}^N$ , for every  $x \in \mathbf{R}^N$  a subspace  $\mathbf{V}(x)$  of  $\mathbf{R}^N$ , such that, for every  $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$  and every  $\omega \in \mathcal{O}(\mathbf{R}^N)$ :*

a) *if  $F(u, \omega) < +\infty$ , then  $u(x) \in \mathbf{V}(x)$ , for q.e.  $x \in \omega$ ,*

b) *if  $u(x) \in \mathbf{V}(x)$ , for q.e.  $x \in \omega$*

$$F(u, \omega) = \int_{\omega} a_{ij} u_i u_j d\lambda. \quad (5)$$

**Remark 8** *Let  $F \in \mathbb{F}$ ,  $\lambda \in \mathcal{M}_0$  be the associated measure and  $\Lambda$  be the set defined as  $\Lambda = \cup_{\omega \in A(F)} \omega$ , where*

$$A(F) = \{\omega \in \mathcal{O}(\mathbf{R}^N) \mid F(\cdot, \omega) < +\infty, \text{ for q.e. } x \in \omega\}.$$

*We define the matrix  $\mu^\bullet = (\mu_{ij}) = (a_{ij}\lambda)_{i,j=1,\dots,N} + \infty_{\mathbf{R}^N \setminus \Lambda} Id$  of measures, and, for every  $x \in \mathbf{R}^N$ , the subspace  $\mathbf{V}(x)$  through*

$$\mathbf{V}(x) = \begin{cases} \mathbf{R}^N & \text{if } x \in \Lambda, \\ \{0\} & \text{if } x \in \mathbf{R}^N \setminus \Lambda. \end{cases} \quad (6)$$

*For every  $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$  and every  $\omega \in \mathcal{O}(\mathbf{R}^N)$ , one has, using the preceding definition of  $\mu^\bullet$*

$$\int_{\omega} u_i u_j d\mu_{ij} = \begin{cases} \int_{\omega} a_{ij} u_i u_j d\lambda & \text{if } \omega \subset \Lambda, \\ \int_{\omega \cap \Lambda} a_{ij} u_i u_j d\lambda & \text{if } \begin{cases} u(x) = 0, \forall x \in \omega \cap \mathbf{R}^N \setminus \Lambda \\ \text{and } \text{Cap}(\omega \cap \mathbf{R}^N \setminus \Lambda) > 0, \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

*Thanks to (6), this expression can be written as*

$$\int_{\omega} u_i u_j d\mu_{ij} = \begin{cases} \int_{\omega} a_{ij} u_i u_j d\lambda & \text{if } u(x) \in \mathbf{V}(x), \text{ for q.e. } x \in \omega, \\ +\infty & \text{otherwise.} \end{cases}$$

*We can thus write the functional  $F$  defined in (5) as*

$$F(u, \omega) = \int_{\omega} u_i u_j d\mu_{ij}.$$

### 3 Study of the problem (1)

We here suppose that the "outer" boundary  $\Gamma_{2,\varepsilon}$  of  $\Sigma_\varepsilon$  can be defined as

$$\Gamma_{2,\varepsilon} = \{(s, t) \mid s \in \Gamma_2, t = -\varepsilon h_\varepsilon(s)\},$$

where  $h_\varepsilon$  is a locally Lipschitz continuous function satisfying

$$\|h_\varepsilon\|_{\mathbf{L}^\infty(\Gamma_2)} \leq C, \forall \varepsilon > 0,$$

for some constant  $C$  independent of  $\varepsilon$ . The Lipschitz continuity of  $h_\varepsilon$  ensures the almost everywhere existence of a unit outer normal vector to  $\Gamma_{2,\varepsilon}$ , thanks to Rademacher's Theorem, and ensures the

existence of an extension of every function of  $\mathbf{H}^1(\Omega_\varepsilon, \mathbf{R}^N)$  in a function of  $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ . Let us define the functional spaces

$$\begin{aligned} \mathbf{L}^2(\mathbf{R}^N, \text{div}) &= \left\{ u \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \mid \text{div}(u) = 0 \text{ in } \mathbf{R}^N \right\}, \\ \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \text{div}) &= \left\{ u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N) \mid \text{div}(u) = 0 \text{ in } \mathbf{R}^N, \right. \\ &\quad \left. u = 0 \text{ on } \Gamma_1 \right\}, \\ \mathbf{H}_{\Gamma_1}^1(\Omega, \text{div}) &= \left\{ u \in \mathbf{H}^1(\Omega, \mathbf{R}^N) \mid \text{div}(u) = 0 \text{ in } \Omega, u = 0 \text{ on } \Gamma_1 \right\}, \\ \mathbf{V}_{\Gamma_1}(\Omega) &= \mathbf{L}^2(\mathbf{R}^N, \text{div}) \cap \mathbf{H}_{\Gamma_1}^1(\Omega, \text{div}), \\ \mathbf{V}_{0, \Gamma_1}(\Omega) &= \mathbf{H}_{\Gamma_1}^1(\Omega, \text{div}) \cap \left\{ u \in \mathbf{H}^1(\Omega, \mathbf{R}^N) \mid u \cdot n = 0 \text{ on } \Gamma_2 \right\}. \end{aligned} \quad (7)$$

In (1), let us replace throughout this section the homogeneous Dirichlet boundary condition  $u^\varepsilon = 0$ , on  $\partial\Omega_\varepsilon$  by a combination between the homogeneous Dirichlet boundary condition  $u^\varepsilon = 0$ , on  $\Gamma_{2, \varepsilon} \cap \omega$ , for a given  $\omega \in \mathcal{O}(\mathbf{R}^N)$ , and homogeneous Neumann boundary conditions on  $\Gamma_{2, \varepsilon} \setminus (\Gamma_{2, \varepsilon} \cap \omega)$ . We introduce the functional space adapted to (1), with these modified boundary conditions

$$\mathbf{V}_{0, \omega}(\Omega_\varepsilon) = \left\{ v \in \mathbf{H}^1(\Omega_\varepsilon, \mathbf{R}^N) \mid \text{div}(v) = 0 \text{ in } \Omega_\varepsilon, \right. \\ \left. v = 0 \text{ on } \Gamma_1 \cup (\Gamma_{2, \varepsilon} \cap \omega) \right\}.$$

The variational formulation of (1) can be written as

$$\begin{aligned} \forall \varphi \in \mathbf{V}_{0, \omega}(\Omega_\varepsilon) : \nu \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \varphi dx + \nu \varepsilon \int_{\Sigma_\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi dx \\ + \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \varphi dx = \int_{\Omega_\varepsilon} f \cdot \varphi dx. \end{aligned} \quad (8)$$

Thanks to [15], for example, we deduce that (1) has a unique solution  $(u^\varepsilon, p^\varepsilon)$  belonging to the space  $\mathbf{V}_{0, \omega}(\Omega_\varepsilon) \times \mathbf{L}^2(\Omega_\varepsilon)/\mathbf{R}$ .

**Proposition 9** *The solution  $(u^\varepsilon, p^\varepsilon)$  of (1) satisfies the following estimates*

$$\begin{aligned} \sup_{\varepsilon} \left( \int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \right) &< +\infty, \\ \sup_{\varepsilon} \int_{\mathbf{R}^N} |u^\varepsilon|^2 dx &< +\infty, \\ \sup_{\varepsilon} \|p^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon)/\mathbf{R}} &< +\infty. \end{aligned}$$

**Proof.** 1. Taking  $u^\varepsilon$  as test-function in (8), we obtain

$$\begin{aligned} \nu \int_{\Omega} |\nabla u^\varepsilon|^2 dx + \nu \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \\ = \int_{\Omega} f \cdot u^\varepsilon dx + \int_{\Sigma_\varepsilon} f \cdot u^\varepsilon dx \\ \leq \|f\|_{\mathbf{L}^\infty(\mathbf{R}^N, \mathbf{R}^N)} \|u^\varepsilon\|_{\mathbf{L}^1(\Omega_\varepsilon, \mathbf{R}^N)} \\ \leq \|f\|_{\mathbf{L}^\infty(\mathbf{R}^N, \mathbf{R}^N)} C(\Omega) \|\nabla u^\varepsilon\|_{\mathbf{L}^1(\Omega, \mathbf{R}^N)}, \end{aligned}$$

using Poincaré's inequality. Cauchy-Schwarz' inequality implies

$$\begin{aligned} \int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \\ \leq C(f, \Omega) \left( \left( \int_{\Omega} |\nabla u^\varepsilon|^2 dx \right)^{1/2} + \left( \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \right)^{1/2} \right), \end{aligned}$$

whence, using the trivial inequality  $(a + b)^2 \leq 2(a^2 + b^2)$

$$\int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \leq C \Rightarrow \|\nabla u^\varepsilon\|_{\mathbf{L}^1(\Omega_\varepsilon, \mathbf{R}^N)} \leq C.$$



The continuous embedding from  $\mathbf{W}_{\Gamma_1}^{1,1}(\Omega_\varepsilon, \mathbf{R}^N)$  to  $\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)$  implies the existence of a constant  $C$  independent of  $\varepsilon$  such that

$$\int_{\Omega_\varepsilon} |u^\varepsilon|^2 dx \leq C.$$

2. Let us define the zero mean value pressure  $\overline{p^\varepsilon} = p^\varepsilon - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p^\varepsilon dx$ , and let  $\psi_\varepsilon$  be the solution of the following problem (see [15])

$$\begin{cases} \operatorname{div}(\psi_\varepsilon) &= \overline{p^\varepsilon} & \text{in } \Omega_\varepsilon, \\ \psi_\varepsilon &= 0 & \text{on } \Gamma_1 \cup (\Gamma_2 \cap \omega), \\ \|\nabla \psi_\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)} &\leq C(\Omega) \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}, \end{cases} \quad (9)$$

for some constant  $C(\Omega)$  independent of  $\varepsilon$ . Multiplying (1)<sub>1,2</sub> by  $\psi_\varepsilon$  and using Green's formula, one obtains

$$\begin{aligned} & \nu \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon dx + \nu \varepsilon \int_{\Sigma_\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon dx + \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \psi_\varepsilon dx \\ &= \int_{\Omega_\varepsilon} f \cdot \psi_\varepsilon dx + \int_{\Omega_\varepsilon} (\overline{p^\varepsilon})^2 dx. \end{aligned}$$

Because

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} f \cdot \psi_\varepsilon dx \right| &\leq \|f\|_{\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)} \|\psi_\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)} \\ &\leq C \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)} \\ \left| \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \psi_\varepsilon dx \right| &\leq C \|\psi_\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)} \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}^2 \\ &\leq C \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)} \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}^2 \\ \left| \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon dx \right| &\leq C \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)} \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}, \\ \left| \int_{\Sigma_\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi_\varepsilon dx \right| &\leq C \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)} \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}, \end{aligned}$$

thanks to (9)<sub>3</sub> and using Poincaré's inequality, we obtain

$$\|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}^2 \leq C \left( \|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon, \mathbf{R}^N)}^2 + 1 \right) \|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)},$$

which proves the third estimate. ■

**Remark 10** We can observe that, when we impose an homogeneous Dirichlet boundary condition on the whole  $\Gamma_{2,\varepsilon}$ , for example when  $\omega = \mathbf{R}^N$ , the above estimates can be obtained in a simpler way, assuming only that  $f \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$ .

## 4 Convergence

Every function  $u \in \mathbf{H}_{\Gamma_1}^1(\Omega_\varepsilon, \operatorname{div})$  can be extended in a function of the space  $\mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div})$ , still denoted  $u$  (see [16, Theorem 4.3.3], for example). We define the functional  $\Phi^\varepsilon$  on  $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$  associated to (1), with the above-described modified boundary conditions on  $\Gamma_{2,\varepsilon}$  through

$$\Phi^\varepsilon(u) = \begin{cases} \nu \int_{\Omega} |\nabla u|^2 dx + \nu \varepsilon \int_{\mathbf{R}^N \setminus \Omega} |\nabla u|^2 dx & \text{if } u \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div}), \\ +\infty & \text{otherwise} \end{cases} \quad (10)$$

and the functional  $\Phi^0$  defined on  $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$  through

$$\Phi^0(u) = \begin{cases} \nu \int_{\Omega} |\nabla u|^2 dx & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From the estimates given in Proposition 9, we can deduce that the asymptotic behaviour of the problem (1) is obtained when studying the  $\Gamma$ -limit of the associated energy functional for the following topology.

**Definition 11** *A sequence  $(u_\varepsilon)_\varepsilon$   $\tau$ -converges to  $u$ , if it converges to  $u$  in the strong topology of  $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$  and if  $\sup_\varepsilon \Phi^\varepsilon(u_\varepsilon) < +\infty$ .*

We first present the  $\Gamma$ -convergence result for  $(\Phi^\varepsilon)_\varepsilon$ .

**Proposition 12** *When  $\varepsilon$  goes to 0, the sequence  $(\Phi^\varepsilon)_\varepsilon$   $\Gamma$ -converges to  $\Phi^0$ , in the topology  $\tau$ .*

**Proof.** Step 1: verification of the  $\Gamma$ -limsup. Take any  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and consider the set  $\Omega^{0,\varepsilon} = \Omega \cup \partial\Omega \cup \Sigma^{0,\varepsilon}$ , with

$$\Sigma^{0,\varepsilon} = \{x \in \mathbf{R}^N \mid 0 < d(x, \partial\Omega) < \sqrt{\varepsilon}\},$$

where  $d(x, \partial\Omega)$  denotes the euclidean distance between  $x$  and the boundary  $\partial\Omega$ . Let  $u^{1,\varepsilon}$  be such that  $\operatorname{div}(u^{1,\varepsilon}) = 0$  in  $\mathbf{R}^N$  and

$$\|u - u^{1,\varepsilon}\|_{\mathbf{L}^2(\mathbf{R}^N \setminus \Omega^{0,\varepsilon}, \mathbf{R}^N)} < \varepsilon.$$

We define  $\bar{u}^{1,\varepsilon}$  through

$$\bar{u}^{1,\varepsilon} = \begin{cases} u^{1,\varepsilon} & \text{in } \mathbf{R}^N \setminus \Omega^{0,\varepsilon}, \\ 0 & \text{on } \partial\Omega^{0,\varepsilon}. \end{cases}$$

We then take a nonnegative and smooth function  $\rho_\varepsilon \in \mathbf{C}_c^\infty(\mathbf{R}^N)$  with support in  $B(0, \varepsilon)$  and satisfying  $\int_{\mathbf{R}^N} \rho_\varepsilon(x) dx = 1$ . We define the function  $\bar{u}^{0,\varepsilon}$  through  $\bar{u}^{0,\varepsilon} = (\rho_\varepsilon * \bar{u}^{1,\varepsilon})|_{\mathbf{R}^N \setminus \overline{\Omega^{0,\varepsilon}}}$ . There exists  $\hat{u} \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)$  such that  $\operatorname{curl}(\hat{u}) = u$  in  $\mathbf{R}^N$  (see [15], for example). We finally define the function  $u^{0,\varepsilon}$  through

$$u^{0,\varepsilon} = \begin{cases} \bar{u}^{0,\varepsilon} & \text{in } \mathbf{R}^N \setminus \overline{\Omega^{0,\varepsilon}}, \\ \operatorname{curl}\left(\hat{u} \frac{\sqrt{\varepsilon} - d(x, \partial\Omega)}{\sqrt{\varepsilon}}\right) & \text{in } \Sigma^{0,\varepsilon}, \\ u & \text{in } \Omega. \end{cases}$$

We immediately satisfy that  $u^{0,\varepsilon} \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div})$ , that the sequence  $(u^{0,\varepsilon})_\varepsilon$  converges to  $u$  in the topology  $\tau$  and that

$$\limsup_{\varepsilon \rightarrow 0} \Phi^\varepsilon(u^{0,\varepsilon}) \leq \nu \int_{\Omega} |\nabla u|^2 dx = \Phi^0(u).$$

Step 2: verification of the  $\Gamma$ -liminf. We take any sequence  $(u_\varepsilon)_\varepsilon$  contained in  $\mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div})$  which converges to  $u$  in the topology  $\tau$ . We trivially have

$$\Phi^0(u) \leq \liminf_{\varepsilon \rightarrow 0} \Phi^0(u_\varepsilon) \leq \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(u_\varepsilon),$$

thanks to the lower semi-continuity property of  $\Phi^0$  for the weak topology of  $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ . ■

We define the functional  $G^\varepsilon$  on  $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{O}(\mathbf{R}^N)$  through

$$G^\varepsilon(u, \omega) = \begin{cases} \Phi^\varepsilon(u) + F^\varepsilon(u, \omega) & \text{if } u \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \operatorname{div}), \\ +\infty & \text{otherwise,} \end{cases}$$

where  $F^\varepsilon$  is defined in (4). Our main result is the following.

**Theorem 13** *There exist a rich family  $\mathcal{R} \subset \mathcal{B}(\mathbf{R}^N)$  and a symmetric matrix  $\mu^\bullet = (\mu_{ij})_{i,j=1,\dots,N}$  of Borel measures having their support contained in  $\Gamma_2$ , which are absolutely continuous with respect to the above-defined capacity  $\operatorname{Cap}$ , and satisfying  $\mu_{ij}(B) \zeta_i \zeta_j \geq 0$ ,  $\forall \zeta \in \mathbf{R}^N$ ,  $\forall B \in \mathcal{B}(\mathbf{R}^N)$ , such that, for every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and every  $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbf{R}^N)$*

$$\left(\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} G^\varepsilon\right)(u, \omega) = \nu \int_{\Omega} |\nabla u|^2 + \int_{\Gamma_2 \cap \omega} u_i u_j d\mu_{ij} =: G^0(u, \omega),$$

where the  $\Gamma$ -limit is taken with respect to the topology  $\tau$ .

**Proof.** The upper and lower  $\Gamma$ -limits of the sequence  $(G^\varepsilon)_\varepsilon$ , with respect to the topology  $\tau$ , exist, which are respectively defined through

$$\forall u \in \mathbf{V}_{\Gamma_1}(\Omega), \forall B \in \mathcal{B}(\mathbf{R}^N) : \begin{cases} G^s(u, B) &= \inf_{u_\varepsilon \xrightarrow{\tau} u} \limsup_{\varepsilon \rightarrow 0} G^\varepsilon(u_\varepsilon, B), \\ G^i(u, B) &= \inf_{u_\varepsilon \xrightarrow{\tau} u} \liminf_{\varepsilon \rightarrow 0} G^\varepsilon(u_\varepsilon, B). \end{cases} \quad (11)$$

Because  $F^\varepsilon$  takes nonnegative values and thanks to Proposition 12, we observe that, for every  $B \in \mathcal{B}(\mathbf{R}^N)$ , one has

$$G^s(., B) \geq \Phi^0(.) ; G^i(., B) \geq \Phi^0(.).$$

Let us define the functionals  $F^s$  and  $F^i$  on  $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{B}(\mathbf{R}^N)$  through

$$(F^0)^\alpha(u, B) = \begin{cases} G^\alpha(u, B) - \Phi^0(u) & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

with  $\alpha = s, i$ . Let  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and  $(u_\varepsilon)_\varepsilon \subset \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \text{div})$  be such that  $(u_\varepsilon)_\varepsilon$  converges to  $u$  in the topology  $\tau$ . We define  $z_\varepsilon = u_\varepsilon - u$ . Thus  $(z_\varepsilon)_\varepsilon \subset \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$  and  $(z_\varepsilon)_\varepsilon$  converges to 0 in the topology  $\tau$ . Replacing  $u_\varepsilon$  by  $z_\varepsilon + u$  in (11), one obtains, using the quadratic property of  $\Phi^\varepsilon$

$$\begin{aligned} (F^0)^s(u, B) &= \inf_{z_\varepsilon \xrightarrow{\tau} 0} \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(u + z_\varepsilon, B)), \\ (F^0)^i(u, B) &= \inf_{z_\varepsilon \xrightarrow{\tau} 0} \liminf_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(u + z_\varepsilon, B)). \end{aligned}$$

The functionals  $(F^0)^s$  and  $(F^0)^i$  satisfy the following properties.

1. For every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ ,  $(F^0)^s(u, .)$  and  $(F^0)^i(u, .)$  are nonnegative measures, because  $F^\varepsilon(u + z_\varepsilon, .)$  is a measure for every  $\varepsilon > 0$  and for every sequence  $(z_\varepsilon)_\varepsilon \subset \mathbf{V}_{\Gamma_1}(\Omega)$  which converges to 0 in the topology  $\tau$ .
2.  $(F^0)^s(., B)$  and  $(F^0)^i(., B)$  are lower semi-continuous on  $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ , when equipped with its strong topology, because  $G^s(., B)$ ,  $G^i(., B)$  and  $\Phi^0$  are lower semi-continuous as upper, lower, or  $\Gamma$ -limits of functionals which are lower semi-continuous for this strong topology.
3. Let  $\omega \in \mathcal{O}(\mathbf{R}^N)$  and  $u, v \in \mathbf{V}_{\Gamma_1}(\Omega)$  be such that  $u|_\omega = v|_\omega$ . Then  $(F^0)^s(u, \omega) = (F^0)^s(v, \omega)$  and  $(F^0)^i(u, \omega) = (F^0)^i(v, \omega)$ , because  $F^\varepsilon(u + z_\varepsilon, \omega) = F^\varepsilon(v + z_\varepsilon, \omega)$ , for every sequence  $(z_\varepsilon)_\varepsilon$  such that  $u + z_\varepsilon \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \text{div})$ , for every  $\varepsilon > 0$ .
4. Take any  $\varphi \in \mathbf{C}^1(\mathbf{R}^N)$  such that  $0 \leq \varphi \leq 1$ ,  $u, v \in \mathbf{V}_{\Gamma_1}(\Omega)$  and  $B \in \mathcal{B}(\mathbf{R}^N)$ . One has, for every sequence  $(z_\varepsilon)_\varepsilon \subset \mathbf{V}_{\Gamma_1}(\Omega)$  converging to 0 in the topology  $\tau$

$$\begin{aligned} F^\varepsilon(z_\varepsilon + \varphi u + (1 - \varphi)v, B) &= F^\varepsilon((z_\varepsilon + u)\varphi + (1 - \varphi)(z_\varepsilon + v), B) \\ &\leq F^\varepsilon(z_\varepsilon + u, B) + F^\varepsilon(z_\varepsilon + v, B), \end{aligned}$$

because  $F^\varepsilon$  is  $\mathbf{C}^1$ -convex. Because  $\Phi^\varepsilon$  takes nonnegative values, for every  $\varepsilon > 0$ , one has

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + \varphi u + (1 - \varphi)v, B)) \\ \leq \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + u, B) + \Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + v, B)) \\ \leq \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + u, B)) \\ \quad + \limsup_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + v, B)). \end{aligned}$$

Taking the infimum over all sequences  $(z_\varepsilon)_\varepsilon \subset \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$  which converge to 0 in the topology  $\tau$ , one obtains

$$(F^0)^s(\varphi u + (1 - \varphi)v, B) \leq (F^0)^s(u, B) + (F^0)^s(v, B).$$

We prove in a similar way that  $(F^0)^s$  is convex. Thus  $(F^0)^s$  is  $\mathbf{C}^1$ -convex.

Thanks to the compacity theorem of [10], there exist a subsequence  $(\varepsilon_k)_k$  and a dense and countable family  $\mathcal{D} \subset \mathcal{B}(\mathbf{R}^N)$  such that, for every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and every  $B \in \mathcal{D}$

$$\left( \Gamma\text{-}\lim_{k \rightarrow +\infty} G^{\varepsilon_k} \right) (u, B) = G^0(u, B),$$

where the  $\Gamma$ -limit is taken with respect to the topology  $\tau$ . We then define the functional  $F^0$  on  $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{D}$  as

$$F^0(u, B) = \begin{cases} G^0(u, B) - \Phi^0(u) & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \quad (12)$$

We have  $F^0 = (F^0)^s = (F^0)^i$  on  $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{D}$ . We then extend  $F^0$  on  $\mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{B}(\mathbf{R}^N)$  defining

$$F^0(u, B) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} (F^0)^s(u, D) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} (F^0)^i(u, D). \quad (13)$$

We define the family  $\mathcal{R}(F)$  of Borel subsets of  $\mathbf{R}^N$  through

$$\mathcal{R}(F) = \left\{ B \in \mathcal{B}(\mathbf{R}^N) \mid \forall u \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N) : (F^0)^s_+(u, B) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} (F^0)^s(u, D) = \inf_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} (F^0)^s(u, D) = (F^0)^s_-(u, B) \right\}.$$

Then we prove (see [5, Proposition 14.14]) that  $\mathcal{R}(F^0)$  is a rich family in  $\mathcal{B}(\mathbf{R}^N)$  and  $F^0 = (F^0)^s = (F^0)^s_+ = (F^0)^s_- = (F^0)^i_+ = (F^0)^i_- = (F^0)^i$  on  $\mathcal{R}(F^0)$ . One obtains, for every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and every  $B \in \mathcal{R}(F^0)$

$$\begin{aligned} F^0(u, B) &= \inf_{z_{\varepsilon_k} \xrightarrow{\tau} 0} \limsup_{k \rightarrow +\infty} (\Phi^{\varepsilon_k}(z_{\varepsilon_k}) + F^{\varepsilon_k}(u + z_{\varepsilon_k}, B)) \\ &= \inf_{z_{\varepsilon_k} \xrightarrow{\tau} 0} \liminf_{k \rightarrow +\infty} (\Phi^{\varepsilon_k}(z_{\varepsilon_k}) + F^{\varepsilon_k}(u + z_{\varepsilon_k}, B)). \end{aligned}$$

Let now  $\varepsilon'$  denote any subsequence of  $\varepsilon$ . Thanks to the above method, there exist a subsequence  $(\varepsilon'_k)_k$ , a functional  $\mathcal{F}^0$  and a rich family  $\mathcal{R}(\mathcal{F}^0)$  such that, for every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and every  $B \in \mathcal{R}(\mathcal{F}^0)$

$$\begin{aligned} \mathcal{F}^0(u, B) &= \inf_{z_{\varepsilon'_k} \xrightarrow{\tau} 0} \limsup_{k \rightarrow +\infty} (\Phi^{\varepsilon'_k}(z_{\varepsilon'_k}) + F^{\varepsilon'_k}(u + z_{\varepsilon'_k}, B)) \\ &= \inf_{z_{\varepsilon'_k} \xrightarrow{\tau} 0} \liminf_{k \rightarrow +\infty} (\Phi^{\varepsilon'_k}(z_{\varepsilon'_k}) + F^{\varepsilon'_k}(u + z_{\varepsilon'_k}, B)). \end{aligned}$$

Because  $\mathcal{R}(F^0) \cap \mathcal{R}(\mathcal{F}^0)$  is still a rich family, one has

$$\forall u \in \mathbf{V}_{\Gamma_1}(\Omega), \forall B \in \mathcal{R} : F^0(u, \cdot) = \mathcal{F}^0(u, \cdot), \text{ on } \mathcal{R}(F^0) \cap \mathcal{R}(\mathcal{F}^0).$$

Because the countable intersection of rich families is a rich family too, one can repeat the above reasoning and deduce the existence of a rich family  $\mathcal{R}$  in  $\mathcal{B}(\mathbf{R}^N)$  on which the above limits coincide. One thus obtains, for every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and every  $B \in \mathcal{R}$

$$\left( \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} G^\varepsilon \right) (u, \omega) = \Phi^0(u) + F^0(u, B), \quad (14)$$

where the  $\Gamma$ -limit is taken with respect to the topology  $\tau$ .

Thanks to the above properties 1., 2., 3. and 4. and to the relations (12) and (13),  $F^0$  belongs to  $\mathbb{F}$ . Because  $\Phi^\varepsilon$  and  $F^\varepsilon$  are quadratic, thanks to Corollary 7 and to Remark 8, there exist  $\lambda \in \mathcal{M}_0$  finite, a

symmetric matrix  $(a_{ij})_{i,j=1,\dots,N}$  of Borel functions from  $\mathbf{R}^N$  to  $\mathbf{R}$  with  $a_{ij}(x)\zeta_i\zeta_j \geq 0$ ,  $\forall \zeta \in \mathbf{R}^N$  and for q.e.  $x \in \mathbf{R}^N$ , such that, for every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and every  $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbf{R}^N)$

$$F^0(u, \omega) = \int_{\omega} u_i u_j d\mu_{ij},$$

with  $\mu^\bullet = (\mu_{ij})_{i,j=1,\dots,N} = (a_{ij}\lambda)_{i,j=1,\dots,N} + \infty_{\mathbf{R}^N \setminus \Lambda} Id$ , where  $\Lambda$  is defined as in Remark 8.

Let us now precise the support of  $\mu^\bullet$ . For every  $u, v \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^N, \text{div})$ , such that  $v|_{\Omega} = u|_{\Omega}$ , one has

$$F^0(u, \mathbf{R}^N) = \int_{\mathbf{R}^N} v_i v_j d\mu_{ij},$$

because  $F^0$  is local ( $\mathbf{R}^N$  belongs to  $\mathcal{R}$  because every rich family is dense, and every dense family contains  $\mathbf{R}^N$ ). One deduces that  $\text{supp}(\mu^\bullet) \subset \Omega \cup \Gamma_2$ . Thanks to (14), one has

$$0 \leq \int_{\mathbf{R}^N} u_i u_j d\mu_{ij} + \Phi^0(u) \leq \liminf_{\varepsilon \rightarrow 0} (\Phi^\varepsilon(u) + F^\varepsilon(u, \mathbf{R}^N)). \quad (15)$$

Taking  $u \in \mathbf{H}_0^1(\Omega, \text{div}) = \{u \in \mathbf{H}_0^1(\Omega, \mathbf{R}^N) \mid \text{div}(u) = 0\}$ , then, for every  $\varepsilon > 0$ ,  $F^\varepsilon(u, \mathbf{R}^N) = 0$ , and  $\liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(u) = \Phi^0(u)$ . One deduces, using (15), that  $\int_{\Omega} u_i u_j d\mu_{ij} = 0$ , and thus that  $\text{supp}(\mu^\bullet) \subset \Gamma_2$ , which ends the proof. ■

**Remark 14** 1. We thus get Navier's wall law at the zeroth-order limit of the problem (1).

2. Theorem 13 can be extended to every kind of obstacle functional in  $\mathbb{F}$ , using Theorem 6 for the integral representation. One can define, for example, sequences of obstacle functionals on  $\mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{O}(\mathbf{R}^N)$  of the kind

$$(F^\varepsilon)^+(u, \omega) = \begin{cases} 0 & \text{if } \tilde{u} \geq 0 \text{ q.e. on } \Gamma_{2,\varepsilon} \cap \omega, \\ +\infty & \text{otherwise,} \end{cases}$$

the limit  $(F^0)^+$  of which is defined on  $\mathbf{V}_{\Gamma_1}(\Omega) \times (\mathcal{R}^+ \cap \mathcal{O}(\mathbf{R}^N))$  (for some rich family  $\mathcal{R}^+$ ) as

$$(F^0)^+(u, \omega) = \int_{\omega \cap \Gamma_2} u_i^+ u_j^+ d\mu_{ij},$$

where  $u_i^+ = \max(0, u_i)$ ,  $i = 1, \dots, N$ .

3. One proves that  $\mu_{ij} \in \mathbf{H}^{-1/2}(\Gamma_2)$ ,  $\forall i, j = 1, \dots, N$ , where  $\mu_{ij}$  is the measure defined in Theorem 13. One first observes that the measure  $\lambda$  defined in Theorem 6 belongs to  $\mathbf{H}^{-1/2}(\Gamma_2)^+$ .  $\lambda$  is indeed finite. Because for every compact subset  $K \subset \Gamma_2$ , one has  $\lambda(K) < +\infty$ , hence  $\lambda$  is a Radon nonnegative measure. Moreover, because  $\lambda$  is absolutely continuous with respect to the capacity  $Cap$ , we deduce from [6, Theorem 2.2], the existence of a Radon measure  $\varkappa \in \mathbf{H}^{-1/2}(\Gamma_2)$  and of a Borel function  $f : \Gamma_2 \rightarrow [0, +\infty[$  such that  $f = \frac{d\lambda}{d\varkappa}$ .

Let us come back to the study of problem (1). The solution  $u^\varepsilon$  of (1), with the homogeneous Dirichlet boundary conditions on  $\partial\Omega_\varepsilon$  is also the solution of the minimization problem

$$\inf_{v \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)} \left( G^\varepsilon(v, \mathbf{R}^N) + 2 \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot v dx - 2 \int_{\Omega_\varepsilon} f \cdot v dx \right). \quad (16)$$

From Theorem 13, one deduces the following asymptotic behaviour of the solution of (1).

**Corollary 15** *The solution  $(u^\varepsilon, p^\varepsilon)$  of (1), is such that  $(u^\varepsilon)_\varepsilon$  converges to  $u^0$  in the topology  $\tau$  and  $\left((p^\varepsilon)|_\Omega\right)_\varepsilon$  converges to  $p^0$  in the strong topology of  $\mathbf{L}^2(\Omega)/\mathbf{R}$ , where  $(u^0, p^0)$  belongs to  $\mathbf{V}_{0,\Gamma_1}(\Omega) \times \mathbf{L}^2(\Omega)/\mathbf{R}$  and is the solution of the limit minimization problem*

$$\inf_{v \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)} \left( G^0(v, \mathbf{R}^N) + 2 \int_\Omega (u^0 \cdot \nabla) u^0 \cdot v dx - 2 \int_\Omega f \cdot v dx \right), \quad (17)$$

or of the limit problem with Navier law

$$\begin{cases} -\nu \Delta u^0 + (u^0 \cdot \nabla) u^0 + \nabla p^0 &= f & \text{in } \Omega, \\ \operatorname{div}(u^0) &= 0 & \text{in } \Omega, \\ u^0 &= 0 & \text{on } \Gamma_1, \\ u^0 \cdot n &= 0 & \text{on } \Gamma_2, \\ (I - n \otimes n) \nu \frac{\partial u^0}{\partial n} + \mu^\bullet u^0 &= 0 & \text{on } \Gamma_2. \end{cases} \quad (18)$$

**Proof.** We first observe that, for every sequence  $(v_\varepsilon)_\varepsilon$  converging to  $v$  in the topology  $\tau$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f \cdot v_\varepsilon dx = \int_\Omega f \cdot v dx,$$

Thanks to the properties of the  $\Gamma$ -convergence,  $(u^\varepsilon)_\varepsilon$  converges to  $u^0$  in the topology  $\tau$ , with  $u^0 \in \mathbf{V}_{\Gamma_1}(\Omega)$ , and

$$\lim_{\varepsilon \rightarrow 0} G^\varepsilon(u^\varepsilon, \mathbf{R}^N) = G^0(u^0, \mathbf{R}^N) = \nu \int_\Omega |\nabla u^0|^2 dx + \int_{\Gamma_2} (u^0)_i (u^0)_j d\mu_{ij}.$$

Then

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot v_\varepsilon dx = \int_\Omega (u^0 \cdot \nabla) u^0 \cdot v dx,$$

for every sequence  $(v_\varepsilon)_\varepsilon$  converging to  $v$  in the topology  $\tau$ . For every  $\varphi \in \mathbf{C}^1(\mathbf{R}^N)$ , one has

$$\left| \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx \right| \leq \left( \int_{\Sigma_\varepsilon} |\nabla \varphi|^2 dx \right)^{1/2} \left( \int_{\mathbf{R}^N} |u^\varepsilon|^2 dx \right)^{1/2},$$

and thus  $\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx = 0$ . Because  $\operatorname{div}(u^\varepsilon) = \operatorname{div}(u^0) = 0$ , and  $u^\varepsilon = 0$ , q.e. on  $\Gamma_2$ , one has

$$0 = \int_{\Omega_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx = \int_\Omega u^\varepsilon \cdot \nabla \varphi dx + \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla \varphi dx.$$

Taking the limit of this equality, we obtain

$$0 = \int_\Omega u^0 \cdot \nabla \varphi dx = \int_{\Gamma_2} u^0 \cdot n \varphi d\Gamma_2,$$

which proves that  $u^0 \cdot n = 0$  on  $\Gamma_2$ . Thus  $u^0 \in \mathbf{V}_{0,\Gamma_1}(\Omega)$  is the solution of the problem (17). The variational formulation of (17) can be written as

$$\begin{aligned} \forall \varphi \in \mathbf{V}_{0,\Gamma_1}(\Omega) : & \int_\Omega (-\nu \Delta u^0 + (u^0 \cdot \nabla) u^0) \cdot \varphi dx \\ & + \int_{\Gamma_2} \nu \frac{\partial u^0}{\partial n} \cdot \varphi d\Gamma_2 + \int_{\Gamma_2} (u^0)_i \varphi_j d\mu_{ij} = \int_\Omega f \cdot \varphi dx. \end{aligned}$$

There exists  $p_0 \in L^2(\Omega)/\mathbf{R}$  such that  $-\nu \Delta u^0 + (u^0 \cdot \nabla) u^0 - f = -\nabla p_0$ . Thanks to Proposition 9, the sequence  $\left((p^\varepsilon)|_\Omega\right)_\varepsilon$  converges to  $p^0$  in the strong topology of  $\mathbf{L}^2(\Omega)/\mathbf{R}$ . Because  $\varphi \cdot n = 0$  on  $\Gamma_2$ , with  $n = (0, 0, 1)$ , one has:  $\nu \frac{\partial u^0}{\partial n} \cdot \varphi = (Id - n \otimes n) \nu \frac{\partial u^0}{\partial n} \cdot \varphi$ , which ends the proof. ■

## 5 Special cases

We intend to specialize the general result obtained in Theorem 13, in two cases where the boundary  $\Gamma_{2,\varepsilon}$  can be defined through some Lipschitz continuous function.

### 5.1 Periodic case

In this section, we suppose that  $\Omega \subset \{x_3 > 0\}$  with  $\partial\Omega \cap \{x_3 = 0\} = \Gamma_2$ ,  $\Gamma_2$  containing 0. We define  $Y = (-1/2, 1/2)^2$  and consider a  $Y$ -periodic function  $h \in \mathbf{C}_c^2(Y, \mathbf{R}_+)$ . For every  $k \in \mathbf{Z}^2$ , we define  $Y_\varepsilon^k = (-\varepsilon/2, \varepsilon/2)^2 + (k_1\varepsilon, k_2\varepsilon)$ , and let  $I_\varepsilon = \{k \in \mathbf{Z}^2 \mid Y_\varepsilon^k \subset \Gamma_2\}$ . We define  $h_\varepsilon$  on  $\Gamma_2$  through

$$h_\varepsilon(x') = \begin{cases} h\left(\frac{x'}{\varepsilon}\right) & \text{if there exists } k \in I_\varepsilon \text{ such that } x' = (x_1, x_2) \in Y_\varepsilon^k, \\ 0 & \text{otherwise} \end{cases}$$

and  $\Sigma_\varepsilon$  through

$$\Sigma_\varepsilon = \{x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon h_\varepsilon(x') < x_3 < 0\}.$$

Thanks to Theorem 13, there exist a rich family  $\mathcal{R} \subset \mathcal{B}(\mathbf{R}^3)$ , a symmetric matrix  $(\mu_{ij})_{i,j=1,\dots,N}$  of Borel measures having the same support contained in  $\Gamma_2$ , absolutely continuous with respect to the capacity  $Cap$ , and satisfying  $\mu_{ij}(B) \zeta_i \zeta_j \geq 0, \forall \zeta \in \mathbf{R}^3, \forall B \in \mathcal{B}(\mathbf{R}^3)$ , such that, for every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and every  $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbf{R}^3)$

$$\inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \mid u + z_\varepsilon = 0 \text{ on } \{x_3 = -\varepsilon h_\varepsilon(x')\} \cap \omega \text{ and } z_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\} = \int_{\omega \cap \Gamma_2} u_i u_j d\mu_{ij}, \quad (19)$$

where  $\Phi^\varepsilon$  is the energy functional defined in (10).

Because the lower boundary  $\Gamma_{2,\varepsilon}$  of  $\Sigma_\varepsilon$ , defined through the equality  $\Gamma_{2,\varepsilon} = \{(x', x_3) \mid x_3 = -\varepsilon h_\varepsilon(x')\}$ , has a periodic structure, the measures  $\mu_{ij}, i, j = 1, \dots, N$ , are invariant under translations on  $\Gamma_2$ . This implies  $\mu_{ij} = K_{ij} dx'$ , where  $K_{ij}, i, j = 1, 2, 3$ , are constants in  $\bar{\mathbf{R}}$  satisfying  $K_{ij} \zeta_i \zeta_j \geq 0, \forall \zeta \in \mathbf{R}^3$ .

The purpose of this section is to identify these constants  $K_{ij}, i, j = 1, 2, 3$ . We observe that we do not have to determine  $K_{i3}, i = 1, 2, 3$ , because, in the limit problem, one has  $u \cdot n = u \cdot e_3 = u_3 = 0$ .

**Theorem 16** *The limit Navier wall law of the limit problem (18) is in this case*

$$\frac{\partial (u^0)_m}{\partial x_3} = c_m (u^0)_m, \text{ on } \Gamma_2, m = 1, 2,$$

where the constants  $c_m$  are defined in (21).

**Proof.** We define the set  $Z_h = \{x \mid x' \in Y, -h(x') < x_3 < 0\}$  and consider in  $Z_h$  the local Stokes problems for  $m = 1, 2$

$$\begin{cases} -\Delta w^m + \nabla q^m &= e^m & \text{in } Z_h, \\ \operatorname{div}(w^m) &= 0 & \text{in } Z_h, \\ w^m &= e^m & \text{on } \{x_3 = -h(x')\}, \\ w^m &= 0 & \text{on } \{x_3 = 0\}, \\ w^m, q^m && Y\text{-periodic,} \end{cases} \quad (20)$$

where  $e^m$  is the  $m$ -th vector of the canonical basis of  $\mathbf{R}^3$ . Lax-Milgram' Theorem implies that (20) has a unique solution  $(w^m, q^m)$  with

$$\begin{aligned} w^m &\in \mathbf{V}(Z_h) = \left\{ u \in \mathbf{H}^1(Z_h, \mathbf{R}^3) \mid \operatorname{div}(u) = 0 \text{ in } Z_h, \right. \\ &\quad \left. u = 0 \text{ on } \{x_3 = 0\}, u \text{ } Y\text{-periodic} \right\} \\ q^m &\in \mathbf{L}^2(Z_h)/\mathbf{R}, q^m \text{ } Y\text{-periodic.} \end{aligned}$$

Let  $z_h = \max_{x' \in Y} h(x')$  and choose  $H > z_h$ . We define

$$\tilde{Z}_h = \{x \mid x' \in Y, -H < x_3 < -h(x')\}$$

and consider in  $\tilde{Z}_h$  problems similar to (20) except that we impose  $\tilde{w}^m = e^m$  on  $\{x_3 = -h(x')\}$  and  $\tilde{w}^m = 0$  on  $\{x_3 = -H\}$ . Let us define

$$\begin{aligned} \tilde{\Sigma}_\varepsilon &= \{x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon H < x_3 < -\varepsilon h_\varepsilon(x')\}, \\ B_\varepsilon &= \{x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, -\varepsilon H < x_3 < 0\} \end{aligned}$$

and the functions  $(w^{\varepsilon m}, q^{\varepsilon m})$  and  $(\tilde{w}^{\varepsilon m}, \tilde{q}^{\varepsilon m})$  through

$$\begin{cases} w^{\varepsilon m}(x) &= w^m\left(\frac{x}{\varepsilon}\right), & q^{\varepsilon m}(x) &= q^m\left(\frac{x}{\varepsilon}\right), \\ \tilde{w}^{\varepsilon m}(x) &= \tilde{w}^m\left(\frac{x}{\varepsilon}\right), & \tilde{q}^{\varepsilon m}(x) &= \tilde{q}^m\left(\frac{x}{\varepsilon}\right). \end{cases}$$

We finally build the function  $z_\varepsilon^{0m}$ , on  $B_\varepsilon$ , through

$$z_\varepsilon^{0m}(x) = \begin{cases} w^{\varepsilon m}(x) & \text{if } x \in \Sigma_\varepsilon, \\ e^m & \text{on } \{x_3 = -\varepsilon h_\varepsilon(x')\}, \\ \tilde{w}^{\varepsilon m}(x) & \text{on } \tilde{\Sigma}_\varepsilon. \end{cases}$$

Because  $h = 0$  on  $\partial Y$ , one can suppose that  $z_\varepsilon^{0m} = 0$  on  $\partial\Gamma_2 \times (-\varepsilon H, 0)$ . This implies that  $z_\varepsilon^{0m} \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^3, \text{div})$  and  $z_\varepsilon^{0m} = 0$  on  $\partial B_\varepsilon$ . Moreover

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^N} |z_\varepsilon^{0m}|^2 dx &= \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} |z_\varepsilon^{0m}|^2 dx \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} \int_{-\varepsilon H}^0 |z_\varepsilon^{0m}|^2 dx \\ &= \lim_{\varepsilon \rightarrow 0} \left( \sum_{k \in I_\varepsilon} \varepsilon^3 \int_Y \int_{-H}^{-h(x')} |\tilde{w}^m(x)|^2 dx \right. \\ &\quad \left. + \sum_{k \in I_\varepsilon} \varepsilon^3 \int_Y \int_{-h(x')}^0 |w^m(x)|^2 dx \right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0m}) &= \lim_{\varepsilon \rightarrow 0} \nu \varepsilon \int_{\Sigma_\varepsilon} |\nabla z_\varepsilon^{0m}|^2 dx \\ &= \nu \sum_{k \in I_\varepsilon} \varepsilon^2 \int_Y \int_{-h(x')}^0 |\nabla w^m(x)|^2 dx \\ &= \nu |\Gamma_2| c_m, \end{aligned}$$

with

$$c_m = \int_{Z_h} |\nabla w^m|^2 dx. \quad (21)$$

Taking  $u = -e^m$  on  $\Sigma_\varepsilon$ , in (19), one obtains

$$\begin{aligned} K_{mm} |\Gamma_2| &= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \mid z_\varepsilon = e^m \text{ on } \{x_3 = -\varepsilon h_\varepsilon(x')\}, \right. \\ &\quad \left. z_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\} \\ &\leq \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0m}) = \nu c_m |\Gamma_2|. \end{aligned}$$

This implies

$$K_{mm} |\Gamma_2| \leq \nu c_m |\Gamma_2|. \quad (22)$$



Take any sequence  $(z_\varepsilon)_\varepsilon \subset \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^3, \text{div})$  such that  $z_\varepsilon = e^m$  on the surface  $\{x_3 = -\varepsilon h_\varepsilon(x')\}$  and  $(z_\varepsilon)_\varepsilon$  converges to 0 in the topology  $\tau$ . We write the subdifferential inequality

$$\Phi^\varepsilon(z_\varepsilon) \geq \Phi^\varepsilon(z_\varepsilon^{0m}) + 2\nu\varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0m}) dx. \quad (23)$$

We observe that

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0m}) dx &= -\varepsilon \int_{\Sigma_\varepsilon} \Delta z_\varepsilon^{0m} \cdot (z_\varepsilon - z_\varepsilon^{0m}) dx \\ &\quad - \varepsilon \int_{\Gamma_2} \frac{\partial z_\varepsilon^{0m}}{\partial n} \cdot (z_\varepsilon - z_\varepsilon^{0m}) d\Gamma_2. \end{aligned}$$

Using the regularity (at least  $\mathbf{H}^2$ ) of  $w^m$ , we obtain

$$\varepsilon \Delta z_\varepsilon^{0m} \xrightarrow{\varepsilon \rightarrow 0} \mathbf{1}_{\Gamma_2} \int_{Z_h} \Delta w^m(x) dx,$$

where the convergence takes place in the weak topology of  $\mathbf{L}^2(\mathbf{R}^3, \mathbf{R}^3)$  and  $\mathbf{1}_{\Gamma_2}$  is the characteristic function of  $\Gamma_2$ . Then

$$\begin{aligned} &\left| \varepsilon \int_{\Gamma_2} \frac{\partial z_\varepsilon^{0m}}{\partial n} \cdot (z_\varepsilon - z_\varepsilon^{0m}) d\Gamma_2 \right| \\ &\leq \left( \int_{\Gamma_2} \left| \frac{\partial w^m}{\partial n} \right|^2 d\Gamma_2 \right)^{1/2} \left( \int_{\mathbf{R}^3} |z_\varepsilon - z_\varepsilon^{0m}|^2 dx \right)^{1/2}. \end{aligned}$$

Because  $(z_\varepsilon - z_\varepsilon^{0m})_\varepsilon$  converges to 0 in the strong topology  $\mathbf{L}^2(\mathbf{R}^3, \mathbf{R}^3)$ , we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0m}) dx = 0.$$

Taking the  $\liminf$  in (23), one obtains

$$\liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0m}) = \nu c_m |\Gamma_2|.$$

In this last inequality, taking the infimum with respect to all sequences  $(z_\varepsilon)_\varepsilon$  satisfying the imposed conditions, one obtains:  $K_{mm} |\Gamma_2| \geq \nu c_m |\Gamma_2|$ . This inequality and (22) imply:  $K_{mm} = \nu c_m$ . Taking now  $u = -(e^1 + e^2)$  on  $\Sigma_\varepsilon$  in (19), one obtains

$$\begin{aligned} (K_{11} + 2K_{12} + K_{22}) |\Gamma_2| &= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \mid z_\varepsilon = e^1 + e^2 \text{ on } \{x_3 = -\varepsilon h_\varepsilon(x')\}, z_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \right\} \\ &\leq \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{01} + z_\varepsilon^{02}). \end{aligned}$$

Because  $\int_{Z_h} \nabla w^1 \cdot \nabla w^2 dz = 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{01} + z_\varepsilon^{02}) = \nu |\Gamma_2| (c_1 + c_2).$$

This implies:  $K_{12} \leq 0$ , through the above expression of  $K_{mm}$ . Writing a subdifferential inequality as in (23), one obtains:  $K_{12} \geq 0$ , which implies:  $K_{12} = 0$ . ■

## 5.2 Case where $h_\varepsilon$ is independent of $\varepsilon$

As in the previous section, we still suppose that  $\Omega \subset \{x_3 > 0\}$  and  $\partial\Omega \cap \{x_3 = 0\} = \Gamma_2$ . But, we here suppose that the boundary  $\Gamma_{2,\varepsilon}$  is given as

$$\Gamma_{2,\varepsilon} = \{(x', x_3) \mid x_3 = -\varepsilon h(x')\}$$

where  $h$  is a Lipschitz continuous function satisfying  $h(x') > 0, \forall x' \in \Gamma_2$ . We have the following result.

**Theorem 17** *Under the preceding hypothesis, the Navier wall law is in this case*

$$(Id - n \otimes n) \frac{\partial u^0}{\partial n} + \frac{u^0}{h} = 0, \text{ on } \Gamma_2.$$

**Proof.** Thanks to Theorem 13, there exist a rich family  $\mathcal{R}_{\Gamma_2} \subset \mathcal{B}(\Sigma)$ , a symmetric matrix  $(\mu_{ij})_{i,j=1,\dots,N}$  of Borel measures having their support contained in  $\Gamma_2$ , which are absolutely continuous with respect to the capacity  $Cap$ , and satisfying  $\mu_{ij}(B) \zeta_i \zeta_j \geq 0, \forall \zeta \in \mathbf{R}^3, \forall B \in \mathcal{B}(\Sigma)$ , such that, for every  $u \in \mathbf{V}_{\Gamma_1}(\Omega)$  and every  $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$

$$\int_{\omega} u_i u_j d\mu_{ij} = \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon) \mid u + z_\varepsilon = 0 \text{ on } \{x_3 = -\varepsilon h(x')\} \cap \omega, z_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\}. \quad (24)$$

Take  $u = -e^1$  on  $\{x_3 = -\varepsilon h(x')\}$ . Then choose  $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$ , an open subset  $\omega^\varepsilon$  of  $\mathbf{R}^2$  such that  $\omega^\varepsilon \setminus \bar{\omega} = \{x' \in \mathbf{R}^2 \mid 0 < d(x', \partial\omega) < \varepsilon\}$  and  $\varphi^\varepsilon \in \mathbf{C}^1(\mathbf{R}^2)$  with  $0 \leq \varphi^\varepsilon \leq 1$  such that

$$\begin{cases} \varphi^\varepsilon &= 1 & \text{in } \omega, \\ \varphi^\varepsilon &= 0 & \text{on } \partial\omega_\varepsilon. \end{cases}$$

We define the function  $w^{1\varepsilon}$  through

$$\begin{cases} (w^{1\varepsilon})_1(x) &= \frac{x_3}{\varepsilon h(x')} \varphi^\varepsilon(x'), \\ (w^{1\varepsilon})_2(x) &= 0, \\ (w^{1\varepsilon})_3(x) &= \frac{\varepsilon}{2} \left( \frac{\partial h}{\partial x_1}(x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') h(x') \right) \\ &\quad + \frac{(x_3)^2}{2} \left( \frac{1}{\varepsilon h(x')} \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') - \frac{\varphi^\varepsilon(x')}{\varepsilon h^2(x')} \frac{\partial h}{\partial x_1}(x') \right). \end{cases}$$

One has  $\operatorname{div}(w^{1\varepsilon}) = 0, \forall \varepsilon > 0$ , and  $w^{1\varepsilon} = e^1$  on  $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$ . We now consider the problem

$$\begin{cases} -\Delta \zeta^{1\varepsilon} + \nabla \varpi^{1\varepsilon} &= e^1 & \text{in } \Omega, \\ \operatorname{div}(\zeta^{1\varepsilon}) &= 0 & \text{in } \Omega, \\ \zeta^{1\varepsilon} &= 0 & \text{on } \Gamma_1, \\ \zeta^{1\varepsilon} &= \left( 0, 0, \frac{1}{2} \left( \frac{\partial h}{\partial x_1}(x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') h(x') \right) \right) & \text{on } \Gamma_2. \end{cases} \quad (25)$$

The problem (25) has a unique solution  $(\zeta^{1\varepsilon}, \varpi^{1\varepsilon}) \in \mathbf{H}_{\Gamma_1}^1(\Omega, \operatorname{div}) \times \mathbf{L}^2(\Omega) / \mathbf{R}$ , satisfying

$$\int_{\Omega} |\nabla \zeta^{1\varepsilon}|^2 dx \leq C; \int_{\Omega} |\zeta^{1\varepsilon}|^2 dx \leq C,$$

where  $C$  is a constant independent of  $\varepsilon$ . Let  $H > z_h$ , with  $z_h = \max_{\Gamma_2} h$ . We define the function  $\tilde{w}^{1\varepsilon}$  in  $D_\varepsilon = \{x \mid -H < x_3 < -\varepsilon h(x')\}$  through

$$\left\{ \begin{array}{l} (\tilde{w}^{1\varepsilon})_1(x) = \frac{x_3 + H}{\varepsilon(H - h(x'))} \varphi^\varepsilon(x'), \\ (\tilde{w}^{1\varepsilon})_2(x) = 0, \\ (\tilde{w}^{1\varepsilon})_3(x) = \frac{\varepsilon}{2} \left( \frac{\partial h}{\partial x_1}(x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') (H - h(x')) \right) \\ \quad - \frac{(x_3 + H)^2}{2} \left( \frac{1}{\varepsilon(H - h(x'))} \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') \right. \\ \quad \left. + \frac{\varphi^\varepsilon(x')}{\varepsilon(H - h(x'))^2} \frac{\partial h}{\partial x_1}(x') \right). \end{array} \right.$$

We consider the bounded, smooth and open subset  $\Omega_H = \{x \mid x_3 > -H\}$  and  $\partial\Omega_H \cap \{x \mid x_3 = -H\} = \Gamma_2$ , and the solution  $(\zeta_H^{1\varepsilon}, \omega_H^{1\varepsilon})$  of the problem

$$\left\{ \begin{array}{ll} -\Delta \zeta_H^{1\varepsilon} + \nabla \omega_H^{1\varepsilon} = e^1 & \text{in } \Omega_H, \\ \operatorname{div}(\zeta_H^{1\varepsilon}) = 0 & \text{in } \Omega_H, \\ \zeta_H^{1\varepsilon} = 0 & \text{on } \Omega_H \setminus \Gamma_2, \\ \zeta_H^{1\varepsilon} = \left( 0, 0, \frac{1}{2} \left( \frac{\partial h}{\partial x_1}(x') \varphi^\varepsilon(x') - \frac{\partial \varphi^\varepsilon}{\partial x_1}(x') (H - h(x')) \right) \right) & \text{on } \Gamma_2. \end{array} \right.$$

Let us define the function  $z_0^{1,\varepsilon}$  through

$$z_\varepsilon^{0,1} = \begin{cases} \varepsilon \zeta^{1\varepsilon} & \text{in } \Omega, \\ w^{1\varepsilon} & \text{in } \Sigma_\varepsilon, \\ \tilde{w}^{1\varepsilon} & \text{in } D_\varepsilon, \\ \varepsilon \zeta_H^{1\varepsilon} & \text{in } \Omega_H. \end{cases}$$

One immediately verifies that  $z_\varepsilon^{0,1} \in \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^3, \operatorname{div})$ ,  $z_\varepsilon^{0,1} = e^1$  on the surface  $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$ ,  $(z_\varepsilon^{0,1})_\varepsilon$  converges to 0 in the strong topology of  $\mathbf{L}^2(\mathbf{R}^3, \mathbf{R}^3)$  and

$$\lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0,1}) = \lim_{\varepsilon \rightarrow 0} \nu \varepsilon \int_{\omega^\varepsilon \times (-\varepsilon h(x'), 0)} |\nabla z_\varepsilon^{0,1}|^2 dx = \nu \int_\omega \frac{dx'}{h(x')}.$$

One thus deduces from (24) within this context

$$\mu_{11}(\omega) \leq \nu \int_\omega \frac{dx'}{h(x')}.$$

Furthermore, taking  $(z_\varepsilon)_\varepsilon \subset \mathbf{H}_{\Gamma_1}^1(\mathbf{R}^3, \operatorname{div})$ ,  $z_\varepsilon = e^1$  on  $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$ ,  $(z_\varepsilon)_\varepsilon$  converges to 0 in the topology  $\tau$ , and using the subdifferential inequality

$$\begin{aligned} \Phi^\varepsilon(z_\varepsilon) &\geq \Phi^\varepsilon(z_\varepsilon^{0,1}) \\ &\quad + \nu \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0,1} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0,1}) dx + \nu \int_\Omega \nabla z_\varepsilon^{0,1} \cdot \nabla (z_\varepsilon - z_\varepsilon^{0,1}) dx, \end{aligned}$$

we prove that  $\mu_{11}(\omega) \geq \nu \int_\omega dx'/h(x')$ . This implies the equality:  $\mu_{11}(\omega) = \nu \int_\omega dx'/h(x')$  and, since this equality is true for every  $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$ , we obtain  $\mu_{11} = \nu dx'/h(x')$ .

Choosing now  $u = -e^2$  on  $\Sigma_\varepsilon$ , we can build a test-function  $z_\varepsilon^{0,2}$  in a similar way and prove:  $\mu_{22} = \nu dx'/h(x')$ .

Finally, taking  $u = -(e^1 + e^2)$  on  $\Sigma_\varepsilon$ , we consider the sequence  $(z_\varepsilon^0)_\varepsilon$  defined through:  $z_\varepsilon^0 = z_\varepsilon^{0,1} + z_\varepsilon^{0,2}$ . One deduces from the above computations that

$$\lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^0) = \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon(z_\varepsilon^{0,1} + z_\varepsilon^{0,2}) = 2\nu \int_\omega \frac{dx'}{h(x')}$$

and, as in the periodic case, that  $\mu_{12} = 0$ . The boundary conditions on  $\Gamma_2$  can thus be written as

$$\begin{cases} (u^0)_3 &= 0, \\ \frac{\partial (u^0)_m}{\partial x_3} &= \frac{1}{h} (u^0)_m, \quad m = 1, 2, \end{cases}$$

which ends the proof. ■

**Remark 18** In a general way, if  $\Sigma_\varepsilon = \{\sigma + tn \mid \sigma \in \Gamma_2, -\varepsilon h(\sigma) < t < 0\}$ , with  $h$  positive and Lipschitz continuous on  $\Gamma_2$ , we can prove that the limit law is

$$\begin{cases} (Id - n \otimes n) \frac{\partial u^0}{\partial n} + \frac{u^0}{u^0 \cdot n} &= 0, \\ u^0 \cdot n &= 0. \end{cases}$$

## 6 Optimal control problem

For a given real  $m > 0$ , we consider the set  $\Xi_m$  of all matrices  $\mathbf{h} = \text{Diag}(h_i)_{i=1,\dots,N}$  of functions  $h_i : \Gamma_2 \rightarrow [0, +\infty]$ ,  $d\Gamma_2$ -measurable and such that

$$\int_{\Gamma_2} h_i d\Gamma_2 = m, \quad \forall i = 1, \dots, N.$$

We suppose that  $\partial\Omega$  is  $\mathbf{C}^2$  and consider the Navier-Stokes problem, with Navier wall law, according to Theorem 17

$$\begin{cases} -\nu \Delta u^h + (u^h \cdot \nabla) u^h + \nabla p^h &= f & \text{in } \Omega, \\ \operatorname{div}(u^h) &= 0 & \text{in } \Omega, \\ \mathbf{h} (Id - n \otimes n) \frac{\partial u^h}{\partial n} + u^h &= 0 & \text{on } \Gamma_2, \\ u^h \cdot n &= 0 & \text{on } \Gamma_2, \\ u^h &= 0 & \text{on } \Gamma_1, \end{cases} \quad (26)$$

which has a unique solution  $(u^h, p^h) \in \mathbf{V}_{0,\Gamma_1}(\Omega) \times \mathbf{L}^2(\Omega)/\mathbf{R}$ . We define the functional  $\mathbf{F}$  defined on  $\Xi_m \times \mathbf{H}_{\Gamma_1}^1(\Omega, \operatorname{div})$  and associated to (26) through

$$\mathbf{F}(\mathbf{h}, u) = \begin{cases} \frac{\nu}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \sum_{i=1}^N \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2 \\ \quad + \int_\Omega (u^h \cdot \nabla) u^h \cdot u dx - \int_\Omega f \cdot u dx & \text{if } u \in \mathbf{V}_{0,\Gamma_1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We consider the optimal control problem (3), which means that the cost functional is here taken as the global energy. We observe that

$$\mathbf{F}(\mathbf{h}, u^h) = - \int_\Omega f \cdot u^h dx.$$

This implies that the minimization of  $\mathbf{F}$ , with respect to  $u$  on the set  $\mathbf{V}_{0,\Gamma_1}(\Omega)$ , is equivalent to the maximization of the work of the external forces on this set. The problem (3) has a unique minimizer when Poincaré's inequality

$$\left( \int_{\Gamma_2} |u_i| d\Gamma_2 \right)^2 \leq \int_{\Gamma_2} h_i d\Gamma_2 \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2,$$

becomes an equality, for every  $i = 1, \dots, N$ , that is when

$$h_i^m = m \frac{|u_i^m|_{\Gamma_2}}{\int_{\Gamma_2} |u_i^m| d\Gamma_2},$$

where  $(u^m, p^m)$  is the solution of

$$\left\{ \begin{array}{l} -\nu \Delta u^m + (u^m \cdot \nabla) u^m + \nabla p^m = f \quad \text{in } \Omega, \\ \operatorname{div} (u^m) = 0 \quad \text{in } \Omega, \\ u^m \cdot n = 0 \quad \text{on } \Gamma_2, \\ u^m = 0 \quad \text{on } \Gamma_1, \\ \\ (Id - n \otimes n) \frac{\partial u^m}{\partial n} \\ + \frac{1}{m} \left( \begin{array}{c} \operatorname{sign}((u^m)_1(x)) \int_{\Gamma_2} |(u^m)_1| d\Gamma_2 \\ \vdots \\ \operatorname{sign}((u^m)_N(x)) \int_{\Gamma_2} |(u^m)_N| d\Gamma_2 \end{array} \right) = 0 \quad \text{on } \Gamma_2. \end{array} \right.$$

Trivially, the study of the  $\Gamma$ -convergence of the sequence of the energies associated to (3), when  $m$  goes to 0 and relatively to the weak topology of  $\mathbf{H}^1(\Omega, \mathbf{R}^N)$ , will lead to the following conclusions:  $(u^m)_m$  converges to  $u^0$  in the weak topology of  $\mathbf{H}^1(\Omega, \mathbf{R}^N)$ ,  $(p^m)_m$  converges to  $p^0$  in the strong topology of  $\mathbf{L}^2(\Omega)/\mathbf{R}$ , where  $(u^0, p^0)$  is the solution of the problem

$$\left\{ \begin{array}{l} -\nu \Delta u^0 + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f \quad \text{in } \Omega, \\ \operatorname{div} (u^0) = 0 \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (27)$$

In order to study the asymptotic behavior of  $\left((u^m/m)|_{\Gamma_2}\right)_m$ , we introduce the following linearized perturbation of the Navier-Stokes problem (27)

$$\left\{ \begin{array}{l} -\nu \Delta u^{0,m} + \nabla p^{0,m} = f - (u^m \cdot \nabla) u^m \quad \text{in } \Omega, \\ \operatorname{div} (u^{0,m}) = 0 \quad \text{in } \Omega, \\ u^{0,m} = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (28)$$

The problem (28) is a Stokes system, the source term of which is  $f - (u^m \cdot \nabla) u^m$ . Consider now the functional  $I_m$  defined on  $\mathbf{V}_{0,\Gamma_1}(\Omega)$  through

$$\begin{aligned} I_m(v) &= \frac{m\nu}{2} \int_{\Omega} |\nabla v|^2 dx + \frac{1}{2} \sum_{i=1}^N \left( \int_{\Gamma_2} |v_i| d\Gamma_2 \right)^2 \\ &\quad + \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \cdot v d\Gamma_2. \end{aligned}$$

$I_m$  has a unique minimizer  $(v^m, q^m) \in \mathbf{V}_{0,\Gamma_1}(\Omega) \times \mathbf{L}^2(\Omega) / \mathbf{R}$  which is the solution of the problem

$$\left\{ \begin{array}{lcl} -\nu m \Delta v^m + \nabla q^m & = & 0 \quad \text{in } \Omega, \\ \operatorname{div}(v^m) & = & 0 \quad \text{in } \Omega, \\ v^m \cdot n & = & 0 \quad \text{on } \Gamma_2, \\ v^m & = & 0 \quad \text{on } \Gamma_1, \\ (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} + m (Id - n \otimes n) \frac{\partial v^m}{\partial n} \\ + \left( \begin{array}{c} \operatorname{sign}((v^m)_1) \int_{\Gamma_2} |(v^m)_1| d\Gamma_2 \\ \vdots \\ \operatorname{sign}((v^m)_N) \int_{\Gamma_2} |(v^m)_N| d\Gamma_2 \end{array} \right) & = & 0 \quad \text{on } \Gamma_2. \end{array} \right.$$

We observe that the couple  $(v^m, q^m)$  defined through

$$v^m = \frac{u^m - u^{0,m}}{m}; \quad q^m = p^m - p^{0,m},$$

is the minimizer of  $I_m$ . For every  $\varphi \in \mathbf{H}^{1/2}(\Gamma_2, \mathbf{R}^N)$ , there exists a unique extension  $v_\varphi \in \mathbf{V}_{0,\Gamma_1}(\Omega)$  of  $\varphi$  defined through

$$\int_{\Omega} |\nabla v_\varphi|^2 dx = \inf_{\{w \in \mathbf{V}_{0,\Gamma_1}(\Omega) | w|_{\Gamma_2} = \varphi\}} \int_{\Omega} |\nabla w|^2 dx.$$

Let us denote  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$  the space of finite Radon measures on  $\Gamma_2$  with values in  $\mathbf{R}^N$ . We consider the functional  $J_m$  defined on  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$  through

$$J_m(\varphi) = \begin{cases} \frac{m\nu}{2} \int_{\Omega} |\nabla v_\varphi|^2 dx + \frac{1}{2} \sum_{i=1}^N \left( \int_{\Gamma_2} |\varphi_i| d\Gamma_2 \right)^2 \\ + \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \cdot \varphi d\Gamma_2 & \text{if } \varphi \in \mathbf{H}^{1/2}(\Gamma_2, \mathbf{R}^N) \\ & \text{and } \varphi \cdot n = 0 \text{ on } \Gamma_2, \\ +\infty & \text{otherwise.} \end{cases}$$

Then  $(v^m)_{|\Gamma_2}$  is the unique minimizer of  $J_m$ .

**Proposition 19** *One has the following properties.*

1.  $\sup_m \sum_{i=1}^n \left( \int_{\Gamma_2} |v_i^m| d\Gamma_2 \right) < +\infty$ .
2. The sequence  $(J_m)_m$   $\Gamma$ -converges, when  $m$  tends to 0 and with respect to the weak\* topology of  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ , to the functional  $J$  defined from  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$  to  $\mathbf{R}$  through

$$J(\lambda) = \sum_{i=1}^N (|\lambda_i|(\Gamma_2))^2 + \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^0}{\partial n} d\lambda,$$

where  $|\lambda_i|(\Gamma_2)$  is the total variation of  $\lambda_i$  on  $\Gamma_2$ .

**Proof.** 1. Remark that a regularity property of the boundary  $\partial\Omega$  implies that

$$\sup_m \left\| (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \right\|_{\mathbf{L}^\infty(\Gamma_2, \mathbf{R}^N)} < +\infty.$$

One thus obtains

$$J_m \left( (v^m)_{|\Gamma_2} \right) \geq \frac{1}{2} \sum_{i=1}^N \left( \int_{\Gamma_2} |v_i^m| d\Gamma_2 \right)^2 - \frac{C}{2} \sum_{i=1}^N \left( \int_{\Gamma_2} |v_i^m| d\Gamma_2 \right).$$

Moreover

$$\sup_m J_m \left( (v^m)_{|\Gamma_2} \right) \leq \sup_m J_m(0) = 0 \Rightarrow \sup_m \sum_{i=1}^N \left( \int_{\Gamma_2} |v_i^m| d\Gamma_2 \right) \leq C.$$

This implies the existence of a subsequence of  $\left( (v^m)_{|\Gamma_2} \right)_m$ , still denoted  $\left( (v^m)_{|\Gamma_2} \right)_m$ , which converges to some  $\lambda$  in the weak\* topology of  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ .

2. Choose any sequence  $(\varphi^m)_m \subset \mathbf{H}^{1/2}(\Gamma_2, \mathbf{R}^N)$ , satisfying  $\varphi^m \cdot n = 0$ , on  $\Gamma_2$  and converging to  $\lambda$  in the weak\* topology of  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ . The functional  $\mu \mapsto |\mu|$ , where  $|\mu|$  is the total variation of  $\mu$ , being lower semi-continuous on  $\mathcal{M}(\Gamma_2)$ , one has

$$\liminf_{m \rightarrow 0} \int_{\Gamma_2} |\varphi_i^m| d\Gamma_2 \geq |\lambda_i|(\Gamma_2).$$

Thanks to the regularity of the boundary,  $\left( (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \right)_m$  uniformly converges to  $(Id - n \otimes n) \frac{\partial u^0}{\partial n}$ , hence

$$\liminf_{m \rightarrow 0} \int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \cdot \varphi^m d\Gamma_2 \geq \int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\lambda_i.$$

This implies

$$\liminf_{m \rightarrow 0} J_m(\varphi^m) \geq J(\lambda). \quad (29)$$

In order to prove the  $\Gamma$ -lim sup property, let us suppose that  $\Omega \subset \{x_N < 0\}$  and  $\partial\Omega \cap \{x_N = 0\} = \Gamma_2$  (in fact using a system of local coordinates, one can then study the case of every smooth surface  $\Gamma_2$ ). We define  $x' = (x_1, \dots, x_{N-1})$  and the nonnegative and smooth function  $\rho_\varepsilon$  through

$$\rho_\varepsilon(x') = \begin{cases} \frac{C}{\varepsilon^{N-1}} \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |x'|^2}\right) & \text{if } |x'| < \varepsilon, \\ 0 & \text{if } |x'| \geq \varepsilon, \end{cases}$$

where

$$C = \left( \int_{B_{N-1}(0,1)} \exp\left(\frac{-1}{1 - |\zeta|^2}\right) d\zeta \right)^{-1}.$$

Let  $(\omega_{[1/\varepsilon]})_\varepsilon$ , where  $[1/\varepsilon]$  denotes the entire part of  $1/\varepsilon$ , be a sequence of open subsets of  $\Gamma_2$  such that

$$\begin{cases} \omega_1 \subset \omega_2 \subset \dots \subset \omega_{[1/\varepsilon]} \subset \dots \subset \Gamma_2, \\ \bigcup_{\varepsilon} \omega_{[1/\varepsilon]} = \Gamma_2, \\ d(\omega_{[1/\varepsilon]}, \partial\Gamma_2) = \varepsilon. \end{cases}$$

We associate the partition of unity  $(\eta_\varepsilon)_\varepsilon$  through

$$\begin{cases} \eta_\varepsilon \in \mathbf{C}_c^\infty(\omega_{[1/\varepsilon]}), \\ \eta_\varepsilon(x') = 1 \text{ in } \omega_{[1/\varepsilon]-1} \text{ ( } [1/\varepsilon] - 1 = [1/\varepsilon'], \text{ with } \varepsilon' = \frac{\varepsilon}{1-\varepsilon} \text{)}, \\ 0 \leq \eta_\varepsilon(x') \leq 1, \forall x' \in \Gamma_2, \forall \varepsilon > 0. \end{cases}$$

For  $\lambda = (\lambda_1, \dots, \lambda_{N-1}, 0) \in \mathcal{M}(\Gamma_2, \mathbf{R}^N)$ , we define the vectorial measure  $\lambda^\varepsilon$  through  $\lambda^\varepsilon = (\lambda * \rho_\varepsilon) \eta_\varepsilon$ . We observe that  $\lambda^\varepsilon \in \mathbf{C}_c^\infty(\Gamma_2, \mathbf{R}^N)$  and

$$\begin{cases} \lambda^\varepsilon & \xrightarrow{\varepsilon \rightarrow 0} \lambda & w^*-\mathcal{M}(\Gamma_2, \mathbf{R}^N), \\ |\nabla \lambda^\varepsilon|(x') & \leq \frac{C}{\varepsilon^N} & \forall x' \in \Gamma_2. \end{cases}$$

We build the function  $w^\varepsilon$

$$\begin{cases} (w^\varepsilon)_i(x) &= \frac{\varepsilon - x_N}{\varepsilon} (\lambda^\varepsilon)_i(x') & i = 1, \dots, N-1, \forall x \in \Omega, \\ (w^\varepsilon)_N(x) &= \frac{\operatorname{div}(\lambda^\varepsilon(x'))}{2} \left( \frac{(\varepsilon - x_N)^2}{\varepsilon} - \varepsilon \right). \end{cases}$$

We immediately observe that  $w^\varepsilon \in \mathbf{H}^1(\Omega, \mathbf{R}^N)$  and

$$\begin{cases} \operatorname{div}(w^\varepsilon) &= 0 & \text{in } \Omega, \\ (w^\varepsilon)_N &= 0 & \text{on } \Gamma_2, \\ w^\varepsilon &= 0 & \text{on } \Gamma_1, \end{cases}$$

that is  $w^\varepsilon \in \mathbf{V}_{0, \Gamma_1}(\Omega)$ , for every  $\varepsilon > 0$ . We now define

$$\begin{cases} \varepsilon &= m^{\frac{1}{4N}}, \\ w^m &= w^{m^{\frac{1}{4N}}}, \\ \lambda^m &= \lambda^{m^{\frac{1}{4N}}}. \end{cases}$$

One has

$$\begin{cases} m \int_{\Omega} |\nabla w^m|^2 dx &\leq C\sqrt{m}, \\ J_m(\lambda^m) &= I_m(v_{\lambda^m}) \leq I_m(w^m), \end{cases}$$

hence

$$\limsup_{m \rightarrow 0} J_m(\lambda^m) \leq \limsup_{m \rightarrow 0} I_m(w^m) = J(\lambda).$$

This inequality and (29) end the proof. ■

One has the following result.

**Theorem 20** *Let*

$$\begin{aligned} M_i &= \max_{\sigma \in \Gamma_2} \left| \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i(\sigma) \right|, \\ K_i^\pm &= \left\{ \sigma \in \Gamma_2 \mid \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i(\sigma) = \pm M_i \right\}. \end{aligned}$$

*We have the following properties.*

1. *When  $m$  goes to 0, the sequence  $\left( (u^m/m)_{|\Gamma_2} \right)_m$  converges in the weak\* topology of the space  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$  to a vectorial measure  $\lambda = (\lambda_i)_{i=1, \dots, N}$  such that  $\operatorname{supp}(\lambda_i) \subseteq K_i^+ \cup K_i^-$ , with  $\lambda_i$  positive on  $K_i^-$  and negative on  $K_i^+$ ,  $i = 1, \dots, N$ .*
2.  $\int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\lambda_i = -M_i$ ,  $i = 1, \dots, N$ .
3.  $\lim_{m \rightarrow 0} \int_{\Gamma_2} |u_i^m/m| d\Gamma_2 = |\lambda_i|(\Gamma_2) = M_i$ ,  $i = 1, \dots, N$ .



4. When  $m$  goes to 0, the sequence  $(h_i^m/m)_m$  converges in the weak\* topology of  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$  to a measure  $\bar{\lambda}_i$  such that  $\text{supp}(\bar{\lambda}_i) \subseteq K_i^+ \cup K_i^-$ ,  $\bar{\lambda}_i$  is positive on  $K_i^-$  and negative on  $K_i^+$ , and  $|\bar{\lambda}_i|(\Gamma_2) = 1$ ,  $i = 1, \dots, N$ .

**Proof.** One deduces from Proposition 19 and from the properties of the  $\Gamma$ -convergence that  $\left((v^m)_{|\Gamma_2}\right)_m = \left((u^m/m)_{|\Gamma_2}\right)_m$  converges in the weak\* topology of  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ , when  $m$  goes to 0, to a measure  $\lambda = (\lambda_i)_{i=1, \dots, N}$  such that  $J(\lambda) = \min_{v \in \mathcal{M}(\Gamma_2, \mathbf{R}^N)} J(v)$ . Define

$$\mathcal{M}_1(\Gamma_2, \mathbf{R}^N) = \{\mu \in \mathcal{M}(\Gamma_2, \mathbf{R}^N) \mid |\mu_i|(\Gamma_2) = 1, i = 1, \dots, N\}$$

and consider the functional  $\tilde{J}$  defined from  $[0, +\infty[^N \times \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)$  to  $\mathbf{R}$  through

$$\begin{aligned} \tilde{J}((t_1, \dots, t_N), (\mu_1, \dots, \mu_N)) &= J((t_1\mu_1, \dots, t_N\mu_N)) \\ &= \frac{1}{2} \sum_{i=1}^N (t_i)^2 + \sum_{i=1}^N t_i \int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\mu_i. \end{aligned}$$

One has

$$\min_{v \in \mathcal{M}(\Gamma_2, \mathbf{R}^N)} J(v) = \min_{\mu \in \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)} \min_{t_i \geq 0, i=1, \dots, N} \tilde{J}((t_1, \dots, t_N), (\mu_1, \dots, \mu_N)). \quad (30)$$

The minimum of (30) with respect to  $t = (t_1, \dots, t_N)$  exists if

$$\int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\mu_i \leq 0, \forall i = 1, \dots, N.$$

Let us now find the minimum with respect to  $\mu \in \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)$ . One has

$$-\int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\mu_i \geq -M_i,$$

for every  $\mu \in \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)$  such that

$$\int_{\Gamma_2} \left( (Id - n \otimes n) \frac{\partial u^0}{\partial n} \right)_i d\mu_i \leq 0, \forall i = 1, \dots, N,$$

the minimum being reached in the case of equality, that is if and only if  $\text{supp}(\mu_i) \subset K_i^+ \cup K_i^-$ . One has  $\lambda_i = M_i \mu_i$ ,  $i = 1, \dots, N$ . Remarking that  $\bar{\lambda}_i = \mu_i$ , one observes that  $(h_i^m/m)_m$  converges in the weak\* topology of  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ , when  $m$  tends to 0, to  $\bar{\lambda}_i$ , and the same result occurs for the sequence  $\left((|u_i^m|_{\Gamma_2}) / \int_{\Gamma_2} |u_i^m| d\Gamma_2\right)_m$ . The sequence  $(h_i^m/m)_m$  converges in  $\mathcal{M}(\Gamma_2, \mathbf{R}^N)$ -weak\* to a probability measure  $\bar{\lambda}_i$  ( $\bar{\lambda}_i(\Gamma_2) = 1$ ) with support in the set of points of  $\Gamma_2$  where the shear motions, given through  $(Id - n \otimes n) \frac{\partial u^0}{\partial n}$ , are large for the limit flow described through (27). ■

**Remark 21** We thus think that, inside this flow, a thin boundary layer of thinness  $mh_i$  occurs in the  $i$ -th direction with a probability  $\bar{\lambda}_i$  (for every  $i$ ).

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