

A new approach for regularization of inverse problems in image processing

I. Souopgui^{1,2}, E. Kamgnia², **F.-X. Le Dimet**¹, A. Vidard¹

(1) INRIA / LJK Grenoble

(2) University of Yaounde I



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Ingredients

Physical system

$$\begin{aligned}\mathcal{M}: \mathcal{V} &\rightarrow \mathcal{Y} \\ \mathbf{v} &\mapsto \mathbf{y} = \mathcal{M}(\mathbf{v})\end{aligned}$$

- $\mathbf{y} \in \mathcal{Y}$, the system state
- $\mathbf{v} \in \mathcal{V}$, the control variable
- \mathcal{M} , model mapping \mathcal{V} to \mathcal{Y}

Observation

\mathcal{Y}°

- $\mathbf{y}^{\circ} \in \mathcal{O}$, observed state

Observation system

$$\begin{aligned}\mathcal{H}: \mathcal{Y} &\rightarrow \mathcal{O} \\ \mathbf{y} &\mapsto \mathcal{H}(\mathbf{y})\end{aligned}$$

- \mathcal{H} observation operator mapping \mathcal{Y} to \mathcal{O}

Definition

Giving observed state \mathbf{y}^o ,

Inverse problem (unconstrained)

Find $\mathbf{v}^* = \text{MinArg}(J(\mathbf{v}))$, $\mathbf{v} \in \mathcal{V}$ where

$$J(\mathbf{v}) = J_o(\mathbf{v}) = \frac{1}{2} \|\mathcal{H}(\mathcal{M}(\mathbf{v})) - \mathbf{y}^o\|_0^2 \quad (1)$$

under adequate conditions, the solution \mathbf{v}^* is given by the Euler-Lagrange Equation $\nabla J(\mathbf{v}^*) = 0$

Problems

- ill-posedness \Rightarrow use *a priori* knowledges;
- ill-conditioning \Rightarrow use preconditioning.

A priori knowledges

For a priori knowledge \mathcal{A} , set $J = J_o + J_{\mathcal{A}}$
where $J_{\mathcal{A}}$ is defined to force the solution to satisfy \mathcal{A}

Use of a priori informations

- Background \mathbf{v}^b and background error covariance \mathbf{B}

$$J_b = \frac{1}{2} \alpha_b \|\mathbf{v} - \mathbf{v}^b\|_{\mathbf{B}^{-1}}^2 \quad (2)$$

- Regularity of the solution : Φ -smooth (minimum gradient)

$$J_r = \frac{1}{2} \alpha_r \|\Phi(\mathbf{v})\|^2 \quad (3)$$

Φ function of the derivatives of \mathbf{v}

Vector fields regularization

first order regularization : first order derivatives of \mathbf{v}

$$\Phi^{(1)} \left(\frac{\partial v_i}{\partial x_j} \right)_{1 \leq i, j \leq n},$$

- gradient penalization : $J_{\nabla}(\mathbf{v}) = \frac{1}{2} \alpha_{\nabla} \int_{\Omega} \sum_{i=1}^n \|\nabla v_i\|^2 dx$

second order regularization : second order derivatives of \mathbf{v}

$$\Phi^{(2)} \left(\frac{\partial^2 v_i}{\partial x_j \partial x_k} \right)_{1 \leq i, j, k \leq n},$$

- Suter regularization :

$$J_{suter}(\mathbf{v}) = \frac{1}{2} \int_{\Omega} \alpha_{\nabla div} \|\nabla div(\mathbf{v})\|^2 + \alpha_{\nabla curl} \|\nabla curl(\mathbf{v})\|^2 dx$$

⇒ difficult to defined optimal weighting parameter(s)

Notations and definition

Let :

- $\mathbf{v}(\mathbf{x})$ be an incomplete/inconsistent control variable, with $\mathbf{x} \in \Omega$ the physical space
- $\Phi(\mathbf{v})$ regularization operator as defined previously
- $\varphi(\mathbf{x})$ a scalar positive trust function given the quality of \mathbf{v} at \mathbf{x}

{ small value meaning bad/lack/inconsistent control variable
large value for good quality control variable

we define restored control variable $\mathbf{u}^* = \text{MinArg}(\varepsilon(\mathbf{u}))$, $\mathbf{u} \in \mathcal{V}$

$$\varepsilon(\mathbf{u}) = \frac{1}{2} \int_{\Omega} \|\Phi(\mathbf{u}(\mathbf{x}))\|^2 + \varphi(\mathbf{x}) \|\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})\|^2 d\mathbf{x} \quad (4)$$

$$\varepsilon(\mathbf{u}) = \frac{1}{2} \int_{\Omega} \|\Phi(\mathbf{u}(\mathbf{x}))\|^2 + \varphi(\mathbf{x}) \|\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})\|^2 d\mathbf{x}$$

ε is minimized by setting \mathbf{u} to be :

- close to \mathbf{v} when φ is large (\mathbf{v} has adequate properties)
- Φ – regular when φ is small (otherwise)

Under adequate conditions $\text{MinArg}(\varepsilon)$ is given by the Euler-Lagrange condition

$$\nabla_{\mathbf{u}}\varepsilon(\mathbf{u}) = 0 \quad (5)$$

Gateaux derivatives development leads to

$$\nabla_{\mathbf{u}}\varepsilon(\mathbf{u}) = \Phi^* \circ \Phi(\mathbf{u}(\mathbf{x})) + \varphi(\mathbf{x})(\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})) \quad (6)$$

Gradient penalization : mathematical expression

$$J_{\nabla}(\mathbf{v}) = \frac{1}{2}\alpha_{\nabla} \int_{\Omega} \sum_{i=1}^n \|\nabla v_i\|^2 dx$$

Applied as smoothing operator, we get

$$\Phi_{\nabla}^* \circ \Phi_{\nabla} = -\Delta, \text{ with boundary conditions : } \nabla u_i \perp \nu \text{ on } \partial\Omega$$

$$\Rightarrow \nabla \varepsilon_{\nabla}(u_i) = -\Delta u_i(\mathbf{x}) + \varphi(\mathbf{x})(u_i(\mathbf{x}) - v_i(\mathbf{x})), 1 \leq i \leq n \quad (7)$$

Numerical implementation

Generalized diffusion implementation

Classical implementation : given $\nabla \varepsilon$, use descent-type algorithms.

Problem : solve the Euler-Lagrange equation

$$\Delta u_i - \varphi(\mathbf{x})(u_i(\mathbf{x}) - v_i(\mathbf{x})) = 0, \quad 1 \leq i \leq n \quad (8)$$

considers u_i as a function of time and solve the equivalent problem

$$\frac{\partial}{\partial t} u_i(\mathbf{x}, t) = \Delta u_i(\mathbf{x}, t) - \varphi(\mathbf{x})(u_i(\mathbf{x}, t) - v_i(\mathbf{x})), \quad 1 \leq i \leq n \quad (9)$$

known as the generalized diffusion equations.

As diffusion operator, it can directly be used in background covariance [see Weaver et al.]

optical flow of Horn and Shunck : luminance conservation

$$\frac{df}{dt} = 0 \quad (10)$$

$f(\mathbf{x}, t)$ noted f is the luminance function.

For geophysical fluid images, the mass conservation equation is more adequate [Fitzpatrick 1985]

$$\frac{df}{dt} + f(\nabla \cdot \mathbf{v}) = 0 \quad (11)$$

$\mathbf{v}(\mathbf{x})$ is the velocity at \mathbf{x}

given the luminance function $f(\mathbf{x}, 0) = f^0(\mathbf{x})$ at time 0, solution to equations (10) or (11) defines $f(\mathbf{x}, t)$ as function of the static velocity field $\mathbf{v}(\mathbf{x})$

$$\begin{aligned} \mathcal{M} : \mathcal{V} &\rightarrow \mathcal{F} \\ \mathbf{v} &\mapsto \mathbf{f} = \mathcal{M}(\mathbf{v}) \end{aligned}$$

motion estimation is reduced to the inverse problem :

$$\delta \mathbf{v}^* = \text{MinArg}(J(\delta \mathbf{v}))$$

with

$$J(\delta \mathbf{v}) = \frac{1}{2} \|\mathcal{M}(\mathbf{v}^b + \delta \mathbf{v}) - \mathbf{f}^o\|_{\mathcal{F}}^2 + \frac{1}{2} \|\delta \mathbf{v}\|_{\mathcal{V}}^2 \quad (12)$$

Aperture problem :

only motion along the normal to iso-contours can be inferred \Rightarrow
use regularization

Trust function for motion estimation

Proposition: define trust function φ

to have large values on discontinuities (contours) for motion component along the normal to the contour, and small values in homogeneous areas.

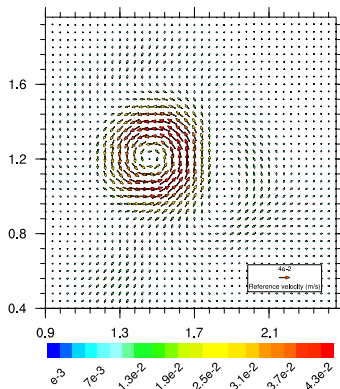
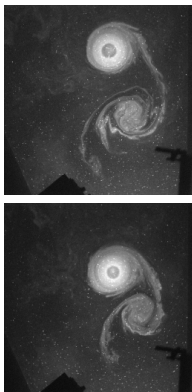
Example : set φ to be the contours map c^1 or c^2 defined as

$$c^1(\mathbf{x}, f) = \|\nabla_{\mathbf{x}} f(\mathbf{x})\|^2$$

$$c^2(\mathbf{x}, f) = \|\nabla_{\mathbf{x}}(\mathbf{G}_{\sigma}(\mathbf{x}) * f(\mathbf{x}))\|^2$$

Twin experiments

Direct Image sequences assimilation [Titaud et al 2009] \Rightarrow true initial state (velocity fields)

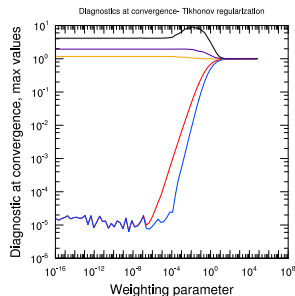
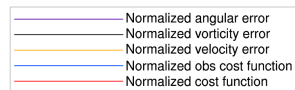
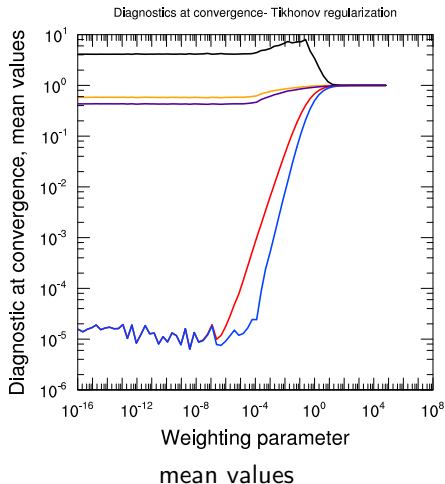


Images from [J.-B. Flór (LEGI) and I. Eames, 2002]

Error analysis : Tikhonov regularization

Evolution of diagnostic functions with respect to the weighting parameter

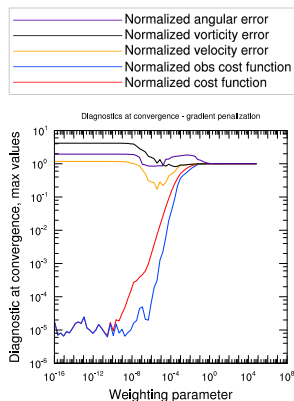
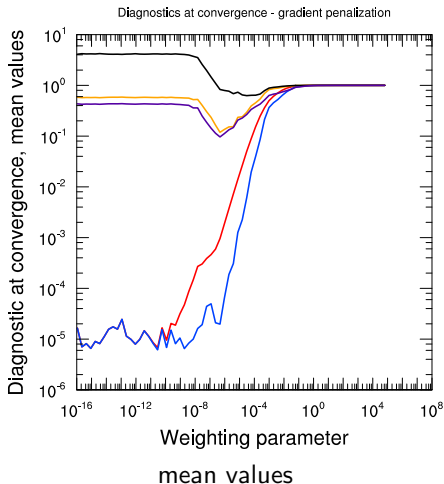
α



Error analysis : gradient penalization

Evolution of diagnostic functions with respect to the weighting parameter

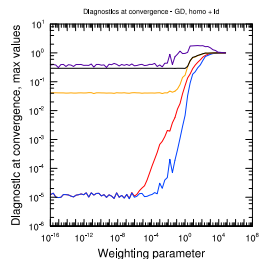
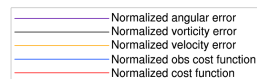
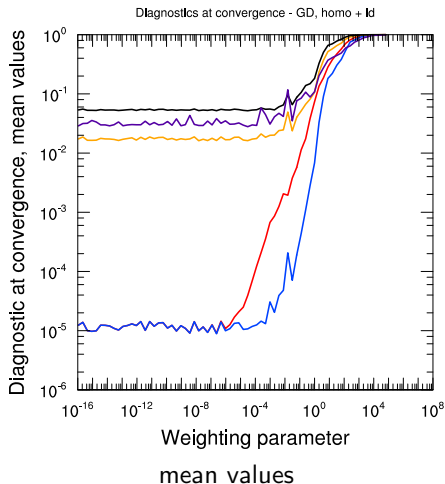
$\alpha \nabla$



Error analysis : Generalised diffusion

Evolution of diagnostic functions with respect to the weighting parameter

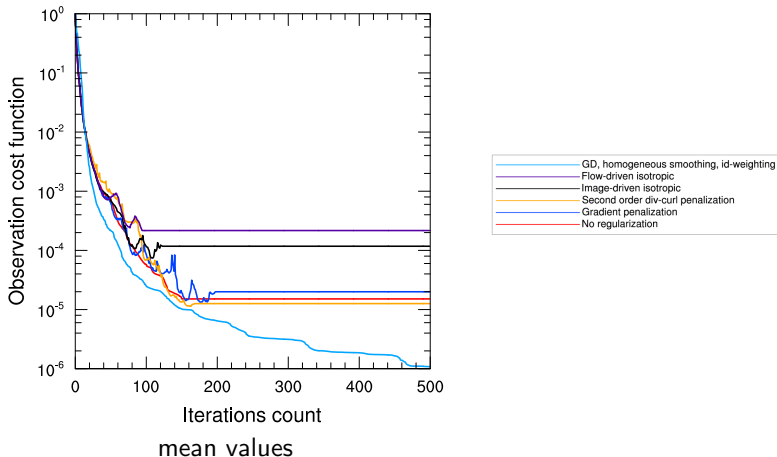
α_{GD}



Error analysis : Comparison - cost function

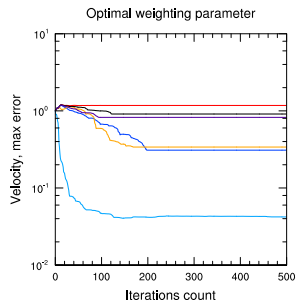
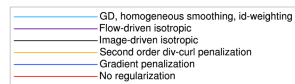
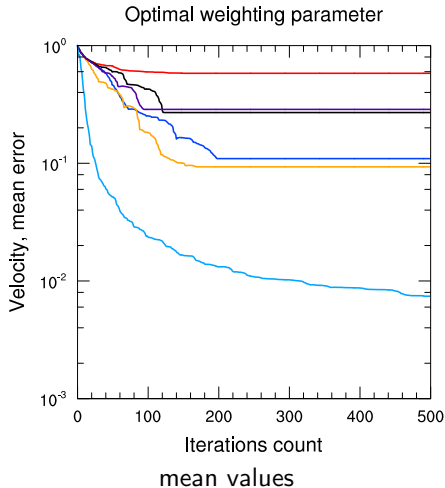
Evolution of the observation cost function with minimization iterations

Optimal weighting parameter



Error analysis : Comparison - velocity error

Evolution of the velocity error with minimization iterations

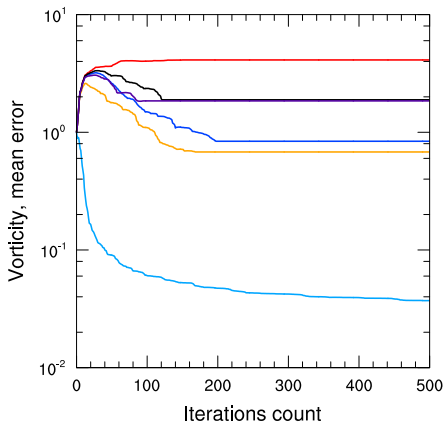


max values

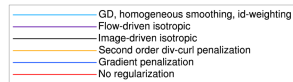
Error analysis : Comparison - vorticity error

Evolution of the vorticity error with minimization iterations

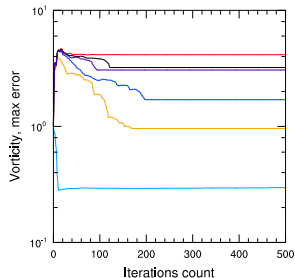
Optimal weighting parameter



mean values



Optimal weighting parameter

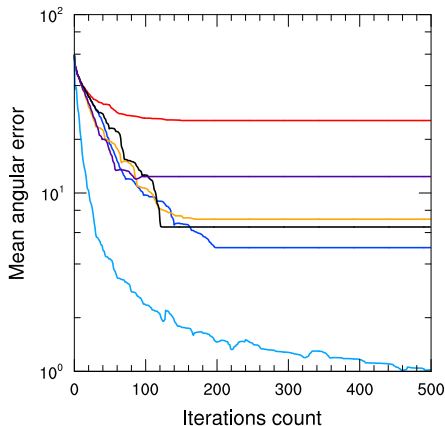


max values

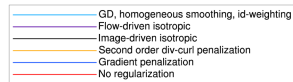
Error analysis : Comparison - angular error

Evolution of the angular error with minimization iterations

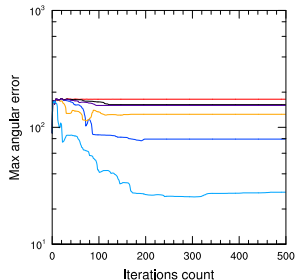
Optimal weighting parameter



mean values

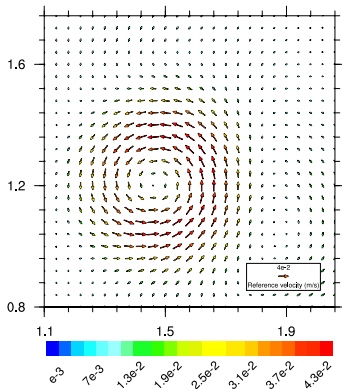


Optimal weighting parameter

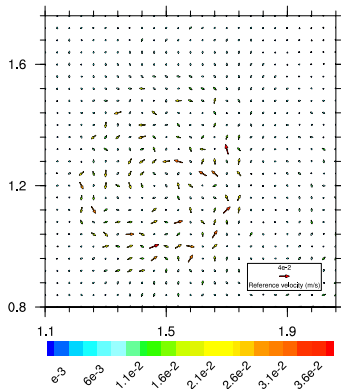


max values

Analysis : vector field

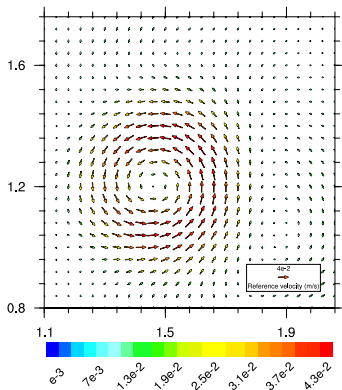


true

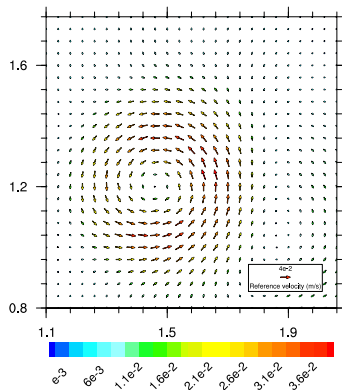


Tikhonov regularization

Analysis : vector field

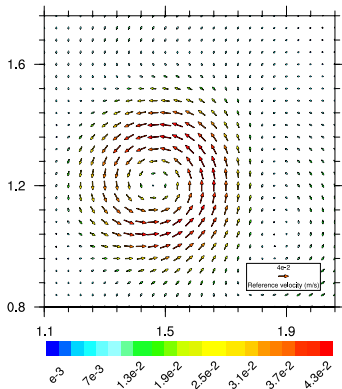


true

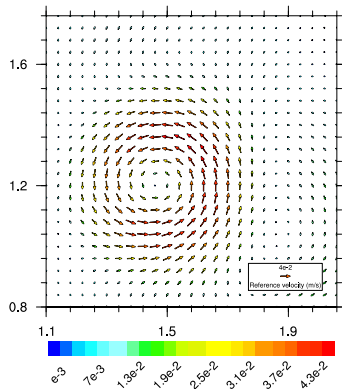


gradient penalization

Analysis : vector field



true



Generalised Diffusion

Conclusion

Conclusion

Inverse problems :

- ill-posed \Rightarrow use regularization
- ill-conditioned \Rightarrow use preconditioner

proposed : promising approach for regularization of inverse problems.