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ABSTRACT. Optical flow motion estimation from two images is limited by the aperture problem. A method to deal with this problem is to use regularization techniques. Usually, one adds a regularization term with appriopriate weighting parameter to the optical flow cost funtion. Here, we suggest a new approach to regularization for optical flow motion estimation. In this approach, all the regularization informations are used in the definition of an appropriate norm for the cost function via a trust function to be defined, one don't ever need weighting parameter. A simple derivation of such a trust function from images is proposed and a comparison with usual approaches is presented. These results show the superiority of such approach over usual ones.

RÉSUMÉ. L'estimation du mouvement par flot optique est sujet au problème d'ouverture. Pour cela, on a recours aux techniques de régularisation. De façon usuelle, Cela se caractérise par l'ajout d'un terme de régularisation pondéré à la fonction coût du flot optique. Dans ce papier, nous proposons une nouvelle approche pour la régularisation des méthodes de flot optique. Toute l'information de régularisation est utilisée pour définir une norme appropriée à la fonction coût par l'intermédiaire d'une fonction de confiance qui permet de se passer du paramètre de poids. Nous proposons une dérivation simplifiée de la fonction de confiance à partir des images et présentons les résultats comparés avec les méthodes usuelles. Ces résultats montrent la supériorité de la nouvelle approche

KEYWORDS : Motion estimation, optical flow, image, regularization

MOTS-CLÉS : Estimation du mouvement, flow optique, image, régularisation

1. Introduction

Motion estimation is an example of inverse problem in computer vision and images processing. The expression inverse problem is used as opposite to direct problem. Given a complete description of the behavior of a physical system in terms of mathematical models and physical parameters, the state of the system can be computed using the mathematical model; this is known as the forward (direct, modeling or simulation) problem. The inverse problem consists in using given measurements of the system's state to infer the values of the parameters characterizing the model. In motion estimation the inverse problem consists in determining motion vectors that describe the transformation from one 2D image to another. Motion estimation is affected by ill-posedness as general inverse problem. Due to the ill-posedness, one has recourse to a priori informations on the solution while solving inverse problems. A priori informations include but are not limited to

– background and background errors covariance

– regularity of the solution

These informations are generally used as constraints to get the appropriate solution when optimization techniques are used to solve an inverse problem. In usual cases, constrains are turned into penalization of some characteristics of the solution. A common constraint is the regularity of the solution leading to regularization techniques for inverse problems. Until now, regularization is generally used as penalization while solving inverse problems. This practice is affected by two principal problems: - as the cost function is composite, the convergence rate of optimization algorithms decreases - when adequate regularization functions are defined, one have to define balance parameters between regularization functions and the objective function to minimize. The determination of the optimal weighting parameter requires second order analysis. Here, we suggest a new approach for regularization of ill-posed inverse problems. We introduce an observation based trust function that is used to define an appropriate norm for the cost function. This approach don't need extra terms in the cost function, and of course is not affected nor by the ill-convergence due to composite cost function, nor by the choice of weighting parameters.

The present document is organized as followed : in section (2), we present inverse problems in a general framework, the use of a priori informations while solving inverse problems. In section (3), we present regularization methods for inverse problems; we emphasize on vector fields regularization. In section (4), we present the derivation of the new approach and comparisons with classical methods.

2. Inverse problems

2.1. Definition of inverse problems

2.1.1. Direct problem

Given a physical system whose the state $\mathbf{y} \in \mathcal{Y}$ can be defined as a function of a so called control variable $\mathbf{v} \in \mathcal{V}$

$$\begin{array}{rccc} \mathcal{M}: & \mathcal{V} & \rightarrow & \mathcal{Y} \\ & \mathbf{v} & \mapsto & \mathbf{y} = \mathcal{M}(\mathbf{v}) \end{array} \qquad [1]$$

The model $\mathcal{M}$ (that link the control space $\mathcal{V}$ to the state space $\mathcal{Y}$) defines the direct problem. Given a realization of the control variable $\mathbf{v}$, this problem has a unique solution in the deterministic case. It is common to have not a realization of the control variable, but observations of the system state. The problem of inferring the control variable from observations is known as an inverse problem.

2.1.2. Inverse problem

The inverse problem associated to the direct problem (equation 1) is defined in term of optimization problem as followed :

$$\text{find } \mathbf{v}^* = \text{ArgMin}(J(\mathbf{v})), \mathbf{v} \in \mathcal{V} \qquad [2]$$

where

$$J(\mathbf{v}) = J_o(\mathbf{v}) = \frac{1}{2}\|\mathcal{M}(\mathbf{v}) - \mathbf{y^o}\|^2_{\mathcal{O}} \qquad [3]$$

$\|.\|_{\mathcal{O}}$ is the appropriate norm (taking into account observations covariance errors) in the observation space $\mathcal{O}$

The problem defined by (equation 2) is known as the unconstrained inverse problem.

The existence and uniqueness of the solution to the unconstrained problem (equation 2) is guaranteed if J is strictly convex and lower semi continuous with

$$\lim_{\|\mathbf{v}\| \to +\infty} J(\mathbf{v}) \to +\infty$$

under these conditions, if J is differentiable, then the solution to the unconstrained inverse problem (equation 2) is also the solution of the Euler-Lagrange equation

$$\nabla J(\mathbf{v}) = \mathbf{0} \qquad [4]$$

To address the ill-posedness, one uses of all a priori knowledge of the properties of the solution.

2.2. Use of a priori knowledges in solving inverse problems

A priori knowledges are a set of constraints on the solution of the inverse problems. These constraints define a subset $\mathcal{W} \subset \mathcal{V}$ of admissible candidates leading to a constraint problem defined as

$$\begin{aligned} \text{find} \quad & \mathbf{v}^* = \text{ArgMin}(J(\mathbf{v})) \\ & \mathbf{v} \in \mathcal{W} \end{aligned} \qquad [5]$$

Here, we are interrested in cases where the set of admissible solutions can mathematically be defined as $\mathcal{W} = \{\mathbf{v} \in \mathcal{V} / g(\mathbf{v}) = 0\}$, the function g being to define. In this case, the constraint problem can be reduced to the unconstrained penalized problem

$$\text{find } \mathbf{v}^*_\epsilon = \text{ArgMin}(J_o(\mathbf{v}) + \frac{1}{\epsilon_c} J_c(\mathbf{v})), \mathbf{v} \in \mathcal{V} \qquad [6]$$

where J_o is the observation cost function defined by (equation 3) and the constraint cost function J_c is defined as

$$J_c(\mathbf{v}) = \frac{1}{2}\|g(\mathbf{v})\|^2 \qquad [7]$$

The solution $\mathbf{v}^*_{\epsilon_\mathbf{c}} \to \mathbf{v}^*$ when $\epsilon_c \to 0$. Instead of using parameter ϵ_c and let it go to zero, one can use a multiplicative parameter α_c and let it go to infinity. We are going to consider this case in the remainder part of the document. *It is known that pure penalization as defined above is not numerically efficient; it is better to used augmented Lagrangian algorithms see Glowinski et Le Tallec [2]*

Developpement here will be limited to background informations and the regularity of the solution. In this cases, the goal is usually not to find the exact solution $\mathbf{v} \in \mathcal{W}$, but to find the solution that realizes the best fit between the observation cost function and the constraint cost function. This is choosing the best parameter ϵ_c or α_c.

2.2.1. Background and background errors covariance

If one gets from some previous process an approximation of the control state and the associate covariance error also known as background and background covariance errors, one may asks to the computed solution to be closed to this background. This can be defined in term of penalization as

$$\text{find } \mathbf{v}^*_{\alpha_\mathbf{b}} = \text{ArgMin}(J_o(\mathbf{v}) + \alpha_b J_b(\mathbf{v})), \mathbf{v} \in \mathcal{V} \tag{8}$$

where α_b is the weighting parameter associated to the background part of the cost function defined as

$$J_b(\mathbf{v}) = \frac{1}{2}\|\mathbf{v} - \mathbf{v^b}\|^2_\mathcal{V} \tag{9}$$

well known as Tikhonov regularization [1]. $\mathbf{v^b}$ is the background knowledge of the solution, and $\|\cdot\|_\mathcal{V}$ the appropriate norm defined in term of the background covariance errors. This norm will be analyzed in the preconditioning section. Background informations are very important in solving inverse problems; this is a simple way to address the ill-posedness of the problem. Even in the case where there is no background information, it is a usual practice to consider the zero background constraining the solution to have small norm. In real live applications, background comes from previous analysis; this is the case of forecast centers.

It is common to define the cost function J in term of the increment $\delta\mathbf{v} = \mathbf{v} - \mathbf{v^b}$ leading to incremental problem,

$$J(\delta\mathbf{v}) = \frac{\mathbf{1}}{\mathbf{2}}\|\mathcal{M}(\mathbf{v^b} + \delta\mathbf{v}) - \mathbf{y^o}\|^\mathbf{2}_\mathcal{O} + \frac{\mathbf{1}}{\mathbf{2}}\alpha_\mathbf{b}\|\delta\mathbf{v}\|^\mathbf{2}_\mathcal{V} \tag{10}$$

2.2.2. Regularity of the solution

Sometime, the physics of the problem defines the regularity of admissible solutions (eg. irrotational or divergence free flow.) These are constraints defines as functions of the derivatives of the control variable. In these case, one defines the penalized problem

$$\text{find } \delta\mathbf{v}^*_{\alpha_\mathbf{r}} = \text{ArgMin}(J_o(\delta\mathbf{v}) + \alpha_b J_b(\delta\mathbf{v}) + \alpha_r J_r(\delta\mathbf{v})), \mathbf{v} \in \mathcal{V} \tag{11}$$

where α_r is the weighting parameter associated to the regularization part of the cost function defined in terms of the derivatives of the control variable. Regularization will be explored in more details in section (3.)

3. Vector fields regularization

As we said previously, regularization is a class of a priori knowledges used to address the ill-posedness while solving inverse problem. One adds regularization terms J_r to the cost function. The function J_r is based on the derivatives of $\mathbf{v}$. The order of the derivatives used in the definition of J_r defines the order of the regularization. We will name m-order regularization those involving up to $m-$order derivatives.

It is useful to give some specifications of the notations defined in section (2), especially for the control space.

3.1. Notations

Let Ω be an open subset of R^m ($\Omega \subset R^m$), this is the physical space of the system, we are interested in control spaces defined as $\mathcal{V} = (L^2(\Omega))^n$. control states are then defined as $\mathbf{v} \in \mathcal{V} = (L^2(\Omega))^n$, $\mathbf{v}(\mathbf{x}) = (v_i(\mathbf{x}))_{1\leq i\leq n}$ and $\mathbf{x} = (x_i)_{1\leq j\leq m} \in \Omega$

3.2. First order methods

The first order regularization methods define J_r as a function of the first order derivatives of $\mathbf{v}$:

$$J_r(\mathbf{v}) = J_r\left(\frac{\partial v_i}{\partial x_j}\right)_{1\leq i,j\leq n,} \qquad [12]$$

The most used of first order regularization methods is the gradient penalization. It has been used by Horn and Schunck in the formulation of optical flow [3] for motion estimation. The regularization function of Horn and Schunck is defined as follow:

$$J_{grad}(\mathbf{v}) = \frac{1}{2}\int_\Omega \sum_{i=1}^{n} \|\nabla v_i\|^2 d\mathbf{x} \qquad [13]$$

For incompressible fluid or irrotational flow, it is common to penalize the divergence or the curl of the vector field leading to divergence penalization

$$J_{div}(\mathbf{v}) = \frac{1}{2}\int_\Omega \|div(\mathbf{v})\|^2 d\mathbf{x} \qquad [14]$$

for incompressible fluid flow and curl penalization

$$J_{curl}(\mathbf{v}) = \frac{1}{2}\int_\Omega \|curl(\mathbf{v})\|^2 d\mathbf{x} \qquad [15]$$

for irrotational flow

3.3. Second order methods

The second order regularization methods are based on the second order derivatives of $\mathbf{v}$.

$$J_r(\mathbf{v}) = J_r\left(\frac{\partial^2 v_i}{\partial x_j \partial x_k}\right)_{1\leq i,j,k\leq n,} \qquad [16]$$

An example based on the first order derivatives of div and $curl$ is the regularization of Suter [8] defined as followed :

$$J_{suter}(\mathbf{v}) = \frac{1}{2}\int_\Omega \alpha\|\nabla div(\mathbf{v})\|^2 + \beta\|\nabla curl(\mathbf{v})\|^2 d\mathbf{x} \qquad [17]$$

Higher order derivatives of v can also be used for regularization; for example (17) has been generalized by Chen and Suter [7] using m-order derivatives of div and $curl$.

$$J_m(\mathbf{v}) = \frac{1}{2}\int_\Omega \alpha\|\nabla^m div(\mathbf{v})\|^2 + \beta\|\nabla^m curl(\mathbf{v})\|^2 d\mathbf{x} \qquad [18]$$

4. Turning regularization functions into covariance operators

4.1. Regularization operator out of an optimization process : case of gradient penalization

4.1.1. definition

Let :

– $\mathbf{v}(\mathbf{x})$ be an incomplete/inconsistent state of the studied system with $\mathbf{x} \in \mathbf{\Omega}$ the space on which the system is defined

– $\varphi(\mathbf{x})$ a scalar positive trust function given the quality of the state $\mathbf{v}$ at $\mathbf{x}$

$$\begin{cases} \text{small value meaning bad/lack/inconsistent state} \\ \text{big value for good quality state} \end{cases}$$

we define a restoration of v as the minimum argument of the function

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\int_\Omega \sum_{i=1}^{n} \|\nabla u_i(x)\|^2 + \varphi(\mathbf{x})\|\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})\|^2 d\mathbf{x} \qquad [19]$$

The minimization of ε is achieved by setting $\mathbf{u}$ to be closed to $\mathbf{v}$ when φ is large ($\mathbf{v}$ is of good quality) and smooth (small gradient norm) when φ is small ($\mathbf{v}$ is not of good quality)

4.1.2. Practical use

Under the conditions given in section (2), MinArg(ε) equation (19) is defined by the Euler-Lagrange condition

$$\nabla_{\mathbf{u}}\varepsilon(\mathbf{u}) = 0 \qquad [20]$$

The difficulty with nonlinear problems is to express $\nabla\varepsilon$. When $\nabla\varepsilon$ is expressed, it can be used in descent type algorithms to solve the minimization problem. $\nabla\varepsilon$ can be obtained by making explicit the linear dependency of the Gateaux derivatives $\hat{\varepsilon}$ with respect to the gradient. Development based on vector calculus leads to

$$\nabla\varepsilon^{grad}(\mathbf{u}) = -\Delta\mathbf{u}(\mathbf{x}) + \varphi(\mathbf{x})(\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})), 1 < i < n \qquad [21]$$

Instead of using classical descent type algorithm to get the solution of the problem, u_i can be considered as a function of time and the solution obtained by solving (22) according to development in [6],

$$\frac{\partial}{\partial t}u_i(\mathbf{x},t) \;=\; \nabla^2 u_i(\mathbf{x},t) - \varphi(\mathbf{x})(u_i(\mathbf{x},t) - v_i(\mathbf{x}))), \;\; 1 \le i \le n \qquad [22]$$

the set of equations (22) are known as the generalized diffusion equations. The diffusion operator $\mathcal{L}$ giving the solution $\mathbf{u}^* = \mathcal{L}(\mathbf{v}) = \text{minArg}(\varepsilon)$ can then used as a covariance operator to define the appropriate norm for the cost function of the inverse problem.

4.1.3. Generalization

If the regularization term is defined as $J_r(\mathbf{v}) = \|\mathbf{\Phi}(\mathbf{v})\|$, the function ε can be generalized as

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\int_{\Omega} \|\Phi(\mathbf{u}(\mathbf{x}))\|^2 + \varphi(\mathbf{x})\|\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})\|^2 d\mathbf{x} \qquad [23]$$

The minimum of ε is achieved by setting $\mathbf{u}$ to be closed to $\mathbf{v}$ when φ is large ($\mathbf{v}$ is of good quality) and Φ-smooth when φ is small ($\mathbf{v}$ is not of good quality)

4.2. Application to motion estimation

We performed a set of motion estimation's twin experiments in order to analyze the behavior of the approach we introduced here. We use images from experimental study of the drift of a vortex on a turntable [4]. The zonal component $v_1 = v_1(x, y)$ and meridional component $v_2 = v_2(x, y)$ of the true current velocity are computed by direct image sequence assimilation (DISA) [5]. The trust function is defined as the edge map (see [6] for details on edge map function) of the first image We use second order analysis to define optimal parameter for gradient, first order div-curl and second order div-curl (Suter) regulization. These optimal parameters are then used to make a set of experiments. Figure (5) shows the evolution of the normalized root mean square error (RMSE, log coordinates) on velocity and vorticity with the minimization iterations. The graphic clearly shows that th approach introduced here is better than the others and their combination. With this new approach, velocity error decreases from 100% to 10% after 40 iterations while classical regularization need more than 200 iteration to get the same result. Furthermore, with the new approach, velocity error can be reduced to less than 1% while for the other methods, the best result is affected by about 10% of error. Vorticity error is reduce to 4% with the new approach and only to 40% with classical regularization methods.

5. Conclusion

We introduced here a new formalism for taking into account a priori knowledges on the regularity of the solution while solving inverse problems. This new formalism is based on the generalized diffusion equations. Preliminary results shows the superiority of this formalism over classical methods that include regularity informations as penalization in the cost function.

6. References

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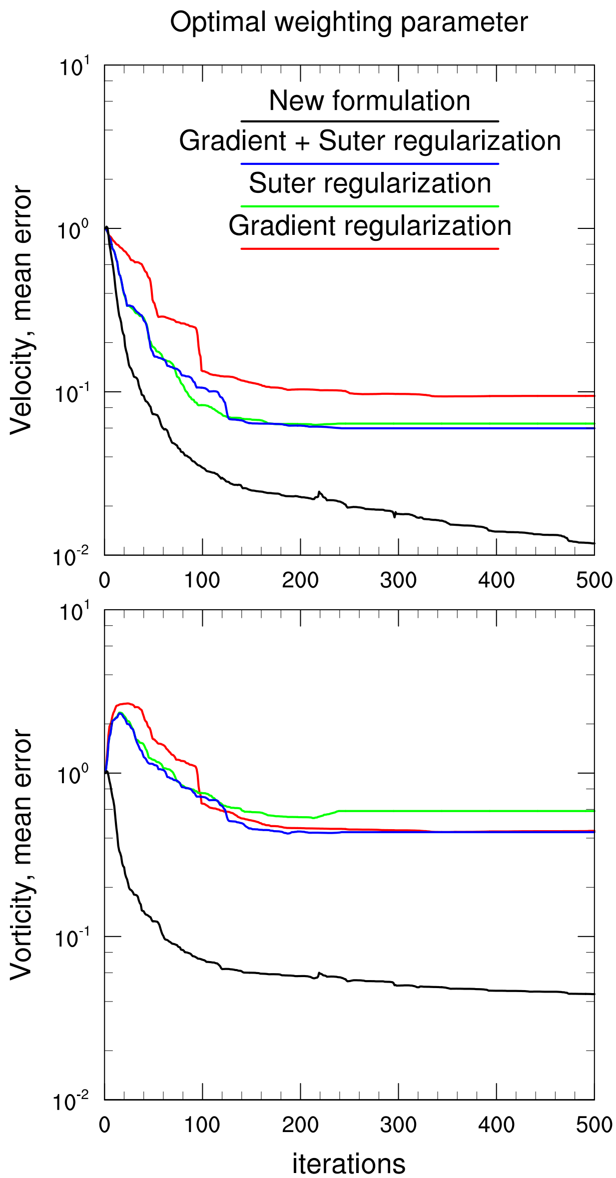

Figure 1. *Comparison of normalized RMSE on velocity and vorticity, classical regularization (optimal parameters) and new formulation; evolution with the minimization's iterations*

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