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Conway polynomials of two-bridge links

Pierre-Vincent Koseleff & Daniel Pecker

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Abstract

We give necessary conditions for a polynomial to be the Conway polynomial of a two-bridge link. As a consequence, we obtain simple proofs of the classical theorems of Murasugi and Hartley. We give a modulo 2 congruence for links, which implies the classical modulo 2 Murasugi congruence for knots. We also give sharp bounds for the coefficients of the Conway and Alexander polynomials of a two-bridge link. These bounds improve and generalize those of Nakanishi and Suketa.

Keywords: two-bridge link, Conway polynomial, Alexander polynomial

MSC2000: 57M25

1 Introduction

In this paper, we study the problem of determining whether a given polynomial is the Conway polynomial of a two-bridge link (or knot), or equivalently, if it is a Euler continuant polynomial.

For small degrees, this problem can be solved by an exhaustive search of possible two-bridge links (see Algorithm section 5). Here, however, we give necessary conditions on the coefficients of the polynomial, which can be tested for high degree polynomials.

We shall use the Siebenmann description of the Conway polynomial of a two-bridge link. Conway polynomials of links (or knots) are written as

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k} z^{m-2k}.$$

We obtain the following inequalities:

THEOREM 2.6. *For $k \geq 0$,*

$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m|.$$

If equality holds for some positive integer $k < \lfloor \frac{m}{2} \rfloor$, then it holds for all integers. In this case, the link is isotopic to a link of Conway form $C(2, -2, 2, \dots, (-1)^{m+1}2)$ or $C(2, 2, \dots, 2)$, up to mirror symmetry.

When $|c_m| \neq 1$, we have the following sharper bounds:

THEOREM 2.8. Let $g \geq 1$ be the greatest prime divisor of c_m , and $m \geq 2k \geq 2$. Then

$$|c_{m-2k}| \leq \left(\binom{m-k-1}{k} + \frac{1}{g} \left(\binom{m-k-1}{k-1} - 1 \right) \right) |c_m| + 1.$$

Equality holds for links of Conway forms $C(2g, 2, 2, \dots, 2)$ and $C(2g, -2, 2, \dots, (-1)^{m+1} 2)$.

Our inequalities refine those of Nakanishi and Suketa for Alexander polynomials of two-bridge knots (see [22, theorems 2 and 3]). Moreover, they are sharp and hold for any k .

It is convenient to write Conway polynomials in terms of Fibonacci polynomials f_k defined by:

$$f_0 = 0, f_1 = 1, f_{n+2}(z) = z f_{n+1}(z) + f_n(z).$$

We obtain an extension to links of both the Murasugi alternating theorem [19, 20], and the Hartley trapezoidal theorem [8], see also [10].

THEOREM 2.9. Let K be a two-bridge link (or knot). Let

$$\nabla_K = c_m \left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m-2i+1} \right), \quad \alpha_0 = 1$$

be its Conway polynomial written in the Fibonacci basis. Then we have

1. $\alpha_j \geq 0$, $j = 0, \dots, \lfloor \frac{m}{2} \rfloor$.
2. If $\alpha_i = 0$ for some $i > 0$ then $\alpha_j = 0$ for $j \geq i$.

We also obtain:

THEOREM 4.1. Let $\nabla(z) \in \mathbf{Z}[z]$ be the Conway polynomial of a rational link (or knot). There exists a Fibonacci polynomial $f_D(z)$ such that $\nabla(z) \equiv f_D(z) \pmod{2}$.

This provides a simple proof of a congruence of Murasugi [21] for two-bridge knots. Moreover, we deduce a new congruence for the Hosokawa polynomials of two-bridge links.

We give a simple method (Algorithm 4.8) that determines the integer D such that $\nabla(z) \equiv f_D(z) \pmod{2}$. This is used to test when $\nabla(z) \equiv 1 \pmod{2}$, which is a necessary condition to be a Lissajous knot.

We give examples showing that the conditions on Conway coefficients are sharper than the conditions on the Alexander coefficients deduced from them.

We conclude our paper with the following convexity conjecture:

Let $P(t) = a_0 - a_1(t+t^{-1}) + a_2(t^2+t^{-2}) - \dots + (-1)^n a_n(t^n+t^{-n})$ be the Alexander polynomial of a rational knot. Then there exists an integer $k \leq n$ such that (a_0, \dots, a_k) is convex and (a_k, \dots, a_n) is concave.

We have tested this conjecture for all two-bridge knots with 20 crossings or fewer.

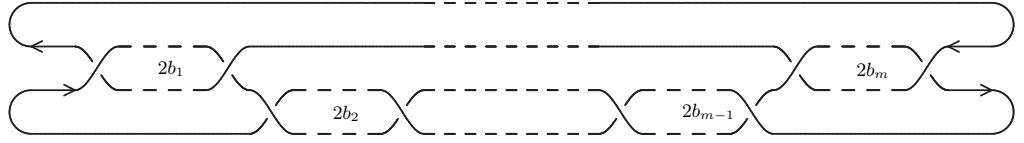


Figure 1: Oriented two-bridge links (m odd)

2 Conway polynomial

Any oriented two-bridge link can be put in the form shown in Figure 1. It will be denoted by $C(2b_1, 2b_2, \dots, 2b_m)$ with $b_i \neq 0$ for all i , including the indicated orientation (see [14, p. 26], [15, 12]). This is a two-component link if and only if m is odd.

Its Conway polynomial ∇_m is then given by the Siebenmann method (see [23, 6]).

Theorem 2.1 (Siebenmann, [6]) *Let $\nabla_m = \nabla_m(z)$ be the Conway polynomial of the oriented two-bridge link (or knot) of Conway form $C(2b_1, -2b_2, \dots, (-1)^{m+1}2b_m)$. Let $\nabla_{-1} = 0$, $\nabla_0 = 1$. Then*

$$\nabla_m = b_m z \nabla_{m-1} + \nabla_{m-2}, \quad (1)$$

for $m \geq 1$.

When $z = 1$, this is the classical Euler continuant polynomial.

The Fibonacci polynomials will be useful in studying these Conway polynomials.

Definition 2.2 (Fibonacci Polynomials) *Let $f_m(z)$ be the polynomials defined by:*

$$f_0 = 0, f_1 = 1, f_{n+2}(z) = z f_{n+1}(z) + f_n(z), \quad m \in \mathbf{Z}. \quad (2)$$

We have $f_{-m}(z) = (-1)^{m+1} f_m(z)$.

Let us recall some basic facts about Fibonacci polynomials.

Lemma 2.3 *For $m \geq 0$:*

$$f_{m+1}(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} z^{m-2k}.$$

Proof. By induction on m . The result is clear for $m = 1$ and for $m = 2$. Let us suppose the result true for $m - 1$ and m . By induction, the coefficient of z^{m-2k} is $\binom{m-1-k}{k}$ in $z f_m(z)$, and $\binom{m-1-k}{k-1}$ in $f_{m-1}(z)$. Consequently, the coefficient of z^{m-2k} in $f_{m+1}(z)$ is

$$\binom{m-1-k}{k} + \binom{m-1-k}{k-1} = \binom{m-k}{k}.$$

□

Remark 2.4 *The Fibonacci polynomials can be read on the diagonals of Pascal's triangle. When $z = 1$, we recover the classical Lucas identity*

$$F_m = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k},$$

where F_m are the Fibonacci numbers.

We shall need the following more explicit notation for Conway polynomials:

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k}(b_1, \dots, b_m) z^{m-2k}.$$

The next result gives some properties of $c_{m-2k}(b_1, \dots, b_m)$, viewed as a polynomial in m variables.

- Proposition 2.5**
1. *The polynomial $c_{m-2k}(b_1, \dots, b_m)$ is the sum of all monomials $\frac{b_1 \cdots b_m}{b_{i_1} b_{i_1+1} \cdots b_{i_k} b_{i_k+1}}$, where $i_h + 1 < i_{h+1}$.*
 2. *The number of these monomials is $\binom{m-k}{k}$. They are relatively prime if $k \neq 0$.*
 3. *Let $m \geq 2k \geq 4$. For any j , the number of these monomials which are relatively prime to b_j is at least $\binom{m-1-k}{k-1}$. Furthermore these monomials are relatively prime.*

Proof.

1. This is a classical property of the Euler continuant.
2. This number is $c_{m-2k}(1, 1, \dots, 1)$, which is a coefficient of the Fibonacci polynomial

$$f_{m+1}(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k}(1, 1, \dots, 1) z^{m-2k} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} z^{m-2k}.$$

3. Let $1 \leq j \leq m$ and $\mathbf{b} = (1, \dots, 1, 0, 1, \dots, 1)$ where $b_j = 0$, and $b_k = 1$ for $k \neq j$. Let us define the polynomials g_n , for $n \leq m$ by $g_n(z) = \nabla_n(\mathbf{b})(z)$. The number of our monomials is the coefficient $c_{m-2k}(\mathbf{b})$ of $g_m(z)$.

If $j = 1$, we have $g_1 = 0$, $g_2 = 1$ and therefore $g_n = f_{n-1}$, $n \geq 1$.

If $j > 1$, we have

$$g_1 = f_2, \dots, g_{j-1} = f_j, g_j = f_{j-1}, \dots, g_{n+1} = z g_n + g_{n-1}, n \geq j.$$

Let us write $p(z) \succeq q(z)$ when each coefficient of p is greater than or equal to the corresponding coefficient of q . We have $f_{k+2} \succeq f_k$, and then an simple induction shows that $g_m \succeq f_{m-1}$.

To conclude the proof, it is sufficient to verify that for any $i \neq j$, there is a monomial which is prime to the monomial b_i . This is clear since $m \geq 4$.

□

Theorem 2.6 For $k \geq 0$,

$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m|.$$

If equality holds for some positive integer $k < \lfloor \frac{m}{2} \rfloor$, then it holds for all integers. In this case, the link is isotopic to a link of Conway form $C(2, -2, 2, \dots, (-1)^{m+1}2)$ or $C(2, 2, \dots, 2)$, up to mirror symmetry.

Proof. By Proposition 2.5, the number of monomials of $c_{m-2k}(b_1, \dots, b_m)$ is $\binom{m-k}{k}$. The result follows since no monomial is greater than $|c_m| = |b_1 \cdots b_m|$.

If equality holds for some positive integer $k < \lfloor \frac{m}{2} \rfloor$, then for all i, j , $b_i b_{i+1} = b_j b_{j+1} = \pm 1$, which implies the result. □ To prove the refined inequalities of Theorem 2.8, we shall use the following

lemma, which generalizes the inequality $a + b \leq ab + 1$, valid for positive integers.

Lemma 2.7 Let $p_i(\mathbf{x}), i \in \mathcal{S}$ be relatively prime divisors of $p(\mathbf{x}) = x_1 x_2 \cdots x_m$.

Let $\mathbf{b} = (b_1, \dots, b_m)$ be a m -tuple of positive integers. Then

$$\sum_{i \in \mathcal{S}} p_i(\mathbf{b}) \leq (\text{card}(\mathcal{S}) - 1)p(\mathbf{b}) + 1. \quad (3)$$

Proof. We do not suppose the p_i distinct. Let us prove the result by induction on $k = \text{card}(\mathcal{S})$. The result is clear if $k = 1$, we have $p_1 = \pm 1$, and the inequality is $\pm 1 \leq 1$.

If all the $p_i = 1$, the result is clear. Otherwise, let x_h be a divisor of some p_i .

Let $\mathcal{S}_1 = \{i \in \mathcal{S} : x_h | p_i\}$, and $\mathcal{S}_2 = \mathcal{S} - \mathcal{S}_1$. We have $k = k_1 + k_2$, where $k_j = \text{card}(\mathcal{S}_j)$. Let $q_j = \text{GCD}\{p_i, i \in \mathcal{S}_j\}$, then q_1 and q_2 are coprime, and $q_1 q_2$ is a divisor of p .

By induction we obtain for $j = 1, 2$:

$$\sum_{i \in \mathcal{S}_j} p_i(\mathbf{b}) \leq q_j(\mathbf{b}) \left((k_j - 1) \frac{p(\mathbf{b})}{q_j(\mathbf{b})} + 1 \right).$$

Adding these two inequalities we get

$$\begin{aligned} \sum_{i \in \mathcal{S}} p_i(\mathbf{b}) &\leq (k_1 + k_2 - 1)p(\mathbf{b}) + q_1(\mathbf{b}) + q_2(\mathbf{b}) - p(\mathbf{b}) \\ &\leq (k_1 + k_2 - 1)p(\mathbf{b}) + q_1(\mathbf{b})q_2(\mathbf{b}) - p(\mathbf{b}) + 1, \end{aligned}$$

which proves the result, since $q_1(\mathbf{b})q_2(\mathbf{b}) \leq p(\mathbf{b})$. □

With this lemma we can prove:

Theorem 2.8 Let $m \geq 2k > 0$, and $g \geq 1$ be the greatest prime divisor of c_m . Then

$$|c_{m-2k}| \leq \left(\binom{m-k-1}{k} + \frac{1}{g} \left(\binom{m-k-1}{k-1} - 1 \right) \right) |c_m| + 1.$$

Equality holds for links of Conway form $C(2g, -2, \dots, (-1)^{m-1}2)$.

Proof. If $k = 1$, by Proposition 2.5 there are $m - 1$ monomials in the polynomial $c_{m-2}(b_1, \dots, b_m)$. Then, using Lemma 2.7 and the notation $|\mathbf{b}| = (|b_1|, \dots, |b_m|)$, we get

$$|c_{m-2}| = |c_{m-2}(\mathbf{b})| \leq c_{m-2}(|\mathbf{b}|) \leq (m-2)c_m(|\mathbf{b}|) + 1 = (m-2)|c_m| + 1.$$

Now, suppose $k \geq 2$. Let g be the greatest prime divisor of the integer $c_m = b_1 \cdots b_m$, and suppose that $g \mid b_j$. Let N be the number of monomials of $c_{m-2k}(b_1, \dots, b_m)$ that are prime to the monomial b_j . By Proposition 2.5, these monomials are relatively prime, and $N \geq \binom{m-1-k}{k-1}$. Using Lemma 2.7 we obtain:

$$\begin{aligned} |c_{m-2k}| &\leq (N-1) \frac{|c_m|}{|b_j|} + 1 + \left(\binom{m-k}{k} - N \right) |c_m| \\ &\leq \left(\frac{N-1}{g} + \left(\binom{m-k}{k} - N \right) \right) |c_m| + 1 \\ &= \left(\binom{m-k}{k} - N \left(1 - \frac{1}{g} \right) - \frac{1}{g} \right) |c_m| + 1 \\ &\leq \left(\binom{m-k}{k} - \binom{m-1-k}{k-1} \left(1 - \frac{1}{g} \right) - \frac{1}{g} \right) |c_m| + 1 \\ &= \left(\binom{m-1-k}{k} + \frac{1}{g} \left(\binom{m-1-k}{k-1} - 1 \right) \right) |c_m| + 1. \end{aligned}$$

For links of Conway form $C(2g, -2, \dots, (-1)^{m+1}2)$, we have $\mathbf{b} = (g, 1, \dots, 1)$, $N = \binom{m-1-k}{k-1}$, $c_m = g$, and $c_{m-2k} = g \binom{m-1-k}{k} + \binom{m-1-k}{k-1}$, and equality holds throughout.

For links of Conway form $C(2g, 2, \dots, 2)$, we get $c_{m-2k} = (-1)^{\lfloor \frac{m}{2} \rfloor + k} \left(g \binom{m-1-k}{k} + \binom{m-1-k}{k-1} \right)$.

□ Now, we will express the Conway polynomials of two-bridge links in terms of Fibonacci

polynomials, and show that their coefficients are alternating.

Theorem 2.9 *Let K be a two-bridge link (or knot). Let*

$$\nabla_K = c_m \left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m-2i+1} \right), \quad \alpha_0 = 1$$

be its Conway polynomial written in the Fibonacci basis. Then we have

1. $\alpha_j \geq 0$, $j = 0, \dots, \lfloor \frac{m}{2} \rfloor$.
2. If $\alpha_i = 0$ for some $i > 0$ then $\alpha_j = 0$ for $j \geq i$.

Proof. Let $K = C(2b_1, -2b_2, \dots, (-1)^{m+1}2b_m)$, with $b_i \neq 0$ for all i , and let ∇_n be the polynomials obtained in the Siebenmann method.

$$\text{We have } \nabla_0 = f_1, \nabla_1 = b_1 f_2, \nabla_2 = b_1 b_2 \left(f_3 - \left(1 - \frac{1}{b_1 b_2} \right) f_1 \right).$$

Let us show by induction that if

$$\nabla_m = b_1 \cdots b_m \left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m+1-2i} \right), \quad \nabla_{m-1} = b_1 \cdots b_{m-1} \left(\sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \beta_i f_{m-2i} \right)$$

then $\alpha_j \geq \beta_j \geq 0$, and if $\alpha_i = 0$ for some i , then $\alpha_j = 0$ for $j \geq i$.

The result is true for $m = 2$ from the expressions of ∇_1 and ∇_2 . Using $zf_{m+1-2i} = f_{m+2-2i} - f_{m-2i}$ and $\nabla_{m+1} = b_{m+1}z\nabla_m + \nabla_{m-1}$, we deduce that

$$\nabla_{m+1} = b_1 \cdots b_{m+1} \left(\sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} (-1)^i \gamma_i f_{m+2-2i} \right),$$

where $\gamma_0 = 1$ and

$$\gamma_i = \alpha_i + (\alpha_{i-1} - \beta_{i-1}) + \left(1 - \frac{1}{b_m b_{m+1}}\right) \beta_{i-1}, \quad i = 1, \dots, \lfloor \frac{m+1}{2} \rfloor. \quad (4)$$

As $|b_m b_{m+1}| \geq 1$, we deduce by induction that $\gamma_i \geq \alpha_i \geq 0$.

Furthermore, if $\gamma_i = 0$, then by Formula (4) $\alpha_i = 0$, and then, by induction, $\alpha_j = \beta_j = 0$ for $j \geq i$. Finally, by Formula (4), we get $\gamma_j = 0$ for $j \geq i$. \square

Remark 2.10 *It is interesting to look at the condition 2. of Theorem 2.9. Let us give a direct proof of it in the case $m = 4$. The polynomial ∇_3 has only two terms and*

$$\begin{aligned} \nabla_4 = b_1 b_2 b_3 b_4 f_5 - (3b_1 b_2 b_3 b_4 - b_1 b_2 - b_1 b_4 - b_3 b_4) f_3 \\ + (2b_1 b_2 b_3 b_4 - b_1 b_2 - b_1 b_4 - b_3 b_4 + 1) f_1. \end{aligned}$$

Suppose that the second coefficient of ∇_4 is equal to zero. Using Lemma 2.7, we get

$$3|b_1 b_2 b_3 b_4| \leq |b_1 b_2| + |b_1 b_4| + |b_3 b_4| \leq 2|b_1 b_2 b_3 b_4| + 1,$$

and therefore $b_1 = b_2 = b_3 = b_4 = \pm 1$, which implies that $\nabla_4 = \pm f_5$. This shows that the point 2. is true for $m = 4$.

3 Applications to Alexander polynomials of knots

In this section, we will see that our necessary conditions on Conway coefficients are improvements of the classical bounds of [22] on Alexander coefficients of two-bridge knots. For simplicity, we shall restrict ourselves to knots. Conway and Alexander polynomials of a knot K will be denoted by

$$\nabla_K(z) = 1 + \tilde{c}_1 z^2 + \cdots + \tilde{c}_n z^{2n}$$

and

$$\Delta_K(t) = a_0 - a_1(t + t^{-1}) + \cdots + (-1)^n a_n(t^n + t^{-n}).$$

The Alexander polynomial $\Delta_K(t)$ is deduced from the Conway polynomial:

$$\Delta_K(t) = \nabla_K\left(t^{1/2} - t^{-1/2}\right).$$

It is often normalized so that a_n is positive. Thanks to this formula, it is not difficult to deduce the Alexander polynomial from the Conway polynomial. If we use the Fibonacci basis, it is even easier to deduce the Conway polynomial of a knot from its Alexander polynomial.

Lemma 3.1 *If $z = t^{1/2} - t^{-1/2}$, and $n \in \mathbf{Z}$ is an integer, we have the identity*

$$f_{n+1}(z) + f_{n-1}(z) = (t^{1/2})^n + (-t^{-1/2})^n,$$

where $f_k(z)$ are Fibonacci polynomials.

Proof. Let $A = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$ be the (polynomial) Fibonacci matrix. If $z = t^{1/2} - t^{-1/2}$, the eigenvalues of A are $t^{1/2}$ and $-t^{-1/2}$, and consequently $\text{tr } A^n = (t^{1/2})^n + (-t^{-1/2})^n$. On the other hand, we have $A^n = \begin{bmatrix} f_{n+1}(z) & f_n(z) \\ f_n(z) & f_{n-1}(z) \end{bmatrix}$, and then $\text{tr } A^n = f_{n+1}(z) + f_{n-1}(z)$. \square

Remark 3.2 *The Lucas polynomials ℓ_n are defined by $\ell_n = f_{n+1} + f_{n-1}$. They satisfy $\ell_0 = 2$, $\ell_1 = z$, $\ell_{n+1} = z\ell_n + \ell_{n-1}$. From Lemma 2.3 we recover the classical result:*

$$\ell_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} z^{n-2j}. \quad (5)$$

From Lemma 3.1, we immediately deduce:

Corollary 3.3 *Let the Laurent polynomial $P(t)$ be defined by*

$$P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \dots + (-1)^n a_n(t^n + t^{-n}).$$

We have

$$P(t) = \sum_{k=0}^n (-1)^k (a_k - a_{k+1}) f_{2k+1}(z),$$

where $z = t^{1/2} - t^{-1/2}$, and $a_{n+1} = 0$.

We deduce a useful formula:

$$f_{2n+1}(t^{1/2} - t^{-1/2}) = (t^n + t^{-n}) - (t^{n-1} + t^{1-n}) + \dots + (-1)^n. \quad (6)$$

Now, we shall show that Theorem 2.9 implies both Murasugi and Hartley theorems for two-bridge knots:

Theorem 3.4 (Murasugi (1958), Hartley (1979)) *Let*

$$P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \dots + (-1)^n a_n(t^n + t^{-n}), \quad a_n > 0$$

be the Alexander polynomial of a two-bridge knot. There exists an integer $k \leq n$ such that $a_0 = a_1 = \dots = a_k > a_{k+1} > \dots > a_n$.

Proof. Let K be a two-bridge knot and $\nabla(z) = \alpha_0 f_1 - \alpha_1 f_3 + \cdots + (-1)^n \alpha_n f_{2n+1}$ be its Conway polynomial written in the Fibonacci basis. By Theorem 2.9, $\alpha_n \alpha_k \geq 0$ for all k , and if $\alpha_i = 0$ for some i then $\alpha_j = 0$ for $j \leq i$.

Let $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$, $a_n > 0$ be the Alexander polynomial of K . We have $\Delta(t) = \varepsilon \nabla(t^{1/2} - t^{-1/2})$, where $\varepsilon = \pm 1$, and then, by Corollary 3.3, $\varepsilon \alpha_k = a_k - a_{k+1}$.

We deduce that $\varepsilon \alpha_n = a_n > 0$, and then $a_k - a_{k+1} = \varepsilon \alpha_k \geq 0$ for all k .

Consequently we obtain $a_0 \geq a_1 \geq \cdots \geq a_n > 0$.

Furthermore, if $a_k = a_{k-1}$ for some k , then $\alpha_{k-1} = 0$, and consequently $\alpha_{j-1} = 0$ for all $j \leq k$. This implies that for all $j \leq k$, $a_j = a_{j-1}$, which concludes the proof. \square Now, we shall give

explicit formulas for Alexander coefficients in terms of Conway coefficients.

Lemma 3.5 *Let us denote $u_i = \ell_{2i} = t^i + t^{-i}$. We have*

$$z^{2m} = \left(t^{1/2} - t^{-1/2}\right)^{2m} = \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} u_{m-k} + (-1)^m \binom{2m}{m}.$$

Proof. By induction. We have $z^2 = u_1 - u_0$, and the result is true for $m = 1$. Suppose the result true for m , we have

$$z^{2(m+1)} = z^{2m}(u_1 - u_0) = \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} u_{m-k}(u_1 - u_0) + (-1)^m \binom{2m}{m} u_0(u_1 - u_0).$$

Using the relations $u_i u_j = u_{i+j} + u_{i-j}$ and $u_0 = 2$, the rest of the proof is straightforward. \square

Proposition 3.6 *Let $Q(z) = \tilde{c}_0 + \tilde{c}_1 z^2 + \cdots + \tilde{c}_n z^{2n}$ be a polynomial. We have*

$$Q(t^{1/2} - t^{-1/2}) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),$$

where

$$a_{n-j} = \sum_{k=0}^j (-1)^{n-k} \tilde{c}_{n-k} \binom{2n-2k}{j-k}. \quad (7)$$

Proof. It is sufficient to prove this formula for the monomials z^{2m} , which is done using our lemma. \square

Remark 3.7 *By considering Formula (6) for the polynomial $f_{2n+1} = \sum_{k=0}^n \binom{2n-k}{k} z^{2n-2k}$, we deduce the identity*

$$1 = \sum_{k=0}^j (-1)^k \binom{2n-k}{k} \binom{2n-2k}{j-k}, \quad n, j \geq 0. \quad (8)$$

Remark 3.8 Fukuhara [7] gives a converse formula for the c_k in terms of the a_k , which can be easily deduced from Remark 3.2:

$$\tilde{c}_{n-j} = \sum_{k=0}^j (-1)^{n-k} a_{n-k} \frac{2n-2k}{2n-j-k} \binom{2n-j-k}{2n-2j}. \quad (9)$$

We shall not use this formula. Nevertheless, we note that it implies a nice identity:

$$\binom{2n-j}{j} = \sum_{k=0}^j (-1)^k \frac{2n-2k}{2n-j-k} \binom{2n-j-k}{2n-2j}. \quad (10)$$

From the bounds we obtained for Conway coefficients we can deduce an improvement of the bounds of Nakanishi and Suketa ([22]) for Alexander coefficients.

Theorem 3.9 We have the following sharp inequalities (where all the a_i are positive):

1. $a_{n-j} \leq a_n \left(\sum_{k=0}^j \binom{2n-2k}{j-k} \binom{2n-k}{k} \right)$.
2. $2a_n - 1 \leq a_{n-1} \leq (4n-2)a_n + 1$.
3. $a_{n-2} \leq (8n^2 - 15n + 8)a_n + 2n - 1$, if $a_n \neq 1$.

Proof. The first two bounds were given in [22] and the third one is an improvement. These three bounds are sharp.

1. Using Formula (7) and Theorem 2.6, we obtain

$$|a_{n-j}| \leq \sum_{k=0}^j |\tilde{c}_{n-k}| \binom{2n-2k}{j-k} \leq |a_n| \sum_{k=0}^j \binom{2n-k}{k} \binom{2n-2k}{j-k}. \quad (11)$$

2. We have $|\tilde{c}_{n-1}| \leq \binom{2n-2}{1} |\tilde{c}_n| + 1$ by Theorem 2.8, and $a_{n-1} = \tilde{c}_{n-1} - \binom{2n}{1} \tilde{c}_n$ by Proposition 3.6. We thus deduce

$$|a_{n-1}| \leq \binom{2n}{1} |\tilde{c}_n| + \binom{2n-2}{1} |\tilde{c}_n| + 1 = (4n-2)|a_n| + 1. \quad (12)$$

We also have

$$|a_{n-1}| \geq \binom{2n}{1} |\tilde{c}_n| - |\tilde{c}_{n-1}| \geq \binom{2n}{1} |\tilde{c}_n| - \binom{2n-2}{1} |\tilde{c}_n| - 1 = 2|a_n| - 1.$$

3. From Proposition 3.6 and Theorem 2.8, we get

$$\begin{aligned} |a_{n-2}| &\leq \binom{2n}{2} |\tilde{c}_n| + \binom{2n-2}{1} |\tilde{c}_{n-1}| + \binom{2n-4}{0} |\tilde{c}_{n-2}| \\ &\leq \binom{2n}{2} |\tilde{c}_n| + \binom{2n-2}{1} \left(\binom{2n-2}{1} |\tilde{c}_n| + 1 \right) + \left(\binom{2n-3}{2} + \frac{1}{g} \left(\binom{2n-3}{1} - 1 \right) \right) |\tilde{c}_n| + 1 \\ &= (8n^2 - 16n + 10 + \frac{2(n-2)}{g}) |a_n| + 2n - 1. \end{aligned}$$

If $a_n \neq 1$ then $g \geq 2$, and we obtain

$$|a_{n-2}| \leq |a_n| (8n^2 - 15n + 8) + 2n - 1. \quad (13)$$

The upper bounds (11) and (12) are attained by the knots $C(2, 2, \dots, 2)$. The bound (13) is attained for the knot $C(4, 2, 2, 2, \dots, 2)$. \square

Remarks 3.10 1. Let us look at the proof of inequality (13) if $g = 1$ and $a_n = 1$. We get

$$a_{n-2} \leq 8n^2 - 12n + 5.$$

that is the first inequality (11) when $j = 2$.

2. If $g \geq 3$, the inequality (13) can be improved:

$$a_{n-2} \leq (8n^2 - 16n + 10 + \frac{2(n-2)}{g})a_n + 2n - 1.$$

3. For $j = 3$ we obtain

$$\begin{aligned} a_{n-3} &\leq 2/3 (2n - 3) (8n^2 - 24n + 25) a_n + \frac{(3n-5)(2n-5)}{g} a_n + n(2n - 3) \\ &\leq 1/6 (64n^3 - 270n^2 + 413n - 225) a_n + n(2n - 3). \end{aligned}$$

4. Since the inequalities on Conway coefficients are simpler and stronger, we shall not give the inequalities on Alexander coefficients for $j \geq 4$. Furthermore, if we want to apply our bounds to the Alexander polynomials, we first compute

$$\tilde{c}_{n-j} = \sum_{k=0}^j (-1)^{n-k} a_{n-k} \frac{2n-2k}{2n-j-k} \binom{2n-j-k}{2n-2j},$$

using Remark 3.8 and test if $|\tilde{c}_{n-j}| \leq \binom{2n-j}{j} |\tilde{c}_n|$, which is stronger than the inequality (11), or if $|\tilde{c}_{n-j}| \leq \left(\binom{2n-j-1}{j} + \frac{1}{g} \left(\binom{2n-j-1}{j-1} - 1 \right) \right) |\tilde{c}_n| + 1$. The cost of these evaluations is less than the cost of the evaluations of the inequalities of Theorem 3.9. They are also sharper.

Our last example shows an infinity of polynomials satisfying all the known necessary conditions, but which are not the Alexander polynomial of a two-bridge knot.

4 Modulo 2 polynomials

Theorem 4.1 Let ∇_m be the Conway polynomial of a two-bridge link. Then there exists a Fibonacci polynomial f_D such that $\nabla_m \equiv f_D \pmod{2}$.

Proof. Let us write $(a, b) \equiv (c, d) \pmod{2}$ when $a \equiv c \pmod{2}$ and $b \equiv d \pmod{2}$. We will show by induction on m that there exist integers D and $\varepsilon = \pm 1$ such that $(\nabla_{m-1}, \nabla_m) \equiv (f_{D-\varepsilon}, f_D) \pmod{2}$.

The result is true for $m = 0$ as $(\nabla_{-1}, \nabla_0) = (0, 1) = (f_0, f_1)$, that is $D = \varepsilon = 1$.

Suppose that $(\nabla_{m-1}, \nabla_m) \equiv (f_{D-\varepsilon}, f_D) \pmod{2}$, with $\varepsilon = \pm 1$ for some $m \geq 0$. Then we have $\nabla_{m+1} = b_{m+1}z\nabla_m + \nabla_{m-1}$.

If $b_{m+1} \equiv 0 \pmod{2}$ then $\nabla_{m+1} \equiv \nabla_{m-1} \equiv f_{D-\varepsilon} \pmod{2}$ and $(\nabla_m, \nabla_{m+1}) \equiv (f_D, f_{D-\varepsilon})$. If $b_{m+1} \equiv 1 \pmod{2}$ then $\nabla_{m+1} \equiv z f_D + f_{D-\varepsilon} \equiv f_{D+\varepsilon} \pmod{2}$ and $(\nabla_m, \nabla_{m+1}) \equiv (f_D, f_{D+\varepsilon})$. \square

Example 4.2 (The torus links $T(2, m)$) *The Conway polynomial of the torus link $T(2, m)$ is the Fibonacci polynomial $f_m(z)$ (see [13, 17]).*

Consequently, Theorem 4.1 gives in fact a characterization of modulo 2 Conway polynomials of two-bridge links.

Then, we deduce a simple proof of an elegant criterion from Murasugi ([21, 3])

Corollary 4.3 (Murasugi (1971)) *Let $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \dots + (-1)^n a_n(t^n + t^{-n})$ be the Alexander polynomial of a two-bridge knot. There exists an integer $k \leq n$ such that a_0, a_1, \dots, a_k are odd, and a_{k+1}, \dots, a_n are even.*

Proof. If K is a two-bridge knot, its Conway polynomial is a modulo 2 Fibonacci polynomial f_{2k+1} . By Corollary 3.3 we have $f_{2k+1}(t^{1/2} - t^{-1/2}) = (t^k + t^{-k}) - (t^{k-1} + t^{1-k}) + \dots + (-1)^k$, and the result follows. \square

Remark 4.4 *This congruence may be used as a simple criterion to prove that some knots cannot be two-bridge knots. There is a more efficient criterion by Kanenobu [11, 24] using the Jones and Q polynomials.*

There is an analogous result for two-component links

Corollary 4.5 (Modulo 2 Hosokawa polynomials of two-bridge links) *Let $\Delta(t) = (t^{1/2} - t^{-1/2})(a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \dots + (-1)^n a_n(t^n + t^{-n}))$ be the Alexander polynomial of a two-component two-bridge link. Then all the coefficients a_i are even or there exists an integer $k \leq n$ such that $a_k, a_{k-2}, a_{k-4}, \dots$ are odd, and the other coefficients are even.*

Proof. If K is a two-component two-bridge link, its Conway polynomial is an odd Fibonacci polynomial modulo 2, that is of the form $f_{2h}(z)$. An easy induction shows that

$$f_{4k}(t^{1/2} - t^{-1/2}) = (t^{1/2} - t^{-1/2})(1 + u_2 + u_4 + \dots + u_{2k})$$

and

$$f_{4k+2}(t^{1/2} - t^{-1/2}) = (t^{1/2} - t^{-1/2})(u_1 + u_3 + \dots + u_{2k+1}),$$

where $u_j = t^j + t^{-j}$, and the result follows. \square

Remark 4.6 *This rectifies Satz 4 in [14, p. 186].*

Example 4.7 *Fibonacci links, introduced by J. C. Turner ([25]) are the two-bridge links of Conway form $C(n, n, \dots, n)$, where n is a fixed integer. Their modulo 2 Conway and Alexander polynomials are computed in [17] (see also [16]).*

Following the proof of Theorem 4.1, we propose an algorithm for the determination of D such that $\nabla_K \equiv f_D \pmod{2}$.

Algorithm 4.8 Let K be a two-bridge link (or knot) with Conway form $C(2b_1, 2b_2, \dots, 2b_m)$. Let us define the sequences of integers ε_i and D_i , $i = 0, \dots, m$, by

$$\varepsilon_0 = 1, D_0 = 1, \varepsilon_{i+1} = -(-1)^{b_{i+1}}\varepsilon_i, D_{i+1} = D_i + \varepsilon_{i+1}.$$

Then the modulo 2 Conway polynomial of K is the Fibonacci polynomial $f_D(z)$, where $D = |D_m|$.

Example 4.9 Consider the two-bridge knot $K = S(\frac{1828139}{1042750})$. One can write

$$\frac{1828139}{1042750} = [2b_1, \dots, 2b_{10}] = [2, -4, -20, 2, -2, -12, -2, 4, -12, -4].$$

Our algorithm gives $D = D_{10} = 3$. Consequently, the modulo 2 Conway polynomial of K is $f_3(z) = z^2 + 1$. Its modulo 2 Alexander polynomial is then $1 - (t + t^{-1})$.

We see that the Alexander (and Conway) polynomial of our knot is not congruent to 1 modulo 2. Hence, by a theorem of V. F. R. Jones, J. Przytycki and C. Lamm ([9, 18]), it cannot be a Lissajous knot.

Using Algorithm 4.8 we easily obtain (in Table 1) the number of two-bridge knots with Conway polynomial congruent to 1 modulo 2 (compare [2]) The condition $|D| = 1$, that is $\nabla_K \equiv 1 \pmod{2}$

Crossing Number	3	4	5	6	7	8	9	10	11	12
Two-bridge	1	1	2	3	7	12	24	45	91	176
$\nabla(t) \equiv 1$	0	0	1	1	2	4	8	13	26	51

Crossing Number	13	14	15	16	17	18	19	20	21	22
Two-bridge	352	693	1387	2752	5504	10965	21931	43776	87552	174933
$\nabla(t) \equiv 1$	97	185	365	705	1369	2675	5233	10211	20011	39221

Table 1: The number of two-bridge knots, and two-bridge knots with Conway polynomial congruent to 1 modulo 2.

or equivalently $\Delta_K \equiv 1 \pmod{2}$ is a necessary condition for a two-bridge knot to be Lissajous.

5 Experiments

The following example shows an infinite family of polynomials satisfying all the necessary conditions except the equality case of Theorem 2.6.

Example 5.1 Consider the polynomial $P(z) = f_{m+1}(z) - 2dz^2$, $m = 4n \geq 4$, $d \neq 0$. All its coefficients, except one, satisfy $c_{m-2k} = \binom{m-k}{k}$. By Theorem 2.6, it is not the Conway polynomial of a two-bridge knot. Hence, the corresponding Alexander polynomial

$$\Delta(t) = 4d + 1 - (2d + 1)u_1 + u_2 - u_3 + \dots + u_{2n},$$

where $u_i = t^i + t^{-i}$ is not the Alexander polynomial of a two-bridge knot. Nevertheless, it satisfies all the necessary conditions of Hartley and Murasugi. If $0 < d < \frac{1}{2}n(n + 1)$, it also satisfies the bounds of Theorems 2.6 and 2.8, and then the Nakanishi and Suketa bounds.

Our next example shows that all our necessary conditions are not sufficient.

Example 5.2 Let $p \geq 7$ be an odd prime, and let $n \geq 3$ be an odd integer. Let us define the Conway polynomial $\nabla(z)$ (such that $\nabla(0) = 1$) by

$$\nabla(z) = pf_{2n+1}(z) - (p+3)f_{2n-1}(z) + 4f_{2n-3}(z).$$

This satisfies the Murasugi congruence, the trapezoidal conditions of Theorem 2.9, and the inequalities of Theorems 2.6 and 2.8.

If it were the Conway polynomial of a two-bridge knot, then there would exist integers b_1, \dots, b_{2n} , such that $\tilde{c}_n = b_1 b_2 \cdots b_{2n} = p$, and

$$\tilde{c}_{n-1} = \tilde{c}_n \left(\frac{1}{b_1 b_2} + \frac{1}{b_2 b_3} + \cdots + \frac{1}{b_{2n-1} b_{2n}} \right) \equiv 0, \pm 1, \pm 2 \pmod{2}.$$

Since $\tilde{c}_{n-1} = (2n-2)p - 3$ and $p \geq 7$, this is impossible, and therefore $\nabla(z)$ is not the Conway polynomial of a two-bridge knot.

These simple examples motivate us to compare the efficiency of our several criteria on the Conway polynomials of the first knots and links. Some non two-bridge links have two-bridged Conway polynomials. This means that their Conway polynomial is also the Conway polynomial of a two-bridge link. We will summarize of our results by considering the first 12965 knots with 13 crossings or fewer and the 1424 multi-component links with 11 crossings or fewer. We obtained their Conway polynomial using the data lists of KnotInfo [4] and KnotAtlas [1].

Two-bridged Conway polynomials

First of all, there is a method to determine whether a given polynomial is a two-bridged Conway polynomial. The Conway polynomial of a knot is even, whereas the Conway polynomial of a two-component link is odd.

We shall first consider the case of knots. Let K be the knot $C(2b_1, -2b_2, \dots, 2b_{2n-1}, -2b_{2n})$. The continued fraction $\frac{\alpha}{\beta} = [2b_1, -2b_2, \dots, 2b_{2n-1}, -2b_{2n}]$ is such that $0 < |\beta| < \alpha$, β is even, and α is odd. Using the Siebenmann method, we compute its Conway polynomial

$$\nabla_K = c_n z^{2n} + c_{n-1} z^{2n-2} + \cdots + c_1 z^2 + 1.$$

We have $|c_n| = |b_1 \cdots b_{2n}|$ and $\alpha = |P(2i)|$ (see [6]).

We thus deduce the following algorithm.

Algorithm 5.3 (IsTwoBridged)

Input: $P = c_n z^{2n} + c_{n-1} z^{2n-2} + \cdots + c_1 z^2 + 1$ in $\mathbf{Z}[z]$.

Output: The sequence of two-bridge knots $K = S\left(\frac{\alpha}{\beta}\right)$ such that $P = \nabla_K$.

-
1. Compute $\alpha = |P(2i)|$.
 2. For any even integer β such that $1 \leq \beta < \alpha$,

- (a) compute the continued fraction decomposition $\frac{\alpha}{\beta} = [2q_1, \dots, 2q_{2\ell}]$, $q_i \neq 0$,
- (b) test if $\ell = n$ and $q_1 \cdots q_{2n} = c_n$,
- (c) compute ∇_K where $K = S(\frac{\alpha}{\beta})$ and compare it with P .

For the two-bridge two-component links, the method is exactly the same except that P is odd, and $\frac{\alpha}{\beta} = [2b_1, -2b_2, \dots, 2b_{2n+1}]$ with α even and β odd. In this case, the classical Schubert fraction is not $\frac{\alpha}{\beta}$ but $\frac{\alpha}{\beta + \alpha}$ (see [12]).

Example 5.4 Let $P = 2880z^{10} + 4944z^8 + 2304z^6 + 158z^4 - 61z^2 + 1$. Our Algorithm `IsTwoBridged` easily finds the fractions with positive even denominators

$$\frac{1828139}{1042750} = [2, -4, -20, 2, -2, -12, -2, 4, -12, -4], \quad \frac{1828139}{447486} = [4, 12, -4, 2, 12, 2, -2, 20, 4, -2].$$

They correspond to the same two-bridge knot K . Consequently, K and \overline{K} are the only two-bridge knots such that $\nabla_K = P$.

Two-bridge knots

We used the Algorithm 5.3 to select non two-bridged Conway polynomials among the Conway polynomials of all knots with 13 crossings or fewer. Among the 12965 prime knots with 13 or fewer crossing, we found 10104 non two-bridged Conway polynomials. It is remarkable that Theorem 4.1 on the modulo 2 congruence detects 79 % of these polynomials. If we select knots that satisfy this criterion and the Murasugi and Hartley conditions (Theorem 2.9), we still detect non two-bridged Conway polynomials. In this case the most efficient criterion is the equality case in Theorem 2.6.

For example, the knot $K11n109$ has Conway polynomial $-z^6 + z^4 + \underline{6}z^2 + 1 = -f_7 + 6f_5 - 6f_3 + 2f_1$. It satisfies all conditions except the equality condition of Theorem 2.6 ($f_7 = t^6 + 5t^4 + \underline{6}t^2 + 1$).

There are exactly 3 Conway polynomials

$$\begin{aligned} \nabla_{K13n1862} &= \nabla_{K13n2935} = 1 + 8z^2 + 3z^4 - z^6, \\ \nabla_{K13n2089} &= \nabla_{K13n3038} = 1 + 8z^2 + 5z^4 - z^6, \\ \nabla_{K13n3508} &= 1 + 10z^2 + 5z^4 - z^6, \end{aligned}$$

whose corresponding Alexander polynomials satisfy the Nakanishi-Suketa conditions (11,13) but not those of Theorem 2.6.

The knot $K13n3010$ has Conway polynomial $\nabla = 1 + 10z^2 + 4z^4 - 2z^6$. It satisfies all conditions except those of Theorem 2.8.

Nevertheless, the inequalities on the Conway coefficients are better because they define a polyhedron of volume much smaller than the polyhedron defined by the bounds on the Alexander coefficients.

Multi-component links

We have used the data base of KnotAtlas ([1]) that contains many invariants of the first 1424 multi-component links with 11 crossings or fewer. We deduced the Conway polynomial from the Homfly polynomial. Using our algorithm, we detected 1131 multi-component links whose Conway polynomials are not two-bridged.

Here again, the most efficient condition is the modulo 2 congruence, that detects 86 % of the non two-bridged polynomials. If we consider the 5 criteria together, we detect 1009 among the 1131 non two-bridged polynomials. These polynomials were detected only by using Theorems 4.1 and 2.9.

Conjecture

We observed a trapezoidal property for the Conway polynomials of two-bridged links with crossings fewer than 20 (their number is 131 839).

Conjecture 5.5 *Let $\nabla_m = b_1 \cdots b_m \left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m+1-2i} \right)$, $\alpha_0 = 1$, be the Conway polynomial of a two-bridge link written in the Fibonacci basis. Then there exists $n \leq \lfloor \frac{m}{2} \rfloor$ such that*

$$0 \leq \alpha_0 \leq \alpha_1 \leq \alpha_n \geq \alpha_{n+1} \geq \cdots \geq \alpha_{\lfloor \frac{m}{2} \rfloor} \geq 0.$$

If this conjecture is true, it would imply the following property of Alexander polynomials: *Let $P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ be the Alexander polynomial of a two-bridge knot. Then there exists an integer $k \leq n$ such that (a_0, \dots, a_k) is convex and (a_k, \dots, a_n) is concave.*

Note that this property detects 670 non two-bridged Conway polynomials among the knots with 13 crossings or fewer and 107 among the multi-component links with 11 crossings or fewer. It still detects non two-bridged polynomials among the knots that do not satisfy the modulo 2 congruence Theorem 4.1.

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