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# Conway polynomials of two-bridge links

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## Abstract

We give necessary conditions for the Conway polynomial of a two-bridge link. We apply our conditions to the Alexander polynomial of a two-bridge knot, and obtain simple proofs of the classical theorems of Murasugi and Hartley. We also give sharp bounds for the coefficients of the Conway and Alexander polynomials of a two-bridge link. These bounds improve and generalize those of Nakanishi and Suketa. The efficiency of these conditions are compared on a large number of knots and links.

**MSC2000:** 57M25

**Keywords:** two-bridge link, Conway polynomial, Alexander polynomial.

## 1 Introduction

In this paper, we shall study the problem: given a polynomial, is it the Conway polynomial of a two-bridge link (or knot)? For small degrees, this can be solved by an exhaustive search of possible two-bridge links (see algorithm section 5.1). Here, we rather give necessary conditions on the coefficients of the polynomial, that can be tested also for high degree polynomials.

We shall use L. Siebenmann description of the Conway polynomial of a two-bridge link.

Conway polynomials of links (or knots) are written as

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k} z^{m-2k}.$$

We obtain the following inequalities:

**Theorem 6.** For  $k \geq 0$ ,

$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m|.$$

If equality holds for some positive integer  $k < \lfloor \frac{m}{2} \rfloor$ , then it holds for all integers. In this case, the link is isotopic to a link of Conway form  $C(2, -2, 2, \dots, (-1)^{m+1}2)$  or  $C(2, 2, \dots, 2)$ , up to mirror symmetry.

When  $|c_m| \neq 1$ , we have the following sharper bounds:

**Theorem 8.** *Let  $g \geq 1$  be the greatest prime divisor of  $c_m$ , and  $m \geq 2k \geq 2$ . Then*

$$|c_{m-2k}| \leq \left( \binom{m-k-1}{k} + \frac{1}{g} \left( \binom{m-k-1}{k-1} - 1 \right) \right) |c_m| + 1.$$

Equality holds for links of Conway forms  $C(2g, 2, 2, \dots, 2)$  and  $C(2g, -2, 2, \dots, (-1)^{m+1}2)$ .

Our inequalities refine those of Nakanishi and Suketa for Alexander polynomials of two-bridge knots (theorems 2 and 3 in [25]). Moreover, they are sharp and hold for any  $k$ .

It is convenient to write Conway polynomials in terms of Fibonacci polynomials  $f_k$  defined by:

$$f_0 = 0, f_1 = 1, f_{n+2}(z) = z f_{n+1}(z) + f_n(z).$$

We obtain an extension to links of both the alternating Murasugi theorem [22, 23], and the trapezoidal Hartley theorem [10].

**Theorem 9.** *Let  $K$  be a two-bridge link (or knot). Let*

$$\nabla_K = c_m \left( \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m-2i+1} \right), \quad \alpha_0 = 1$$

be its Conway polynomial written in the Fibonacci basis. Then we have

1.  $\alpha_j \geq 0$ ,  $j = 0, \dots, \lfloor \frac{m}{2} \rfloor$ .
2. If  $\alpha_i = 0$  for some  $i > 0$  then  $\alpha_j = 0$  for  $j \geq i$ .

We also obtain:

**Theorem 21.** *Let  $\nabla(z) \in \mathbf{Z}[z]$  be the Conway polynomial of a rational link (or knot). There exists a Fibonacci polynomial  $f_D(z)$  such that  $\nabla(z) \equiv f_D(z) \pmod{2}$ .*

This provides a simple proof of a congruence of Murasugi [24] for two-bridge knots. Moreover, we obtain a similar congruence for the Hosokawa polynomials of two-bridge links.

We give a simple algorithm that determines the integer  $D$  such that  $\nabla(z) \equiv f_D(z) \pmod{2}$ . This is applied to test when  $\nabla(z) \equiv 1 \pmod{2}$ , which is a necessary condition to be a Lissajous knot.

We give examples showing that the conditions on Conway coefficients are sharper than the conditions on Alexander coefficients deduced from them.

We conclude our paper with experiments comparing the different necessary conditions. Our results on the 12965 knots with 13 crossings or fewer and the 1424 multi-component links with 11 crossings or fewer lead us to the following convexity conjecture:

Let  $P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \dots + (-1)^n a_n(t^n + t^{-n})$  be the Alexander polynomial of a rational knot. There exists an integer  $k \leq n$  such that  $(a_0, \dots, a_k)$  is convex and  $(a_k, \dots, a_n)$  is concave.

We have tested this conjecture for all two-bridge knots with 20 crossings or fewer. We also give a similar conjecture for two-bridge two-component links.

## 2 Conway polynomial

Every oriented two-bridge link can be put in the form shown in figure 1. It will be denoted by  $C(2b_1, 2b_2, \dots, 2b_m)$  with  $b_i \neq 0$  for all  $i$ , including the indicated orientation (see [17] p.26, [18, 14]). This is a two-component link if and only if  $m$  is odd.

Its Conway polynomial  $\nabla_m$  is then given by Siebenmann method (see [26, 7]).

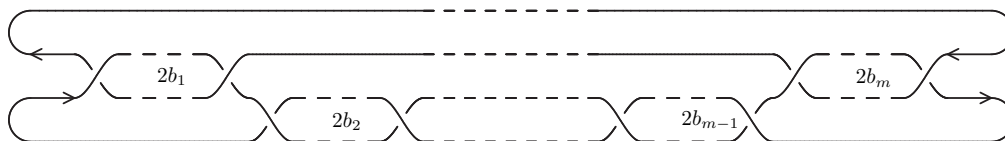


Figure 1: Oriented two-bridge links ( $m$  odd)

**Theorem 1 (Siebenmann, [7]).** Let  $\nabla_m = \nabla_m(z)$  be the Conway polynomial of the oriented two-bridge link (or knot) of Conway form  $C(2b_1, -2b_2, \dots, (-1)^{m+1}2b_m)$ . Let  $\nabla_{-1} = 0$ ,  $\nabla_0 = 1$ . Then

$$\nabla_m = b_m z \nabla_{m-1} + \nabla_{m-2}, \quad (1)$$

for  $m \geq 1$ .

When  $z = 1$ , this is the classical Euler continuant polynomial.

The Fibonacci polynomials will be useful to study these Conway polynomials.

**Definition 2 (Fibonacci Polynomials).** Let  $f_m(z)$  be the polynomials defined by:

$$f_0 = 0, f_1 = 1, f_{n+2}(z) = z f_{n+1}(z) + f_n(z), \quad m \in \mathbf{Z}. \quad (2)$$

We have  $f_{-m}(z) = (-1)^{m+1} f_m(z)$ .

Let us recall some basic facts about Fibonacci polynomials.

**Lemma 3.** For  $m \geq 0$ :

$$f_{m+1}(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} z^{m-2k}.$$

*Proof.* By induction on  $m$ . The result is clear for  $m = 1$  and for  $m = 2$ . Let us suppose the result true for  $m - 1$  and  $m$ . By induction, the coefficient of  $z^{m-2k}$  is  $\binom{m-1-k}{k}$  in  $zf_m(z)$ , and  $\binom{m-1-k}{k-1}$  in  $f_{m-1}(z)$ . Consequently, the coefficient of  $z^{m-2k}$  in  $f_{m+1}(z)$  is

$$\binom{m-1-k}{k} + \binom{m-1-k}{k-1} = \binom{m-k}{k}.$$

□

**Remark 4.** The Fibonacci polynomials can be read on the diagonals of Pascal's triangle. When  $z = 1$ , we recover the classical Lucas identity

$$F_m = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k},$$

where  $F_m$  are the Fibonacci numbers.

We shall need the following more explicit notation for Conway polynomials:

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k}(b_1, \dots, b_m) z^{m-2k}.$$

The next result gives some properties of  $c_{m-2k}(b_1, \dots, b_m)$ , viewed as a polynomial in  $m$  variables.

**Proposition 5.**

1. The polynomial  $c_{m-2k}(b_1, \dots, b_m)$  is the sum of all monomials  $\frac{b_1 \cdots b_m}{b_{i_1} b_{i_1+1} \cdots b_{i_k} b_{i_k+1}}$ , where  $i_h + 1 < i_{h+1}$ .
2. The number of these monomials is  $\binom{m-k}{k}$ . They are relatively prime if  $k \neq 0$ .
3. Let  $m \geq 2k \geq 4$ . For any  $j$ , the number of these monomials which are relatively prime to  $b_j$  is at least  $\binom{m-1-k}{k-1}$ . Furthermore these monomials are relatively prime.

*Proof.*

1. This is a classical property of the Euler continuant.
2. This number is  $c_{m-2k}(1, 1, \dots, 1)$ , which is a coefficient of the Fibonacci polynomial

$$f_{m+1}(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k}(1, 1, \dots, 1) z^{m-2k} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} z^{m-2k}.$$

3. Let  $1 \leq j \leq m$  and  $\mathbf{b} = (1, \dots, 1, 0, 1, \dots, 1)$  where  $b_j = 0$ , and  $b_k = 1$  for  $k \neq j$ . Let us define the polynomials  $g_n$ , for  $n \leq m$  by  $g_n(z) = \nabla_n(\mathbf{b})(z)$ . The number of our monomials is the coefficient  $c_{m-2k}(\mathbf{b})$  of  $g_m(z)$ .

If  $j = 1$ , we have  $g_1 = 0$ ,  $g_2 = 1$  and therefore  $g_n = f_{n-1}$ ,  $n \geq 1$ .

If  $j > 1$ , we have

$$g_1 = f_2, \dots, g_{j-1} = f_j, g_j = f_{j-1}, \dots, g_{n+1} = z g_n + g_{n-1}, \quad n \geq j.$$

Let us write  $p(z) \succeq q(z)$  when each coefficient of  $p$  is greater than or equal to the corresponding coefficient of  $q$ . We have  $f_{k+2} \succeq f_k$ , and then an easy induction shows that  $g_m \succeq f_{m-1}$ .

To conclude the proof, it is enough to verify that for any  $i \neq j$ , there is a monomial which is prime to the monomial  $b_i$ . This is clear since  $m \geq 4$ .  $\square$

**Theorem 6.** For  $k \geq 0$ ,

$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m|.$$

If equality holds for some positive integer  $k < \lfloor \frac{m}{2} \rfloor$ , then it holds for all integers. In this case, the link is isotopic to a link of Conway form  $C(2, -2, 2, \dots, (-1)^{m+1}2)$  or  $C(2, 2, \dots, 2)$ , up to mirror symmetry.

*Proof.* By proposition 5, the number of monomials of  $c_{m-2k}(b_1, \dots, b_m)$  is  $\binom{m-k}{k}$ . The result follows since each monomial is not greater than  $|c_m| = |b_1 \cdots b_m|$ .

If equality holds for some positive integer  $k < \lfloor \frac{m}{2} \rfloor$ , then for all  $i, j$ ,  $b_i b_{i+1} = b_j b_{j+1} = \pm 1$ , which implies the result.  $\square$

To prove the refined inequalities of theorem 8, we shall use the following lemma, which generalizes the inequality  $a + b \leq ab + 1$ , valid for positive integers.

**Lemma 7.** Let  $p_i(\mathbf{x}), i \in \mathcal{S}$  be relatively prime divisors of  $p(\mathbf{x}) = x_1 x_2 \cdots x_m$ .

Let  $\mathbf{b} = (b_1, \dots, b_m)$  be a  $m$ -tuple of positive integers. Then

$$\sum_{i \in \mathcal{S}} p_i(\mathbf{b}) \leq (\text{card}(\mathcal{S}) - 1) p(\mathbf{b}) + 1. \quad (3)$$

*Proof.* We do not suppose the  $p_i$  distinct. Let us prove the result by induction on  $k = \text{card}(\mathcal{S})$ . The result is clear if  $k = 1$ , we have  $p_1 = \pm 1$ , and the inequality is  $\pm 1 \leq 1$ .

If all the  $p_i = 1$ , the result is clear. Otherwise, let  $x_h$  be a divisor of some  $p_i$ .

Let  $\mathcal{S}_1 = \{i \in \mathcal{S}, x_h | p_i\}$ , and  $\mathcal{S}_2 = \mathcal{S} - \mathcal{S}_1$ . We have  $k = k_1 + k_2$ , where  $k_j = \text{card}(\mathcal{S}_j)$ . Let  $q_j = \text{GCD}\{p_i, i \in \mathcal{S}_j\}$ , then  $q_1$  and  $q_2$  are coprime, and  $q_1 q_2$  is a divisor of  $p$ .

By induction we obtain for  $j = 1, 2$ :

$$\sum_{i \in \mathcal{S}_j} p_i(\mathbf{b}) \leq q_j(\mathbf{b}) \left( (k_j - 1) \frac{p(\mathbf{b})}{q_j(\mathbf{b})} + 1 \right).$$

Adding these two inequalities we get

$$\begin{aligned} \sum_{i \in \mathcal{S}} p_i(\mathbf{b}) &\leq (k_1 + k_2 - 1)p(\mathbf{b}) + q_1(\mathbf{b}) + q_2(\mathbf{b}) - p(\mathbf{b}) \\ &\leq (k_1 + k_2 - 1)p(\mathbf{b}) + q_1(\mathbf{b})q_2(\mathbf{b}) - p(\mathbf{b}) + 1, \end{aligned}$$

which proves the result, since  $q_1(\mathbf{b})q_2(\mathbf{b}) \leq p(\mathbf{b})$ .  $\square$

With this lemma we can prove:

**Theorem 8.** *Let  $m \geq 2k > 0$ , and  $g \geq 1$  be the greatest prime divisor of  $c_m$ . Then*

$$|c_{m-2k}| \leq \left( \binom{m-k-1}{k} + \frac{1}{g} \left( \binom{m-k-1}{k-1} - 1 \right) \right) |c_m| + 1.$$

*Equality holds for links of Conway form  $C(2g, -2, \dots, (-1)^{m-1}2)$ .*

*Proof.* If  $k = 1$ , by proposition 5 there are  $m-1$  monomials in the polynomial  $c_{m-2}(b_1, \dots, b_m)$ . Then, using lemma 7 and the notation  $|\mathbf{b}| = (|b_1|, \dots, |b_m|)$ , we get

$$|c_{m-2}| = |c_{m-2}(\mathbf{b})| \leq c_{m-2}(|\mathbf{b}|) \leq (m-2)c_m(|\mathbf{b}|) + 1 = (m-2)|c_m| + 1.$$

Now, suppose  $k \geq 2$ . Let  $g$  be the greatest prime divisor of the integer  $c_m = b_1 \cdots b_m$ , and suppose that  $g \mid b_j$ . Let  $N$  be the number of monomials of  $c_{m-2k}(b_1, \dots, b_m)$  that are prime to the monomial  $b_j$ . By proposition 5, these monomials are relatively prime, and  $N \geq \binom{m-1-k}{k-1}$ . Using lemma 7 we obtain:

$$\begin{aligned} |c_{m-2k}| &\leq (N-1) \frac{|c_m|}{|b_j|} + 1 + \left( \binom{m-k}{k} - N \right) |c_m| \\ &\leq \left( \frac{N-1}{g} + \left( \binom{m-k}{k} - N \right) \right) |c_m| + 1 \\ &= \left( \binom{m-k}{k} - N \left( 1 - \frac{1}{g} \right) - \frac{1}{g} \right) |c_m| + 1 \\ &\leq \left( \binom{m-k}{k} - \binom{m-1-k}{k-1} \left( 1 - \frac{1}{g} \right) - \frac{1}{g} \right) |c_m| + 1 \\ &= \left( \binom{m-1-k}{k} + \frac{1}{g} \left( \binom{m-1-k}{k-1} - 1 \right) \right) |c_m| + 1. \end{aligned}$$

For links of Conway form  $C(2g, -2, \dots, (-1)^{m+1}2)$ , we have  $\mathbf{b} = (g, 1, \dots, 1)$ ,  $N = \binom{m-1-k}{k-1}$ ,  $c_m = g$ , and  $c_{m-2k} = g \binom{m-1-k}{k} + \binom{m-1-k}{k-1}$ , and equality holds everywhere.

For links of Conway form  $C(2g, 2, \dots, 2)$ , we get  $c_{m-2k} = (-1)^{\lfloor \frac{m}{2} \rfloor + k} \left( g \binom{m-1-k}{k} + \binom{m-1-k}{k-1} \right)$ , which proves the result.  $\square$

Now, we will express the Conway polynomials of two-bridge links in terms of Fibonacci polynomials, and show that their coefficients are alternating.

**Theorem 9.** *Let  $K$  be a two-bridge link (or knot). Let*

$$\nabla_K = c_m \left( \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m-2i+1} \right), \quad \alpha_0 = 1$$

be its Conway polynomial written in the Fibonacci basis. Then we have

1.  $\alpha_j \geq 0$ ,  $j = 0, \dots, \lfloor \frac{m}{2} \rfloor$ .
2. If  $\alpha_i = 0$  for some  $i > 0$  then  $\alpha_j = 0$  for  $j \geq i$ .

*Proof.* Let  $K = C(2b_1, -2b_2, \dots, (-1)^{m+1} 2b_m)$ , with  $b_i \neq 0$  for all  $i$ , and let  $\nabla_n$  be the polynomials obtained in the Siebenmann method.

$$\text{We have } \nabla_0 = f_1, \nabla_1 = b_1 f_2, \nabla_2 = b_1 b_2 \left( f_3 - \left(1 - \frac{1}{b_1 b_2}\right) f_1 \right).$$

Let us show by induction that if

$$\nabla_m = b_1 \cdots b_m \left( \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m+1-2i} \right), \quad \nabla_{m-1} = b_1 \cdots b_{m-1} \left( \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \beta_i f_{m-2i} \right)$$

then  $\alpha_j \geq \beta_j \geq 0$ , and if  $\alpha_i = 0$  for some  $i$ , then  $\alpha_j = 0$  for  $j \geq i$ .

The result is true for  $m = 2$  from the expressions of  $\nabla_1$  and  $\nabla_2$ . Using  $z f_{m+1-2i} = f_{m+2-2i} - f_{m-2i}$  and  $\nabla_{m+1} = b_{m+1} z \nabla_m + \nabla_{m-1}$ , we deduce that

$$\nabla_{m+1} = b_1 \cdots b_{m+1} \left( \sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} (-1)^i \gamma_i f_{m+2-2i} \right),$$

where  $\gamma_0 = 1$  and

$$\gamma_i = \alpha_i + (\alpha_{i-1} - \beta_{i-1}) + \left(1 - \frac{1}{b_m b_{m+1}}\right) \beta_{i-1}, \quad i = 1, \dots, \lfloor \frac{m+1}{2} \rfloor. \quad (4)$$

As  $|b_m b_{m+1}| \geq 1$ , we deduce by induction that  $\gamma_i \geq \alpha_i \geq 0$ .

Furthermore, if  $\gamma_i = 0$ , then by formula (4)  $\alpha_i = 0$ , and then, by induction,  $\alpha_j = \beta_j = 0$  for  $j \geq i$ . Finally, by formula (4), we get  $\gamma_j = 0$  for  $j \geq i$ .  $\square$

**Remark 10.** It is interesting to look at the condition 2. of theorem 9. Let us give a direct proof of it in the case  $m = 4$ . The polynomial  $\nabla_3$  has only two terms and

$$\begin{aligned} \nabla_4 &= b_1 b_2 b_3 b_4 f_5 - (3b_1 b_2 b_3 b_4 - b_1 b_2 - b_1 b_4 - b_3 b_4) f_3 \\ &\quad + (2b_1 b_2 b_3 b_4 - b_1 b_2 - b_1 b_4 - b_3 b_4 + 1) f_1. \end{aligned}$$



Suppose that the second coefficient of  $\nabla_4$  is equal to zero. Using lemma 7, we get

$$3|b_1b_2b_3b_4| \leq |b_1b_2| + |b_1b_2| + |b_3b_4| \leq 2|b_1b_2b_3b_4| + 1,$$

and therefore  $b_1 = b_2 = b_3 = b_4 = \pm 1$ , which implies that  $\nabla_4 = \pm f_5$ . This shows that the point 2. is true for  $m = 4$ .

### 3 Applications to Alexander polynomials of knots

In this paragraph, we will see that our necessary conditions on Conway coefficients are improvements of the classical bounds of [25] on Alexander coefficients of two-bridge knots. For simplicity, we shall restrict ourselves to knots. Conway and Alexander polynomials of a knot  $K$  will be denoted by

$$\nabla_K(z) = 1 + \tilde{c}_1z^2 + \cdots + \tilde{c}_nz^{2n}$$

and

$$\Delta_K(t) = a_0 - a_1(t + t^{-1}) + \cdots + (-1)^n a_n(t^n + t^{-n}).$$

The Alexander polynomial  $\Delta_K(t)$  is deduced from the Conway polynomial:

$$\Delta_K(t) = \nabla_K(t^{1/2} - t^{-1/2}).$$

It is often normalized so that  $a_n$  is positive. Thanks to this formula, it is not difficult to deduce the Alexander polynomial from the Conway polynomial. If we use the Fibonacci basis, it is even easier to deduce the Conway polynomial of a knot from its Alexander polynomial.

**Lemma 11.** *If  $z = t^{1/2} - t^{-1/2}$ , and  $n \in \mathbf{Z}$  is an integer, we have the identity*

$$f_{n+1}(z) + f_{n-1}(z) = (t^{1/2})^n + (-t^{-1/2})^n,$$

where  $f_k(z)$  are Fibonacci polynomials.

*Proof.* Let  $A = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$  be the (polynomial) Fibonacci matrix. If  $z = t^{1/2} - t^{-1/2}$ , the eigenvalues of  $A$  are  $t^{1/2}$  and  $-t^{-1/2}$ , and consequently  $\text{tr } A^n = (t^{1/2})^n + (-t^{-1/2})^n$ . On the other hand, we have  $A^n = \begin{bmatrix} f_{n+1}(z) & f_n(z) \\ f_n(z) & f_{n-1}(z) \end{bmatrix}$ , and then  $\text{tr } A^n = f_{n+1}(z) + f_{n-1}(z)$ .  $\square$

**Remark 12.** The Lucas polynomials  $\ell_n$  are defined by  $\ell_n = f_{n+1} + f_{n-1}$ . They satisfy  $\ell_0 = 2$ ,  $\ell_1 = z$ ,  $\ell_{n+1} = z\ell_n + \ell_{n-1}$ . From Lemma 3 we recover the classical result:

$$\ell_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} z^{n-2j}. \quad (5)$$

From Lemma 11, we immediately deduce:

**Corollary 13.** *Let the Laurent polynomial  $P(t)$  be defined by*

$$P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}).$$

We have

$$P(t) = \sum_{k=0}^n (-1)^k (a_k - a_{k+1}) f_{2k+1}(z),$$

where  $z = t^{1/2} - t^{-1/2}$ , and  $a_{n+1} = 0$ .

We deduce a useful formula:

$$f_{2n+1}(t^{1/2} - t^{-1/2}) = (t^n + t^{-n}) - (t^{n-1} + t^{1-n}) + \cdots + (-1)^n. \quad (6)$$

Now, we shall show that theorem 9 implies both Murasugi and Hartley theorems for two-bridge knots:

**Theorem 14 (Murasugi (1958), Hartley (1979)).** *Let*

$$P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}), \quad a_n > 0$$

*be the Alexander polynomial of a two-bridge knot. There exists an integer  $k \leq n$  such that  $a_0 = a_1 = \cdots = a_k > a_{k+1} > \cdots > a_n$ .*

*Proof.* Let  $K$  be a two-bridge knot and  $\nabla(z) = \alpha_0 f_1 - \alpha_1 f_3 + \cdots + (-1)^n \alpha_n f_{2n+1}$  be its Conway polynomial written in the Fibonacci basis. By theorem 9,  $\alpha_n \alpha_k \geq 0$  for all  $k$ , and if  $\alpha_i = 0$  for some  $i$  then  $\alpha_j = 0$  for  $j \leq i$ .

Let  $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ ,  $a_n > 0$  be the Alexander polynomial of  $K$ . We have  $\Delta(t) = \varepsilon \nabla(t^{1/2} - t^{-1/2})$ , where  $\varepsilon = \pm 1$ , and then, by corollary 13,  $\varepsilon \alpha_k = a_k - a_{k+1}$ .

We deduce that  $\varepsilon \alpha_n = a_n > 0$ , and then  $a_k - a_{k+1} = \varepsilon \alpha_k \geq 0$  for all  $k$ .

Consequently we obtain  $a_0 \geq a_1 \geq \cdots \geq a_n > 0$ .

Furthermore, if  $a_k = a_{k-1}$  for some  $k$ , then  $\alpha_{k-1} = 0$ , and consequently  $\alpha_{j-1} = 0$  for all  $j \leq k$ . This implies that for all  $j \leq k$ ,  $a_j = a_{j-1}$ , which concludes the proof.  $\square$

Now, we shall give explicit formulas for Alexander coefficients in terms of Conway coefficients.

**Lemma 15.** *Let us denote  $u_i = \ell_{2i} = t^i + t^{-i}$ . We have*

$$z^{2m} = (t^{1/2} - t^{-1/2})^{2m} = \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} u_{m-k} + (-1)^m \binom{2m}{m}.$$

*Proof.* By induction. We have  $z^2 = u_1 - u_0$ , and the result is true for  $m = 1$ . Suppose the result true for  $m$ , we have

$$z^{2(m+1)} = z^{2m}(u_1 - u_0) = \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} u_{m-k}(u_1 - u_0) + (-1)^m \binom{2m}{m} u_0(u_1 - u_0).$$

Using the relations  $u_i u_j = u_{i+j} + u_{i-j}$  and  $u_0 = 2$ , the rest of the proof is straightforward.  $\square$

**Proposition 16.** *Let  $Q(z) = \tilde{c}_0 + \tilde{c}_1 z^2 + \dots + \tilde{c}_n z^{2n}$  be a polynomial. We have*

$$Q(t^{1/2} - t^{-1/2}) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \dots + (-1)^n a_n(t^n + t^{-n}),$$

where

$$a_{n-j} = \sum_{k=0}^j (-1)^{n-k} \tilde{c}_{n-k} \binom{2n-2k}{j-k}. \quad (7)$$

*Proof.* It is enough to prove this formula for the monomials  $z^{2m}$ , which is done using our lemma.  $\square$

**Remark 17.** By considering the formula (6) for the polynomial  $f_{2n+1} = \sum_{k=0}^n \binom{2n-k}{k} z^{2n-2k}$ , we deduce the identity

$$1 = \sum_{k=0}^j (-1)^k \binom{2n-k}{k} \binom{2n-2k}{j-k}, \quad n, j \geq 0. \quad (8)$$

**Remark 18.** Fukuhara [8] gives a converse formula for the  $c_k$  in terms of the  $a_k$ , that can be easily deduced from remark 12:

$$\tilde{c}_{n-j} = \sum_{k=0}^j (-1)^{n-k} a_{n-k} \frac{\binom{2n-2k}{2n-j-k}}{\binom{2n-2j}{2n-2j}}. \quad (9)$$

We shall not use this formula. Nevertheless, we remark that it implies a nice identity:

$$\binom{2n-j}{j} = \sum_{k=0}^j (-1)^k \frac{\binom{2n-2k}{2n-j-k}}{\binom{2n-2j}{2n-2j}}. \quad (10)$$

From the bounds we obtained for Conway coefficients we deduce an improvement of the bounds of Nakanishi and Suketa ([25]) for Alexander coefficients.

**Theorem 19.** *We have the following sharp inequalities (where all the  $a_i$  are positive):*

1.  $a_{n-j} \leq a_n \left( \sum_{k=0}^j \binom{2n-2k}{j-k} \binom{2n-k}{k} \right)$ .
2.  $2a_n - 1 \leq a_{n-1} \leq (4n - 2)a_n + 1$ .
3.  $a_{n-2} \leq (8n^2 - 15n + 8)a_n + 2n - 1$ , if  $a_n \neq 1$ .

*Proof.* The first two bounds were given in [25] and the third one is an improvement. These three bounds are sharp.

1. Using the expression (7) and theorem 6, we obtain

$$|a_{n-j}| \leq \sum_{k=0}^j |\tilde{c}_{n-k}| \binom{2n-2k}{j-k} \leq |a_n| \sum_{k=0}^j \binom{2n-k}{k} \binom{2n-2k}{j-k}. \quad (11)$$

2. We have  $|\tilde{c}_{n-1}| \leq \binom{2n-2}{1} |\tilde{c}_n| + 1$  by theorem 8, and  $a_{n-1} = \tilde{c}_{n-1} - \binom{2n}{1} \tilde{c}_n$  by proposition 16. We thus deduce

$$|a_{n-1}| \leq \binom{2n}{1} |\tilde{c}_n| + \binom{2n-2}{1} |\tilde{c}_n| + 1 = (4n - 2) |a_n| + 1. \quad (12)$$

We also have

$$|a_{n-1}| \geq \binom{2n}{1} |\tilde{c}_n| - |\tilde{c}_{n-1}| \geq \binom{2n}{1} |\tilde{c}_n| - \binom{2n-2}{1} |\tilde{c}_n| - 1 = 2 |a_n| - 1.$$

3. From proposition 16 and theorem 8, we get

$$\begin{aligned} |a_{n-2}| &\leq \binom{2n}{2} |\tilde{c}_n| + \binom{2n-2}{1} |\tilde{c}_{n-1}| + \binom{2n-4}{0} |\tilde{c}_{n-2}| \\ &\leq \binom{2n}{2} |\tilde{c}_n| + \binom{2n-2}{1} \left( \binom{2n-2}{1} |\tilde{c}_n| + 1 \right) + \left( \binom{2n-3}{2} + \frac{1}{g} \left( \binom{2n-3}{1} - 1 \right) \right) |\tilde{c}_n| + 1 \\ &= (8n^2 - 16n + 10 + \frac{2(n-2)}{g}) |a_n| + 2n - 1. \end{aligned}$$

If  $a_n \neq 1$  then  $g \geq 2$ , and we obtain

$$|a_{n-2}| \leq |a_n| (8n^2 - 15n + 8) + 2n - 1. \quad (13)$$

The upper bounds (11) and (12) are attained by the knots  $C(2, 2, \dots, 2)$ . The bound (13) is attained for the knot  $C(4, 2, 2, 2, \dots, 2)$ .  $\square$

### Remarks 20.

1. Let us look at the proof of inequality (13) if  $g = 1$  and  $a_n = 1$ . We get

$$a_{n-2} \leq 8n^2 - 12n + 5.$$

that is the first inequality (11) when  $j = 2$ .

2. If  $g \geq 3$ , the inequality (13) can be improved:

$$a_{n-2} \leq (8n^2 - 16n + 10 + \frac{2(n-2)}{g})a_n + 2n - 1.$$

3. For  $j = 3$  we obtain

$$\begin{aligned} a_{n-3} &\leq 2/3 (2n - 3) (8n^2 - 24n + 25) a_n + \frac{(3n-5)(2n-5)}{g} a_n + n(2n - 3) \\ &\leq 1/6 (64n^3 - 270n^2 + 413n - 225) a_n + n(2n - 3). \end{aligned}$$

4. Since the inequalities on Conway coefficients are simpler and stronger, we shall not give the inequalities on Alexander coefficients for  $j \geq 4$ . Furthermore, if we want to apply our bounds to the Alexander polynomials, we first compute

$$\tilde{c}_{n-j} = \sum_{k=0}^j (-1)^{n-k} a_{n-k} \frac{2n-2k}{2n-j-k} \binom{2n-j-k}{2n-2j},$$

using remark 18 and test if  $|\tilde{c}_{n-j}| \leq \binom{2n-j}{j} |\tilde{c}_n|$ , which is stronger than the inequality (11), or if  $|\tilde{c}_{n-j}| \leq \left( \binom{2n-j-1}{j} + \frac{1}{g} \left( \binom{2n-j-1}{j-1} - 1 \right) \right) |c_n| + 1$ . The cost of these evaluations is less than the cost of the evaluations of the inequalities of theorem 19. They are also sharper.

Our last example shows an infinity of polynomials satisfying all the known necessary conditions, but which are not the Alexander polynomial of a two-bridge knot.

## 4 Modulo 2 polynomials

**Theorem 21.** *Let  $\nabla_m$  be the Conway polynomial of a two-bridge link. Then there exists a Fibonacci polynomial  $f_D$  such that  $\nabla_m \equiv f_D \pmod{2}$ .*

*Proof.* Let us write  $(a, b) \equiv (c, d) \pmod{2}$  when  $a \equiv c \pmod{2}$  and  $b \equiv d \pmod{2}$ . We will show by induction on  $m$  that there exist integers  $D$  and  $\varepsilon = \pm 1$  such that  $(\nabla_{m-1}, \nabla_m) \equiv (f_{D-\varepsilon}, f_D) \pmod{2}$ .

The result is true for  $m = 0$  as  $(\nabla_{-1}, \nabla_0) = (0, 1) = (f_0, f_1)$ , that is  $D = \varepsilon = 1$ .

Suppose that  $(\nabla_{m-1}, \nabla_m) \equiv (f_{D-\varepsilon}, f_D) \pmod{2}$ , with  $\varepsilon = \pm 1$  for some  $m \geq 0$ . Then we have  $\nabla_{m+1} = b_{m+1}z\nabla_m + \nabla_{m-1}$ .

If  $b_{m+1} \equiv 0 \pmod{2}$  then  $\nabla_{m+1} \equiv \nabla_{m-1} \equiv f_{D-\varepsilon} \pmod{2}$  and  $(\nabla_m, \nabla_{m+1}) \equiv (f_D, f_{D-\varepsilon})$ . If  $b_{m+1} \equiv 1 \pmod{2}$  then  $\nabla_{m+1} \equiv z f_D + f_{D-\varepsilon} \equiv f_{D+\varepsilon} \pmod{2}$  and  $(\nabla_m, \nabla_{m+1}) \equiv (f_D, f_{D+\varepsilon})$ .  $\square$

Following this proof we propose an algorithm at the end of the paper for the determination of  $D$  such that  $\nabla_K \equiv f_D \pmod{2}$ . The condition  $|D| = 1$ , that is  $\nabla_K \equiv 1 \pmod{2}$  or

equivalently  $\Delta_K \equiv 1 \pmod{2}$  is a necessary condition for a two-bridge knot to be Lissajous.

**Example 22 (The torus links  $T(2, m)$ ).** The Conway polynomial of the torus link  $T(2, m)$  is the Fibonacci polynomial  $f_m(z)$  (see [15, 20]). Consequently, theorem 21 gives in fact a characterization of modulo 2 Conway polynomials of two-bridge links.

Then, we deduce a simple proof of a beautiful criterion due to Murasugi ([24, 4])

**Corollary 23 (Murasugi (1971)).** *Let  $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \dots + (-1)^n a_n(t^n + t^{-n})$  be the Alexander polynomial of a two-bridge knot. There exists an integer  $k \leq n$  such that  $a_0, a_1, \dots, a_k$  are odd, and  $a_{k+1}, \dots, a_n$  are even.*

*Proof.* If  $K$  is a two-bridge knot, its Conway polynomial is a modulo 2 Fibonacci polynomial  $f_{2k+1}$ . By corollary 13 we have  $f_{2k+1}(t^{1/2} - t^{-1/2}) = (t^k + t^{-k}) - (t^{k-1} + t^{1-k}) + \dots + (-1)^k$ , and the result follows.  $\square$

**Remark 24.** This congruence may be used as a simple criterion to prove that some knots cannot be two-bridge knots. There is a more efficient criterion by Kanenobu [13, 27] using the Jones and Q polynomials.

There is an analogous result for two-component links

**Corollary 25 (Modulo 2 Hosokawa polynomials of two-bridge links).**

*Let  $\Delta(t) = (t^{1/2} - t^{-1/2})(a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \dots + (-1)^n a_n(t^n + t^{-n}))$  be the Alexander polynomial of a two-component two-bridge link. There exists an integer  $k \leq n$  such that  $a_k, a_{k-2}, a_{k-4}, \dots$  are odd, and the other coefficients are even.*

*Proof.* If  $K$  is a two-component two-bridge link, its Conway polynomial is an odd Fibonacci polynomial modulo 2, i.e. of the form  $f_{2h}(z)$ . An easy induction shows that

$$f_{4k}(t^{1/2} - t^{-1/2}) = (t^{1/2} - t^{-1/2})(1 + u_2 + u_4 + \dots + u_{2k})$$

and

$$f_{4k+2}(t^{1/2} - t^{-1/2}) = (t^{1/2} - t^{-1/2})(u_1 + u_3 + \dots + u_{2k+1}),$$

where  $u_j = t^j + t^{-j}$ , and the result follows.  $\square$

**Example 26.** Fibonacci links, introduced by J. C. Turner ([29]) are the two-bridge links of Conway form  $C(n, n, \dots, n)$ , where  $n$  is a fixed integer. Their modulo 2 Conway and Alexander polynomials are computed in [20] (see also [19]).

## 5 Experiments

The following example shows an infinite family of polynomials satisfying all the necessary conditions except the equality case of theorem 6.

**Example 27.** Consider the polynomial  $P(z) = f_{m+1}(z) - 2dz^2$ ,  $m = 4n \geq 4$ ,  $d \neq 0$ . All its coefficients, except one, satisfy  $c_{m-2k} = \binom{m-k}{k}$ . By theorem 6, it is not the Conway polynomial of a two-bridge knot. Hence, the corresponding Alexander polynomial

$$\Delta(t) = 4d + 1 - (2d + 1)u_1 + u_2 - u_3 + \cdots + u_{2n},$$

where  $u_i = t^i + t^{-i}$  is not the Alexander polynomial of a two-bridge knot. Nevertheless, it satisfies all the necessary conditions of Hartley and Murasugi. If  $0 < d < \frac{1}{2}n(n+1)$ , it also satisfies the bounds of theorems 6 and 8, and then the Nakanishi and Suketa bounds.

Our next example shows that all our necessary conditions are not sufficient.

**Example 28.** Let  $p \geq 7$  be an odd prime, and let  $n \geq 3$  be an odd integer. Let us define the Conway polynomial  $\nabla(z)$  (such that  $\nabla(0) = 1$ ) by

$$\nabla(z) = pf_{2n+1}(z) - (p+3)f_{2n-1}(z) + 4f_{2n-3}(z).$$

It satisfies the Murasugi congruence, the trapezoidal conditions of theorem 9, and the inequalities of theorems 6 and 8.

If it was the Conway polynomial of a two-bridge knot, then there would exist integers  $b_1, \dots, b_{2n}$ , such that  $\tilde{c}_n = b_1 b_2 \cdots b_{2n} = p$ , and

$$\tilde{c}_{n-1} = \tilde{c}_n \left( \frac{1}{b_1 b_2} + \frac{1}{b_2 b_3} + \cdots + \frac{1}{b_{2n-1} b_{2n}} \right) \equiv 0, \pm 1, \pm 2 \pmod{2}.$$

Since  $\tilde{c}_{n-1} = (2n-2)p - 3$  and  $p \geq 7$ , this is impossible, and then  $\nabla(z)$  is not the Conway polynomial of a two-bridge knot.

These simple examples motivate us to compare the efficiency of our several criterions on the Conway polynomials of the first knots and links. Some non two-bridge links have two-bridged Conway polynomials. It means that their Conway polynomial is also the Conway polynomial of a two-bridge link. We will give a summary of our results by considering the first 12965 knots with 13 crossings or fewer and the 1424 multi-component links with 11 crossings or fewer. We obtained their Conway polynomial using the data lists of KnotInfo [5] and KnotAtlas [1].

### 5.1 Two-bridged Conway polynomials

First of all, there is a method to determine if a given polynomial is a two-bridged Conway polynomial. We will not discuss in details the complexity of this algorithm, but it is fast enough to test our data in a few seconds.

The Conway polynomial of a knot is even, whereas the Conway polynomial of a two-component link is odd.

We shall first consider the case of knots. Let  $K$  be the knot  $C(2b_1, -2b_2, \dots, 2b_{2n-1}, -2b_{2n})$ . The continued fraction  $\frac{\alpha}{\beta} = [2b_1, -2b_2, \dots, 2b_{2n-1}, -2b_{2n}]$  is such that  $0 < |\beta| < \alpha$ ,  $\beta$  is even, and  $\alpha$  is odd. Using Siebenmann method, we calculate its Conway polynomial

$$\nabla_K = c_n z^{2n} + c_{n-1} z^{2n-2} + \dots + c_1 z^2 + 1.$$

We have  $|c_n| = |b_1 \cdots b_{2n}|$  and  $\alpha = |P(2i)|$  (see [7]).

We thus deduce the following algorithm.

**Algorithm 29** (IsTwoBridged).

**Input:**  $P = c_n z^{2n} + c_{n-1} z^{2n-2} + \dots + c_1 z^2 + 1$  in  $\mathbf{Z}[z]$ .

**Output:** The sequence of two-bridge knots  $K = S(\frac{\alpha}{\beta})$  such that  $P = \nabla_K$ .

1. Compute  $\alpha = |P(2i)|$ .
2. For any even integer  $\beta$  such that  $1 \leq \beta < \alpha$ ,
  - (a) compute the continued fraction decomposition  $\frac{\alpha}{\beta} = [2q_1, \dots, 2q_\ell]$ ,  $q_i \neq 0$ ,
  - (b) test if  $\ell = n$  and  $q_1 \cdots q_{2n} = c_n$ ,
  - (c) compute  $\nabla_K$  where  $K = S(\frac{\alpha}{\beta})$  and compare it with  $P$ .

For the two-bridge two-component links, the method is exactly the same except that  $P$  is odd, and  $\frac{\alpha}{\beta} = [2b_1, -2b_2, \dots, 2b_{2n+1}]$  with  $\alpha$  even and  $\beta$  odd. In this case, the classical Schubert fraction is not  $\frac{\alpha}{\beta}$  but  $\frac{\alpha}{\beta + \alpha}$ .

### Complexity of the algorithm

First, note that  $\alpha = |P(2i)| \leq \sum_{k=0}^n |c_{n-k}| 4^{n-k} \leq \frac{1}{3} \|P\| 4^{n+1}$ , where  $\|P\| = \max\{|c_0|, \dots, |c_n|\}$ .

Conversely, the following lemma allows us to bound the length of the continued fractions and therefore the degrees of the polynomials  $\nabla_K$ .

**Lemma 30.** Let  $\frac{\alpha}{\beta} = [2q_1, \dots, 2q_m]$ ,  $q_i \neq 0$ . Then  $m \leq |\alpha| - 1$ .

*Proof.* We shall prove our lemma by induction on  $m$ . First, it is not difficult to prove that  $|\alpha| > |\beta|$ . If  $m = 1$ , the result is clear. Let us suppose the result true for  $m - 1$ , and consider  $\frac{\alpha}{\beta} = [2q_1, \dots, 2q_m]$ ,  $q_i \neq 0$ . Consider  $\frac{\alpha'}{\beta'} = [2q_2, \dots, 2q_m]$ ,  $q_i \neq 0$ . By induction we have  $m - 1 \leq |\alpha'| - 1$ . On the other hand, we have

$$\frac{\alpha}{\beta} = [2q_1, \frac{\alpha'}{\beta'}] = 2q_1 + \frac{\beta'}{\alpha'} = \frac{2q_1 \alpha' + \beta'}{\alpha'}.$$



We thus have  $\beta = \alpha'$ ,  $|\alpha| \geq |2q_1\alpha'| - |\beta'| \geq 2|\alpha'| - (|\alpha'| - 1) = |\alpha'| + 1 \geq m + 1$ .  $\square$

**Remark 31.** The rational number  $\frac{m+1}{m}$  has the continued fraction expansion of length  $m$ :  $[2, -2, \dots, (-1)^{m-1}2]$ . This shows that the inequality of lemma 30 is sharp.

Lemma 30 shows that the computation of the continued fraction expansion  $\frac{\alpha}{\beta} = [2q_1, \dots, 2q_m]$  requires less than  $\alpha$  Euclidean divisions). We thus compute all the continued fractions  $\frac{\alpha}{\beta}$  in  $\mathcal{O}(\alpha^2)$  steps.

The computation of  $\nabla_K$  requires  $m = 2n$  polynomial multiplications of form  $(P, Q) \mapsto q_i z P + Q$ . Furthermore, by theorem 6 the coefficients of these polynomials are bounded by  $|c_n| F_{2n}$ .

We conclude that the algorithm `IsTwoBridged` has an exponential complexity  $\mathcal{O}(\|P\| 4^{3n} F_{2n})$ .

**Example 32.** Let  $P = 2880z^{10} + 4944z^8 + 2304z^6 + 158z^4 - 61z^2 + 1$ . Our algorithm `IsTwoBridged` easily finds the fractions with positive even denominators

$$\frac{1828139}{1042750} = [2, -4, -20, 2, -2, -12, -2, 4, -12, -4], \frac{1828139}{447486} = [4, 12, -4, 2, 12, 2, -2, 20, 4, -2].$$

They correspond to the same two-bridge knot  $K$ . Consequently,  $K$  and  $\overline{K}$  are the only two-bridge knots such that  $\nabla_K = P$ .

**Remark 33.** Using theorem 6, it is possible to improve our complexity estimation. We have  $|c_{n-k}| \leq |c_n| \binom{2n-k}{k}$  and by lemma 3

$$\alpha = |P(2i)| \leq |c_n| \sum_{k=0}^n \binom{2n-k}{k} 2^{2n-2k} = |c_n| f_{2n+1}(2).$$

We also deduce the following general result on continued fractions.

**Corollary 34.** Let  $\frac{\alpha}{\beta} = [2q_1, \dots, 2q_m]$ ,  $q_i \neq 0$  where  $(\alpha, \beta) = 1$  and  $\alpha > 0$ . Then we have  $\alpha \leq |q_1 \cdots q_m| f_{m+1}(2)$ ,  $|\beta| \leq |q_2 \cdots q_m| f_m(2)$ . Equality holds for  $\frac{\alpha}{\beta} = \frac{f_{m+1}(2)}{f_m(2)} = [2, \dots, 2]$ .

*Proof.* Consider the link  $K = C(2q_1, -2q_2, \dots, (-1)^{m-1}2q_m)$ . Its Conway polynomial  $\nabla_m(z)$  is the numerator of the continued fraction  $[q_1 z, q_2 z, \dots, q_m z]$  and we have

$$\alpha = |\nabla_m(q_1, \dots, q_m)(2i)|, \beta = \pm |\nabla_{m-1}(q_2, \dots, q_m)(2i)|.$$

Using lemma 3 and theorem 6, we get

$$\alpha \leq |c| \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} 2^{m-2k} = |c| f_{m+1}(2),$$

where  $c = q_1 \cdots q_m$ . We also get  $|\beta| \leq |q_2 \cdots q_m| f_m(2)$ .  $\square$

## 5.2 Two-bridge knots

We first use the algorithm 29 to select non two-bridged Conway polynomials among the Conway polynomials of all knots with 13 crossings or fewer.

Crossing Number	1	2	3	4	5	6	7	8	9	10	11	12	13
Knots	0	0	1	1	2	3	7	21	49	165	552	2176	9988
Two-bridge Knots	0	0	1	1	2	3	7	12	24	45	91	176	352
Two-bridged $\nabla$	0	0	1	1	2	3	7	12	26	75	203	604	1927
Non two-bridged $\nabla$	0	0	0	0	0	0	0	9	23	90	349	1572	8061

Table 1: Number of knots having two-bridged Conway polynomials

We now compare our conditions only on the first 10104 non two-bridged Conway polynomials. It is remarkable that theorem 21 on the modulo 2 congruence detects 79 % of these polynomials (see table 2).

Crossing Number	8	9	10	11	12	13	$\leq 13$
Non two-bridged $\nabla$	9	23	90	349	1572	8061	10104
Detected by theorem 9	1	3	13	42	184	994	1237
Detected by theorem 21	9	21	71	281	1203	6437	8022
Not detected by theorems 9 and 21	0	2	15	56	320	1428	1821

Table 2: Efficiency of theorems 9 and 21 in the detection of non two-bridged Conway polynomials

Using the other conditions on these 10104 non two-bridged Conway polynomials, we obtain

- 8022 do not satisfy theorem 21 on modulo 2 congruence.
- 1237 do not satisfy theorem 9 which is equivalent to both Hartley condition and alternating Murasugi theorem for Alexander polynomials.
- 29 do not satisfy the inequality (11).
- 57 do not satisfy the inequality (13).
- 172 do not satisfy theorem 6.
- 333 do not satisfy the equality condition in theorem 6.
- 197 do not satisfy theorem 8

It appears that the congruence modulo 2 is the more accurate criterion. If we select knots that satisfy this criterion and the Murasugi and Hartley conditions (theorem 9), we still detect non two-bridged Conway polynomials. In this case the most accurate criterion is the equality case in theorem 6.

For example, the knot  $K11n109$  has Conway polynomial  $-z^6 + z^4 + \underline{6}z^2 + 1 = -f_7 + 6f_5 - 6f_3 + 2f_1$ . It satisfies all conditions but the equality condition of theorem 6 ( $f_7 = t^6 + 5t^4 + \underline{6}t^2 + 1$ ).

There are 3 Conway polynomials

$$\begin{aligned}\nabla_{K13n1862} = \nabla_{K13n2935} &= 1 + 8z^2 + 3z^4 - z^6, \\ \nabla_{K13n2089} = \nabla_{K13n3038} &= 1 + 8z^2 + 5z^4 - z^6, \\ \nabla_{K13n3508} &= 1 + 10z^2 + 5z^4 - z^6,\end{aligned}$$

whose corresponding Alexander polynomials satisfy the Nakanishi-Suketa conditions (11,13) but not theorem 6.

The knot  $K13n3010$  has Conway polynomial  $\nabla = 1 + 10z^2 + 4z^4 - 2z^6$ . It satisfies all conditions but theorem 8.

### 5.3 Multi-component links

We have used the data base of KnotAtlas ([1]) that contains many invariants of the first 1424 multi-component links with 11 crossings or fewer. We deduced the Conway polynomial from the Homfly polynomial. Using our algorithm, we detected 1131 multi-component links whose Conway polynomials are not two-bridged. We obtain:

Crossing Number	1	2	3	4	5	6	7	8	9	10	11	$\leq 11$
Multi-component links	0	1	0	1	1	6	9	29	83	287	1007	1424
Two-bridge links	0	1	0	1	1	3	3	8	12	27	45	101
Two-bridged $\nabla$	0	1	0	1	1	3	5	13	22	67	180	293
Non two-bridged $\nabla$	0	0	0	0	0	3	4	16	61	220	827	1131

Table 3: Number of multi-component links having two-bridged Conway polynomials

Among these 1131 polynomials:

- 968 do not satisfy theorem 21 on modulo 2 congruence.
- 168 do not satisfy theorem 9 which is equivalent, in the case of knots, to both Hartley condition and alternating Murasugi theorem for Alexander polynomials.
- 28 do not satisfy theorem 6.
- 53 do not satisfy the equality condition in theorem 6.
- 32 do not satisfy theorem 8

Here again, the most accurate condition is the modulo 2 congruence, that detects 86 % of the non two-bridged polynomials. If we consider these 5 criterions together, we detect 1009 among the 1131 non two-bridged polynomials. These polynomials were detected only by using theorems 21 and 9. We suppose that links that are detected by theorem 6 or 8 but not by theorem 21 nor 9 appear when their number of crossings is higher.

### 5.4 Conjecture

We observed some trapezoidal property for the Conway polynomials of two-bridged links with crossings fewer than 20 (their number is 131839).

**Conjecture 35.** *Let  $\nabla_m = b_1 \cdots b_m \left( \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m+1-2i} \right)$ ,  $\alpha_0 = 1$ , be the Conway polynomial of a two-bridge link. Then there exists  $n \leq \lfloor \frac{m}{2} \rfloor$  such that*

$$0 \leq \alpha_0 \leq \alpha_1 \leq \alpha_n \geq \alpha_{n+1} \geq \cdots \geq \alpha_{\lfloor \frac{m}{2} \rfloor} \geq 0.$$

If this conjecture was true, it would imply the following property of Alexander polynomials: *Let  $P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$  be the Alexander polynomial of a two-bridge knot. There exists an integer  $k \leq n$  such that  $(a_0, \dots, a_k)$  is convex and  $(a_k, \dots, a_n)$  is concave.*

Note that this property detects 670 non two-bridged Conway polynomials among the knots with 13 crossings or fewer and 107 among the multi-component links with 11 crossings or fewer. It still detects non two-bridged polynomials among the knots that do not satisfy the modulo 2 congruence theorem 21.

### 5.5 Two-bridge Lissajous knots

Using theorem 21, we deduce an algorithm to compute the integer  $D$  such that  $\nabla_m \equiv f_D \pmod{2}$ .

**Algorithm 36.** *Let  $K$  be a two-bridge link (or knot) with Conway form  $C(2b_1, 2b_2, \dots, 2b_m)$ . Let us define the sequences of integers  $\varepsilon_i$  and  $D_i$ ,  $i = 0, \dots, m$ , by*

$$\varepsilon_0 = 1, D_0 = 1, \varepsilon_{i+1} = -(-1)^{b_{i+1}} \varepsilon_i, D_{i+1} = D_i + \varepsilon_{i+1}.$$

*Then the modulo 2 Conway polynomial of  $K$  is the Fibonacci polynomial  $f_D(z)$ , where  $D = |D_m|$ .*

Note that if  $K = S\left(\frac{\alpha}{\beta}\right)$ , where  $\frac{\alpha}{\beta}$  is the Schubert form of  $K$ , then its Conway form  $C(2b_1, 2b_2, \dots, 2b_m)$  has length  $m \leq \alpha - 1$ .

**Example 37.** Consider from the example 32, two-bridge knot  $K = S\left(\frac{1828139}{1042750}\right)$ . One can write

$$\frac{1828139}{1042750} = [2b_1, \dots, 2b_{10}] = [2, -4, -20, 2, -2, -12, -2, 4, -12, -4].$$

Using a formula of Stoimenow, we see that the crossing number of  $K$  is 59 (see [28]). Our algorithm gives

$i$	0	1	2	3	4	5	6	7	8	9	10
$b_i$		1	-2	-10	1	-1	-6	-1	2	-6	-2
$(-1)^{b_i}$		-1	1	1	-1	-1	1	-1	1	1	1
$\varepsilon_i$	1	1	-1	1	1	1	-1	-1	1	-1	1
$D_i$	1	2	1	2	3	4	3	2	3	2	3

Consequently, the modulo 2 Conway polynomial of  $K$  is  $f_3(z) = z^2 + 1$ . Its modulo 2 Alexander polynomial is then  $1 - (t + t^{-1})$ .

We see that the Alexander (and Conway) polynomial of our knot is not congruent to 1 modulo 2. Hence, by a theorem of V. F. R. Jones, J. Przytycki and C. Lamm ([11], [21]), it cannot be a Lissajous knot.

A much more complicated computation gives the Conway polynomial of  $K$ :

$$\nabla_K(z) = 2880z^{10} + 4944z^8 + 2304z^6 + 158z^4 - 61z^2 + 1.$$

$\nabla_K$  may be expressed in terms of Fibonacci polynomials and we obtain:

$$\begin{aligned} \nabla_K(z) &= 2880f_{11} - 20976f_9 + 68496f_7 - 128482f_5 + 140969f_3 - 62886f_1 \\ &\equiv f_3 \pmod{2}. \end{aligned}$$

Note that this conversion in terms of Chebyshev polynomials is fast (see [3]). We could also have computed directly  $\nabla_K \pmod{2}$  using Siebenmann method modulo 2.

Using algorithm 36 we easily obtain (in table 4) the number of two-bridge knots with Conway polynomial congruent to 1 modulo 2 (compare [2])

Crossing Number	3	4	5	6	7	8	9	10	11	12
Two-bridge	1	1	2	3	7	12	24	45	91	176
$\nabla(t) \equiv 1$	0	0	1	1	2	4	8	13	26	51

Crossing Number	13	14	15	16	17	18	19	20	21	22
Two-bridge	352	693	1387	2752	5504	10965	21931	43776	87552	174933
$\nabla(t) \equiv 1$	97	185	365	705	1369	2675	5233	10211	20011	39221

Table 4: The number of two-bridge knots, and two-bridge knots with Conway polynomial congruent to 1 modulo 2.

## 6 Conclusion

In this paper, we gave a theorem on Conway polynomials of two-bridge links implying both the famous Murasugi and Hartley theorems. Our proof may be considered as a simple proof of these classical theorems obtained by writing Conway polynomials in the Fibonacci basis (see also [4, 9, 12]). We have experimentally noticed that these Conway coefficients seem to be first increasing, then decreasing. It would be interesting to study this problem.

We have found an elementary proof of the Murasugi congruence for two-bridge knots. Furthermore our result on Conway polynomials is also valid for links.

We gave a simple algorithm to compute the modulo 2 Conway polynomial of a two-bridge link. This algorithm can be applied to the recent study of Lissajous knots. V. F. R. Jones, J. Przytycki (1998) and C. Lamm (1997) showed that the Alexander polynomial of a two-bridge Lissajous knot must be congruent to 1 modulo 2 ([11, 21]). This property is the main tool used by A. Boocher, J. Daigle, J. Hoste and W. Zheng (2009) to prove that some two-bridge knots cannot be Lissajous ([2]). Our algorithm provides a fast method to compute modulo 2 Alexander polynomials.

Finally, we obtain sharp inequalities for the coefficients of Conway polynomials of two-bridge links. These inequalities imply sharp bounds for Alexander coefficients generalizing the bounds of Nakanishi and Suketa [25]. Nevertheless, the inequalities on the Conway coefficients are better because they define a polyhedron of volume much smaller than the polyhedron defined by the bounds on the Alexander coefficients.

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