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# PARTIAL REGULARITY OF FINITE MORSE INDEX SOLUTIONS TO THE LANE-EMDEN EQUATION

JUAN DÁVILA<sup>1,3</sup>, LOUIS DUPAIGNE<sup>2,4</sup> AND ALBERTO FARINA<sup>2,5</sup>

<sup>1</sup>Departamento de Ingeniería Matemática and CMM, Universidad de Chile  
Casilla 170/3 Correo 3, Santiago Chile

<sup>2</sup> LAMFA, UMR CNRS 6140, Université Picardie Jules Verne  
33, rue St Leu, 80039 Amiens, France

<sup>3</sup>jdavila@dim.uchile.cl

<sup>4</sup> louis.dupaigne@math.cnrs.fr

<sup>5</sup> alberto.farina@u-picardie.fr

ABSTRACT. We prove regularity and partial regularity results for finite Morse index solutions  $u \in H^1(\Omega) \cap L^p(\Omega)$  to the Lane-Emden equation  $-\Delta u = |u|^{p-1}u$  in  $\Omega$ .

## 1. INTRODUCTION

Given an open set  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 1$ , and  $p > 1$ , consider the Lane-Emden equation

$$(1.1) \quad -\Delta u = |u|^{p-1}u \quad \text{in } \Omega.$$

We are interested in the classical question of regularity of solutions to (1.1). Namely, given a class of weak solutions  $\mathcal{C}$ , we ask: what is the largest exponent  $p > 1$ , such that

$$(1.2) \quad u \in \mathcal{C} \implies u \in C^2(\Omega) ?$$

Consider first  $\mathcal{C} = L^p_{loc}(\Omega)$  and assume (1.1) is understood in the sense of distributions. Then, as follows from a well-known bootstrap argument, (1.2) holds true for all  $p < p_0(N)$ , where

$$p_0(N) = \begin{cases} +\infty, & \text{if } 1 \leq N \leq 2, \\ \frac{N}{N-2}, & \text{if } 3 \leq N. \end{cases}$$

The exponent  $p_0(N)$  is sharp. Indeed, for all  $p > p_0(N)$ ,  $u(x) = c_{N,p}|x|^{-\frac{2}{p-1}}$  is a singular solution belonging to  $\mathcal{C}$ . A (radial) singular solution also exists if  $p = p_0(N)$ , see [2, 14].

Consider next the case  $\mathcal{C} = H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ . Then (1.2) holds true for all  $p \leq p_S(N)$  where

$$p_S(N) = \begin{cases} +\infty, & \text{if } 1 \leq N \leq 2, \\ \frac{N+2}{N-2}, & \text{if } 3 \leq N. \end{cases}$$

When  $p < p_S(N)$ , the proof uses the same bootstrap argument and the extra assumption that  $u \in H_{loc}^1(\Omega)$  for the initial step. See [6] for the critical case  $p = p_S(N)$ . The Sobolev exponent  $p_S(N)$  is again sharp in the considered class, using the same counter-example.

Restrict at last to the class  $\mathcal{C}$  of energy solutions having finite Morse index, i.e.,  $u \in \mathcal{C}$  if  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  and the maximal dimension of a vector space  $X \subset C_c^1(\Omega)$  such that

$$Q_u(\varphi) := \int_{\Omega} |\nabla \varphi|^2 dx - p \int_{\Omega} |u|^{p-1} \varphi^2 dx < 0, \quad \text{for all } \varphi \in X \setminus \{0\}$$

is an integer  $k$ , called the Morse index of  $u$ . If  $k = 0$ , we say that  $u$  is stable. Note that this class of weak solutions is a natural choice, since any  $C^2$  solution to (1.1) is bounded on any open set  $\omega \subset\subset \Omega$  and so must have finite Morse index on  $\omega$ . We obtain the following result.

**Theorem 1.1.** *Let  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  be a solution to (1.1) of finite Morse index. If  $p < p_c(N)$ , where*

$$p_c(N) = \begin{cases} +\infty, & \text{for } 1 \leq N \leq 10, \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}, & \text{for } 11 \leq N, \end{cases}$$

then,  $u \in C^2(\Omega)$ .

Under the stronger assumptions that  $\Omega$  is smoothly bounded,  $u$  is stable, and  $u|_{\partial\Omega} = 1$ , the smoothness of  $u$  was first proved in [7, 13]. At no surprise, the proof of Theorem 1.1 is by bootstrap (using the additional information that  $u$  has finite Morse index in the initial step). The Joseph-Lundgren exponent  $p_c(N)$  is again sharp in the considered class, using the same counter-example.

In the supercritical cases ( $p \geq p_0(N)$ ,  $p > p_S(N)$ ,  $p \geq p_c(N)$ ), solutions can be singular at a point, as discussed earlier, but also on larger sets. E.g. if there exists an integer  $k$  such that  $0 \leq k \leq N-1$  and  $k < N - 2\frac{p}{p-1}$ , then

$$(1.3) \quad u(x) = C_{N,p,k} (x_1^2 + \dots + x_{N-k}^2)^{-\frac{1}{p-1}}$$

is a solution having  $k$ -dimensional singular set. See also an example in [18] where the Hausdorff dimension of the singular set is not an integer.

Nevertheless, if solutions are assumed to be positive and stationary, the singular set can be estimated as follows.

**Theorem 1.2** ([16]). *Let  $N \geq 3$  and  $p \geq p_S(N)$ . Let  $u \in H_{loc}^1(\Omega) \cap L_{loc}^{p+1}(\Omega)$  be a positive weak solution to (1.1). Assume in addition that  $u$  is stationary. Then,  $u \in C^2(\Omega \setminus \Sigma)$ , where  $\Sigma$  is a closed set of Hausdorff dimension bounded above by*

$$\mathcal{H}_{dim}(\Sigma) \leq N - 2\frac{p+1}{p-1}.$$

The precise definition of stationary solution can be found in [16]. Smooth solutions (and limits thereof in the  $H^1(\Omega) \cap L^{p+1}(\Omega)$  topology) are stationary. When the solution has finite Morse index, we prove that the singular set is in fact much smaller.

**Theorem 1.3.** *Let  $N \geq 11$  and  $p > p_c(N)$ . Let  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  be a positive solution to (1.1) of finite Morse index. Then,  $u \in C^2(\Omega \setminus \Sigma)$ , where  $\Sigma$  is a closed set of Hausdorff dimension bounded above by*

$$\mathcal{H}_{dim}(\Sigma) \leq N - 2\frac{p+\gamma}{p-1},$$

with  $\gamma = 2p + 2\sqrt{p(p-1)} - 1$ .

**Remark 1.4.** *The dimension of the singular set computed in Theorem 1.3 is optimal at least when it is an integer. Indeed, the solution given by (1.3) is stable in  $\mathbb{R}^N$  if  $p > p_c(N-k)$ , while  $p_c(N-k)$  solves*

$$N - 2\frac{p+\gamma}{p-1} = k.$$

**Remark 1.5.** *If  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  is a positive solution with finite Morse index, we prove in the next section that for any point  $x \in \Omega$ , there exists a ball  $B = B(x, r)$  such that  $u$  is the limit of  $C^2$  solutions in the  $H^1(B) \cap L^{p+1}(B)$  topology. Hence,  $u$  is stationary in  $B$ .*

*Theorem 1.3 remains valid for sign-changing solutions, provided they are stationary.*

We discuss at last the question of universal *a priori* estimates.

**Theorem 1.6.** *Assume  $1 < p < p_c(N)$  and  $p \neq p_S(N)$ . Assume  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  is a solution to (1.1) of finite Morse index  $m$ . Then, there exists a constant  $C$  depending on  $N, p, m$  only, such that for all  $x \in \Omega$ ,*

$$(1.4) \quad |u(x)| + |\nabla u(x)|^{\frac{2}{p+1}} \leq C \text{dist}(x, \partial\Omega)^{-\frac{2}{p-1}}.$$

**Remark 1.7.** *Assume  $p = p_S(N)$ . Then, (1.4) remains true for stable solutions (see [10]). However, the estimate is false for solutions of finite Morse index, since for  $\lambda > 0$ ,*

$$u_\lambda(x) = \left( \frac{\lambda \sqrt{N(N-2)}}{\lambda^2 + |x|^2} \right)^{\frac{N-2}{2}}$$

*provides an unbounded family of solutions of constant Morse index.*

The universal estimate (1.4) was first proved for positive solutions and  $1 < p < p_S(N)$  (see [4, 8, 11, 19]), with a constant  $C$  independent of the Morse index  $m$ . Note however that for such  $p$ , there do exist sign-changing solutions of arbitrary large Morse index, for which the dependance of the constant  $C$  to  $m$  must be kept (see [1, 3, 17]). Estimate (1.4) was then proved in [10] for  $C^2$  solutions which are stable, for the full range  $1 < p < p_c(N)$ .

We provide at last a universal estimate for  $C^2$  solutions of the more general problem:

$$(1.5) \quad -\Delta u = f(u) \quad \text{in } \Omega$$

where  $f \in C^1(\mathbb{R}, \mathbb{R})$  behaves like a power of  $u$  at infinity. More precisely,

**Theorem 1.8.** *Suppose*

$$(1.6) \quad \lim_{t \rightarrow \pm\infty} \frac{f'(t)}{p|t|^{p-1}} = a$$

for some  $a > 0$  and  $1 < p < p_c(N)$  and  $p \neq p_S(N)$ . Let  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  be a solution of (1.5) of finite Morse index  $m$ . Then, there exists a constant  $C$  depending on  $N, f, m$  only, such that for all  $x \in \Omega$ ,

$$|u(x)| + |\nabla u(x)|^{\frac{2}{p+1}} \leq C(1 + \text{dist}(x, \partial\Omega)^{-\frac{2}{p-1}}).$$

Theorem 1.8 was proved in [20] for positive solutions,  $1 < p < p_S(N)$ , and with a constant  $C$  independent of the Morse index  $m$ . In the case  $p = p_S(N)$ , the theorem remains valid for stable solutions, but fails for solutions of finite Morse index (see the counter-example in Remark 1.7). Similar statements can be derived for the nonlinearity  $f(u) = e^u$ , as we shall demonstrate in a future publication.

## 2. PRELIMINARY RESULTS

### 2.1. Reduction to the case of stable solutions.

**Proposition 2.1.** *Let  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  be a solution to (1.1) with finite Morse index. Then, for every  $x_0 \in \Omega$ , there exists  $r_0 > 0$  such that  $u$  is stable in  $B(x_0, r_0)$ .*

**Proof.** When  $N = 1$ , any function  $u \in H_{loc}^1(\Omega)$  is locally bounded by Morrey's inequality. In particular, the linearized operator  $L = -\Delta - p|u|^{p-1}$  has positive principal eigenvalue in any sufficiently small ball, whence  $u$  is stable on such a ball. Assume now  $N \geq 2$ . We may always assume that  $B(0, 1) \subset \Omega$  and it suffices to prove that  $u$  is stable near the origin. Assume first that  $u$  has Morse index 1. Either  $u$  is stable in  $B(0, 1/n)$  for some  $n \geq 2$  and we are done. Or, for all  $n \geq 2$ , there exists a direction  $\varphi_n \in C_c^1(B(0, 1/n))$  such that  $Q_u(\varphi_n) < 0$ . Since  $u$  has index 1, this implies that  $u$  is stable in  $B(0, 1) \setminus \overline{B(0, 1/n)}$ . This being true for all  $n \geq 2$ , we deduce that  $u$  is stable in  $B(0, 1) \setminus \{0\}$ . In fact, since  $N \geq 2$ , points have zero Newtonian capacity and so  $u$  is stable in  $B(0, 1)$ . So, every solution of index 1 is stable in a neighborhood of 0. Take now a solution  $u$  of index  $k \geq 2$ . Working exactly as above, we deduce that  $u$  has index  $k - 1$  in some ball  $B(0, r_1)$ . Working inductively on  $k$ , we deduce that  $u$  is stable in some ball  $B(0, r_k)$ .  $\square$

### 2.2. Approximation of singular stable solutions.

**Lemma 2.2.** *Suppose  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  is a nonnegative stable weak solution to (1.1). Then, there exists a sequence of nonnegative stable solutions  $u_n \in C^2(\Omega)$  to (1.1), such that  $u_n \nearrow u$  a.e. and in  $H_{loc}^1(\Omega)$ .*

**Proof.** The proof is a refinement of a concave truncation technique found in [5].

Let us first observe that since  $u \in H_{loc}^1(\Omega)$  and  $u$  solves (1.1), we have  $u \in L_{loc}^{p+1}(\Omega)$ . Take now  $\omega \subset \subset \Omega$  with smooth boundary, so that  $u \in H^1(\omega) \cap L^{p+1}(\omega)$ . We are going to produce a sequence  $u_n$  converging to  $u$  in  $H^1(\omega)$ . By a standard diagonal argument, we then reach the desired conclusion.

In the sequel, we write  $\omega = \Omega$  for notational convenience. Given  $c > 0$ , consider the function

$$\phi_c(t) = \left(c + t^{-(p-1)}\right)^{-\frac{1}{p-1}}, \quad \text{defined for } t > 0.$$

We set also  $\phi_c(0) = 0$ . Then,  $\phi_c$  is increasing, concave, and smooth for  $t > 0$ . In addition,  $\phi_c(t) \nearrow t$  as  $c \searrow 0^+$ , and  $\phi_c(t) \leq t$ , for all  $t \geq 0$ . Also, if  $c > 0$ , then  $\phi_c$ ,

$\phi'_c$  are uniformly bounded. We have

$$\phi'_c(t) = \frac{\phi_c(t)^p}{t^p} \quad \forall t > 0.$$

Let  $w_c$  denote the unique solution to

$$\begin{cases} -\Delta w_c = 0 & \text{in } \Omega \\ w_c = \phi_c(u) & \text{on } \partial\Omega. \end{cases}$$

Then,  $w_c \geq 0$ ,  $w_c \in L^\infty(\Omega) \cap H^1(\Omega)$ . Moreover,  $w_c$  is non-increasing with respect to  $c$ . We claim that  $w_c \rightarrow w$  in  $H^1(\Omega)$  as  $c \rightarrow 0$ , where  $w$  is the solution to

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ w = u & \text{on } \partial\Omega. \end{cases}$$

To see this, consider the problem

$$(2.1) \quad \begin{cases} -\Delta v = (v + w_c)^p & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $w_c \in L^\infty(\Omega)$ , (2.1) has a minimal nonnegative solution  $v_c$ , which can be constructed by the method of sub and super-solutions, as follows. Note that  $\underline{v} = 0$  is a sub-solution, since  $w_c \geq 0$ . Moreover, by Kato's inequality,  $\bar{v} = \phi_c(u) - w_c$  is a bounded super-solution:

$$-\Delta(\phi_c(u) - w_c) = -\Delta\phi_c(u) \geq -\phi'_c(u)\Delta u = \phi_c(u)^p = (\phi_c(u) - w_c + w_c)^p.$$

In particular, (2.1) has a minimal nonnegative solution  $v_c$ . This minimal solution is bounded and by elliptic regularity,  $v_c$  belongs to  $C^{1,\alpha}(\bar{\Omega})$ . Moreover,  $v_c$  is stable in the sense that

$$p \int_{\Omega} (v_c + w_c)^{p-1} \varphi^2 dx \leq \int_{\Omega} |\nabla \varphi|^2 dx, \quad \text{for all } \varphi \in C_c^1(\Omega).$$

Since  $v_c$  is minimal and  $w_c$  is non-increasing with respect to  $c$ , we deduce that  $v_c$  is also non-increasing with respect to  $c$ . It follows that  $v(x) = \lim_{c \rightarrow 0} v_c(x)$  is well-defined for all  $x \in \Omega$ . Since  $v_c \in C^1(\bar{\Omega})$ , we have

$$\int_{\Omega} |\nabla v_c|^2 dx = \int_{\Omega} (v_c + w_c)^p v_c dx \leq \int_{\Omega} u^{p+1} dx.$$

In particular,  $v_c$  is bounded in  $H_0^1(\Omega)$ . It follows that  $v_c \rightharpoonup v$  weakly in  $H_0^1(\Omega)$ . Multiplying (2.1) by  $\varphi \in C_c^\infty(\Omega)$ , integrating, and passing to the limit as  $c \rightarrow 0$ , we see that  $v$  is a weak solution to

$$(2.2) \quad \begin{cases} -\Delta v = (v + w)^p & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Let  $\varphi_k \in C_c^{0,1}(\Omega)$  be a sequence such that  $\varphi_k \rightarrow v$  in  $H_0^1(\Omega)$ . Since  $v \geq 0$  we can assume  $\varphi_k \geq 0$ . We can also assume that  $\varphi_k \rightarrow v$  a.e. in  $\Omega$ . Multiplying (2.2) by  $\varphi_k$  and integrating, we obtain

$$\int_{\Omega} \nabla v \nabla \varphi_k dx = \int_{\Omega} (v + w)^p \varphi_k dx$$

By Fatou's lemma,

$$\int_{\Omega} (v + w)^p v dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \nabla v \nabla \varphi_k dx = \int_{\Omega} |\nabla v|^2 dx.$$

By monotone convergence,

$$\lim_{c \rightarrow 0} \int_{\Omega} |\nabla v_c|^2 dx = \lim_{c \rightarrow 0} \int_{\Omega} (v_c + w_c)^p v_c dx = \int_{\Omega} (v + w)^p v dx.$$

Hence,

$$\lim_{c \rightarrow 0} \int_{\Omega} |\nabla v_c|^2 dx = \int_{\Omega} (v + w)^p v dx \leq \int_{\Omega} |\nabla v|^2 dx.$$

Since  $v_c \rightharpoonup v$  weakly in  $H_0^1(\Omega)$ , the reverse inequality

$$\int_{\Omega} |\nabla v|^2 dx \leq \liminf_{c \rightarrow 0} \int_{\Omega} |\nabla v_c|^2 dx$$

also holds, which proves that  $v_c \rightarrow v$  in  $H_0^1(\Omega)$ .

We claim that  $u = v + w$ , from which Lemma 2.2 follows. By construction,  $v = \lim v_c \leq \lim(\phi_c(u) - w_c) = u - w$ . We need thus only prove that  $u \leq v + w$ . Note that  $\tilde{v} = u - w$  solves (2.2). Let  $z = \tilde{v} - v \geq 0$ . Then,  $z \in H_0^1(\Omega)$ , and since  $u$  is stable,

$$(2.3) \quad p \int_{\Omega} (\tilde{v} + w)^{p-1} (\tilde{v} - v)^2 dx \leq \int_{\Omega} |\nabla(\tilde{v} - v)|^2 dx.$$

Now,  $\tilde{v} - v$  satisfies

$$\int_{\Omega} \nabla(\tilde{v} - v) \nabla \varphi dx = \int_{\Omega} ((\tilde{v} + w)^p - (v + w)^p) \varphi dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

We would like to take  $\varphi = \tilde{v} - v$ . First, we claim that we can take  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ . These functions can be approximated in  $H_0^1(\Omega)$  by functions in  $C_c^\infty(\Omega)$  with a uniform bound. Then, take  $\varphi = \min(\tilde{v} - v, t)$ ,  $t > 0$ , which belongs to  $H_0^1(\Omega) \cap L^\infty(\Omega)$ . We get

$$\int_{[\tilde{v}-v \leq t]} |\nabla(\tilde{v} - v)|^2 dx = \int_{\Omega} ((\tilde{v} + w)^p - (v + w)^p) \min(\tilde{v} - v, t) dx.$$

Now let  $t \rightarrow \infty$ . Then,

$$\int_{\Omega} |\nabla(\tilde{v} - v)|^2 dx = \int_{\Omega} ((\tilde{v} + w)^p - (v + w)^p) (\tilde{v} - v) dx$$

Combined with (2.3) we find

$$\int_{\Omega} (\tilde{v} - v) [p(\tilde{v} + w)^{p-1} (\tilde{v} - v) - (\tilde{v} + w)^p + (v + w)^p] dx \leq 0.$$

By convexity,  $p(\tilde{v} + w)^{p-1} (\tilde{v} - v) - (\tilde{v} + w)^p + (v + w)^p \geq 0$  with strict inequality, unless  $\tilde{v} \equiv v$ .  $\square$

**2.3. Some well-known ingredients.** Proofs of all the results in this section can be found in [9]. We begin with a so-called  $\varepsilon$ -regularity result for weak solutions to (1.1) in Morrey spaces. Recall the following definition.

**Definition 2.3.** *Let  $\Omega$  be a bounded open set of  $\mathbb{R}^N$ ,  $N \geq 1$ . Given  $p > 1$  and  $\lambda \in [0, N]$ , the Morrey space  $L^{p,\lambda}(\Omega)$  is the set of functions  $u$  in  $L^p(\Omega)$  such that the following norm is finite:*

$$\|u\|_{L^{p,\lambda}(\Omega)}^p = \sup_{x_0 \in \Omega, r > 0} r^{-\lambda} \int_{B(x_0, r) \cap \Omega} |u|^p dx < \infty.$$

Then,

**Theorem 2.4** ([12,16]). *Let  $N \geq 3$ ,  $p > 1$ , and  $\lambda = N - 2\frac{p+1}{p-1}$ . Let also  $B(x_0, r_0)$  be a ball. There exists  $\varepsilon = \varepsilon(N, p) > 0$  such that for any weak solution  $u \in H^1(B(x_0, r_0)) \cap C(\overline{B(x_0, r_0)})$  to (1.1) satisfying*

$$(2.4) \quad \|u\|_{L^{p+1, \lambda}(B(x_0, r_0))} \leq \varepsilon,$$

there holds

$$\|u\|_{L^\infty(B(x_0, r_0/2))} \leq \left(\frac{4}{r_0}\right)^{\frac{2}{p-1}}.$$

Also recall the following classical result from geometric measure theory.

**Theorem 2.5.** *Let  $\Omega$  denote an open set of  $\mathbb{R}^N$ ,  $N \geq 1$ ,  $u$  a function in  $L^1_{loc}(\Omega)$  and  $0 \leq s < N$ . Set*

$$E_s = \left\{ x \in \Omega : \limsup_{r \rightarrow 0^+} r^{-s} \int_{B_r(x)} |u(y)| dy > 0 \right\}.$$

Then,

$$\mathcal{H}^s(E_s) = 0,$$

where  $\mathcal{H}^s$  denotes the Hausdorff measure of dimension  $s$ .

The next ingredient in the proof of Theorem 1.3 is the following monotonicity formula.

**Theorem 2.6** ([15]). *Let  $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$  denote a stationary weak solution to (1.1). For  $x \in \Omega$ ,  $r > 0$ , such that  $B(x, r) \subset \Omega$ , consider the energy  $\mathcal{E}_u(x, r)$  given by*

$$(2.5) \quad \mathcal{E}_u(x, r) = r^{-\mu} \int_{B(x, r)} \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) dx + \frac{r^{-\mu-1}}{p-1} \int_{\partial B(x, r)} |u|^2 d\sigma,$$

where

$$\mu = N - 2\frac{p+1}{p-1}.$$

Then,

- $\mathcal{E}_u(x, r)$  is nondecreasing in  $r$ .
- $\mathcal{E}_u(x, r)$  is continuous in  $x \in \Omega$  and  $r > 0$ .

**Remark 2.7** ([15]). *The energy  $\mathcal{E}_u(x, r)$  can be equivalently written as*

$$(2.6) \quad \mathcal{E}_u(x, r) = \frac{p-1}{p+3} r^{-\mu} \int_{B(x, r)} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} |u|^{p+1} \right) dy + \frac{1}{p+3} \frac{d}{dr} \left( r^{-\mu} \int_{\partial B(x, r)} |u|^2 d\sigma \right).$$

We shall use at last the following capacity estimate.

**Proposition 2.8** ([10]). *Let  $\Omega$  be an open set of  $\mathbb{R}^N$ ,  $p > 1$ . Let  $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$  denote a stable solution to (1.1). Then, for any  $\gamma \in [1, 2p+2\sqrt{p(p-1)}-1)$ , any  $\psi \in C^1_c(\Omega)$ ,  $0 \leq \psi \leq 1$ , and any integer  $m \geq \max\left\{\frac{p+\gamma}{p-1}, 2\right\}$ , there exists a constant  $C_{p, m, \gamma} > 0$  such that*

$$\int_{\Omega} \left( \left| \nabla \left( |u|^{\frac{\gamma-1}{2}} u \right) \right|^2 + |u|^{p+\gamma} \right) \psi^{2m} dx \leq C_{p, m, \gamma} \int_{\Omega} |\nabla \psi|^{2\left(\frac{p+\gamma}{p-1}\right)} dx.$$

In the case where  $u \in C^2(\Omega)$ , the proof of this result is given in [10]. This proof can be adapted to the case  $u \in H_{loc}^1(\Omega) \cap L_{loc}^p(\Omega)$  as follows: multiply (1.1) with  $|T_k(u)|^{\gamma-1} u \varphi^2$ , where  $T_k(s) = \max(-k, \min(u, k))$  and  $\varphi \in C_c^2(\Omega)$  and apply stability with test function  $|T_k(u)|^{\frac{\gamma-1}{2}} u \varphi$ .

### 3. PROOFS OF THEOREMS 1.1 AND 1.3.

**Proof of Theorem 1.1.** Thanks to Proposition 2.8,  $u \in L_{loc}^{p+\gamma}(\Omega)$  for all  $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ . Using elliptic estimates and a standard bootstrap argument, we deduce that  $u \in C^2(\Omega)$ , provided  $N \leq 10$  or  $N \geq 11$  and  $p < p_c(N)$ .  $\square$

**Proof of Theorem 1.3 .** By Proposition 2.1, we may assume that  $u$  is a nonnegative stable weak solution to (1.1). Given  $\varepsilon > 0$ , define

$$\Sigma_\varepsilon = \left\{ x \in \Omega : \forall r > 0 \int_{B(x,r)} (u^{p+1} + |\nabla u|^2) dx \geq \varepsilon r^{N-2\frac{p+1}{p-1}} \right\}$$

**Step 1.** There exists a fixed value of  $\varepsilon > 0$  such that for every  $x \notin \Sigma_\varepsilon$ ,  $u$  is bounded (hence  $C^2$ ) in a neighborhood of  $x$ .

To see this, let  $x_0 \notin \Sigma_\varepsilon$ : there exists  $r_0 > 0$  such that

$$r_0^{-\mu} \int_{B(x_0, r_0)} (u^{p+1} + |\nabla u|^2) dx < \varepsilon,$$

where  $\mu = N - 2\frac{p+1}{p-1}$ . By (2.5), for  $r < r_0$ ,

$$\begin{aligned} \mathcal{E}_u(x_0, r) &\leq r^{-\mu} \int_{B(x_0, r)} \frac{1}{2} |\nabla u|^2 dy + \frac{r^{-\mu-1}}{p-1} \int_{\partial B(x_0, r)} u^2 d\sigma \\ &\leq r^{-\mu} \int_{B(x_0, r_0)} \frac{1}{2} |\nabla u|^2 dy + \frac{r^{-\mu-1}}{p-1} \int_{\partial B(x_0, r)} u^2 d\sigma \\ &\leq \frac{\varepsilon}{2} \left( \frac{r}{r_0} \right)^{-\mu} + \frac{r^{-\mu-1}}{p-1} \int_{\partial B(x_0, r)} u^2 d\sigma \end{aligned}$$

Integrating between  $r = r_0/2$  and  $r_0$ , and recalling that  $\mathcal{E}_u(x, r)$  is nondecreasing in  $r$ , we deduce that

$$\begin{aligned} \frac{r_0}{2} \mathcal{E}_u(x_0, r_0/2) &\leq 2^{\mu-2} \varepsilon r_0 + \frac{1}{p-1} \int_{r_0/2}^{r_0} r^{-\mu-1} \left( \int_{\partial B(x_0, r)} u^2 d\sigma \right) dr \\ &\leq C \varepsilon r_0 + C r_0^{-\mu-1} \int_{B(x_0, r_0)} u^2 dy \\ &\leq C \varepsilon r_0 + C r_0^{-\mu-1} \left( \int_{B(x_0, r_0)} u^{p+1} dy \right)^{\frac{2}{p+1}} r_0^{N(1-\frac{2}{p+1})} \\ &< C \varepsilon r_0. \end{aligned}$$

Hence,

$$\mathcal{E}_u(x_0, r_0/2) < C \varepsilon.$$

Since  $\mathcal{E}_u$  is continuous in  $x$ , there exists  $r_1 < r_0/2$  such that  $\mathcal{E}_u(x, r_0/2) < 2C\varepsilon$ , for  $x \in B(x_0, r_1)$ . Since  $\mathcal{E}_u$  is nonincreasing in  $r$ , we deduce that for all  $x \in B(x_0, r_1)$  and all  $r < r_1$ ,

$$(3.1) \quad \mathcal{E}_u(x, r) < 2C\varepsilon.$$

Now take an approximating sequence  $u_n$  given by Lemma 2.2. Integrating (2.6) between 0 and  $r_2 < r_1$ , we find

$$\begin{aligned} \frac{p-1}{p+3} \int_0^{r_2} r^{-\mu} \left( \int_{B(x,r)} \left( \frac{1}{2} |\nabla u_n|^2 + \frac{1}{p+1} u_n^{p+1} \right) dy \right) dr + \\ + \frac{r_2^{-\mu}}{p+3} \int_{\partial B(x,r_2)} u_n^2 d\sigma \leq r_2 \mathcal{E}_{u_n}(x, r_2). \end{aligned}$$

It follows that

$$\begin{aligned} Cr_2 \mathcal{E}_u(x, r_2) &\geq \int_0^{r_2} \left( r^{-\mu} \int_{B(x,r)} u^{p+1} dy \right) dr \\ &\geq \int_{r_2/2}^{r_2} \left( r^{-\mu} \int_{B(x,r)} u^{p+1} dy \right) dr \end{aligned}$$

By the fundamental theorem of calculus, we deduce that there exists  $r_3 \in (r_2/2, r_2)$  such that

$$C\mathcal{E}_u(x, r_2) \geq r_3^{-\mu} \int_{B(x,r_3)} u^{p+1} dy \geq r_2^{-\mu} \int_{B(x,r_2/2)} u^{p+1} dy$$

Apply now (3.1). Then,

$$r^{-\mu} \int_{B(x,r)} u^{p+1} dy \leq C\varepsilon,$$

for all  $x \in B(x_0, r_1)$  and all  $r < r_1/2$ . Taking  $\varepsilon$  sufficiently small, it follows from Theorem 2.4 that  $(u_n)$  is uniformly bounded near  $x_0$  and so,  $u$  is  $C^2$  in a neighborhood of  $x_0$ .

**Step 2.** For all  $\gamma \geq 1$ , there exists  $\varepsilon' > 0$  such that

$$\Sigma_{\varepsilon'} \supseteq \tilde{\Sigma}_{\varepsilon'} := \left\{ x \in \Omega : \forall r > 0 \quad \int_{B(x,r)} u^{p+\gamma} dx \geq \varepsilon' r^{N-2\frac{p+\gamma}{p-1}} \right\}.$$

Indeed, suppose  $x \notin \tilde{\Sigma}_{\varepsilon'}$ . Then,

$$\int_{B(x,r)} u^{p+\gamma} dx < \varepsilon' r^{N-2\frac{p+\gamma}{p-1}}$$

for some  $r > 0$ . By Hölder's inequality,

$$\begin{aligned} (3.2) \quad \int_{B(x,r)} u^{p+1} dx &\leq C \left( \int_{B(x,r)} u^{p+\gamma} dx \right)^{\frac{p+1}{p+\gamma}} r^{N(1-\frac{p+1}{p+\gamma})} \\ &< C \left( \varepsilon' r^{N-2\frac{p+\gamma}{p-1}} \right)^{\frac{p+1}{p+\gamma}} r^{N(1-\frac{p+1}{p+\gamma})} = C(\varepsilon')^{\frac{p+1}{p+\gamma}} r^{N-2\frac{p+1}{p-1}}. \end{aligned}$$

Take a function  $\varphi \in C_c^2(\Omega)$  and multiply the Lane-Emden equation (1.1) by  $u\varphi^2$ . Then,

$$\int_{\Omega} |\nabla u|^2 \varphi^2 dx + \int_{\Omega} u \nabla u \cdot \nabla \varphi^2 dx = \int_{\Omega} u^{p+1} \varphi^2 dx$$

i.e.

$$\int_{\Omega} |\nabla u|^2 \varphi^2 dx = \int_{\Omega} u^{p+1} \varphi^2 dx + \frac{1}{2} \int_{\Omega} u^2 \Delta \varphi^2 dx$$

Choose now  $\varphi$  such that  $\varphi = 1$  in  $B(x, r/2)$ ,  $\varphi = 0$  outside  $B(x, r)$ , and  $|\Delta \varphi^2| \leq C/r^2$ . Then,

$$\int_{B(x, r/2)} |\nabla u|^2 dx \leq C \int_{B(x, r)} u^{p+1} dx + \frac{C}{r^2} \int_{B(x, r)} u^2 dx$$

We estimate

$$\begin{aligned} \frac{1}{r^2} \int_{B(x, r)} u^2 dx &\leq \frac{C}{r^2} \left( \int_{B(x, r)} u^{p+\gamma} dx \right)^{\frac{2}{p+\gamma}} r^{1-\frac{2}{p+\gamma}} \\ &< \frac{C}{r^2} \left( \varepsilon' r^{n-2\frac{p+\gamma}{p-1}} \right)^{\frac{2}{p+\gamma}} r^{1-\frac{2}{p+\gamma}} = C(\varepsilon')^{\frac{2}{p+\gamma}} r^{N-2\frac{p+1}{p-1}}. \end{aligned}$$

Using (3.2), we deduce that

$$\int_{B(x, r/2)} (u^{p+1} + |\nabla u|^2) dx < C(\varepsilon')^{\frac{2}{p+\gamma}} r^{N-2\frac{p+1}{p-1}}.$$

Choosing  $\varepsilon'$  such that  $C(\varepsilon')^{\frac{2}{p+\gamma}} \leq \varepsilon$ , we deduce that  $x \notin \Sigma_{\varepsilon}$ . And so,  $\tilde{\Sigma}_{\varepsilon'} \subset \Sigma_{\varepsilon}$ .

**Step 3.** By the capacity estimate (Proposition 2.8),  $u \in L_{loc}^{p+\gamma}(\Omega)$  if  $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ . By Theorem 2.5 It follows that for  $\varepsilon' > 0$  small,

$$\mathcal{H}^{N-2\frac{p+1}{p-1}}(\tilde{\Sigma}_{\varepsilon'}) = 0.$$

This being true for all  $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ , Theorem 1.3 follows.  $\square$

#### 4. PROOF OF THE APRIORI ESTIMATES

**Proof of Theorem 1.6.** The proof of (1.4) is the same as the one given in [20], except for the use of Theorem 2 of [10] stating that there are no entire solutions of finite Morse index if  $p$  is in the range of Theorem 1.1.  $\square$

**Proof of Theorem 1.8.** By Theorem 1.1, any finite Morse index solution to (1.1) is  $C^2$ , provided  $p < p_c(N)$ . Working by contradiction, as in the proof of Theorem 2.3 of [20], we can find a sequence  $(u_k)$  of solutions of (1.5) (with Morse index at most  $m$ ) and a sequence of points  $(x_k)$  such that by setting

$$\begin{aligned} \lambda_k &= (|u_k(x_k)|^{\frac{p-1}{2}} + |\nabla u_k(x_k)|^{\frac{p-1}{p+1}})^{-1} \\ v_k(y) &= \lambda_k^{\frac{2}{p-1}} u_k(x_k + \lambda_k y) \end{aligned}$$

we have  $\lambda_k \rightarrow 0$ ,

$$-\Delta v_k = f_k(v_k) \quad \text{in } B(0, k)$$

where

$$f_k(v) = \lambda_k^{\frac{2p}{p-1}} f(\lambda_k^{-\frac{2}{p-1}} v),$$

and

$$|v_k|^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}} \leq 2 \quad \text{in } B(0, k)$$

$$|v_k(0)|^{\frac{p-1}{2}} + |\nabla v_k(0)|^{\frac{p-1}{p+1}} = 1.$$

Note that  $f_k(v_k)$  and  $\nabla[f_k(v_k)]$  are both uniformly bounded. Then, up to subsequence  $v_k \rightarrow v$  in the  $C_{loc}^1(\mathbb{R}^N)$  topology,  $f_k(v_k) \rightarrow g$  in the  $C_{loc}^{0,\alpha}(\mathbb{R}^N)$  topology (for some  $\alpha \in (0, 1)$ ) and  $-\Delta v = g$  in the sense of distributions. By standard elliptic estimates  $v$  is then a classical  $C_{loc}^{2,\alpha}(\mathbb{R}^N)$  solution of  $-\Delta v = g$  in  $\mathbb{R}^N$ .

We claim that  $v$  satisfies

$$(4.1) \quad -\Delta v = a|v|^{p-1}v \quad \text{in } \mathbb{R}^N.$$

To this end it is enough to prove that  $g = a|v|^{p-1}v$  in  $\mathbb{R}^N$ . The assumption (1.6) implies

$$(4.2) \quad \lim_{t \rightarrow \pm\infty} \frac{f(t)}{|t|^{p-1}t} = a.$$

Therefore, on the open set  $[v \neq 0]$ ,  $f_k(v_k(x)) \rightarrow a|v(x)|^{p-1}v(x)$  pointwise, hence  $g = |v|^{p-1}v$  on  $[v \neq 0]$  and also on  $\overline{[v \neq 0]}$  by continuity. If  $y \notin \overline{[v \neq 0]}$  then  $v$  is zero in a neighborhood  $U_y$  of  $y$  and hence  $0 = -\Delta v = g$  in  $U_y$ , giving in particular that  $g(y) = 0$ .

It remains to verify that  $v$  has Morse index at most  $m$ . We first prove that  $\lim_{k \rightarrow +\infty} f'_k(v_k(y)) = ap|v(y)|^{p-1}$  pointwise in  $\mathbb{R}^N$ . This is clearly true for  $y \in [v \neq 0]$ , thanks to (1.6). On the other hand, for  $y \in [v = 0]$ , the desired conclusion holds true since  $\limsup_{k \rightarrow +\infty} |f'_k(v_k(y))| = 0$ . Indeed, let us suppose the contrary, then  $\lim_{n \rightarrow +\infty} |f'_{k_n}(v_{k_n}(y))| > 0$  for a sequence  $k_n \nearrow +\infty$ . Since  $f'_{k_n}(v_{k_n}(y)) = \lambda_{k_n}^2 f'(\lambda_{k_n}^{-\frac{2}{p-1}} v_{k_n}(y))$ , the sequence  $\lambda_{k_n}^{-\frac{2}{p-1}} |v_{k_n}(y)|$  must be unbounded. Hence, up to a subsequence,  $\lambda_{k_n}^{-\frac{2}{p-1}} |v_{k_n}(y)| \rightarrow +\infty$  and then, by (1.6),  $|f'_{k_n}(v_{k_n}(y))| \leq C|v_{k_n}(y)|^{p-1} \rightarrow 0$ . A contradiction.

To conclude we work again by contradiction. Assume there exist  $m+1$  linearly independent functions  $\varphi_j \in C_c^\infty(\mathbb{R}^N)$  such that  $Q_v(\varphi_j) < 0$ ,  $j = 1, \dots, m+1$ , where

$$Q_v(\varphi) = \int_{\mathbb{R}^N} (|\nabla \varphi|^2 - ap|v|^{p-1}\varphi^2) dx.$$

Then, since  $\lim_{k \rightarrow +\infty} f'_k(v_k(y)) = ap|v(y)|^{p-1}$  pointwise in  $\mathbb{R}^N$ , we have

$$\lim_{k \rightarrow \infty} \int_{B(0,k)} (|\nabla \varphi|^2 - f'_k(v_k)\varphi^2) dx = Q_v(\varphi),$$

for any  $\varphi \in C_c^\infty(\mathbb{R}^N)$ . Therefore, for  $k$  large enough,  $u_k$  has Morse index greater or equal than  $m+1$ , which is a contradiction.

We have constructed a nontrivial  $C^2$  solution of (4.1) of finite Morse index, which is not possible by Theorem 2 of [10] if  $p \neq p_S(N)$  (and by Theorem 1 of [10] when  $p = p_S(N)$  and  $m = 0$ ).  $\square$

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