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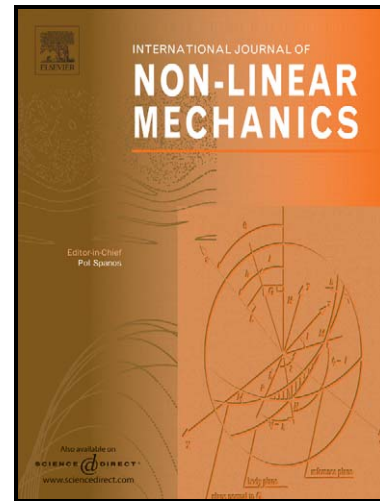
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**A THEORETICAL ANALYSIS  
OF UNSTEADY INCOMPRESSIBLE FLOWS  
FOR TIME-DEPENDENT CONSTITUTIVE EQUATIONS  
BASED ON DOMAIN TRANSFORMATIONS**

by

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## ABSTRACT

In this paper, we propose an analysis that allows calculation of kinematic histories in unsteady problems of continuum mechanics, in relation to the use of memory-integral constitutive equations. Such cases particularly concern flow conditions of processing rheology, requiring evaluation of strain or deformation rate tensors, for viscoelastic incompressible fluids as polymers. In two- and three-dimensional cases, we apply concepts of the Stream-Tube Method (STM) initially given for stationary conditions, where unknown local or global mapping functions are defined instead of classic velocity-pressure formulations, leading to consider the flow parameters in domains where the streamlines and trajectories are parallel straight lines. The approach enables us to provide accurate formulae for evaluating the kinematics histories that can be used later for computing the stresses for a given memory-integral model.

**Keywords:** Memory-integral models, Strain history, Stream-Tube Method, Domain transformation, Unsteady flows, Viscoelasticity

## 1. Introduction

Time-dependent constitutive equations are generally expressed under the general tensorial form

$$\underline{\underline{\tau}}(t) = \underline{\underline{H}}_{-\infty}^t[\underline{\underline{K}}_i(t, t')] \quad (1)$$

where  $\underline{\underline{\tau}}(t)$  denotes the extra-stress tensor at time  $t$ ,  $\underline{\underline{H}}_{-\infty}^t(t)$  is a functional tensor of kinematic tensors  $\underline{\underline{K}}_i(t, t')$  ( $i = 1, \dots, I_0$ ) for times  $t' \leq t$ . Expressing such quantities requires evaluation of time evolution of particles on their pathlines. Equation (1) is generally used on the form of memory-integral expressions [1-8]. Representations of the stress tensor are given with rate constitutive equations as corotational laws [3, 5] expressed with the rate-of-deformation tensor  $\underline{\underline{D}}(t) = (1/2)(\underline{\underline{\nabla}}V + \underline{\underline{\nabla}}V^T)$  in corotational reference frames and, more widely, with codeformational models [1, 2, 4-8] involving deformation tensors as the respective Cauchy and Finger tensors  $\underline{\underline{C}}_i(t')$  and  $\underline{\underline{B}}(t')$  derived from the deformation tensor  $\underline{\underline{F}}_i(t')$  ( $t' \leq t$ ). An example of this type is the popular K-BKZ model (e.g. [5]) expressed by the equation

$$\underline{\underline{\tau}}(t) = \int_{-\infty}^t \left[ \frac{\partial U}{\partial I_1} \underline{\underline{B}}_i(t') - \frac{\partial U}{\partial I_2} \underline{\underline{C}}_i(t') \right] dt' \quad (2)$$

where  $U$  denotes a potential of the invariants  $I_1 = \text{tr}(\underline{\underline{B}})$  and  $I_2 = \text{tr}(\underline{\underline{C}})$ .

Particularly, few numerical papers have concerned time-dependent constitutive equations in non-stationary flow cases [12]. Since the pathlines do not pass through the mesh points of grids defined in numerical applications, significant accuracy problems arise in such cases. Together with fundamental interests, the study of unsteady flows of materials obeying memory-integral viscoelastic constitutive equations is considered in relation to numerous industrial processes of polymers, food fluids, ... as extrusion and injection molding and general two- and three-dimensional cases. At the moment, the numerical simulations involve coupling of sophisticated rheological models and conservation laws in order to determine flow characteristics in geometries that are also complex. However, difficulties arise owing to the convective character of the viscoelastic constitutive equations, since the stress state in a fluid element depends on its strain history. This implies the use of upwind discretizing schemes that require robust and efficient stability techniques ensuring accuracy and convergence of the solving procedures.

Focusing on incompressible materials, the purpose of the present paper is to extend possibilities of the Stream-Tube Method (STM) to evaluation of kinematic quantities in unsteady flows, where streamlines and pathlines are not identical, in relation to time-dependent constitutive equations. The approach is considered through streamline and pathline determination at every time  $t$ , starting from the rest state or from stationary conditions for fluid particles of a domain  $\Omega$  of boundary  $\Gamma$ , given an initial time  $t_0$ .

In STM analysis of stationary cases [13-14] where the concepts of streamlines are used, the fluid dynamics problem is reformulated by using global or local transformations of the physical domain, that must be considered in the governing equations instead of classical primary unknowns as the velocity. Together with the pressure, the local or global unknown transformation functions of the physical domain, ensuring that the mapped streamlines are parallel and straight, must be determined from the governing equations, written with variables of the mapped domains where the computations are performed. It should be pointed out that, in STM, flow with open or closed streamlines can be considered, according to transformations into rectilinear lines summarized in Figure 1a,b [13,14]. From a theoretical viewpoint, Stream-Tube Method has allowed to propose simple and precise formulae to evaluate kinematic and strain histories as already provided in previous papers [16,17] under stationary conditions, for fluids obeying time-dependent constitutive equations. In practice, when performing STM simulations related to steady configurations, the nodes are naturally set on the rectilinear streamlines allowing an optimal accuracy. The calculation of stresses along the mapped closed streamlines avoids problems of numerical diffusion that can occur, in classical methods, from diffusion related to the singularity from the center of rotation of the vortex zone that cannot be improved by convective stabilization techniques. In unsteady cases, such accuracy problems are increased.

Non-stationary flows have been already considered using STM analysis in order to compute start-up and pulsating flows between concentric and eccentric cylinders with purely viscous non-Newtonian constitutive equations [15]. In these cases, streamlines have been determined versus time without referring to the strain history. Concerning approaches related to calculation of strain histories in steady flows, an analysis has been proposed by Adachi [18,19], from concepts referred to Protean coordinates [20], where one coordinate is a streamline. Starting from this study, Clermont [16] applied the theoretical elements of the Stream-Tube Method to evaluate strain rates and strain histories in two- and three-dimensional steady situations, for flows involving only open streamlines. In STM, this latter case corresponds to the use of one or two global transformation functions, in two and three dimensions, respectively.

We wish to develop theoretical tools of the Stream-Tube Method for determining trajectories and histories for moving particles, according to open and closed streamlines of two- and three-dimensional non-stationary flows. In this paper, we first recall briefly the background on global and local transformations related to a bounded flow domain  $\Omega$ , in steady conditions. Then, we go further by proposing an approach towards evaluation of the kinematics history for non-stationary flows, based on elements previously provided for streamlines, in steady conditions, that allows computation of pressure and mapping functions for streamlines and trajectories the transformed of which are parallel straight lines, in rectangles or parallelepipeds .

Global one-to-one transformations  $T_{(t)}$  of a flow domain  $\Omega$  with open streamlines are related to the use of one or two mapping functions, in 2D and 3D cases, respectively [9]. At time  $t$ , the corresponding equations, expressed in Cartesian

coordinates  $X^i$  ( $X^1 = x, X^2 = y, X^3 = z$ ) for the physical domain  $\Omega$  and  $q^i$  ( $q^1 = X, q^2 = Y, q^3 = Z$ ) for its transformed  $\Omega^*$  are given by

$$T_{(t)} : \Omega^* \rightarrow \Omega : (X, Z) \rightarrow (x, z) \quad (3)$$

such that

$$x = f(X, Y) ; z = Z \quad (4)$$

in the planar case and, in the three-dimensional case

$$T_{(t)} : \Omega^* \rightarrow \Omega : (X, Y, Z) \rightarrow (x, y, z) \quad (5)$$

with

$$x = f(X, Y, Z) ; y = g(X, Y, Z) ; z = Z . \quad (6)$$

For the above transformations, we define an upstream reference section  $S_{\text{ref}}$  at  $z = z_{\text{ref}}$  where the kinematics are known, mapped into a section  $S_{\text{ref}}^*$  of  $\Omega^*$ , identical in shape to  $S_{\text{ref}}$ , such that

$$x = f(X, Y, Z_{\text{ref}}) ; y = g(X, Y, Z_{\text{ref}}) ; z_{\text{ref}} = Z_{\text{ref}} \quad (7)$$

The mapped domain  $\Omega^*$  is a straight cylinder of basis  $S_{\text{ref}}^*$ , of transformed streamlines parallel to the mean flow direction.

Similar relations can be obtained with cylindrical coordinates  $(r, \theta, Z)$  as

$$r = f(R, \theta, Z) ; \theta = g(R, \theta, Z) ; z = Z . \quad (8)$$

In the cases investigated, the existence of the reference section  $S_{\text{ref}}$  at  $z = z_{\text{ref}}$ , where the kinematics are known, is assumed [9] to compute the velocity field in the flow domain. The incompressibility condition is automatically verified by the formulation, using expressions involving stream functions.

When flow vortices are expected in domain  $\Omega$ , local mappings  $T_{m(t)} : \Omega_m^* \rightarrow \Omega_m$  are adopted, using domain decomposition such that  $\Omega = \bigcup_{m=1}^{m=M} \Omega_m$ . When using

Cartesian coordinates  $X^i$  ( $X^1 = x, X^2 = y, X^3 = z$ ) in domain  $\Omega$ , we define a coordinate system  $\xi_j$  ( $\xi^1 = X, \xi^2 = Y, \xi^3 = s$ ) in a sub-domain  $\Omega_m^*$ , where the streamlines are parallel segments. The variable  $s$  is defined as the length of a segment of streamline in  $\Omega_m^*$ , being zero at a reference section  $S_m$ , with two branches of the streamline (Fig. 1b). Assuming, in domain  $\Omega_m^*$ , the existence of a reference section  $S_m^*$  identical in shape to the reference section  $S_m$  of the sub-domain  $\Omega_m$  of  $\Omega$ , we can write the correspondence between sub-domain  $\Omega_m$  and a local domain  $\Omega_m^*$  according to the following relations [13,14].

$$x = \alpha_m(X, Y, s); y = \beta_m(X, Y, s); z = \gamma_m(X, Y, s) \quad (9)$$

For each domain  $\Omega^*_m$  we consider a kinematic function  $\phi^*_m$  related to the reference section  $S^*_m$ , leading to write the velocity components in terms of  $\phi^*_m$  and variables  $\xi^j$  [14]. The Jacobian  $\Delta = |\partial(X^i)/\partial(\xi^j)|$ , assumed to be non-zero, is given by the equation

$$\begin{aligned} \Delta = & \alpha'_X (\beta'_Y \gamma'_s - \beta'_s \gamma'_Y) - \beta'_X (\alpha'_Y \gamma'_s - \gamma'_Y \alpha'_s) \\ & + \gamma'_X (\alpha'_Y \beta'_s - \beta'_Y \alpha'_s) \end{aligned} \quad (10)$$

To compute the flow field, the local mapping approach requires to write compatibility equations at the common boundaries of the sub-domains that define the total domain  $\Omega$  together with the classic conservation equations.

## 2. Domain transformation for unsteady flows with open streamlines

### 2.1 Transformation of open streamlines and pathlines

Let us consider a simply-connected bounded domain  $\Omega$  of boundary  $\Gamma$ . In unsteady conditions, the assumption of non existence of secondary flows implies that the pathlines are also open, thus leading to define a mean flow direction  $z$  for both streamlines and pathlines (Fig. 2). An upstream section  $S_{ref}$  at  $z = Z_{ref}(t)$ , where the velocities  $\underline{V}_{ref}$  are known, is assumed. With coordinates  $(X^1, X^2, X^3) \equiv (x, y, z)$  related to the Cartesian basis  $(\underline{\epsilon}_1, \underline{\epsilon}_2, \underline{\epsilon}_3)$ , the three-dimensional velocity vectors expressed by

$$\underline{V}(x, y, z, t) = u(x, y, z, t) \underline{\epsilon}_1 + v(x, y, z, t) \underline{\epsilon}_2 + w(x, y, z, t) \underline{\epsilon}_3 \quad (11)$$

are tangent to the streamlines at a time  $t$  (Fig. 1). Let  $\underline{V}_0$  be the vector of initial velocity conditions in domain  $\Omega$ , limited by the upstream section  $S_{ref}$ . Relations (6) can be defined for the transformation  $T_{s(t)} : \Omega(t)^* \rightarrow \Omega$ , at a given time  $t$  such that,  $\forall t$ , the mapped streamlines are parallel and straight.

According to the assumption of open streamlines, the Jacobian

$$\Delta = \left| \frac{\partial(x, y, z)}{\partial(X, Y, Z)} \right| = f'_x g'_y - g'_x f'_y \quad (12)$$

of the transformation  $T_{s(t)}$  [13,14] is required to be non-zero. The following derivative operators, involving the Jacobian  $\Delta$  are written as

$$\frac{\partial}{\partial x} = \frac{1}{\Delta} \left[ g'_y \frac{\partial}{\partial X} - g'_x \frac{\partial}{\partial Y} \right]$$



$$\frac{\partial}{\partial y} = \frac{1}{\Delta} \left[ f_Y \frac{\partial}{\partial X} - f_X \frac{\partial}{\partial Y} \right] \quad (13)$$

$$\frac{\partial}{\partial z} = \frac{1}{\Delta} \left[ (f_Y g'_z - f_Z g'_Y) \frac{\partial}{\partial X} + (f_Z g'_X - f_X g'_Z) \frac{\partial}{\partial Y} \right] + \frac{\partial}{\partial Z} \quad .$$

Similarly, for the planar case, with Cartesian coordinates  $(x, z)$ , the Jacobian  $\Delta = \left| \frac{\partial(x, z)}{\partial(X, Z)} \right| = f_X$  is assumed to be non-zero and the derivative operators are

$$\frac{\partial}{\partial x} = \frac{1}{f_X} \frac{\partial}{\partial X} \quad ; \quad \frac{\partial}{\partial z} = -\frac{f'_Z}{f_X} \frac{\partial}{\partial X} + \frac{\partial}{\partial Z} \quad (14)$$

According to the non-existence of secondary flows, we can also map the trajectories  $L_P$  into parallel lines of a domain  $\Omega_P^*(t)$  by defining a one-to-one transformation  $T_{P(t)}$ . In the mapped domain, we use a reference section  $S^*_{ref}$  identical to  $S_{ref}$  and write

$$T_{P(t)} : \Omega_P^*(t) \rightarrow \Omega \quad (15)$$

with the following relationships, similar to equations (6)

$$x(t) = \hat{f}(X, Y, Z, t) \quad ; \quad y(t) = \hat{g}(X, Y, Z, t) \quad ; \quad z = Z(t) \quad . \quad (16)$$

The mapping functions  $\hat{f}$  and  $\hat{g}$  are unknown and must be determined, at each time  $t$ . As for the steady case, the assumption of open pathlines requires that the Jacobian  $\Delta(t) = \left| \frac{\partial(x, y, z)}{\partial(X, Y, Z)} \right|(t)$  of the transformation  $T_{P(t)}$ , expressed on the same form than equation (12), is non-zero. The derivative operators are given versus time by relations (13), in terms of the mapping functions  $\hat{f}$  and  $\hat{g}$  .

Referring to the flow direction  $Z$ , the transformed domain  $\Omega_P^*(t)$  is limited downstream by the curve  $\Gamma^*_P(t)$ , defined by the pathline points at time  $t$  (Fig. 3). At all times  $t' \leq t$ , the particles move on the rectilinear lines of domain  $\Omega^*(t)$ , thus allowing more simplicity for particle tracking. At every time  $t$ , the position  $\underline{M}$  of a material point  $\mathbf{M}$  of a pathline originating at the position  $\underline{M}_{ref} \in S_{ref}$  at a reference time  $t_0$ , is given by

$$\underline{M} = \underline{M}_0 + \int_{t_0}^t \underline{V}(\underline{M}, t') dt' \quad . \quad (17)$$

According to pathline properties, the material point  $\mathbf{M}$  moves on the straight line in the mapped domain  $\Omega_P^*(t)$ . The material derivative of a function  $A$  in terms of the transformed variables  $(X, Y, Z)$  of  $\Omega^*(t)$  is expressed by

$$\frac{DA}{Dt} = \frac{\partial A}{\partial X} \frac{DX}{Dt} + \frac{\partial A}{\partial Y} \frac{DY}{Dt} + \frac{\partial A}{\partial Z} \frac{DZ}{Dt} = \frac{\partial A}{\partial Z} \frac{DZ}{Dt} \quad (18)$$

that leads to write

$$\frac{\partial}{\partial Z} = \frac{1}{w(X, Y, Z, t)} \frac{D}{Dt} \quad . \quad (19)$$

## 2.2 Kinematics

At any instant, the velocities can be evaluated by STM formulae with mapping functions for streamlines, as detailed in previous papers [9]. Similarly to the steady flow with open streamlines, we assume a known velocity profile  $w(x, y, z_{ref})$  at section  $S_{ref}$  to which corresponds a kinematic function  $\psi^*(Y, Y)$ . We then get the following relations

$$u = \frac{\hat{f}'_z(X, Y, Z, t) \psi^*(X, Y, t)}{\Delta(X, Y, Z, t)} \quad (20)$$

$$v = \frac{\hat{g}'_z(X, Y, Z, t) \psi^*(X, Y, t)}{\Delta(X, Y, Z, t)} \quad (21)$$

$$w = \frac{\psi^*(X, Y, t)}{\Delta(X, Y, Z, t)} \quad . \quad (22)$$

Using the mapping functions of  $T_{p(t)}$ , we can compute the natural basis  $\underline{b}_i = \frac{\partial M}{\partial X^i}$  as in steady conditions [19], given by

$$\underline{b}_1 = \hat{f}'_x \underline{\epsilon}_1 + \hat{g}'_x \underline{\epsilon}_2 \quad ; \quad \underline{b}_2 = \hat{f}'_y \underline{\epsilon}_1 + \hat{g}'_y \underline{\epsilon}_2 \quad ; \quad \underline{b}_3 = \hat{f}'_z \underline{\epsilon}_1 + \hat{g}'_z \underline{\epsilon}_2 + \underline{\epsilon}_3 \quad (23)$$

allowing the velocity vector to be written as

$$\underline{V}(x, y, z, t) = w(X, Y, Z, t) \underline{b}_3 = \frac{\psi^*(X, Y, t)}{\Delta(X, Y, Z, t)} \underline{b}_3(t) \quad . \quad (24)$$

Problems involving memory-integral rate equations require evaluation of the rate-of-deformation tensor  $\underline{D}(t)$  and of the corotational reference frames  $\underline{\tilde{e}}_i$  at times  $t' \leq t$ , in order to satisfy the principle of material objectivity [5,16]. For steady flows, such problems has been considered in previous papers (e.g. [21]) for axisymmetric flows. Though significant difficulties arise in three-dimensional problems with corotational models in relation to determination of the corotational frames [16], computation of

the tensor  $\underline{\underline{D}}(t)$  can be achieved by STM formulae as previously done with purely viscous models in unsteady situations [15].

Codeformational models can be considered by means of concepts already introduced [5] to compute the strain gradient tensor  $\underline{\underline{F}}_t(t)$  ( $t \leq t$ ). In the following relations, time variables are referred by subscripts  $t$  and  $t'$ . The strain gradient is given in Cartesian coordinates by the matrix

$$\mathbf{F}_{t'(t)} = \begin{bmatrix} \frac{\partial x_t^i}{\partial x_{t'}^j} \end{bmatrix} . \quad (25)$$

The expressions of Cauchy and Finger tensors  $\underline{\underline{C}}_t(t')$  and  $\underline{\underline{B}}(t')$ , generally adopted as kinematic quantities for codeformational models are written as follows.

$$\underline{\underline{C}}_t(t') = \underline{\underline{F}}_t^T(t') \cdot \underline{\underline{F}}_t(t') \quad (26)$$

$$\underline{\underline{B}}(t') = \underline{\underline{C}}_t^{-1}(t') = [\underline{\underline{F}}_t(t')]^{-1} \cdot [\underline{\underline{F}}_t^T(t')]^{-1} . \quad (27)$$

To solve problems related to codeformational models with STM for unsteady flows with open streamlines, we consider pathline mappings versus time, defined by the transformation  $T_{p(t)}$  (equations (15-16)). The previous approach for kinematic histories in steady flow conditions by mapping functions [21] can be adapted to non-stationary flow situations with similar formulae for strain tensors in steady two- and three-dimensional flows without eddies.

Turning back to previous elements from the steady analysis, we use here the transformation  $T_{p(t)}$  of pathlines. According to equation (17), we get

$$X_t^1 = X_t^1 \quad (28)$$

$$X_t^2 = X_t^2 \quad (29)$$

$$X_t^3 = X_t^3 + \int_t^{t'} w(\mathbf{M}, \xi) d\xi . \quad (30)$$

In the natural basis  $b_i$ , the deformation gradient matrix is given by the simple following form in terms of coordinates  $q^i$  ( $q^1 = X$ ,  $q^2 = Y$ ,  $q^3 = Z$ )

$$[\mathbf{F}]_{b_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial q_t^3}{\partial q_t^1} & \frac{\partial q_t^3}{\partial q_t^2} & \frac{\partial q_t^3}{\partial q_t^3} \end{bmatrix} \quad (31)$$

where the unknown components are [16,18]

$$F^{31} = \frac{\partial Z_{t'}}{\partial X_t} = w_{t'} \int_{Z_t}^{Z_{t'}} \frac{1}{(w_{\xi}^2)^2} \frac{\partial w_{\xi}^2}{\partial X_{\xi}} dZ_{\xi} \quad (32)$$

$$F^{32} = \frac{\partial Z_{t'}}{\partial Y_t} = w_{t'} \int_{Z_t}^{Z_{t'}} \frac{1}{(w_{\xi}^2)^2} \frac{\partial w_{\xi}^2}{\partial Y_{\xi}} dZ_{\xi} \quad (33)$$

$$F^{33} = \frac{\partial Z_{t'}}{\partial Z_t} = \frac{w_{t'}}{w_t} \quad (34)$$

In these equations,  $w_{\xi} = w(\mathbf{M}, \xi)$ ,  $w_{t'} = w(\mathbf{M}, t')$  and  $w_t = w(\mathbf{M}, t)$  denote the non-zero velocity component of a material point  $M$  on its pathline at times  $\xi$ ,  $t'$  and  $t$ , respectively.  $(X_{\xi}, Y_{\xi}, Z_{\xi})$ ,  $(X_{t'}, Y_{t'}, Z_{t'})$  and  $(X_t, Y_t, Z_t)$  stand for respective positions of the material point at times  $\xi$ ,  $t'$  and  $t$ .

To compute  $\underline{F}_i(t')$  in the Cartesian basis, the natural basis  $\underline{b}_i$  and its reciprocal basis  $\underline{\chi}_i$  related to the transformation  $T_{p(t)}$  (mapping functions  $\hat{f}(t)$  and  $\hat{g}(t)$  must be formulated [16]. Then, the components of  $\underline{F}_i(t')$  in the basis  $\underline{\mathcal{E}}_i$  are obtained from the following matrix equation

$$[\mathbf{F}]_{\mathcal{E}} = {}^T[\mathbf{A}][\mathbf{F}]_b[\bar{\mathbf{A}}], \quad (35)$$

where the matrices  $\mathbf{A}$  and  $\bar{\mathbf{A}}$  are involved in the expressions related to the natural and reciprocal bases, such that

$$\underline{b}_i = A_i^k \underline{\mathcal{E}}_k \quad , \quad \underline{\chi}^j = \bar{A}_k^j \underline{\mathcal{E}}_k \quad . \quad (36)$$

Thus, the components of the Cauchy and Finger tensors (equations (24-25)) can be computed by using equations (35-36), leading to determine the stress components versus time.

### 3. Domain transformation for unsteady flows with closed streamlines

The fluid flows considered can be created from the rest state or by changes brought by non-stationary conditions applied to a steady flow in domain  $\Omega$ , at an initial time  $t_0$ . When particles of the material move from the rest state, a slow flow assumption without vortices can be made at the beginning. As pointed out previously, the hypothesis of open streamlines and pathlines requires the Jacobians of the domain transformations  $T_s(t)$  and  $T_p(t)$  to be non-zero. From our numerical results obtained for steady flows in ducts and between eccentric cylinders with comparisons with literature data [22,23], a divergence of the algorithms, corresponding to a

Jacobian approaching zero, has been observed for conditions of appearance of circulating regions. Thus, in a non-stationary process, the onset of vortices can be assumed when, at an instant  $t^*$ , the solving process defined with the open streamlines approach diverges, requiring to use STM elements related to flows with closed streamlines. However, we will consider directly the procedure related to flows with closed streamlines. More precisely, the approach concerns more the case of changes of sign for the flow velocities, corresponding to the example of streamlines and pathlines shown in Fig. 4.

### 3.1 Rate constitutive equations

In practice, rate equations as memory-integral corotational models generally expressed in terms of the rate-of-deformation tensor  $\underline{D}(t)$  lead to practically insurmountable difficulties for determining corotational frames in three-dimensional flows. Restricting our study to the two-dimensional case, we define, as in steady conditions, local streamline transformations  $T_s^m(t): \Omega_m^* \rightarrow \Omega_m$  using domain decomposition. In every domain  $\Omega_m^*$ , the transformed streamlines are parallel segments [13,14] perpendicular to a reference section  $S_m^*$  identical in shape to a reference section  $S_m$  of sub-domain  $\Omega_m$  (Fig. 1b), such that the mapped sub-domain is a straight cylinder of basis  $S_m^*$ . In the planar case, we adopt coordinates  $q^i(X^1=x, X^2=z)$  and  $\xi^j(\xi^1=X, \xi^2=s)$  for sub-domains  $\Omega_m$  and  $\Omega_m^*$ , respectively. The variable  $s$  is related to the length of a segment of streamline, referred to the plane  $S_m^*$ , as pointed out in Section 1. The formalism defined for the stationary case can be applied for every time  $t$ , requiring streamline determination and evaluation of the corresponding stress tensor as a function of  $\underline{D}(t')$  ( $t' \leq t$ ). Referring to the Cartesian basis  $\underline{e}_i$ , the tensor components of  $\underline{D}(t)$ , given by the matrix

$$[D(t)] = \begin{bmatrix} \partial u(t)/\partial x & \partial u(t)/\partial z \\ \partial w(t)/\partial x & \partial w(t)/\partial z \end{bmatrix}, \quad (37)$$

can be expressed in terms of the variables  $(X,s)$  of the mapped domain  $\Omega^*$ , using the basic relations of  $T_{m(t)}$ , where the subscript “m” has been removed for reasons of brevity:

$$x = \alpha(X, s); \quad z = \beta(X, s). \quad (38)$$

The Jacobian  $\Delta = |\partial(q^i)/\partial(\xi^j)|$  of  $T_s^m(t)$  is given by

$$\Delta(t) = \alpha'_X \beta'_s - \alpha'_s \beta'_X \quad (39)$$

and the derivative operators are written as

$$\frac{\partial}{\partial x} = \frac{1}{\Delta(t)} \left[ \beta'_s \frac{\partial}{\partial X} - \beta'_X \frac{\partial}{\partial s} \right] \quad ; \quad \frac{\partial}{\partial z} = \frac{1}{\Delta(t)} \left[ -\alpha'_s \frac{\partial}{\partial X} + \alpha'_X \frac{\partial}{\partial s} \right] \quad (40)$$

We use a reference kinematic function  $\phi^*$  at section  $S_m^*$  (where  $s=0$ ) defined by [13]

$$\phi^*(X) = w(X, s=0). \quad (41)$$

When expressing the derivative operators  $\partial/\partial x$  and  $\partial/\partial z$  in terms of  $\partial/\partial X$  and  $\partial/\partial s$  [13], with variables  $(X, s)$  of the mapped sub-domain  $\Omega_m^*$ , the components  $(u, w)$  of the velocity vector  $\underline{V}(x, z, t) = u(x, z, t)\underline{e}_1 + w(x, z, t)\underline{e}_z$  are written as

$$u(t) = \alpha'_s \phi^*(X) / \Delta ; \quad w(t) = \phi^*(X) / \Delta \quad (42)$$

Thus, elements of the matrix of  $\underline{D}(t)$  (equation (37)) can be determined, as functions of variables  $(X, s)$ , from equations (40) and (42), leading to calculation of the stress tensor components at any instant  $t$ .

### 3.2 Codeformational constitutive equations

Even though the flow only involves open streamlines at small times  $t > t_0$  when the fluid moves from the rest state at the initial time  $t_0$ , we focus on the case of closed streamlines and pathlines that may appear in relation to the geometry, the flow rate and the fluid properties. The relative strain gradient is considered for open or closed pathlines. As previously proposed [13,14], we define in domain  $\Omega$  a finite number of sub-domains  $\Omega_m$  ( $m=1, \dots, M$ ) (Fig. 5) where vortices are expected, that may also involve open streamlines, to which are associated reference cross-sections  $S_m$ . To these sub-domains, we specify corresponding domains  $\Omega_m^*$  ( $m=1, \dots, M$ ) where the mapped streamlines are segments parallel to a given direction  $\underline{s}$ , defining a cylinder of basis  $S_m^*$  identical to  $S_m$ . Kinematic functions  $\phi_m^*(S_m^*)$ , a priori unknown, are associated to domains  $\Omega_m^*$  such that

$$\phi_m^*(P \in S_m^*) = \underline{v}(P \in S_m) \cdot \underline{s} \quad (43)$$

As for the previous approach for local transformations in the steady case, we define a new set of variables  $\zeta^{\sharp}(X, Y, s)$ , where:

- $(X, Y)$  are the local variables in the reference section;
- in this case,  $s$  is related to the curvilinear abscissa, that corresponds, differently to the case illustrated in Fig. 1b, to the total length of the streamline limited by two positions  $P^*_o$  and  $P^*$  on this streamline (Fig. 5), such that  $P^*_o \in S_m^*$  (reference section) implies  $s(P^*_o) = 0$ .

At every time  $t$ , the transformation of streamlines

$$T_m^P(t) : \Omega_m^*(t) \rightarrow \Omega_m^*(t) \quad (44)$$

associates a point  $P^*(X, Y, s)$  of  $\Omega_m^*(t)$  to a point  $P(x, y, z)$  of domain  $\Omega_m(t)$  according to the following equations

$$x = \alpha_m(X, Y, s, t); y = \beta_m(X, Y, s, t); z = \gamma_m(X, Y, s, t) \quad , \quad (45)$$

formally the same than equation (9), with a Jacobian given by equation (10), assumed to be non-zero. Denoting by  $\chi$  the corresponding length of the streamline in the physical sub-domain  $\Omega_m(t)$  and omitting the subscript "m" for conciseness, we may write an additional relation between the three local functions  $\alpha$ ,  $\beta$  and  $\gamma$  as

$$\chi = \int_0^s [(\alpha'_s)^2 + (\beta'_s)^2 + (\gamma'_s)^2]^{1/2} ds \quad . \quad (46)$$

A reference kinematic function  $\phi^*(X, Y, t)$ , related to the section  $S_m^*$  of domain  $\Omega_m^*$ , may be defined by

$$\phi^*(X, Y, t) = w(X, Y, s = 0, t) \quad , \quad (47)$$

and the use of derivative operators  $\partial/\partial x$ ,  $\partial/\partial y$  and  $\partial/\partial z$  in terms of  $\partial/\partial X$ ,  $\partial/\partial Y$  and  $\partial/\partial s$  [13] leads to express the velocity components  $(u, v, w)$  as

$$u = \alpha'_s \phi^*(X, Y) / \Delta \quad , \quad v = \beta'_s \phi^*(X, Y) / \Delta \quad , \quad w = \gamma'_s \phi^*(X, Y) / \Delta \quad (48)$$

where  $\Delta$  denotes the Jacobian. These relations verify the mass conservation law.

We now consider local pathline transformations  $T_m^p(t)$ , looking for the position of material points at time  $t$ , in order to determine the kinematics and stresses, still retaining domains  $\Omega_m^*(t)$  and  $\Omega_m(t)$  according to the relation

$$T_m^p(t) : \Omega_m^*(t) \rightarrow \Omega_m(t) \quad (49)$$

These transformations are defined such that the trajectories, in the mapped domain  $\Omega_m^*$ , are segments parallel to a given direction  $\underline{d}$ . For further computational purposes, one can normalize the lengths of the mapped pathlines as done in previous papers that have reported numerical simulations with local transformation functions [14,23]. As for the streamlines, we adopt the reference sections  $S_m^*$  and  $S_m$  and write the following local relationships

$$x = \hat{\alpha}_m(X, Y, \sigma, t) \quad , \quad y = \hat{\beta}_m(X, Y, \sigma, t) \quad , \quad z = \hat{\gamma}_m(X, Y, \sigma, t) \quad . \quad (50)$$

with spatial variables  $X^i \equiv (x, y, z)$  and  $\xi^k \equiv (X, Y, \sigma)$  for domains  $\Omega_m(t)$  and  $\Omega_m^*(t)$ , respectively. The third variable  $\sigma$  is related to the curvilinear abscissa on a pathline, referred to section  $S_m^*$ .

The local domain mapping functions require the Jacobian of the transformation to be non-zero. We retain the same reference kinematic function given by equation (47), leading to write, similarly to equations (48), the velocities in terms of the local pathline transformation functions as follows

$$u = \hat{\alpha}'_s \phi^*(X, Y) / \Delta \quad , \quad v = \hat{\beta}'_s \phi^*(X, Y) / \Delta \quad , \quad w = \hat{\gamma}'_s \phi^*(X, Y) / \Delta \quad . \quad (51)$$

These expressions verify the incompressibility condition. For using memory-integral codeformational models, we need to evaluate the components of the strain tensor  $\underline{\underline{F}}_t(t')$  at times  $t' \leq t$ . The same approach than that proposed for open streamlines can be applied, related to coordinates  $(X^j) \equiv (x, y, z)$  and  $(\xi^j) \equiv (X, Y, \sigma)$  for sub-domains  $\Omega_m$  and  $\Omega_m^*$ , respectively. To express components of the strain tensor for each sub-domain, we make use of coordinates  $(\xi^j)_{t'}$  and  $(\xi^j)_t$  and, similarly to equations (28-30), we get

$$X_{t'} = X_t \quad ; \quad Y_{t'} = Y_t \quad ; \quad \sigma_{t'} = \sigma_t + \int_t^{t'} w(M, \xi) d\xi \quad (52)$$

leading to components of  $\underline{\underline{F}}_t(t')$  of the form given by equation (31). The kinematic terms corresponding to the local sub-domains are computed by relations of the type of equations (25-36), taking into account positions of the material points in the sub-domains under consideration.

#### 4. Concluding remarks

In this paper, we have focused on theoretical elements of stream-tube methods, in order to evaluate accurately strain histories in non-stationary flows of fluids obeying memory-integral models. We have started from results on steady situations to extend the concept of streamline mapping functions to trajectories, by considerations based upon geometry, that have led to their transformations into rectilinear parallel lines, with simple and precise formulae for the strain tensor components versus time. In all cases, the approach defined for the pathlines has been considered by means of relationships similar to those previously developed for stationary flow conditions. It should be pointed out that the possible existence of various vortex regions, notably in complex three-dimensional flows, can lead to insurmountable difficulties for defining reference cross-sections and sub-domains related. However, the present study may provide answers to numerous problems encountered in flows of processing rheology. For flows involving closed streamlines, the determination of streamlines and pathlines versus time are performed by domain decomposition, using reference sections in sub-domains. When starting from stationary flow conditions with eddies, the reference sections can be chosen in regards to the existing vortex flow zones. When the fluid is at rest, these sections should be mostly defined in relation to the shape of the boundaries as conditions encountered in contraction and expansion flow geometries. As pointed out previously, the changes in flows created from the rest state may lead to specify the presence of circulating zones and, consequently, the



physical sub-domains to be adopted for the analysis. Work is continued towards numerical simulations of non-stationary viscoelastic two and three-dimensional flow situations, notably those related to industrial applications as extrusion, injection and journal bearing flows.

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**A THEORETICAL ANALYSIS  
OF UNSTEADY INCOMPRESSIBLE FLOWS  
FOR TIME-DEPENDENT CONSTITUTIVE EQUATIONS  
BASED ON DOMAIN TRANSFORMATIONS**

by

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### Captions for figures

- Fig. 1** Transformation of streamlines in Stream-Tube Method  
(a) global transformation for an open streamline  
(b) local transformation for a closed streamline divided into two branches
- Fig. 2** Open streamlines and pathlines
- Fig. 3** Transformation of trajectories in the case of open streamlines and pathlines
- Fig. 4** Local view of closed streamlines and of a pathline
- Fig. 5** Physical and mapped local sub-domains  $\Omega_m$ ,  $\Omega_{m+1}$ ,  $\Omega_m^*$  and  $\Omega_{m+1}^*$  for streamline and pathline transformations

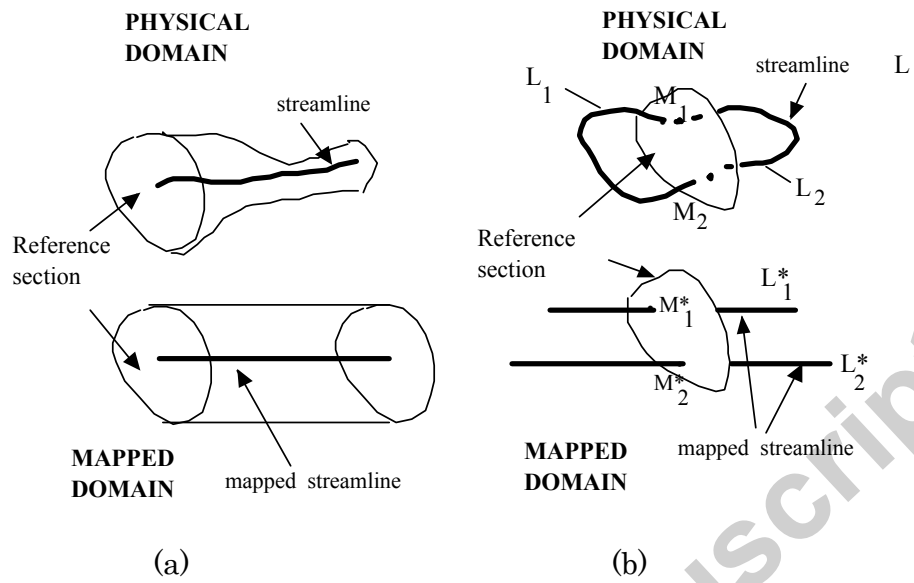


Figure 1

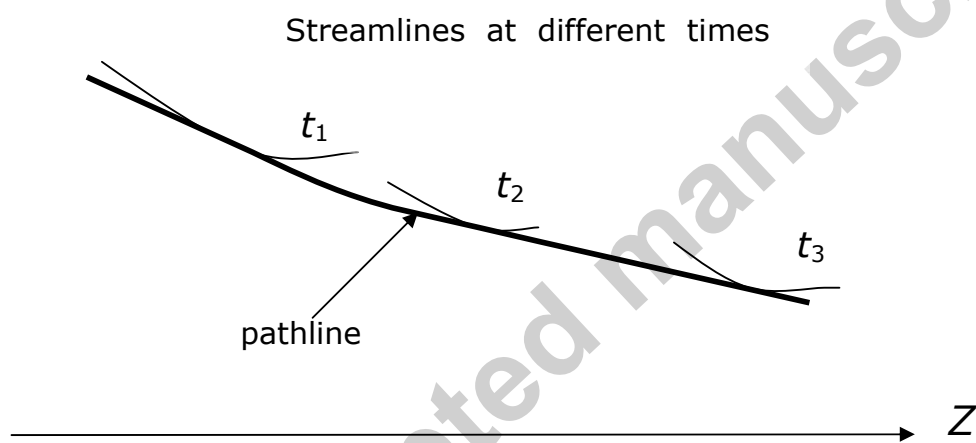


Figure 2

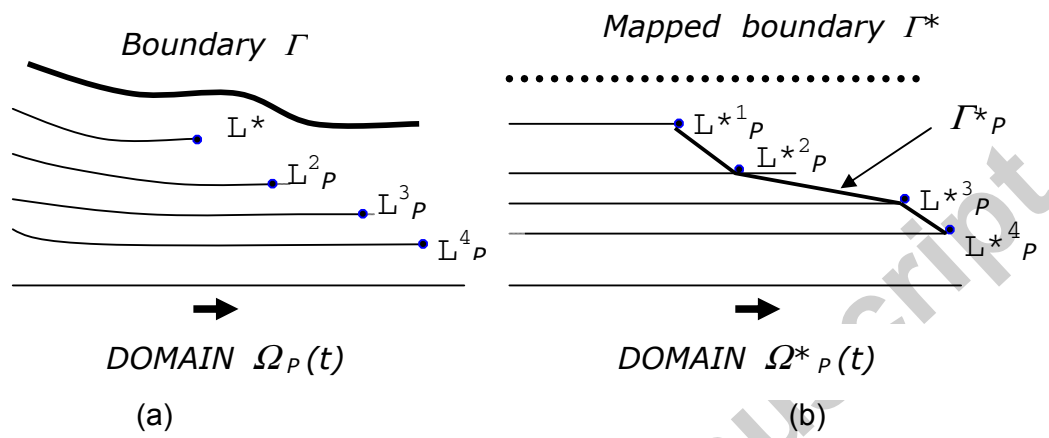


Figure 3



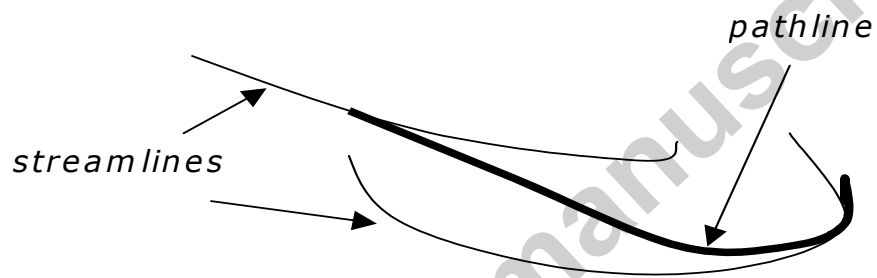


Figure 4

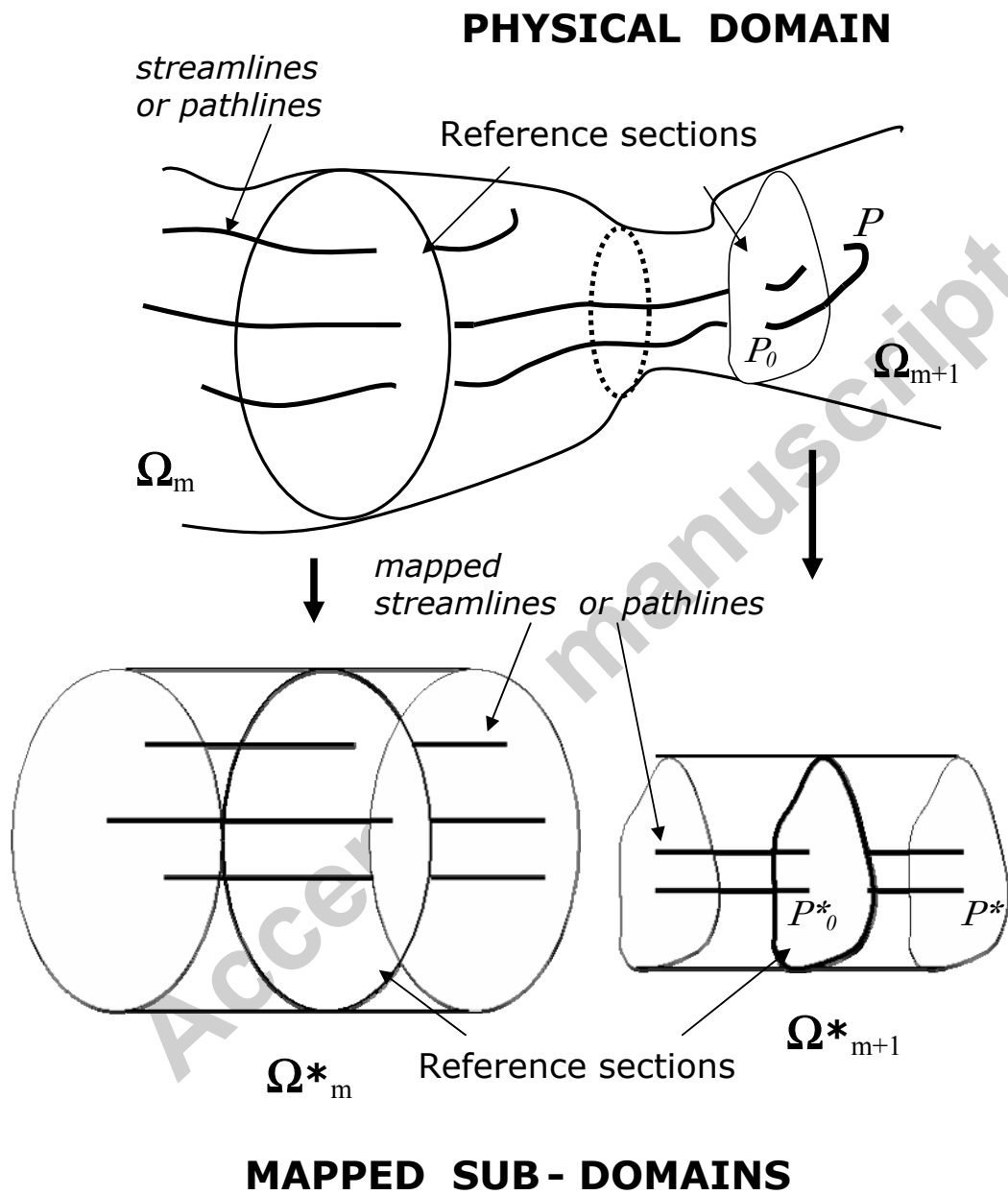


Figure 5