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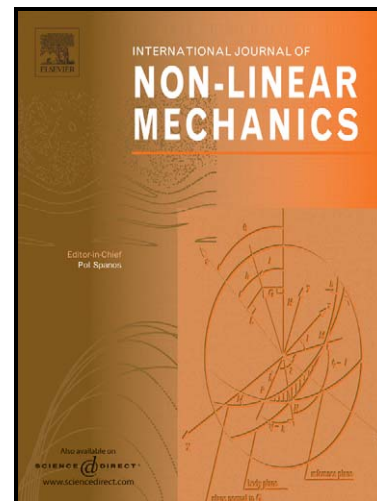
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Non-linear dynamics of curved beams. Part 1, formulation

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Abstract

The non-linear dynamics of elastic beams with uniform initial curvature and double-symmetric cross-section are considered in this work. In particular, the work is divided in two parts. In Part 1, the interest is oriented to the formulation of an accurate model, able to describe the finite dynamics of initially curved beams as to obtain a parameterization of the initial configuration and the weak expression of the equations of motion. To this end, an explicit description of the deformation field and inertia terms is presented. The equations of motion can be used, with slight modifications, for extensible and inextensible, or shear-deformable and shear-indeformable, beams. A description of the free dynamics, of the possible classes of motion under a sinusoidally varying shear tip force, and of bifurcation phenomena is presented in Part 2 for a case-study, together with the results of experimental tests on an aluminum prototype.

Key words: Beams, non-linear dynamics, initial curvature

1 Introduction

The analysis of finite dynamics, even for simple structural systems, is of great interest in different engineering fields. Often a rich and varied response is due to the presence of non-linear terms. In the case of beams, arches and cables, a key role is played by the initial configuration (null, small, or large initial curvature) and by the simplifying assumptions (e.g., internal constraints).

Works devoted to model formulation and to non-linear dynamic analysis of beams are widely present in literature. Non-linear dynamics of straight beams,

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subjected to time-periodic forces, were studied in Bolotin [1], where particular attention was given to flexural-torsional dynamic coupling in the case of thin-walled beams. In Crespo da Silva et al. [2,3,4,5], the formulation of approximate models of Euler-Bernoulli beams, considering or neglecting extensional deformation, was presented. It was proved that, when one end of the beam is free to move, the beam behaves essentially as inextensional. In Simo [6], an accurate formulation of the non-linear model of an elastic beam, as descending from a corresponding 3-dimensional continuum, was presented. In Luongo et al. [7], the steady-state solutions of reduced-order models for planar beams, having different constraint conditions, were analyzed via a perturbation approach. In Antman [8], the formulation of non-linear models and various stability and post-critical analyses were provided for one dimensional bodies using Cosserat rod theory. In Tatone et al. [9], the sensitivity to imperfections for thin-walled beams, where warping is considered, were studied in the case of buckling. In Di Egidio et al. [10], a non-linear model of a thin-walled, non-symmetric open-section beam were studied including both non-linear warping and torsional elongation effects. It was also shown that, in beams with double-symmetric sections, these two last contributions are negligible. In Smoleński [11], the exact non-linear equations for the statics of beams were obtained and numerical computations were performed considering various configurations. In Pai et al. [12], the equations describing the kinematics and statics of initially curved and twisted beams, in case of very large rotations and displacements, were formulated and then solved using multiple shooting method. More recently, in Paolone et al. [13], the stability of thin-walled cantilever beams under static non-conservative forces was studied, using the non-linear Cosserat rod model.

This work is composed of two Parts. In Part 1, an accurate geometrically non-linear 1-dimensional model of initially curved beams is formulated. A double-symmetric, uniform, cross-section is considered. The initial curvature terms, assumed as uniform, are considered as explicit parameters characterizing the initial configuration. The deformation field, the inertia terms, and then the equations of motion are explicitly derived as function of such parameters, without considering the contributions due to the warping (Di Egidio et al. [10]).

These equations describe, with slight modifications, extensional and inextensional, as well as shear-deformable and shear-indeformable, beams. In particular, the model is developed for an extensional, shear-deformable, and rigid-section beam. Then, internal constraints for inextensible and shear-indeformable beam are added directly in weak form, considering the corresponding stress terms as reactive forces.

As an example, a cantilever with distributed and tip mass, and time-periodic shear tip force, is considered.

In Part 2, a description of the results obtained by using the model in a case-study is presented. The free dynamic problem is solved for different initial configurations, in order to discuss the variation of the spectral properties as the initial configuration is varied, enlightening on the possible internal resonance conditions as well as possible veering phenomena among different couples of adjacent modes. The non-linear forced problem is then studied, after use of a Galerkin projection of the continuous problem on a reduced basis. Direct integrations and path-following procedures in the excitation control-parameters plane are applied to the discretized equations. An analysis of the flexural-torsional dynamic instability phenomenon, of bifurcation boundaries, and of post-critical classes of motion for different initial configurations are considered as well.

The results are compared with experimental observations on a straight beam characterized by geometrical imperfections on the initial configuration, in order to evaluate if imperfections, however present in experimental prototypes and in real full-scale cases, could justify the use of a model with some initial curvature to obtain better agreement between analytical results and experimental evidence.

2 The model

2.1 Geometry and configurations

The shape of the beam is a cylinder, occupying the space spanned by the axis line and the cross-sections (Fig. 1a). The axis line is a curve in the 3-D Euclidean configuration space \mathcal{E} . Let \mathcal{I} be a closed set of \mathbb{R} , and $\zeta_1 \in \mathcal{I}$ is chosen as a length abscissa for the axis curve, whose position, in the current configuration, is $\mathbf{x}_o(\zeta_1)$. The time-dependent positions of the points of the curve are oriented, i.e., in correspondence to every position $\mathbf{x}_o(\zeta_1)$, three orthonormal vectors $\{\mathbf{a}_1(\zeta_1), \mathbf{a}_2(\zeta_1), \mathbf{a}_3(\zeta_1)\}$, called *directors*, are considered (Fig. 1b). They belong to the vector translation space \mathcal{V} of \mathcal{E} . These vectors describe the attitude of the rigid plane sections of the beam, which are images of a closed set \mathcal{S} of \mathbb{R}^2 .

The vectors \mathbf{a}_2 and \mathbf{a}_3 are supposed to lay on the sections; therefore, if $(\zeta_2, \zeta_3) \in \mathcal{S}$, the position \mathbf{x} of a point of the beam in the current configuration can be written as

$$\mathbf{x}(\zeta_1, \zeta_2, \zeta_3) = \mathbf{x}_o(\zeta_1) + \zeta_2 \mathbf{a}_2(\zeta_1) + \zeta_3 \mathbf{a}_3(\zeta_1) \quad (1)$$

Equation (1) represents a parameterization of the shape of the beam, denoted as \mathcal{R} , a subset of \mathcal{E} , in the current configuration. In particular, the parameters

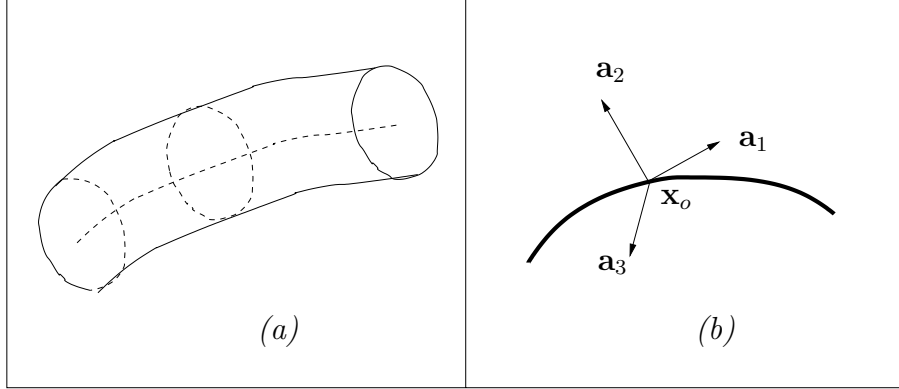


Figure 1. Geometry of the beam in a generic configuration: (a) the cylinder, spanned by the axis curve and the cross-sections; (b) the axis curve and the directors.

that identify the position of a point in \mathcal{R} are $\{\zeta_1, \zeta_2, \zeta_3\}$.

A particular configuration, corresponding to the initial time \bar{t} , is chosen as reference. In this configuration, the shape is denoted by $\bar{\mathcal{R}}$, the axis curve by $\bar{\mathbf{x}}_o(\zeta_1)$, and the directors by $\{\bar{\mathbf{a}}_1(\zeta_1), \bar{\mathbf{a}}_2(\zeta_1), \bar{\mathbf{a}}_3(\zeta_1)\}$. The overbar indicates time-independent terms related to the reference configuration. Both $\{\mathbf{a}_1(\zeta_1), \mathbf{a}_2(\zeta_1), \mathbf{a}_3(\zeta_1)\}$ and $\{\bar{\mathbf{a}}_1(\zeta_1), \bar{\mathbf{a}}_2(\zeta_1), \bar{\mathbf{a}}_3(\zeta_1)\}$ are bases for \mathcal{V} . In analogy with Eq. (1), the parameterization for $\bar{\mathcal{R}}$ is

$$\bar{\mathbf{x}}(\zeta_1, \zeta_2, \zeta_3) = \bar{\mathbf{x}}_o(\zeta_1) + \zeta_2 \bar{\mathbf{a}}_2(\zeta_1) + \zeta_3 \bar{\mathbf{a}}_3(\zeta_1) \quad (2)$$

It can be assumed that the sections, in the reference configuration, are orthogonal to the tangent vector of the axis curve, i.e.

$$\bar{\mathbf{x}}'_o = \bar{\mathbf{a}}_1 \quad (3)$$

where $()'$ stands for differentiation with respect to ζ_1 .

2.2 Initial curvature

The orientation of the directors in the reference configuration is provided by the skew tensor of the initial curvature $\bar{\mathbf{K}}(\zeta_1)$, assumed as known and defined as

$$\bar{\mathbf{a}}'_i(\zeta_1) = \bar{\mathbf{K}}(\zeta_1) \bar{\mathbf{a}}_i(\zeta_1), \quad i = 1, 2, 3 \quad (4)$$

Its representation with respect to the basis $\{\bar{\mathbf{a}}_i\}$ is

$$[\bar{\mathbf{K}}]_{\bar{\mathbf{a}}_i} = \begin{bmatrix} 0 & -\bar{k}_3(\zeta_1) & \bar{k}_2(\zeta_1) \\ \bar{k}_3(\zeta_1) & 0 & -\bar{k}_1(\zeta_1) \\ -\bar{k}_2(\zeta_1) & \bar{k}_1(\zeta_1) & 0 \end{bmatrix} \quad (5)$$

In the following, the initial curvature is assumed to be uniform, i.e., \bar{k}_i do not depend on ζ_1 , for $i = 1, 2, 3$. This provides a limitation to the possible classes of initial shapes; anyhow the obtained set of possible shapes permits one to study several interesting cases. Under such hypothesis, Eq. (4) furnishes

$$\begin{aligned} \bar{\mathbf{a}}'_1(\zeta_1) &= \bar{k}_3\bar{\mathbf{a}}_2(\zeta_1) - \bar{k}_2\bar{\mathbf{a}}_3(\zeta_1) \\ \bar{\mathbf{a}}'_2(\zeta_1) &= -\bar{k}_3\bar{\mathbf{a}}_1(\zeta_1) + \bar{k}_1\bar{\mathbf{a}}_3(\zeta_1) \\ \bar{\mathbf{a}}'_3(\zeta_1) &= \bar{k}_2\bar{\mathbf{a}}_1(\zeta_1) - \bar{k}_1\bar{\mathbf{a}}_2(\zeta_1) \end{aligned} \quad (6)$$

which is a linear ordinary differential equation system with constant coefficient. The necessary initial conditions read

$$\begin{aligned} \bar{\mathbf{a}}_1(0) &= \mathbf{e}_1 \\ \bar{\mathbf{a}}_2(0) &= \mathbf{e}_2 \\ \bar{\mathbf{a}}_3(0) &= \mathbf{e}_3 \end{aligned} \quad (7)$$

asserting, without loss of generality, that the initial section lays on a plane parallel to the natural base $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of \mathcal{V} . The initial value problem (6)-(7) can be solved to obtain the expressions of $\bar{\mathbf{a}}_i$ (see Appendix A). To get information on the axis curve in the reference configuration, Eq. (3) can be integrated, considering $\bar{\mathbf{x}}_o(0) = \mathbf{0}$, i.e., the axis curve starts from the zero position of \mathcal{E} (Appendix A). In this way, the parameterization of $\bar{\mathcal{R}}$ (Eq. (2)) is completely written in terms of the three initial curvature components \bar{k}_i ($i = 1, 2, 3$).

2.3 Deformation

A bijective and smooth transformation

$$\phi: \bar{\mathcal{R}} \rightarrow \mathcal{R} \quad (8)$$

is assumed to deform the initial shape of the beam. In particular this transformation allows one to express the current position as a function of the initial one

$$\begin{aligned} \mathbf{x}(\zeta_1, \zeta_2, \zeta_3) &= \phi(\bar{\mathbf{x}}(\zeta_1, \zeta_2, \zeta_3)) \\ \mathbf{x}_o(\zeta_1) &= \phi(\bar{\mathbf{x}}_o(\zeta_1)) \end{aligned} \quad (9)$$

To take into account the rigidity of the section, the directors $\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3\}$ can only perform a rotation. If \mathbf{R} is a rotation of \mathcal{V} , the evolution of the directors, from the reference configuration, is described by the following relation

$$\mathbf{a}_i(\zeta_1) = \mathbf{R}(\zeta_1)\bar{\mathbf{a}}_i(\zeta_1), \quad i = 1, 2, 3 \quad (10)$$

Hence, Eq. (1) becomes (omitting the independent variables)

$$\mathbf{x} = \mathbf{x}_o + \zeta_2 \mathbf{R} \bar{\mathbf{a}}_2 + \zeta_3 \mathbf{R} \bar{\mathbf{a}}_3 \quad (11)$$

The rotation \mathbf{R} can be written in terms of three functions $\{\vartheta_1(\zeta_1), \vartheta_2(\zeta_1), \vartheta_3(\zeta_1)\}$ describing the finite angles carrying $\bar{\mathbf{a}}_i(\zeta_1)$ on $\mathbf{a}_i(\zeta_1)$. A possible expression¹ for \mathbf{R} with respect to $\bar{\mathbf{a}}_i$ is

$$[\mathbf{R}] = \begin{bmatrix} \cos \vartheta_2 \cos \vartheta_3 & \sin \vartheta_1 \sin \vartheta_2 \cos \vartheta_3 - \cos \vartheta_1 \sin \vartheta_3 & \cos \vartheta_1 \sin \vartheta_2 \cos \vartheta_3 + \sin \vartheta_1 \sin \vartheta_3 \\ \cos \vartheta_2 \sin \vartheta_3 & \cos \vartheta_1 \cos \vartheta_3 + \sin \vartheta_1 \sin \vartheta_2 \sin \vartheta_3 & \cos \vartheta_1 \sin \vartheta_2 \sin \vartheta_3 - \sin \vartheta_1 \cos \vartheta_3 \\ -\sin \vartheta_2 & \sin \vartheta_1 \cos \vartheta_2 & \cos \vartheta_1 \cos \vartheta_2 \end{bmatrix} \quad (12)$$

It is useful to describe the deformation of the beam by means of its gradient. This is defined as the tensor transforming the tangent vectors of the curves passing through $\bar{\mathbf{x}}(\zeta_1, \zeta_2, \zeta_3)$ to the tangent vectors of the corresponding curves passing through $\mathbf{x}(\zeta_1, \zeta_2, \zeta_3)$.

In this way, choosing three curves passing through the position $\bar{\mathbf{x}}(\zeta_1, \zeta_2, \zeta_3)$ as

$$\begin{aligned} \bar{\mathbf{c}}_1(h) &:= \bar{\mathbf{x}}(\zeta_1 + h, \zeta_2, \zeta_3) \\ \bar{\mathbf{c}}_2(h) &:= \bar{\mathbf{x}}(\zeta_1, \zeta_2 + h, \zeta_3) \\ \bar{\mathbf{c}}_3(h) &:= \bar{\mathbf{x}}(\zeta_1, \zeta_2, \zeta_3 + h) \end{aligned} \quad h \in \mathbb{R} \quad (13)$$

and using Eqs. (2), (3) and (6), the tangent vectors in $h = 0$ are

$$\begin{aligned} \left. \frac{d\bar{\mathbf{c}}_1}{dh} \right|_{h=0} &= \bar{\mathbf{x}}' = \bar{\mathbf{a}}_1 + \zeta_2(-\bar{k}_3 \bar{\mathbf{a}}_1 + \bar{k}_1 \bar{\mathbf{a}}_3) + \zeta_3(\bar{k}_2 \bar{\mathbf{a}}_1 - \bar{k}_1 \bar{\mathbf{a}}_2) = \\ &= (1 - \zeta_2 \bar{k}_3 + \zeta_3 \bar{k}_2) \bar{\mathbf{a}}_1 - \zeta_3 \bar{k}_1 \bar{\mathbf{a}}_2 + \zeta_2 \bar{k}_1 \bar{\mathbf{a}}_3 \\ \left. \frac{d\bar{\mathbf{c}}_2}{dh} \right|_{h=0} &= \bar{\mathbf{a}}_2 \\ \left. \frac{d\bar{\mathbf{c}}_3}{dh} \right|_{h=0} &= \bar{\mathbf{a}}_3 \end{aligned} \quad (14)$$

¹ This expression for \mathbf{R} describes the composition of a rotation of amplitude ϑ_3 around $\bar{\mathbf{a}}_3$, followed by a rotation of amplitude ϑ_2 around the new \mathbf{a}_2 , and then by a rotation of amplitude ϑ_1 around the final \mathbf{a}_1 .

The corresponding curves passing through $\mathbf{x}(\zeta_1, \zeta_2, \zeta_3)$ are

$$\begin{aligned} \mathbf{c}_1(h) &:= \mathbf{x}(\zeta_1 + h, \zeta_2, \zeta_3) \\ \mathbf{c}_2(h) &:= \mathbf{x}(\zeta_1, \zeta_2 + h, \zeta_3) \\ \mathbf{c}_3(h) &:= \mathbf{x}(\zeta_1, \zeta_2, \zeta_3 + h) \end{aligned} \quad h \in \mathbb{R} \quad (15)$$

and, using Eqs. (11) and (6), the tangent vectors in $h = 0$ are

$$\begin{aligned} \left. \frac{d\mathbf{c}_1}{dh} \right|_{h=0} &= \mathbf{x}' = \mathbf{x}'_o + \mathbf{R}'(\zeta_2 \bar{\mathbf{a}}_2 + \zeta_3 \bar{\mathbf{a}}_3) + \\ &\quad + \mathbf{R} \left[\zeta_2 (-\bar{k}_3 \bar{\mathbf{a}}_1 + \bar{k}_1 \bar{\mathbf{a}}_3) + \zeta_3 (\bar{k}_2 \bar{\mathbf{a}}_1 - \bar{k}_1 \bar{\mathbf{a}}_2) \right] \\ \left. \frac{d\mathbf{c}_2}{dh} \right|_{h=0} &= \mathbf{R} \bar{\mathbf{a}}_2 \\ \left. \frac{d\mathbf{c}_3}{dh} \right|_{h=0} &= \mathbf{R} \bar{\mathbf{a}}_3 \end{aligned} \quad (16)$$

Therefore, defining the deformation gradient \mathbf{F} as

$$\left. \frac{d\mathbf{c}_i}{dh} \right|_{h=0} = \mathbf{F} \left. \frac{d\bar{\mathbf{c}}_i}{dh} \right|_{h=0}, \quad i = 1, 2, 3 \quad (17)$$

and substituting Eqs. (14) and (16) in Eq. (17), one obtains

$$\begin{aligned} \mathbf{F} \bar{\mathbf{a}}_1 &= \frac{1}{\bar{\kappa}} \left[\mathbf{x}'_o + \mathbf{R}'(\zeta_2 \bar{\mathbf{a}}_2 + \zeta_3 \bar{\mathbf{a}}_3) + (-\zeta_2 \bar{k}_3 + \zeta_3 \bar{k}_2) \mathbf{R} \bar{\mathbf{a}}_1 \right] \\ \mathbf{F} \bar{\mathbf{a}}_2 &= \mathbf{R} \bar{\mathbf{a}}_2 \\ \mathbf{F} \bar{\mathbf{a}}_3 &= \mathbf{R} \bar{\mathbf{a}}_3 \end{aligned} \quad (18)$$

where $\bar{\kappa} := 1 - \zeta_2 \bar{k}_3 + \zeta_3 \bar{k}_2$.

Now the objective is to evaluate the determinant of \mathbf{F} . It represents the current volume of a parallelepiped having unit volume in the initial configuration. To this aim, it is necessary to write in a different form Eq. (18)₁; the comparison of Eqs. (16)₁ with (18)₁, leads one to the following equation

$$\mathbf{F} \bar{\mathbf{a}}_1 = \frac{1}{\bar{\kappa}} [\mathbf{x}' - (-\zeta_3 \bar{k}_1 \mathbf{R} \bar{\mathbf{a}}_2 + \zeta_2 \bar{k}_1 \mathbf{R} \bar{\mathbf{a}}_3)] \quad (19)$$

Calling λ, ν_2, ν_3 the components of \mathbf{x}' with respect to the basis $\{\mathbf{a}_i\}$ (i.e., $\mathbf{x}' = \lambda \mathbf{R} \bar{\mathbf{a}}_1 + \nu_2 \mathbf{R} \bar{\mathbf{a}}_2 + \nu_3 \mathbf{R} \bar{\mathbf{a}}_3$), Eq. (19) becomes

$$\mathbf{F} \bar{\mathbf{a}}_1 = \frac{1}{\bar{\kappa}} \left[\lambda \mathbf{R} \bar{\mathbf{a}}_1 + (\nu_2 + \zeta_3 \bar{k}_1) \mathbf{R} \bar{\mathbf{a}}_2 + (\nu_3 - \zeta_2 \bar{k}_1) \mathbf{R} \bar{\mathbf{a}}_3 \right] \quad (20)$$

Hence, using Eqs. (20) and (18)_{2,3}, the representation of \mathbf{F} on the basis $\{\bar{\mathbf{a}}_i\}$

is

$$[\mathbf{F}]_{\bar{\mathbf{a}}_i} = [\mathbf{R}] \begin{bmatrix} \frac{\lambda}{\bar{\kappa}} & 0 & 0 \\ \frac{\nu_2 + \zeta_3 \bar{k}_1}{\bar{\kappa}} & 1 & 0 \\ \frac{\nu_3 - \zeta_2 \bar{k}_1}{\bar{\kappa}} & 0 & 1 \end{bmatrix} \quad (21)$$

and so

$$\det \mathbf{F} = \frac{\lambda}{\bar{\kappa}} \quad (22)$$

A more explicit expression for Eq. (22) will be provided in the following paragraphs.

2.4 Velocity field

The velocity field is also needed to describe the kinematics of the beam. Eqs. (2) and (10) can be combined to obtain

$$\mathbf{x} - \mathbf{x}_o = \mathbf{R}(\bar{\mathbf{x}} - \bar{\mathbf{x}}_o) \quad (23)$$

Considering that the terms with overbar are time-independent, time differentiation of Eq. (23) provides, using the orthogonality of \mathbf{R} , the expression of the velocity of a generic point of the beam

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_o + \dot{\mathbf{R}}\mathbf{R}^T(\mathbf{x} - \mathbf{x}_o) \quad (24)$$

Hence the velocity field is written as

$$\mathbf{w} = \mathbf{w}_o + \mathbf{W}(\mathbf{x} - \mathbf{x}_o) \quad (25)$$

where $\mathbf{W} := \dot{\mathbf{R}}\mathbf{R}^T$ is the skew angular velocity tensor, $\mathbf{w}(\mathbf{x}) := \dot{\mathbf{x}}$ and $\mathbf{w}_o := \dot{\mathbf{x}}_o$.

2.5 Internal power; stress and strain measures

The expression of the internal power (i.e., the power expended by the internal stress for the gradient of the velocity field) is deduced, in the case of the beam, from the corresponding well-known expression for the 3-D Cauchy continuum, applying the kinematic description of the beam developed in the previous paragraphs.

For a 3-D Cauchy continuum of shape \mathcal{R} , the internal power reads [14]

$$\mathcal{W}^{\text{int}} = - \int_{\mathcal{R}} \mathbf{T} \cdot \mathbf{G} dV \quad (26)$$

where \mathbf{T} is the Cauchy stress tensor, $\mathbf{G} := \nabla \mathbf{w}$ is the gradient of the velocity field, the dot represents the tensor inner product ($\mathbf{T} \cdot \mathbf{G} := \text{tr}(\mathbf{T}^T \mathbf{G})$) and dV is a measure of \mathcal{R} .

In the spirit of the referential description, the integral in Eq. (26) must be evaluated in the reference set $\bar{\mathcal{R}}$. Hence, it becomes

$$\mathcal{W}^{\text{int}} = - \int_{\bar{\mathcal{R}}} \mathbf{T} \cdot \mathbf{G} \det \mathbf{F} d\bar{V} \quad (27)$$

where $d\bar{V}$ is a measure of $\bar{\mathcal{R}}$.

As seen before, the shape $\bar{\mathcal{R}}$ of the beam can be parameterized on $\mathcal{I} \times \mathcal{S}$ and, accordingly, Eq. (27) can be written as

$$\mathcal{W}^{\text{int}} = - \int_{\mathcal{I}} \left[\int_{\mathcal{S}} \mathbf{T} \cdot \mathbf{G} \det \mathbf{F} dA \right] d\zeta_1 \quad (28)$$

where dA is a measure of the area of \mathcal{S} . It can be proved [15] that, stating the principle of material frame-indifference, Eq. (28) is equivalent to

$$\mathcal{W}^{\text{int}} = - \int_{\mathcal{I}} [\mathbf{t} \cdot \mathbf{R}\dot{\boldsymbol{\gamma}} + \mathbf{m} \cdot \mathbf{R}\dot{\boldsymbol{\chi}}] d\zeta_1 \quad (29)$$

where the dot, here, represents the inner product in \mathcal{V} , and

$$\begin{aligned} \mathbf{t} &:= \int_{\mathcal{S}} \mathbf{T} \mathbf{a}_1 dA \\ \mathbf{m} &:= \int_{\mathcal{S}} (\mathbf{x} - \bar{\mathbf{x}}) \times \mathbf{T} \mathbf{a}_1 dA \end{aligned} \quad (30)$$

are respectively the internal force and couple (the symbol \times stands for vector product in \mathcal{V}), and

$$\begin{aligned} \boldsymbol{\gamma} &:= \mathbf{R}^T \mathbf{x}'_o - \bar{\mathbf{x}}'_o \\ \boldsymbol{\chi} &:= \text{ax}(\mathbf{R}^T \mathbf{R}') \end{aligned} \quad (31)$$

are the right strain measures ($\text{ax}(\cdot)$ is the axial vector of a skew tensor).

To express the components of $\boldsymbol{\gamma}$

$$\boldsymbol{\gamma} = \varepsilon \bar{\mathbf{a}}_1 + \gamma_2 \bar{\mathbf{a}}_2 + \gamma_3 \bar{\mathbf{a}}_3 \quad (32)$$

it is useful to introduce the vector $\mathbf{u} \in \mathcal{V}$ describing the translations of the axis curve from the reference configuration and defined such that

$$\mathbf{x}_o = \bar{\mathbf{x}}_o + \mathbf{u} \quad (33)$$

The objective is to express all the kinematic terms as functions of the three components of \mathbf{u} with respect to $\{\bar{\mathbf{a}}_i\}$ and the three angles $\{\vartheta_1, \vartheta_2, \vartheta_3\}$. These six scalar functions are the Lagrangian parameters describing the deformation of the beam.

By differentiating Eq. (33) with respect to ζ_1 and using Eq. (3), one obtains

$$\mathbf{x}'_o = \bar{\mathbf{a}}_1 + \mathbf{u}' \quad (34)$$

Using Eqs. (32) and (34), Eq. (31)₁ becomes

$$\varepsilon \bar{\mathbf{a}}_1 + \gamma_2 \bar{\mathbf{a}}_2 + \gamma_3 \bar{\mathbf{a}}_3 = \mathbf{R}^T \bar{\mathbf{a}}_1 + \mathbf{R}^T \mathbf{u}' - \bar{\mathbf{a}}_1 \quad (35)$$

If $\mathbf{u} = \sum_{i=1}^3 u_i \bar{\mathbf{a}}_i$, the expression for \mathbf{u}' is (see Eq. (4))

$$\mathbf{u}' = \sum_{i=1}^3 \left(u'_i \bar{\mathbf{a}}_i + u_i \bar{\mathbf{K}} \bar{\mathbf{a}}_i \right) \quad (36)$$

and applying the inner product in both the members of Eq. (35) to $\bar{\mathbf{a}}_1$, $\bar{\mathbf{a}}_2$ and $\bar{\mathbf{a}}_3$, alternatively, it follows that

$$\begin{aligned} \varepsilon &= \mathbf{R}^T \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{a}}_1 + \sum_{i=1}^3 \left(u'_i \mathbf{R}_i^T + u_i \mathbf{R}^T \bar{\mathbf{K}} \right) \bar{\mathbf{a}}_i \cdot \bar{\mathbf{a}}_1 - 1 \\ \gamma_2 &= \mathbf{R}^T \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{a}}_2 + \sum_{i=1}^3 \left(u'_i \mathbf{R}_i^T + u_i \mathbf{R}^T \bar{\mathbf{K}} \right) \bar{\mathbf{a}}_i \cdot \bar{\mathbf{a}}_2 \\ \gamma_3 &= \mathbf{R}^T \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{a}}_3 + \sum_{i=1}^3 \left(u'_i \mathbf{R}_i^T + u_i \mathbf{R}^T \bar{\mathbf{K}} \right) \bar{\mathbf{a}}_i \cdot \bar{\mathbf{a}}_3 \end{aligned} \quad (37)$$

Using Eqs. (5) and (12), Eq. (37) eventually becomes

$$\begin{aligned} \varepsilon &= (\bar{k}_2 s_2 + c_2 \bar{k}_3 s_3) u_1 + (-c_2 c_3 \bar{k}_3 - \bar{k}_1 s_2) u_2 + (c_2 c_3 \bar{k}_2 - c_2 \bar{k}_1 s_3) u_3 \\ &\quad + c_2 c_3 (1 + u'_1) + c_2 s_3 u'_2 - s_2 u'_3 - 1 \\ \gamma_2 &= (\bar{k}_3 (c_1 c_3 + s_1 s_2 s_3) - c_2 \bar{k}_2 s_1) u_1 + (c_2 \bar{k}_1 s_1 - \bar{k}_3 (c_3 s_1 s_2 - c_1 s_3)) u_2 \\ &\quad + (\bar{k}_2 (c_3 s_1 s_2 - c_1 s_3) - \bar{k}_1 (c_1 c_3 + s_1 s_2 s_3)) u_3 \\ &\quad + (c_3 s_1 s_2 - c_1 s_3) (1 + u'_1) + (c_1 c_3 + s_1 s_2 s_3) u'_2 + c_2 s_1 u'_3 \\ \gamma_3 &= (\bar{k}_3 (c_1 s_2 s_3 - c_3 s_1) - c_1 c_2 \bar{k}_2) u_1 + (c_1 c_2 \bar{k}_1 - \bar{k}_3 (c_1 c_3 s_2 + s_1 s_3)) u_2 \\ &\quad + (\bar{k}_2 (c_1 c_3 s_2 + s_1 s_3) - \bar{k}_1 (c_1 s_2 s_3 - c_3 s_1)) u_3 \\ &\quad + (c_1 c_3 s_2 + s_1 s_3) (1 + u'_1) + (-c_3 s_1 + c_1 s_2 s_3) u'_2 + c_1 c_2 u'_3 \end{aligned} \quad (38)$$

in which $c_i := \cos \vartheta_i$ and $s_i := \sin \vartheta_i$, $i = 1, 2, 3$.

In Eq. (38), the components of the vector $\boldsymbol{\gamma}$ are expressed in terms of $\mathbf{u} := \{u_1, u_2, u_3\}$, $\boldsymbol{\vartheta} := \{\vartheta_1, \vartheta_2, \vartheta_3\}$ and $\bar{\mathbf{k}} := \{\bar{k}_1, \bar{k}_2, \bar{k}_3\}$.

The components of $\boldsymbol{\chi}$

$$\boldsymbol{\chi} = \chi_1 \bar{\mathbf{a}}_1 + \chi_2 \bar{\mathbf{a}}_2 + \chi_3 \bar{\mathbf{a}}_3 \quad (39)$$

can be expressed in an explicit way from Eq. (31)₂.

To this end, the representation of the tensor $\mathbf{R}^T \mathbf{R}'$ with respect to the vectors $\bar{\mathbf{a}}_i$ ($i = 1, 2, 3$) is needed. In particular, the rotation \mathbf{R} can be written as

$$\mathbf{R} = \sum_{i,j=1}^3 r_{ij} \bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_i \quad (40)$$

where r_{ij} is the (i, j) -th component of the matrix $[\mathbf{R}]$ (Eq. (12)) and \otimes is the tensor product ($\forall \mathbf{v} \in \mathcal{V}$, $(\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_i) \mathbf{v} := (\bar{\mathbf{a}}_j \cdot \mathbf{v}) \bar{\mathbf{a}}_i$).

Using Eq.(4) and the properties of the tensor product, the differentiation of both the members of Eq. (40) with respect to ζ_1 provides

$$\mathbf{R}' = \sum_{i,j=1}^3 (r'_{ij} \bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_i + r_{ij} (\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_i) \bar{\mathbf{K}}^T + r_{ij} \bar{\mathbf{K}} (\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_i)) \quad (41)$$

hence, considering that $\bar{\mathbf{K}}^T = -\bar{\mathbf{K}}$, it follows that

$$[\mathbf{R}^T \mathbf{R}']_{\bar{\mathbf{a}}_i} = [\mathbf{R}]^T [\mathbf{R}'] - [\bar{\mathbf{K}}] + [\mathbf{R}]^T [\bar{\mathbf{K}}] [\mathbf{R}] \quad (42)$$

Thus, the expressions of χ_i are obtained from the matrix on the right hand side of Eq. (42), extracting the axial vector

$$\begin{aligned} \chi_1 &= -\bar{k}_1 + c_2 c_3 \bar{k}_1 - \bar{k}_3 s_2 + c_2 \bar{k}_2 s_3 + \vartheta'_1 - s_2 \vartheta'_3 \\ \chi_2 &= -\bar{k}_2 + c_1 c_3 \bar{k}_2 + c_2 \bar{k}_3 s_1 + c_3 \bar{k}_1 s_1 s_2 - c_1 \bar{k}_1 s_3 + \\ &\quad + \bar{k}_2 s_1 s_2 s_3 + c_1 \vartheta'_2 + c_2 s_1 \vartheta'_3 \\ \chi_3 &= -\bar{k}_3 + c_1 c_2 \bar{k}_3 - c_3 \bar{k}_2 s_1 + c_1 c_3 \bar{k}_1 s_2 + \bar{k}_1 s_1 s_3 + \\ &\quad + c_1 \bar{k}_2 s_2 s_3 - s_1 \vartheta'_2 + c_1 c_2 \vartheta'_3 \end{aligned} \quad (43)$$

These components are explicitly expressed in terms of ϑ and $\bar{\mathbf{k}}$.

Starting from Eq. (22), it can be useful to express the determinant of \mathbf{F} in terms of the strain components. Combining Eqs. (16)₁ and (31)₁, it follows that

$$\mathbf{x}'_o = (1 + \varepsilon) \mathbf{a}_1 + \gamma_2 \mathbf{a}_2 + \gamma_3 \mathbf{a}_3 \quad (44)$$

Hence, evaluating $\lambda = \mathbf{x}' \cdot \mathbf{a}_1$ one gets

$$\lambda = (1 + \varepsilon) + \zeta_2 (\mathbf{R}^T \mathbf{R}' + \bar{\mathbf{K}}) \bar{\mathbf{a}}_2 \cdot \bar{\mathbf{a}}_1 + \zeta_3 (\mathbf{R}^T \mathbf{R}' + \bar{\mathbf{K}}) \bar{\mathbf{a}}_3 \cdot \bar{\mathbf{a}}_1 \quad (45)$$

and after some tensor algebra

$$\lambda = (1 + \varepsilon) - \zeta_2(\chi_3 + \bar{k}_3) + \zeta_3(\chi_2 + \bar{k}_2) \quad (46)$$

Accordingly, an explicit expression for $\det \mathbf{F}$ is derived

$$\det \mathbf{F} = \frac{1 + \varepsilon - \zeta_2(\chi_3 + \bar{k}_3) + \zeta_3(\chi_2 + \bar{k}_2)}{1 - \zeta_2 \bar{k}_3 + \zeta_3 \bar{k}_2} \quad (47)$$

It is worth noticing that, in correspondence to the axis curve, i.e., when considering $\zeta_2 = 0$ and $\zeta_3 = 0$, Eq. (47) reduces to $\det \mathbf{F} = 1 + \varepsilon$, therefore the effect of the initial curvature is present only in the definition of ε .

2.6 Inertia forces

As for the description of the internal stress and strain, the expression of the inertia forces is deduced from the corresponding terms valid for the 3-D Cauchy continuum.

In particular, the expression of the (external) power expended by the inertia forces for the velocity field, for a 3-D Cauchy continuum of shape \mathcal{R} , is

$$\mathcal{W}^{\text{ext}} = - \int_{\mathcal{R}} \rho \ddot{\mathbf{x}} \cdot \mathbf{w} dV \quad (48)$$

where ρ is the volume mass density in the current configuration, $\ddot{\mathbf{x}}$ is the acceleration of a point of the body occupying the position \mathbf{x} , and \mathbf{w} is the velocity field.

The integration in Eq. (48) must be evaluated in the reference shape $\bar{\mathcal{R}}$, so

$$\mathcal{W}^{\text{ext}} = - \int_{\bar{\mathcal{R}}} \rho \ddot{\mathbf{x}} \cdot \mathbf{w} \det \mathbf{F} d\bar{V} \quad (49)$$

Moreover, considering the principle of conservation of mass

$$\bar{\rho} = \rho \det \mathbf{F} \quad (50)$$

$\bar{\rho}$ being the volume mass density in the initial configuration, Eq. (49) becomes

$$\mathcal{W}^{\text{ext}} = - \int_{\bar{\mathcal{R}}} \bar{\rho} \ddot{\mathbf{x}} \cdot \mathbf{w} d\bar{V} \quad (51)$$

From Eq. (23) and by using the orthogonality of \mathbf{R} , the expression for the acceleration is

$$\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_o + \ddot{\mathbf{R}}\mathbf{R}^T(\mathbf{x} - \mathbf{x}_o) \quad (52)$$

Using Eq. (25) and Eq. (52), the external power reads

$$\mathcal{W}^{\text{ext}} = - \int_{\bar{\mathcal{R}}} \bar{\rho} (\ddot{\mathbf{x}}_o + \ddot{\mathbf{R}}\mathbf{R}^T(\mathbf{x} - \mathbf{x}_o)) \cdot (\mathbf{w}_o + \mathbf{W}(\mathbf{x} - \mathbf{x}_o)) d\bar{V} \quad (53)$$

Using the properties of the tensor product, Eq. (53) can be written as

$$\begin{aligned} \mathcal{W}^{\text{ext}} = & - \int_{\bar{\mathcal{R}}} \bar{\rho} [(\ddot{\mathbf{x}}_o + \ddot{\mathbf{R}}\mathbf{R}^T(\mathbf{x} - \mathbf{x}_o)) \cdot \mathbf{w}_o \\ & + [(\mathbf{x} - \mathbf{x}_o) \otimes \ddot{\mathbf{x}}_o + \ddot{\mathbf{R}}\mathbf{R}^T((\mathbf{x} - \mathbf{x}_o) \otimes (\mathbf{x} - \mathbf{x}_o))] \cdot \mathbf{W}] d\bar{V} \end{aligned} \quad (54)$$

Considering $\bar{\mathcal{R}}$ as parameterized on $\mathcal{I} \times \mathcal{S}$ and the axis curve crossing the sections in their centroid ($\int_{\mathcal{S}} (\mathbf{x} - \mathbf{x}_o) dA = \mathbf{0}$), if $\bar{\rho}$ is considered as uniform, Eq. (54) becomes

$$\mathcal{W}^{\text{ext}} = -\bar{\rho} \left[A \int_{\mathcal{I}} \ddot{\mathbf{x}}_o \cdot \mathbf{w}_o d\zeta_1 + \int_{\mathcal{I}} \ddot{\mathbf{R}}\mathbf{R}^T \int_{\mathcal{S}} (\mathbf{x} - \mathbf{x}_o) \otimes (\mathbf{x} - \mathbf{x}_o) dA \cdot \mathbf{W} d\zeta_1 \right] \quad (55)$$

where A is the area of the generic cross-section.

If $\bar{\mathbf{a}}_2$ and $\bar{\mathbf{a}}_3$ are principal axes for the section and I_2, I_3 are the respective principal moments of inertia, Eq. (55) becomes

$$\mathcal{W}^{\text{ext}} = -\bar{\rho} \left[A \int_{\mathcal{I}} \ddot{\mathbf{u}} \cdot \mathbf{w}_o d\zeta_1 + \int_{\mathcal{I}} \ddot{\mathbf{R}} [I_3(\bar{\mathbf{a}}_2 \otimes \bar{\mathbf{a}}_2) + I_2(\bar{\mathbf{a}}_3 \otimes \bar{\mathbf{a}}_3)] \mathbf{R}^T \cdot \mathbf{W} d\zeta_1 \right] \quad (56)$$

being $\ddot{\mathbf{x}}_o = \ddot{\mathbf{u}}$ from Eq. (33).

Thus, the inertia forces are described, by means of Eq. (56), in terms of the six components \mathbf{u} and ϑ . Equation (56) can be written in a more compact form as

$$\mathcal{W}^{\text{ext}} = - \int_{\mathcal{I}} (\mathbf{b} \cdot \dot{\mathbf{u}} + \mathbf{c} \cdot \dot{\boldsymbol{\omega}}) d\zeta_1 \quad (57)$$

where

$$\begin{aligned} \mathbf{b} & := -\bar{\rho} A \ddot{\mathbf{u}} \\ \mathbf{c} & := 2\bar{\rho} \text{ax}(\text{skw}(\ddot{\mathbf{R}} [I_3(\bar{\mathbf{a}}_2 \otimes \bar{\mathbf{a}}_2) + I_2(\bar{\mathbf{a}}_3 \otimes \bar{\mathbf{a}}_3)] \mathbf{R}^T)) \end{aligned} \quad (58)$$

and where $\dot{\mathbf{u}} = \mathbf{w}$, $\dot{\boldsymbol{\omega}} := \text{ax}(\mathbf{W})$.

Therefore, when expressing the inertia forces and couples in terms of volume mass density in the initial configuration $\bar{\rho}$, they do not depend on the initial curvatures.

3 Equations of motion

The objective is to obtain the equations of motion for a beam having double-symmetric cross-section and initial length ℓ , constituted by linear elastic isotropic material, with mass density $\bar{\rho}$ in the reference configuration, carrying possible tip masses m_0 and m_ℓ and subjected to tip forces \mathbf{f}_0 and \mathbf{f}_ℓ , and couples $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_\ell$. The subscripts 0 and ℓ indicate terms relevant to $\zeta_1 = 0$ and $\zeta_1 = \ell$, respectively. The equations of motion are obtained using the Principle of Virtual Power.

The internal power has the form of Eq. (29) and can be written, in scalar form, as

$$\mathcal{W}^{\text{int}} = - \int_{\mathcal{I}} (N\dot{\varepsilon} + T_2\dot{\gamma}_2 + T_3\dot{\gamma}_3 + M_3\dot{\chi}_1 + M_2\dot{\chi}_2 + M_3\dot{\chi}) d\zeta_1 \quad (59)$$

where the act of motion is described by $\dot{\mathbf{u}} := \{\dot{u}_1, \dot{u}_2, \dot{u}_3\}$ and $\dot{\boldsymbol{\vartheta}} := \{\dot{\vartheta}_1, \dot{\vartheta}_2, \dot{\vartheta}_3\}$ after using Eqs. (38) and (43).

The external power contains terms due to distributed and concentrated tip masses and terms due to tip forces and couples

$$\begin{aligned} \mathcal{W}^{\text{ext}} = & - \int_{\mathcal{I}} (\mathbf{b} \cdot \dot{\mathbf{u}} + \mathbf{c} \cdot \dot{\boldsymbol{\omega}}) d\zeta_1 + \mathbf{b}_0 \cdot \dot{\mathbf{u}}_0 + \mathbf{c}_0 \cdot \dot{\boldsymbol{\omega}}_0 + \\ & + \mathbf{b}_\ell \cdot \dot{\mathbf{u}}_\ell + \mathbf{c}_\ell \cdot \dot{\boldsymbol{\omega}}_\ell + \mathbf{f}_0 \cdot \dot{\mathbf{u}}_0 + \mathbf{f}_\ell \cdot \dot{\mathbf{u}}_\ell + \boldsymbol{\mu}_0 \cdot \dot{\boldsymbol{\omega}}_0 + \boldsymbol{\mu}_\ell \cdot \dot{\boldsymbol{\omega}}_\ell \end{aligned} \quad (60)$$

In the following, the case of a clamped-free beam is reported as example. In this case $m_0 = 0$, $\dot{\mathbf{u}}_0 = 0$, and $\dot{\boldsymbol{\omega}}_0 = 0$ are considered. Moreover, a tip force \mathbf{f}_ℓ is assumed parallel to the axis $\mathbf{a}_2(\ell)$, sinusoidally varying in time and having an amplitude f

$$\mathbf{f}_\ell := f \sin(\Omega t) \mathbf{a}_{2\ell} \quad (61)$$

The internal constraints, to describe vanishing of the extension and of the shear deformations, are

$$\begin{aligned} \varepsilon &= 0 \\ \gamma_2 &= 0 \\ \gamma_3 &= 0 \end{aligned} \quad (62)$$

and the corresponding stress quantities $\mathbf{T} := \{N, T_2, T_3\}$ are considered as reactive forces. A linear elastic uncoupled relation is considered to express $\{M_1, M_2, M_3\}$ in terms of the strain components

$$\begin{aligned} M_1 &= GJ\chi_1 \\ M_2 &= EI_2\chi_2 \\ M_3 &= EI_3\chi_3 \end{aligned} \quad (63)$$

For a subsequent Galerkin projection, it is useful to express the internal constraint in weak form by introducing a constraint functional

$$\mathcal{V}^c := \int_{\mathcal{I}} (\varepsilon \tilde{N} + \gamma_2 \tilde{T}_2 + \gamma_3 \tilde{T}_3) d\zeta_1 \quad (64)$$

In order to consider the internal constraint (62), this term needs to vanish for every trial function \tilde{N} , \tilde{T}_2 , \tilde{T}_3 . If the internal constraint is not used, this term is not considered and constitutive relations for \mathbf{T} must be added.

Hence, the boundary value problem governing the dynamics of the beam can be obtained by the vanishing of both the total power and constraint term

$$\begin{aligned} \mathcal{W} &:= \mathcal{W}^{\text{int}} + \mathcal{W}^{\text{ext}} = 0 \\ \mathcal{V}^c &= 0 \end{aligned} \quad (65)$$

for every act of motion, described by the nine independent parameters $\dot{\mathbf{u}}$, $\dot{\vartheta}$, $\tilde{T} := \{\tilde{N}, \tilde{T}_2, \tilde{T}_3\}$, defined in \mathcal{I} , and the six independent boundary parameters \mathbf{u}_ℓ , ϑ_ℓ . Moreover, the six boundary conditions at the clamped end must be added

$$\mathbf{u}(0) = \vartheta(0) = 0 \quad (66)$$

The expressions of the total power and constraint term (65) are manipulated by means of the software Mathematica[®] [16] to obtain the coefficients of $\dot{\mathbf{u}}$, $\dot{\vartheta}$, \tilde{T} and of $\dot{\mathbf{u}}_\ell$, $\dot{\vartheta}_\ell$. The separate vanishing of these coefficients provides the equations of motions and the boundary conditions on the tip section, respectively. The equations of motion appear as

$$\begin{aligned} \mathcal{J}_i(\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, \vartheta, \dot{\vartheta}, \ddot{\vartheta}) + \mathcal{H}_i(\mathbf{u}, \mathbf{u}', \vartheta, \vartheta', \vartheta'', \mathbf{T}, \mathbf{T}', \bar{\mathbf{k}}) &= 0 \quad i = 1, \dots, 6 \\ \varepsilon(\mathbf{u}, \mathbf{u}', \vartheta, \bar{\mathbf{k}}) &= 0 \\ \gamma_2(\mathbf{u}, \mathbf{u}', \vartheta, \bar{\mathbf{k}}) &= 0 \\ \gamma_3(\mathbf{u}, \mathbf{u}', \vartheta, \bar{\mathbf{k}}) &= 0 \end{aligned} \quad (67)$$

where \mathcal{J}_i contain inertia terms while \mathcal{H}_i contain elastic terms. The boundary conditions in $\zeta_1 = 0, \ell$ are of the form

$$\begin{aligned} \mathbf{u}(0) = \vartheta(0) &= 0 \\ \mathcal{K}_i(\mathbf{u}_\ell, \dot{\mathbf{u}}_\ell, \ddot{\mathbf{u}}_\ell, \vartheta_\ell, \dot{\vartheta}_\ell, \ddot{\vartheta}_\ell) + \mathcal{K}_i(\mathbf{u}_\ell, \vartheta_\ell, \vartheta'_\ell, \mathbf{T}_\ell, f, \bar{\mathbf{k}}) &= \mathcal{P}_i \quad i = 1, \dots, 6 \end{aligned} \quad (68)$$

where \mathcal{K}_i contain elastic and parametric-exciting boundary terms while \mathcal{P}_i contain direct-exciting terms. The full expressions of the equations of motion, omitted here for sake of space, are reported in [15].

4 Conclusions

In this work a geometrically non-linear elastic beam, with double-symmetric, rigid planar cross-sections and uniform initial curvature was considered. In Part 1 of this work, the geometry of the beam was defined in terms of the axis curve and directors. The initial curvature parameters were introduced to specify the initial configuration. The description of the deformation and of inertia terms was deduced from a corresponding 3-dimensional Cauchy continuum. The weak form of the equations of motion was obtained. These equations are valid, with slight modifications, both for extensible and inextensible beams, and both for shear-deformable and shear-indeformable beams. The internal constraints were introduced with a relevant power term in the expression of the total power and considering the internal forces as reactive.

Investigations on free and forced dynamics for a case-study, and how these are affected by initial curvatures, are described in Part 2 of this work.

A Expressions of the terms describing the initial configuration

Defining

$$\kappa := \sqrt{\bar{k}_1^2 + \bar{k}_2^2 + \bar{k}_3^2} \quad (\text{A.1})$$

the components of $\bar{\mathbf{a}}_1(\zeta_1)$ on $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are

$$\begin{aligned} \bar{a}_{11}(\zeta_1) &= \frac{1}{\kappa^2} \left[\bar{k}_1^2 + (\bar{k}_2^2 + \bar{k}_3^2) \cos(\kappa\zeta_1) \right] \\ \bar{a}_{12}(\zeta_1) &= \frac{1}{\kappa^3} \left[\kappa \bar{k}_1 \bar{k}_2 (1 - \cos(\kappa\zeta_1)) + \kappa^2 \bar{k}_3 \sin(\kappa\zeta_1) \right] \\ \bar{a}_{13}(\zeta_1) &= \frac{1}{\kappa^3} \left[\kappa \bar{k}_1 \bar{k}_3 (1 - \cos(\kappa\zeta_1)) - \kappa^2 \bar{k}_2 \sin(\kappa\zeta_1) \right] \end{aligned} \quad (\text{A.2})$$

The components of $\bar{\mathbf{a}}_2(\zeta_1)$ on $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are

$$\begin{aligned} \bar{a}_{21}(\zeta_1) &= \frac{1}{\kappa^3} \left[\kappa \bar{k}_1 \bar{k}_2 (1 - \cos(\kappa\zeta_1)) - \kappa^2 \bar{k}_3 \sin(\kappa\zeta_1) \right] \\ \bar{a}_{22}(\zeta_1) &= \frac{1}{\kappa^2} \left[\bar{k}_2^2 + (\bar{k}_1^2 + \bar{k}_3^2) \cos(\kappa\zeta_1) \right] \\ \bar{a}_{23}(\zeta_1) &= \frac{1}{\kappa^3} \left[\kappa \bar{k}_2 \bar{k}_3 (1 - \cos(\kappa\zeta_1)) + \kappa^2 \bar{k}_1 \sin(\kappa\zeta_1) \right] \end{aligned} \quad (\text{A.3})$$

The components of $\bar{\mathbf{a}}_3(\zeta_1)$ on $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are

$$\begin{aligned}\bar{a}_{31}(\zeta_1) &= \frac{1}{\kappa^3} \left[\kappa \bar{k}_1 \bar{k}_3 (1 - \cos(\kappa \zeta_1)) + \kappa^2 \bar{k}_2 \sin(\kappa \zeta_1) \right] \\ \bar{a}_{32}(\zeta_1) &= \frac{1}{\kappa^3} \left[\kappa \bar{k}_2 \bar{k}_3 (1 - \cos(\kappa \zeta_1)) - \kappa^2 \bar{k}_1 \sin(\kappa \zeta_1) \right] \\ \bar{a}_{33}(\zeta_1) &= \frac{1}{\kappa^2} \left[\bar{k}_3^2 + (\bar{k}_1^2 + \bar{k}_2^2) \cos(\kappa \zeta_1) \right]\end{aligned}\quad (\text{A.4})$$

Eventually, the components of $\bar{\mathbf{x}}_o(\zeta_1)$ on $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are

$$\begin{aligned}\bar{x}_{o1}(\zeta_1) &= \frac{1}{\kappa^3} \left[\bar{k}_1^2 \kappa \zeta_1 + (\bar{k}_2^2 + \bar{k}_3^2) \sin(\kappa \zeta_1) \right] \\ \bar{x}_{o2}(\zeta_1) &= \frac{1}{\kappa^3} \left[\kappa \bar{k}_3 (1 - \cos(\kappa \zeta_1)) + \bar{k}_1 \bar{k}_2 (\kappa \zeta_1 - \sin(\kappa \zeta_1)) \right] \\ \bar{x}_{o3}(\zeta_1) &= \frac{1}{\kappa^3} \left[-\kappa \bar{k}_2 (1 - \cos(\kappa \zeta_1)) + \bar{k}_1 \bar{k}_3 (\kappa \zeta_1 - \sin(\kappa \zeta_1)) \right]\end{aligned}\quad (\text{A.5})$$

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