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# INVARIANT TORI FOR THE CUBIC SZEGÖ EQUATION

PATRICK GÉRARD AND SANDRINE GRELLIER

RÉSUMÉ. Nous poursuivons l'étude de l'équation hamiltonienne suivante sur l'espace de Hardy du cercle

$$i\partial_t u = \Pi(|u|^2 u) ,$$

où  $\Pi$  désigne le projecteur de Szegö. Cette équation est un cas modèle d'équation sans aucune propriété dispersive. Dans un travail précédent, nous avons montré qu'elle admettait une paire de Lax et qu'elle était complètement intégrable. Dans cet article, nous construisons les variables action-angle, ce qui nous permet de ramener la résolution explicite de l'équation à un problème de diagonalisation. Une conséquence de cette construction est la solution d'un problème spectral inverse pour les opérateurs de Hankel. Nous établissons également la stabilité des tores invariants correspondants. En outre, des formules explicites de résolution ainsi obtenues, nous déduisons la classification des ondes progressives orbitalement stables et instables.

ABSTRACT. We continue the study of the following Hamiltonian equation on the Hardy space of the circle,

$$i\partial_t u = \Pi(|u|^2 u) ,$$

where  $\Pi$  denotes the Szegö projector. This equation can be seen as a toy model for totally non dispersive evolution equations. In a previous work, we proved that this equation admits a Lax pair, and that it is completely integrable. In this paper, we construct the action-angle variables, which reduces the explicit resolution of the equation to a diagonalisation problem. As a consequence, we solve an inverse spectral problem for Hankel operators. Moreover, we establish the stability of the corresponding invariant tori. Furthermore, from the explicit formulae, we deduce the classification of orbitally stable and unstable traveling waves.

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## 1. INTRODUCTION

1.1. **The cubic Szegő equation.** In the paper [2], we introduced the evolution equation

$$(1) \quad i\partial_t u = \Pi(|u|^2 u) ,$$

posed on the Hardy space of the circle

$$L_+^2 = \left\{ u : u = \sum_{k=0}^{\infty} \hat{u}(k) e^{ik\theta} , \sum_{k=0}^{\infty} |\hat{u}(k)|^2 < +\infty \right\} ,$$

where  $\Pi$  denotes the Szegő projector from  $L^2$  to  $L_+^2$ ,

$$\forall (c_k) \in \ell^2(\mathbb{Z}) , \Pi \left( \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \right) = \sum_{k=0}^{\infty} c_k e^{ik\theta} .$$

If  $L_+^2$  is endowed with the symplectic form

$$\omega(u, v) = 4 \operatorname{Im}(u|v) , \quad (u|v) := \int_{\mathbb{S}^1} u \bar{v} \frac{d\theta}{2\pi} ,$$

this system is formally Hamiltonian, associated to the — densely defined— energy

$$E(u) = \int_{\mathbb{S}^1} |u|^4 \frac{d\theta}{2\pi} .$$

The study of this equation as a toy model of a totally non dispersive Hamiltonian equation is motivated in the introduction of [2], to which we refer for more detail. In [2], we proved that the Cauchy problem for (1) is well-posed in the Sobolev spaces

$$H_+^s = H^s \cap L_+^2$$

for all  $s \geq \frac{1}{2}$ . The unexpected feature of this equation is the existence of a Lax pair, in the spirit of Lax [7] for the Korteweg-de Vries equation, and of Zakharov-Shabat [16] for the one-dimensional cubic nonlinear Schrödinger equation. Let us describe this structure. For every  $u \in H_+^{1/2}$ , we define (see *e.g.* Peller [13], Nikolskii [11]), the Hankel operator of symbol  $u$  by

$$H_u(h) = \Pi(u \bar{h}) , \quad h \in L_+^2 .$$

It is easy to check that  $H_u$  is a  $\mathbb{C}$ -antilinear Hilbert-Schmidt operator and satisfies the following symmetry condition,

$$(H_u(h_1)|h_2) = (H_u(h_2)|h_1) , \quad h_1, h_2 \in L_+^2 .$$

In [2], we proved that there exists a mapping  $u \mapsto B_u$ , valued into  $\mathbb{C}$ -linear skew-symmetric operators on  $L_+^2$ , such that  $u$  is a solution of (1) if and only if

$$(2) \quad \frac{d}{dt} H_u = [B_u, H_u] .$$

An important consequence of this structure is that, if  $u$  is a solution of (1), then  $H_{u(t)}$  is unitarily equivalent to  $H_{u(0)}$ . In particular, the spectrum of the  $\mathbb{C}$ -linear positive self adjoint trace class operator  $H_u^2$  is conserved by the evolution. Moreover, one can prove that

$$(3) \quad B_u = -iT_{|u|^2} + \frac{i}{2}H_u^2 ,$$

where  $T_b$  denotes the Toeplitz operator of symbol  $b$ ,

$$T_b(h) = \Pi(bh) .$$

This special form of  $B_u$  induces another consequence, namely that, for every Borel function  $f$  bounded on the spectrum of  $H_u^2$ , the quantity

$$(4) \quad J[f](u) := (f(H_u^2)(1)|1)$$

is a conservation law. Here  $f(H_u^2)$  is the bounded operator provided by the spectral theorem. Let us mention some particular cases of such conservation laws which are of special interest. If  $\lambda^2$  is an eigenvalue of  $H_u^2$ , denote by  $P$  the orthogonal projector onto the corresponding eigenspace of  $H_u^2$ . Then

$$\|P(1)\|^2 = J[\mathbf{1}_{\{\lambda^2\}}](u) .$$

A special role is also played by

$$(5) \quad J_{2n}(u) = (H_u^{2n}(1)|1) , \quad n \in \mathbb{Z}_+ ,$$

for which  $f(s) = s^n$ , and by their generating function

$$(6) \quad J(x)(u) = 1 + \sum_{n=1}^{\infty} x^n J_{2n}(u) = ((I - xH_u^2)^{-1}(1)|1) .$$

for which  $f(s) = (1 - xs)^{-1}$ . Notice that  $E = 2J_4 - J_2^2$ .

A third consequence of the Lax pair structure is the existence of finite dimensional submanifolds of  $L_+^2$  which are invariant by the flow of (1). By a theorem due to Kronecker [5], the Hankel operator  $H_u$  is of finite rank  $N$  if and only if  $u$  is a rational function of the complex variable  $z$ , with no poles in the unit disc, and of the following form,

$$u(z) = \frac{A(z)}{B(z)} ,$$

with  $A \in \mathbb{C}_{N-1}[z]$ ,  $B \in \mathbb{C}_N[z]$ ,  $B(0) = 1$ ,  $d(A) = N - 1$  or  $d(B) = N$ ,  $A$  and  $B$  have no common factors, and  $B(z) \neq 0$  if  $|z| \leq 1$ . Here  $\mathbb{C}_D[z]$  denotes the class of complex polynomials of degree at most  $D$ , and  $d(A)$  denotes the degree of a polynomial  $A$ . We denote by  $\mathcal{M}(N)$  the set of such functions  $u$ . It is elementary to check that  $\mathcal{M}(N)$  is a  $2N$ -dimensional complex submanifold of  $L_+^2$ . In [2], we proved that the functions  $J_{2n}$ ,  $n = 1, \dots, 2N$ , are in involution on  $\mathcal{M}(N)$ , that their differentials are linearly independent outside a closed subset of measure 0, and that the level sets of  $(J_1, \dots, J_{2N})$  are generically compact in  $\mathcal{M}(N)$ . By the Liouville-Arnold theorem [1], the connected

components of these generic level sets are Lagrangian tori, which are invariant by the flow of (1).

The purpose of this paper is to study these invariant tori in detail by introducing the corresponding action-angle variables. As a consequence, this will provide explicit formulae for the resolution of the Cauchy problem for (1). Notice that similar coordinates were introduced for the Korteweg-de Vries equation by Kappeler-Pöschel [4], and more recently by Kuksin-Perelman [6] as an application of Vey's theorem, and, for the cubic one-dimensional nonlinear Schrödinger equation, by Grébert-Kappeler-Pöschel [3]. Our method here is however completely different, since it is based on specific properties of Hankel operators. We now describe the results in more detail.

**1.2. Action angle variables in the finite rank case.** We denote by  $\mathcal{M}(N)_{\text{gen}}$  the set of  $u \in \mathcal{M}(N)$  such that 1 does not belong to the range of  $H_u$ , and such that the vectors  $H_u^{2k}(1)$ ,  $k = 1, \dots, N$ , are linearly independent. We proved in [2], Theorem 7.1, that  $\mathcal{M}(N)_{\text{gen}}$  is an open subset of  $\mathcal{M}(N)$ , whose complement is of Lebesgue measure 0. Moreover, it can be shown that  $\mathcal{M}(N)_{\text{gen}}$  is the set of  $u$  such that  $H_u^2$  admits exactly  $N$  simple positive eigenvalues  $\lambda_1^2 > \dots > \lambda_N^2$  with the following additional property,

$$(7) \quad \nu_j > 0 \text{ for } j = 1, \dots, N \text{ and } \sum_{j=1}^N \nu_j^2 < 1,$$

where, for each  $j$ , we define the normalization constants

$$(8) \quad \nu_j := \|P_j(1)\|,$$

and where  $P_j$  denotes the orthogonal projector onto the eigenspace  $E_j$  of  $H_u^2$  associated to  $\lambda_j^2$ . Indeed, given an orthonormal basis  $(e_1, \dots, e_N)$  of the range of  $H_u$  such that  $H_u^2 e_j = \lambda_j^2 e_j$ , the modulus of the determinant of the vectors  $H_u^{2k}(1)$ ,  $k = 1, \dots, N$  in this basis is equal to

$$|(1|e_1)| \dots |(1|e_N)| |\det(\lambda_j^{2k})_{1 \leq j, k \leq N}|.$$

Moreover,  $\sum_j \nu_j^2$  is the square of the norm of the orthogonal projection of 1 onto the range of  $H_u$ , hence is  $< 1$  if and only if 1 does not belong to the range of  $H_u$ .

We then define our new variables. The first set of action variables is given by

$$I_j(u) = 2\lambda_j^2, \quad j = 1, \dots, N.$$

We define the first set of angle variables as follows. Using the antilinearity of  $H_u$  there exists an orthonormal basis  $(e_1, \dots, e_N)$  of the range of  $H_u$  such that

$$H_u(e_j) = \lambda_j e_j, \quad j = 1, \dots, N.$$

Notice that the orthonormal system  $(e_1, \dots, e_N)$  is determined by  $u$  up to a change of sign on some of the  $e_j$ , in other words up to the action of  $\{\pm 1\}^N$  acting as a group of isometries. Therefore we can define the angles

$$\varphi_j(u) := \arg(1|e_j)^2 \quad j = 1, \dots, N .$$

Since  $\mathcal{M}(N)_{\text{gen}}$  is a symplectic manifold of real dimension  $4N$ , it remains to define  $N$  other action variables and  $N$  other angle variables. We do the same analysis with the operator  $K_u = H_u T_z$  as the one we did with  $H_u$ . Here  $T_z$  is nothing but the multiplication by  $z$ , namely the shift operator on the Fourier coefficients. We will show that  $K_u^2$ , which turns out to be a self-adjoint positive operator, has  $N$  distinct eigenvalues denoted by  $\mu_1^2 > \mu_2^2 > \dots > \mu_N^2$ . Furthermore, the  $\mu_j^2$  are the  $N$  solutions of the equation in  $\sigma$ ,

$$(9) \quad \sum_{j=1}^N \frac{\lambda_j^2 \nu_j^2}{\lambda_j^2 - \sigma} = 1$$

satisfying

$$(10) \quad \lambda_1^2 > \mu_1^2 > \lambda_2^2 > \mu_2^2 > \dots > \lambda_N^2 > \mu_N^2 > 0.$$

As before, by the antilinearity of  $K_u$  there exists an orthonormal basis  $(f_1, \dots, f_N)$  of the range of  $K_u$  such that

$$K_u(f_m) = \mu_m f_m, \quad m = 1, \dots, N,$$

and  $(f_1, \dots, f_N)$  is determined by  $u$  up to a change of sign on some of the  $f_m$ . We set

$$L_m(u) := 2\mu_m^2, \quad j = 1, \dots, N \text{ and } \theta_m(u) := \arg(u|f_m)^2, \quad m = 1, \dots, N .$$

Define

$$\Omega_N := \{(I_1, \dots, I_N, L_1, \dots, L_N) \in \mathbb{R}^{2N}; I_1 > L_1 > I_2 > \dots > I_N > L_N > 0\} .$$

Our main result reads

**Theorem 1.1.** *The mapping*

$$\chi_N := (I_1, \dots, I_N, L_1, \dots, L_N; \varphi_1, \dots, \varphi_N, \theta_1, \dots, \theta_N)$$

*is a symplectic diffeomorphism from  $\mathcal{M}(N)_{\text{gen}}$  onto  $\Omega_N \times \mathbb{T}^{2N}$ , in the sense that*

$$(11) \quad \chi_{N*}\omega = \sum_{j=1}^N dI_j \wedge d\varphi_j + \sum_{m=1}^N dL_m \wedge d\theta_m$$

As we will see in the proof, a complement to this theorem is an explicit formula giving  $u$  in terms of  $\chi_N(u)$ — see Proposition 3 below. As a first consequence of this result, we obtain an explicit solution to the Cauchy problem for (1) for data in  $\mathcal{M}(N)_{\text{gen}}$ .

**Corollary 1.** *The cubic Szegő equation (1) is equivalent, in the above variables, to the system*

$$\begin{cases} \dot{I}_j = 0, \dot{L}_m = 0 \\ \dot{\varphi}_j = \frac{1}{2}I_j, \dot{\theta}_m = -\frac{1}{2}L_m \end{cases}$$

**1.3. The infinite dimensional case.** Theorem 1.1 and Corollary 1 admit natural generalizations to infinite dimension. In this case, we define the set  $H_{+, \text{gen}}^{1/2}$  as the subset of functions  $u$  in  $H_+^{1/2}$  so that  $H_u^2$  admits only simple positive eigenvalues

$$\lambda_1^2 > \lambda_2^2 > \dots$$

on the closure of its range, and such that, for any  $j \geq 1$ ,

$$\nu_j := \|P_j(1)\| \neq 0.$$

We shall prove that  $H_{+, \text{gen}}^{1/2}$  is a dense  $G_\delta$  set in  $H_+^{1/2}$  and that the motion stays on infinite dimensional invariant tori, leading to almost periodic solutions valued in  $H_+^{1/2}$ . More precisely, denoting by  $(\mu_m^2)_{m \geq 1}$  the sequence of positive eigenvalues of  $K_u^2$ , and observing that

$$\lambda_1^2 > \mu_1^2 > \lambda_2^2 > \mu_2^2 > \dots,$$

we can define as before orthonormal systems  $(e_j)_{j \geq 1}$  and  $(f_m)_{m \geq 1}$ , with

$$H_u(e_j) = \lambda_j e_j, \quad K_u(f_m) = \mu_m f_m.$$

As before, we introduce the following sequences of angles,

$$\varphi_j = \arg(1|e_j)^2, \quad \theta_m = \arg(u|f_m)^2, \quad j, m \geq 1.$$

We then have the following generalization of Theorem 1.1 and of Corollary 1.

**Theorem 1.2.** *The mapping*

$$\chi : u \in H_{+, \text{gen}}^{1/2} \mapsto ((\zeta_j := \lambda_j e^{-i\varphi_j})_{j \geq 1}, (\gamma_m := \mu_m e^{-i\theta_m})_{m \geq 1})$$

*is a homeomorphism onto the subset of  $\ell^2 \times \ell^2$  defined by*

$$\Xi := \{((\zeta_j)_{j \geq 1}, (\gamma_m)_{m \geq 1}) \in \ell^2 \times \ell^2, |\zeta_1| > |\gamma_1| > |\zeta_2| > |\gamma_2| > \dots > 0\}.$$

*Moreover, the evolution of (1) reads through  $\chi$  as*

$$i\dot{\zeta}_j = \lambda_j^2 \zeta_j, \quad i\dot{\gamma}_m = -\mu_m^2 \gamma_m.$$

This theorem is deduced from Theorem 1.1 through an approximation argument by the finite rank case. The convergence of this approximation is a consequence of a compactness result on families of Hankel operators— see Proposition 2 below.

**1.4. Application to inverse problems for Hankel operators.** Theorems 1.1 and 1.2 can be rephrased as solutions to inverse spectral problems on Hankel operators. We denote by  $\mathbf{h}^{1/2}$  the space of sequences  $(c_n)_{n \geq 0}$  of complex numbers such that

$$(12) \quad \sum_{n=0}^{\infty} n |c_n|^2 < \infty ,$$

endowed with its natural norm. Given  $c \in \mathbf{h}^{1/2}$ , we define the operator  $\Gamma_c : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  by

$$\forall x = (x_n)_{n \geq 0} \in \ell^2(\mathbb{N}) , \quad \Gamma_c(x)_n = \sum_{p=0}^{\infty} c_{n+p} x_p .$$

In view of (12), it is clear that  $\Gamma_c$  is Hilbert–Schmidt. We also introduce

$$\tilde{\Gamma}_c := \Gamma_{\tilde{c}}$$

where

$$\forall n \in \mathbb{N} , \quad \tilde{c}_n := c_{n+1} .$$

Our first result concerns the prescription of positive singular values of both  $\Gamma_c$  and  $\tilde{\Gamma}_c$ . Recall that the positive singular values of an operator  $A$  are the positive eigenvalues of the operator  $\sqrt{AA^*}$ .

**Corollary 2.** – *Let  $(\lambda_j)_{1 \leq j \leq N}$ ,  $(\mu_j)_{1 \leq j \leq N}$  be  $N$ -tuples of real numbers satisfying*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_N > \mu_N > 0 .$$

*The set of sequences  $c \in \mathbf{h}^{1/2}$  such that  $\Gamma_c$  has rank  $N$  and admits  $\lambda_j$ ,  $1 \leq j \leq N$ , as simple positive singular values, and such that  $\tilde{\Gamma}_c$  has rank  $N$  and admits  $\mu_j$ ,  $1 \leq j \leq N$ , as simple positive singular values, is a Lagrangian torus of dimension  $2N$ .*

– *Let  $(\lambda_j)_{j \geq 1}$ ,  $(\mu_m)_{m \geq 1}$  be sequences of positive real numbers satisfying*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > 0 , \quad \sum_{j=1}^{\infty} \lambda_j^2 < \infty .$$

*The set of functions  $c \in \mathbf{h}^{1/2}$  such that the positive singular values of  $\Gamma_c$  are  $\lambda_j$ ,  $j \geq 1$ , and are simple, and such that the positive singular values of  $\tilde{\Gamma}_c$  are  $\mu_m$ ,  $m \geq 1$ , and are simple, is an infinite dimensional torus.*

In the particular case of real values sequences  $c$  in  $\mathbf{h}^{1/2}$ ,  $\Gamma_c$  is self-adjoint and Corollary 2 has the following simple reformulation.

**Corollary 3.** – *Let  $\zeta_1, \dots, \zeta_N, \gamma_1, \dots, \gamma_N$  be real numbers such that*

$$|\zeta_1| > |\gamma_1| > |\zeta_2| > |\gamma_2| > \dots > |\zeta_N| > |\gamma_N| > 0 .$$



There exists a unique sequence  $c = (c_n)_{n \geq 0}$  of real numbers such that  $\Gamma_c$  has rank  $N$  with non zero eigenvalues  $\zeta_1, \dots, \zeta_N$ , and such that the selfadjoint operator  $\tilde{\Gamma}_c$  has rank  $N$  with non zero eigenvalues  $\gamma_1, \dots, \gamma_N$ .

– Let  $(\zeta_j)_{j \geq 1}$ ,  $(\gamma_m)_{m \geq 1}$  be two sequences of real numbers such that

$$|\zeta_1| > |\gamma_1| > |\zeta_2| > |\gamma_2| > \dots > 0, \sum_{j=1}^{\infty} \zeta_j^2 < \infty.$$

There exists a unique sequence  $c \in \mathbf{h}^{1/2}$  of real numbers such that the non zero eigenvalues of the selfadjoint operator  $\Gamma_c$  are  $\zeta_j$ ,  $j \geq 1$ , and are simple, and the non zero eigenvalues of the selfadjoint operator  $\tilde{\Gamma}_c$  are  $\gamma_m$ ,  $m \geq 1$ , and are simple.

Notice that, in [14] and [15], Treil proved that any noninvertible nonnegative operator on a Hilbert space, with simple discrete spectrum, and which is either one to one or with infinite dimensional kernel, is unitarily equivalent to the modulus of a Hankel operator. This implies in particular that any decreasing sequence of positive numbers in  $\ell^2$  is the sequence of the positive singular values of a Hilbert-Schmidt Hankel operator. In Corollary 2, we prove that it is possible to prescribe both singular values of  $\Gamma_c$  and of  $\tilde{\Gamma}_c$ , assuming that they are all simple and distinct, and we describe the set of solutions as a torus.

As for Corollary 3, it has to be compared to the result by Megretskii, Peller, Treil [9], who characterized in the widest generality the self-adjoint operators which are unitarily equivalent to Hankel operators. In the special case of Hilbert-Schmidt operators with simple non zero eigenvalues, Corollary 3 establishes that it is possible to impose the spectrum of both  $\Gamma_c$  and  $\tilde{\Gamma}_c$ , and that this completely characterizes the symbol.

Finally, let us emphasize that Corollaries 2 and 3 are completed by an explicit formula which gives the sequences  $c$ , see Remark 3 below.

### 1.5. Stability of invariant tori and instability of traveling waves.

Given  $(I_1, \dots, I_N, L_1, \dots, L_N) \in \Omega_N$ , denote by  $\mathbf{T}(I_1, \dots, I_N, L_1, \dots, L_N)$  the corresponding Lagrangian torus in  $\mathcal{M}(N)_{\text{gen}}$  via  $\chi_N$ . Our next result is a variational characterization of  $\mathbf{T}(I_1, \dots, I_N, L_1, \dots, L_N)$  which implies its stability through the evolution of the cubic Szegö equation, analogously to the result by Lax [8] for KdV. We recall the notation

$$\forall u \in H_+^{1/2}, M(u) = (-i\partial_\theta u|u) = \sum_{k=0}^{\infty} k|\hat{u}(k)|^2.$$

**Theorem 1.3.** For  $n = 1, \dots, 2N$ , define

$$(13) \quad j_{2n} = \sum_{j=1}^N 2^{-n} I_j^n \left(1 - \frac{L_j}{I_j}\right) \prod_{k \neq j} \left(\frac{L_k - I_j}{I_k - I_j}\right).$$

Then  $\mathbf{T}(I_1, \dots, I_N, L_1, \dots, L_N)$  is the set of the solutions in  $H_+^{1/2}$  of the minimization problem

$$\inf\{M(u) : J_{2n}(u) = j_{2n}, n = 1, \dots, 2N\}.$$

Consequently,  $\mathbf{T} := \mathbf{T}(I_1, \dots, I_N, L_1, \dots, L_N)$  is stable under the evolution of (1), in the sense that, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, if

$$\inf_{v \in \mathbf{T}} \|u_0 - v\|_{H^{1/2}} \leq \delta,$$

then the solution  $u$  of (1) with  $u(0) = u_0$  satisfies

$$\sup_{t \in \mathbb{R}} \inf_{v \in \mathbf{T}} \|u(t) - v\|_{H^{1/2}} \leq \varepsilon.$$

Let us mention that there is a similar result for the infinite dimensional tori deduced from Theorem 1.2 — see Remark 4 below.

Our next observation concerns the case  $N = 1$ , where  $\mathbf{T}(I_1, L_1)$  consists exactly of functions

$$(14) \quad u_{\alpha,p}(z) = \frac{\alpha}{1 - pz}$$

where  $|\alpha|$  and  $|p|$  are fixed positive numbers which depend on  $I_1, L_1$ . In [2], it was observed that such functions  $u$  are traveling waves of equation (1), in the sense that there exists  $(\omega, c) \in \mathbb{R}^2$  such that

$$t \mapsto e^{-i\omega t} u_{\alpha,p}(ze^{-ict})$$

is a solution to (1). Moreover, Proposition 5 and Corollary 4 of [2] establish the orbital stability of this traveling wave as the solution of a variational problem, which is exactly the statement of Theorem 1.3 in this case. Therefore it is natural to address the question of orbital stability for all the traveling waves of (1), which were classified in Theorem 1.4 of [2]. The next result gives a complete answer to this question.

**Theorem 1.4.** *If  $u$  is a traveling wave of (1) which is not of the form  $u_{\alpha,p}$  as defined in (14), then  $u$  is orbitally unstable.*

The proof of this theorem is based on the explicit resolution of Equation (1) when the Cauchy data are suitable perturbations of the traveling wave  $u$ .

**1.6. Organization of the paper.** We close this introduction by describing the organization of the paper. In Section 2, we introduce some fundamental tools which will be used in the paper, including the compressed shift operator, a trace formula and a compactness result. In Section 3, we prove Theorem 1.1 on action-angle variables in the finite rank case and its corollary about the explicit solution of (1). Section 4 contains the generalization to infinite dimension stated in Theorem 1.2. Section 5 is devoted to the solution of inverse spectral problems

for Hankel operators as stated in Corollaries 2 and 3. In Section 6, we prove Theorem 1.3 about stability of invariant tori. Finally, Section 7 establishes the orbital instability of traveling waves.

## 2. PRELIMINARIES

**2.1. The compressed shift operator.** We are going to use the well known link between the shift operator and the Hankel operators. Namely, if  $T_z$  denotes the shift operator — the Toeplitz operator of symbol  $z \mapsto z$  —, one can easily check the following identity,

$$(15) \quad H_u T_z = T_z^* H_u.$$

With the notation introduced in the introduction, it reads

$$K_u = T_z^* H_u.$$

Moreover,

$$K_u^2 = H_u T_z T_z^* H_u = H_u (I - (\cdot | 1)) H_u = H_u^2 - (\cdot | u) u .$$

We introduce the compressed shift operator ([11], [12], [13])

$$S := P_u T_z ,$$

where  $P_u$  denotes the orthogonal projector onto the closure of the range of  $H_u$ . By property (15),  $\ker H_u = \ker P_u$  is stable by  $T_z$ , hence

$$S = P_u T_z P_u$$

so that  $S$  is an operator from the closure of the range of  $H_u$  into itself. In the sequel, we shall always denote by  $S$  the induced operator on the closure of the range of  $H_u$ , and by  $S^*$  the adjoint of this operator.

Now observe that operator  $S$  arises in the Fourier series decomposition of  $u$ , namely

$$(16) \quad u(z) = \sum_{n=0}^{\infty} (u|z^n) z^n = \sum_{n=0}^{\infty} (u|T_z^n(1)) z^n = \sum_{n=0}^{\infty} (u|S^n P_u(1)) z^n.$$

As a consequence, we have, for  $|z| < 1$ ,

$$(17) \quad u(z) = (u|(I - \bar{z}S)^{-1} P_u(1)).$$

which makes sense since  $\|S\| \leq 1$ . In the next sections, we shall see how the above formula leads to an inverse formula for the maps  $\chi_N$  and  $\chi$ .

**2.2. A trace formula and a compactness result.** For every integer  $j \geq 1$ , we denote by  $\mathcal{F}_j$  the set of subspaces of  $L_+^2$  of dimension at most  $j$ . Given  $u \in H_+^{1/2}$ , we define  $\lambda_j(u) \geq 0$  by

$$\lambda_j^2(u) = \min_{F \in \mathcal{F}_{j-1}} \max_{h \in F^\perp, \|h\|=1} (H_u^2(h)|h) .$$

The following is a standard fact about nonnegative compact operators.

- If  $H_u^2$  has finite rank  $N$ , then  $\lambda_j^2(u) = 0$  for every  $j > N$ , and  $\lambda_1^2(u) \geq \lambda_2^2(u) \geq \dots \geq \lambda_n^2(u) > 0$  are the positive eigenvalues of  $H_u^2$ , listed according to their multiplicities.
- If  $H_u^2$  has infinite rank, then  $\lambda_1^2(u) \geq \lambda_2^2(u) \geq \dots > 0$  are the positive eigenvalues of  $H_u^2$ , listed according to their multiplicities.

Likewise, we define  $\mu_j(u) \geq 0$  by

$$\mu_j^2(h) = \min_{F \in \mathcal{F}_{j-1}} \max_{h \in F^\perp, \|h\|=1} (K_u^2(h)|h) = \min_{F \in \mathcal{F}_{j-1}} \max_{h \in F^\perp, \|h\|=1} (H_u^2(h)|h) - |(h|u)|^2.$$

From these formulae, it is easy to check that

$$\lambda_1(u) \geq \mu_1(u) \geq \lambda_2(u) \geq \mu_2(u) \geq \dots$$

The following result makes an important connection with function  $J(x)$  introduced in (6).

**Proposition 1.** *For every  $u \in H_+^{1/2}$ , the following identities hold.*

$$(18) \quad \sum_{j=1}^{\infty} \left( \frac{\lambda_j^2}{1 - \lambda_j^2 x} - \frac{\mu_j^2}{1 - \mu_j^2 x} \right) = \frac{J'(x)}{J(x)}, \quad x \notin \left\{ \frac{1}{\lambda_j^2}, \frac{1}{\mu_j^2}, j \geq 1 \right\}.$$

$$(19) \quad J(x) = \prod_{j=1}^{\infty} \frac{1 - \mu_j^2 x}{1 - \lambda_j^2 x}, \quad x \notin \left\{ \frac{1}{\lambda_j^2}, j \geq 1 \right\}.$$

*Proof.* First notice that (19) is a direct consequence of (18) by integration and the fact that  $J(0) = 1$ . It remains to prove (18), which we shall interpret as a trace formula. Indeed, recall that

$$K_u^2(h) = H_u^2(h) - (h|u)u,$$

so that an elementary calculation yields

$$(I - xH_u^2)^{-1}(f) - (I - xK_u^2)^{-1}(f) = \frac{x}{J(x)}(f|(I - xH_u^2)^{-1}(u))(I - xH_u^2)^{-1}(u).$$

Consequently,

$$\text{Tr}((I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}) = \frac{x}{J(x)} \|(I - xH_u^2)^{-1}(u)\|^2.$$

Since, on the one hand,

$$\|(I - xH_u^2)^{-1}(u)\|^2 = ((I - xH_u^2)^{-1}H_u^2(1)|1) = \frac{d}{dx}((I - xH_u^2)^{-1}(1)|1) = J'(x),$$

and on the other hand

$$\begin{aligned} \text{Tr}((I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}) &= x \text{Tr} (H_u^2(I - xH_u^2)^{-1} - K_u^2(I - xK_u^2)^{-1}) \\ &= x \sum_{j=1}^{\infty} \left( \frac{\lambda_j^2}{1 - \lambda_j^2 x} - \frac{\mu_j^2}{1 - \mu_j^2 x} \right), \end{aligned}$$

Formula (18) follows.  $\square$

From the above proposition, we infer the following compactness result, which will be of constant use throughout the paper.

**Proposition 2.** *Let  $(u_p)$  be a sequence of  $H_+^{1/2}$  weakly convergent to  $u$  in  $H_+^{1/2}$ . We assume that*

$$(\lambda_j(u_p))_{j \geq 1} \xrightarrow{p \rightarrow \infty} (\bar{\lambda}_j)_{j \geq 1}, \quad (\mu_j(u_p))_{j \geq 1} \xrightarrow{p \rightarrow \infty} (\bar{\mu}_j)_{j \geq 1} \text{ in } \ell^2,$$

and the following simplicity assumptions :

- If  $j > k$  and  $\bar{\lambda}_j > 0$ , then  $\bar{\lambda}_j > \bar{\lambda}_k$ .
- If  $j > k$  and  $\bar{\mu}_j > 0$ , then  $\bar{\mu}_j > \bar{\mu}_k$ .
- If  $\bar{\lambda}_j > 0$  for some  $j \geq 1$ , then  $\bar{\lambda}_j \neq \bar{\mu}_m$  for every  $m \geq 1$ .

Then, for every  $j \geq 1$ ,  $\lambda_j(u) = \bar{\lambda}_j$ ,  $\mu_j(u) = \bar{\mu}_j$ , and the convergence of  $u_p$  to  $u$  is strong in  $H_+^{1/2}$ .

*Proof.* Firstly, we make a connection between the sequences  $(\bar{\lambda}_j)_{j \geq 1}$ ,  $(\bar{\mu}_j)_{j \geq 1}$  and  $(\lambda_j(u))_{j \geq 1}$ ,  $(\mu_j(u))_{j \geq 1}$  by means of standard functional analysis.

**Lemma 1.** *Let  $(A_p)$  be a sequence of compact selfadjoint nonnegative operators on a Hilbert space  $\mathcal{H}$ , which strongly converges to  $A$ , namely*

$$\forall h \in \mathcal{H}, \quad A_p h \xrightarrow{p \rightarrow \infty} Ah.$$

For every  $j \geq 1$ , denote by  $\mathcal{F}_j$  the set of subspaces of  $\mathcal{H}$  of dimension at most  $j$ , set

$$a_j^{(p)} = \min_{F \in \mathcal{F}_{j-1}} \max_{h \in F^\perp, \|h\|=1} (A_p(h)|h),$$

and assume

$$a_j^{(p)} \rightarrow \bar{a}_j$$

with, if  $j > k$  and  $\bar{a}_j \neq 0$ ,  $\bar{a}_j > \bar{a}_k$ . Then the positive eigenvalues of  $A$  are simple and belong to the limit set  $\{\bar{a}_j\}$ .

*Proof.* Denote by  $(e_j^{(p)})$  an orthonormal basis of  $\ker A_p^\perp$  with  $A_p e_j^{(p)} = a_j^{(p)} e_j^{(p)}$ . For every  $h \in \mathcal{H}$ , we decompose

$$h = \sum_j (h|e_j^{(p)}) e_j^{(p)} + h_0^{(p)}$$

where  $h_0^{(p)} \in \ker A_p$ . Let  $a \in \mathbb{R}$ . Then, passing to the limit in

$$(20) \quad \|(A_p - a)h\|^2 = \sum_j (a_j^{(p)} - a)^2 |(h|e_j^{(p)})|^2 + a^2 \|h_0^{(p)}\|^2,$$

we get

$$\|(A - a)h\| \geq \min(\inf_j |\bar{a}_j - a|, |a|) \|h\|$$

and therefore, if  $a \notin \{\bar{a}_j\} \cup \{0\}$ ,  $a$  is not an eigenvalue of  $A$ . Assume now that  $a = \bar{a}_j$ , and come back to (20). If  $Ah = \bar{a}_j h$ , we infer

$$\sum_{k \neq j} |(h|e_k^{(p)})|^2 + \|h_0^{(p)}\|^2 \rightarrow 0$$

or  $\|h - (h|e_j^{(p)})e_j^{(p)}\|^2 \rightarrow 0$ . Consequently, given eigenvectors  $h_1, h_2$  of  $A$  with eigenvalue  $\bar{a}_j$ , we have

$$|(h_1|h_2)| = \lim |(h_1|e_k^{(p)})| |(h_2|e_k^{(p)})| = \|h_1\| \|h_2\| ,$$

which means that  $\bar{a}_j$  is a simple eigenvalue.  $\square$

Let us return to the proof of Proposition 2. By the Rellich theorem,  $u_p$  tends to  $u$  strongly in  $L_+^2$ , hence, for every  $h \in L_+^2$ , we have

$$(21) \quad H_{u_p}(h) \xrightarrow{p \rightarrow \infty} H_u(h) .$$

Since the norm of  $H_{u_p}$  is bounded by its Hilbert-Schmidt norm, namely the  $H^{1/2}$  norm of  $u_p$ , which is bounded, we conclude that (21) holds uniformly for  $h$  in every compact subset of  $L_+^2$ , hence

$$\forall n \geq 1, H_{u_p}^n(h) \xrightarrow{p \rightarrow \infty} H_u^n(h) .$$

In particular, for every  $n \geq 1$ ,

$$J_{2n}(u_p) := (H_{u_p}^{2n}(1)|1) \xrightarrow{p \rightarrow \infty} (H_u^{2n}(1)|1) := J_{2n}(u) ,$$

and there exists  $C > 0$  such that

$$\forall n \geq 1, \sup_p J_{2n}(u_p) \leq C^n .$$

Choose  $\delta > 0$  such that  $\delta C < 1$ . Then, for every real number  $x$  such that  $|x| < \delta$ , we have, by dominated convergence,

$$J(x)(u_p) := 1 + \sum_{n=1}^{\infty} x^n J_{2n}(u_p) \xrightarrow{p \rightarrow \infty} 1 + \sum_{n=1}^{\infty} x^n J_{2n}(u) := J(x)(u) > 0 .$$

Similarly,

$$J'(x)(u_p) \xrightarrow{p \rightarrow \infty} J'(x)(u) ,$$

and therefore

$$\frac{J'(x)(u_p)}{J(x)(u_p)} \xrightarrow{p \rightarrow \infty} \frac{J'(x)(u)}{J(x)(u)} .$$

On the other hand, in view of the assumption about  $\ell^2$  convergence of  $(\lambda_j(u_p))_{j \geq 1}$  and  $(\mu_j(u_p))_{j \geq 1}$ , we also have, for  $|x| < \delta$ ,

$$\sum_{j=1}^{\infty} \left( \frac{\lambda_j^2(u_p)}{1 - \lambda_j^2(u_p)x} - \frac{\mu_j^2(u_p)}{1 - \mu_j^2(u_p)x} \right) \xrightarrow{p \rightarrow \infty} \sum_{j=1}^{\infty} \left( \frac{\bar{\lambda}_j^2}{1 - \bar{\lambda}_j^2 x} - \frac{\bar{\mu}_j^2}{1 - \bar{\mu}_j^2 x} \right)$$

Using Formula (18) of Lemma 1 above, we infer

$$(22) \quad \sum_{j=1}^{\infty} \left( \frac{\bar{\lambda}_j^2}{1 - \bar{\lambda}_j^2 x} - \frac{\bar{\mu}_j^2}{1 - \bar{\mu}_j^2 x} \right) = \sum_{j=1}^{\infty} \left( \frac{\lambda_j^2(u)}{1 - \lambda_j^2(u)x} - \frac{\mu_j^2(u)}{1 - \mu_j^2(u)x} \right),$$

for  $|x| < \delta$ , and hence for every  $x$  distinct from the poles, by analytic continuation. By the assumption of the proposition, no cancellation can occur in the left hand side of (22), and the poles are all distinct. On the other hand, applying Lemma 1 to  $A_p = H_{u_p}^2$  and to  $A_p = K_{u_p}^2$ , we know that

$$\{\lambda_j^2(u), j \geq 1\} \subset \{\bar{\lambda}_j^2, j \geq 1\}, \quad \{\mu_j^2(u), j \geq 1\} \subset \{\bar{\mu}_j^2, j \geq 1\}$$

and that the multiplicity of positive eigenvalues is 1. Consequently, there is no cancellation in the right hand side of (22) either, and all the poles are simple. We conclude that  $\lambda_j(u) = \bar{\lambda}_j$ ,  $\mu_j(u) = \bar{\mu}_j$  for every  $j \geq 1$ . Moreover,

$$\text{Tr}(H_u^2) = \lim_{p \rightarrow \infty} \text{Tr}(H_{u_p}^2),$$

which, since  $\text{Tr}(H_u^2) \simeq \|u\|_{H^{1/2}}^2$ , implies the strong convergence in  $H^{1/2}$ .  $\square$

### 3. THE ACTION-ANGLE VARIABLES

In this section we prove Theorem 1.1 and its corollaries 1 and 2. The proof of Theorem 1.1 is split into five parts. Firstly, we study the compressed shift operator in connection to the spectral theory of  $K_u^2$ . As a second step, using the compressed shift operator, we prove that the unknown  $u$  can be recovered from  $\chi_N(u)$ , with an explicit formula. The third step is devoted to calculating the Poisson brackets between action functions  $(I, L)$  and angle functions  $(\varphi, \theta)$ , which implies in particular that  $\chi_N$  is a local diffeomorphism. This calculation is achieved thanks to function  $J(x)$ , the Hamiltonian flow of which satisfies a Lax pair structure, as we proved in [2]. The surjectivity of  $\chi_N$  is obtained in the fourth step thanks to a topological argument, while the remaining Poisson brackets are calculated in the fifth step.

#### 3.1. Spectral theory of $K_u^2$ and the compressed shift operator.

As a first step, for  $u \in \mathcal{M}(N)_{\text{gen}}$ , we study the eigenvalues of  $K_u^2$  on the range of  $H_u$ . We first observe that 0 cannot be an eigenvalue. Indeed, otherwise there would exist  $g$  in the range of  $H_u$  such that

$$K_u g = 0 = T_{\bar{z}} H_u g,$$

which means that  $H_u g$  is a non zero constant. This would imply that 1 belongs to the range of  $H_u$ , which contradicts the definition of  $\mathcal{M}(N)_{\text{gen}}$ . On the other hand, if  $g$  is an eigenvector associated to an eigenvalue  $\sigma > 0$ , we have, from the identity  $K_u^2 = H_u^2 - (\cdot | u) u$ ,

$$(23) \quad (H_u^2 - \sigma I)g = (g|u)u.$$

We first claim that  $\sigma$  does not belong to  $\{\lambda_1^2, \dots, \lambda_N^2\}$ . Indeed, assume  $\sigma = \lambda_j^2$  in (23). If  $(g|u) = 0$ , (23) implies that  $g = ke_j$  for some  $k \neq 0$ , therefore  $(g|u) = k\lambda_j(1|e_j)$  and this would contradict the assumption  $\nu_j > 0$  — see (7). If  $(g|u) \neq 0$ , (23) implies that  $u$  belongs to the range of  $H_u^2 - \lambda_j^2 I$ , hence  $u$  is orthogonal to  $e_j$ , which again is in contradiction with the assumption  $\nu_j > 0$ .

Therefore (23) yields

$$g = (g|u)(H_u^2 - \sigma I)^{-1}u ,$$

which is possible if and only if

$$((H_u^2 - \sigma I)^{-1}u|u) = 1 ,$$

or, by decomposing  $u$  on the  $e_j$ 's,

$$\sum_{j=1}^N \frac{\lambda_j^2 \nu_j^2}{\lambda_j^2 - \sigma} = 1 ,$$

which is exactly (9). Notice that, as a function of  $\sigma$ , the left hand side of the above equation increases from  $-\infty$  to  $+\infty$  on each interval between two successive  $\lambda_j^2$ , hence the equation admits exactly  $N$  solutions  $\mu_1^2, \dots, \mu_m^2$ . Summing up, we have proved that the eigenvalues of  $K_u^2$  on the range of  $H_u$  are precisely the  $\mu_m^2$ ,  $m = 1, \dots, N$ , defined by (9), with eigenvectors

$$(24) \quad g_m = (H_u^2 - \mu_m^2 I)^{-1}(u) .$$

Since these eigenvalues are simple, and since  $K_u(g_m)$  is also an eigenvector associated to  $\mu_m^2$ , we have

$$K_u(g_m) = \gamma_m g_m$$

with  $|\gamma_m|^2 = \mu_m^2$ . Then an orthonormal basis  $(f_1, \dots, f_N)$  of the range of  $H_u$  satisfying

$$K_u(f_m) = \mu_m f_m$$

is given by

$$f_m = \frac{\gamma_m^{1/2} g_m}{\mu_m^{1/2} \|g_m\|} ,$$

so that, using that

$$(u|g_m) = (u|(H_u^2 - \mu_m^2 I)^{-1}(u)) = 1 ,$$

in view of (9), we have

$$\theta_m := \arg(u|f_m)^2 = \arg(\bar{\gamma}_m) .$$

Finally, we have proved that

$$K_u(g_m) = \mu_m e^{-i\theta_m} g_m .$$



Next we come to the link with operator  $S$ . Recalling the expression (24) of  $g_m$  and the fact that  $K_u = H_u S$ , we infer, using the injectivity of  $H_u$  on the range of  $H_u$ ,

$$(25) \quad S(g_m) = \mu_m e^{i\theta_m} h_m$$

where

$$h_m := (H_u^2 - \mu_m^2 I)^{-1} P_u(1) .$$

We summarize the above result in the following lemma.

**Lemma 2.** *The sequence  $(g_m)$  defined by (24) is an orthogonal basis of the range of  $H_u$ , on which the compressed shift operator acts as*

$$S(g_m) = \mu_m e^{i\theta_m} h_m , \quad h_m := (H_u^2 - \mu_m^2 I)^{-1} P_u(1) .$$

**3.2. The inverse spectral formula.** We now prove that  $\chi_N$  is one to one, with an explicit formula describing  $u$  in terms of  $\chi_N(u)$ .

**Proposition 3.** *If  $\chi_N(u) = (2\lambda_1^2, \dots, 2\lambda_N^2, 2\mu_1^2, \dots, 2\mu_N^2; \varphi_1, \dots, \varphi_N, \theta_1, \dots, \theta_N)$  then*

$$(26) \quad u(z) = X(I - zA)^{-1} Y$$

where

$$X := (\lambda_j \nu_j e^{-i\varphi_j})_{1 \leq j \leq N} ,$$

$$Y := (\nu_k)^T_{1 \leq k \leq N} ,$$

$A := (A_{j,k})_{1 \leq j,k \leq N}$  is given by

$$A_{j,k} = \sum_{\ell=1}^N \frac{\lambda_k \nu_j \nu_k e^{-i(\varphi_k + \theta_\ell)}}{b_\ell (\lambda_j^2 - \mu_\ell^2) (\lambda_k^2 - \mu_\ell^2)} \mu_\ell ,$$

and

$$(27) \quad \nu_j := \left(1 - \frac{\mu_j^2}{\lambda_j^2}\right)^{1/2} \prod_{k \neq j} \left(\frac{\lambda_j^2 - \mu_k^2}{\lambda_j^2 - \lambda_k^2}\right)^{1/2} ,$$

$$(28) \quad b_\ell = \sum_{j=1}^N \frac{\lambda_j^2 \nu_j^2}{(\lambda_j^2 - \mu_\ell^2)^2} = \frac{1}{\lambda_\ell^2 - \mu_\ell^2} \prod_{k \neq \ell} \frac{\mu_\ell^2 - \mu_k^2}{\mu_\ell^2 - \lambda_k^2} .$$

*Proof.* Our starting point is the formula (17) derived in the last section,

$$u(z) = (u | (I - \bar{z}S)^{-1} P_u(1)) , \quad |z| < 1 .$$

We compute this inner product in the orthonormal basis  $(\tilde{e}_j := e^{i\varphi_j/2} e_j)_{1 \leq j \leq N}$  of the range of  $H_u$ . By definition, we have

$$P_u(1) = \sum_{1 \leq m \leq N} (1 | e_j) e_j = \sum_{1 \leq j \leq N} \nu_j e^{i\varphi_j/2} e_j = \sum_{1 \leq j \leq N} \nu_j \tilde{e}_j$$

and

$$u = H_u(P_u(1)) = \sum_{1 \leq j \leq N} \lambda_j \nu_j e^{-i\varphi_j/2} e_j = \sum_{1 \leq j \leq N} \lambda_j \nu_j e^{-i\varphi_j} \tilde{e}_j .$$

Let us compute  $S(\tilde{e}_k)$ . We expand  $\tilde{e}_k$  in the orthonormal basis  $g_\ell/\|g_\ell\|$ .

$$\tilde{e}_k = \sum_{\ell=1}^N (\tilde{e}_k | g_\ell) \frac{g_\ell}{\|g_\ell\|^2}.$$

Moreover,

$$\|g_\ell\|^2 = \sum_{j=1}^N \frac{\lambda_j^2 \nu_j^2}{(\lambda_j^2 - \mu_\ell^2)^2} := b_\ell.$$

Hence

$$\tilde{e}_k = \sum_{\ell=1}^N \frac{\lambda_k \nu_k e^{i\varphi_k}}{b_\ell (\lambda_k^2 - \mu_\ell^2)} g_\ell$$

and, using Lemma 2,

$$S(\tilde{e}_k) = \sum_{\ell=1}^N \frac{\lambda_k \nu_k e^{i\varphi_k}}{b_\ell (\lambda_k^2 - \mu_\ell^2)} \mu_\ell e^{i\theta_\ell} h_\ell.$$

As  $h_\ell = (H_u^2 - \mu_\ell^2 I)^{-1}(P_u(1))$  we get

$$(S(\tilde{e}_k) | \tilde{e}_j) = \sum_{\ell=1}^N \frac{\lambda_k \nu_k \nu_j e^{i(\varphi_k + \theta_\ell)}}{b_\ell (\lambda_j^2 - \mu_\ell^2) (\lambda_k^2 - \mu_\ell^2)} \mu_\ell.$$

Eventually, we obtain that

$$u(z) = X(I - zA)^{-1}Y$$

where

$$\begin{aligned} X &:= (\lambda_j \nu_j e^{-i\varphi_j})_{1 \leq j \leq N} \\ Y &:= (\nu_k)_{1 \leq k \leq N}^T \end{aligned}$$

and  $A := (A_{j,k})_{1 \leq j, k \leq N}$  with  $A_{j,k} = (\tilde{e}_j | S(\tilde{e}_k))$ .

It remains to compute  $\nu_j$  and  $b_\ell$  in terms of  $\lambda_k, \mu_m$ . To this aim, we shall use the generating function  $J(x)$  defined by (6), which in this case is given by

$$(29) \quad J(x) = 1 + x \sum_{j=1}^N \frac{\lambda_j^2 \nu_j^2}{1 - \lambda_j^2 x} = \prod_{j=1}^N \frac{1 - \mu_j^2 x}{1 - \lambda_j^2 x}.$$

The second identity in (29) is (19). The first one comes from the expansion of  $P_u(1)$  along the orthonormal basis  $(e_1, \dots, e_N)$  :

$$\begin{aligned} J(x) &= ((I - xH_u^2)^{-1}(1) | 1) \\ &= \|1 - P_u(1)\|^2 + (((I - xH_u^2)^{-1}(P_u(1)) | P_u(1)) \\ &= 1 - \sum_{j=1}^N \nu_j^2 + \sum_{j=1}^N \frac{\nu_j^2}{1 - \lambda_j^2 x}. \end{aligned}$$

Notice that these identities are valid for all complex values of  $x$ , except the poles  $\lambda_j^{-2}$ ,  $j = 1, \dots, N$ . The value of  $\nu_j^2$  is then obtained by computing the residue of  $J(x)$  at the pole  $1/\lambda_j^2$ , while the value of  $b_\ell$  is given by

$$b_\ell = \frac{1}{\mu_\ell^4} J' \left( \frac{1}{\mu_\ell^2} \right) .$$

□

We shall now prove that  $\chi_N$  is a diffeomorphism from  $\mathcal{M}(N)_{\text{gen}}$  onto  $\Omega \times \mathbb{T}^{2N}$ . The first step is to prove that  $\chi_N$  is a local diffeomorphism. This will be a consequence of a first set of identities on the Poisson brackets of the actions and the angles.

**3.3. First commutation identities.** First we recall some standard definitions. Given a smooth real-valued function  $F$  on a finite dimensional symplectic manifold  $(\mathcal{M}, \omega)$ , the Hamiltonian vector field of  $F$  is the vector field  $X_F$  on  $\mathcal{M}$  defined by

$$\forall m \in \mathcal{M}, \forall h \in T_m \mathcal{M}, dF(m).h = \omega(h, X_F(m)) .$$

Given two smooth real valued functions  $F, G$ , the Poisson bracket of  $F$  and  $G$  is

$$\{F, G\} = dG.X_F = \omega(X_F, X_G) .$$

The above identity is generalized to complex valued functions  $F, G$  by  $\mathbb{C}$ -bilinearity.

**Proposition 4.** *For any  $j, k \in \{1, \dots, N\}$ , one has*

$$\begin{aligned} \{2\lambda_j^2, \varphi_k\} &= \delta_{jk} \quad , \quad \{2\mu_j^2, \varphi_k\} = 0 \quad , \\ \{2\lambda_j^2, \theta_k\} &= 0 \quad , \quad \{2\mu_j^2, \theta_k\} = \delta_{jk} . \end{aligned}$$

In order to compute for instance  $\{2\mu_j^2, \theta_k\}$  one has to differentiate  $\theta_k$  along the direction of  $X_{\mu_j^2}$ . As the expression of  $X_{\mu_j^2}$  is fairly complicated, we use the "Szegö hierarchy", formed by the sequence of functions  $J_{2n}$ , which we studied in [2]. More precisely, we use the generating function  $J(x)$  given by (29). In the sequel, we shall restrict ourselves to real values of  $x$ , so that  $J(x)$  is a real valued function.

We proved in [2] that the Hamiltonian flow associated to  $J(x)$  as a function of  $u$  has a Lax pair, which we recall in the next statement. We set

$$w(x) := (I - xH_u^2)^{-1}(1) .$$

**Theorem 3.1** (Szegö hierarchy [2], Theorem 8.1 and Corollary 8). *Let  $s > \frac{1}{2}$ . The map  $u \mapsto J(x)$  is smooth on  $H_+^s$  and its Hamiltonian vector field is given by*

$$(30) \quad X_{J(x)}(u) = \frac{x}{2i} w(x) H_u w(x) .$$

Moreover, the equation

$$(31) \quad \partial_t u = X_{J(x)}(u)$$

is equivalent to

$$(32) \quad \partial_t H_u = [B_u^x, H_u] ,$$

with

$$B_u^x(h) = \frac{x}{4i} \left( w(x)\Pi(\overline{w(x)}h) + xH_u w(x)\Pi(\overline{H_u(w)}(x)h) - x(h|H_u w(x))H_u w(x) \right) .$$

**Remark 1.** Notice that, since  $B_u^x$  is skew-adjoint if  $x$  is real, we infer that the spectrum of  $H_u$  is conserved by the Hamiltonian flow of  $J(x)$ . Moreover, since

$$(33) \quad B_u^x(1) = \frac{xJ(x)}{4i} w(x) ,$$

we also deduce that the spectral measure of  $H_u^2$  associated to vector 1 is invariant. Since, by (9), the  $\mu_m^2$  are the solutions in  $\sigma$  of the equation

$$((H_u^2 - \sigma I)^{-1} H_u^2(1)|1) = 1,$$

we conclude that the  $\mu_m^2$ 's are also invariant. We infer that the Poisson brackets of  $J(x)$  with  $\lambda_j^2$  or  $\mu_m^2$  are zero, which implies, in view of the expression (19), that the brackets of  $\lambda_k^2$  or  $\mu_\ell^2$  with  $\lambda_j^2$  or  $\mu_m^2$  are zero.

Thanks to this theorem, we can compute the Poisson brackets of  $J(x)$  with the angles  $\varphi_j$ . The result is stated in the following lemma.

**Lemma 3.**

$$\{J(x), \varphi_j\} = \frac{1}{2} \frac{xJ(x)}{1 - \lambda_j^2 x} .$$

*Proof.* Let us make  $e_j$  evolve according to the Hamiltonian flow of  $J(x)$ . Taking the derivative of  $H_u(e_j) = \lambda_j e_j$ , we get

$$\begin{aligned} \lambda_j \dot{e}_j &= [B_u^x, H_u](e_j) + H_u(\dot{e}_j) \\ &= \lambda_j B_u^x(e_j) - H_u(B_u^x e_j) + H_u(\dot{e}_j) \end{aligned}$$

Hence,  $(H_u - \lambda_j I)(\dot{e}_j - B_u^x e_j) = 0$ , as by assumption  $\ker(H_u - \lambda_j I) = \mathbb{R}e_j$ , there exists  $C_j \in \mathbb{R}$  so that

$$\dot{e}_j = B_u^x e_j + C_j e_j.$$

Using that  $\text{Re}(\dot{e}_j|e_j) = 0$  as  $e_j$  is normalized, and observing that  $i(B_u^x e_j|e_j)$  is real-valued because of the skew-symmetry of  $B_u^x$ , we obtain  $C_j = 0$ . Eventually, we have

$$\dot{e}_j = B_u^x e_j$$

and

$$-4i(1|\dot{e}_j) = (1|4iB_u^x(e_j)) = (4iB_u^x(1)|e_j) = xJ(x)(w(x)|e_j) = \frac{xJ(x)}{1 - \lambda_j^2 x}(1|e_j) .$$

As a consequence

$$\dot{\varphi}_j = \frac{d}{dt} \arg(1|e_j)^2 = \frac{1}{2} \frac{xJ(x)}{1 - \lambda_j^2 x} .$$

□

To compute the bracket with  $\theta_m$ , we are going to use the same method but we have to replace the Hankel operator  $H_u$  by the shifted Hankel operator  $K_u$ . We first establish that there is also a Lax pair associated to  $K_u$ . We obtain it as a corollary of Theorem 3.1.

**Corollary 4.** *The equation*

$$(34) \quad \partial_t u = X_{J(x)}(u) .$$

implies

$$(35) \quad \partial_t K_u = [C_u^x, K_u] ,$$

with

$$C_u^x(h) = \frac{x}{4i} \left( w(x)\Pi(\overline{w(x)h}) + xH_u w(x)\Pi(\overline{H_u(w)(x)h}) \right) .$$

*Proof.* One computes, by using Theorem 3.1,

$$\begin{aligned} \partial_t K_u &= \partial_t(H_u T_z) = [B_u^x, H_u]T_z = B_u^x K_u - H_u B_u^x T_z \\ &= B_u^x K_u - K_u B_u^x + H_u[z, B_u^x] . \end{aligned}$$

By the formula of  $B_u^x$  given in Theorem 3.1, and by the elementary identity

$$\forall g \in L^2, \quad \Pi(zg) - z\Pi(g) = (zg|1),$$

we have

$$[z, B_u^x](h) = -\frac{x}{4i} ((zh|w)w + x(h|H_u w)zH_u w) ,$$

so that

$$\begin{aligned} H_u[z, B_u^x](h) &= \frac{x}{4i} ((w-1|zh)H_u w + x(H_u w|h)K_u H_u w) \\ &= \frac{x}{4i} ((xH_u^2 w|zh)H_u w + x(H_u w|h)K_u H_u w) \\ &= \frac{x}{4i} (x(K_u H_u w|h)H_u w + x(H_u w|h)K_u H_u w) \\ &= \frac{x^2}{4i} ((K_u h|H_u w)H_u w + (H_u w|h)K_u H_u w) \\ &= [D_u^x, K_u](h) \end{aligned}$$

where

$$(36) \quad D_u^x(h) = \frac{x^2}{4i} (h|H_u w)H_u w .$$

Here we have used that, by definition of  $w$ ,  $w-1 = xH_u^2 w$ . Coming back to the above expression for the derivative of  $K_u$ , we obtain the claimed formula with  $C_u^x = B_u^x + D_u^x$ . This completes the proof. □

This Lax pair allow us to obtain the analogous of Lemma 3.

**Lemma 4.**

$$\{J(x), \theta_m\} = -\frac{1}{2} \frac{xJ(x)}{1 - \mu_m^2 x}.$$

*Proof.* Let us look at the evolution of  $f_m$  under the flow of  $X_{J(x)}$ . Let us take the derivative of the equation  $\mu_m f_m = K_u(f_m)$ . We get, using the same arguments as for  $\dot{e}_j$ , that  $\dot{f}_m = C_u^x f_m$ . We obtain

$$\begin{aligned} \frac{d}{dt}(u|f_m) &= (\dot{u}|f_m) + (u|\dot{f}_m) \\ &= ([B_u^x, H_u](1)|f_m) + (u|C_u^x f_m) \\ &= (B_u^x(u)|f_m) - (H_u \left( \frac{xJ(x)}{4i} w \right) |f_m) - (C_u^x u|f_m) \\ &= -(D_u^x(u)|f_m) + \frac{xJ(x)}{4i} (H_u w|f_m) \end{aligned}$$

Using the above Formula 36 for  $D_u^x$ , we get

$$\begin{aligned} \frac{d}{dt}(u|f_m) &= -\frac{x^2}{4i} (H_u^2 w|1) (H_u w|f_m) + \frac{xJ(x)}{4i} (H_u w|f_m) \\ &= -\frac{x}{4i} (w-1|1) (H_u w|f_m) + \frac{xJ(x)}{4i} (H_u w|f_m) \\ &= \frac{x}{4i} (H_u w|f_m) = \frac{x}{4i} (u|f_m) (H_u w|g_m). \end{aligned}$$

At this stage we observe that

$$(H_u^2 - \mu_m^2 I)^{-1} H_u w = (H_u^2 - \mu_m^2 I)^{-1} (I - x H_u^2)^{-1} u = \frac{1}{1 - \mu_m^2 x} (g_m + x H_u w).$$

This yields

$$\begin{aligned} (H_u w|g_m) &= ((H_u^2 - \mu_m^2 I)^{-1} H_u(w)|u) \\ &= \frac{1}{1 - \mu_m^2 x} (1 + x(H_u w|u)) = \frac{J(x)}{1 - \mu_m^2 x}, \end{aligned}$$

it implies

$$\frac{d}{dt}(u|f_m) = (u|f_m) \frac{x}{4i} \frac{J(x)}{1 - \mu_m^2 x}$$

and eventually

$$\frac{d}{dt} \arg(u|f_m)^2 = \frac{d}{dt} \theta_m = -\frac{x}{2} \frac{J(x)}{1 - \mu_m^2 x}.$$

□

From Lemma 3 and Lemma 4 above, we easily deduce Proposition 4. Indeed, from formula (29), we have

$$\begin{aligned} \{J(x), \varphi_j\} &= J(x) \sum_{k=1}^N \left( \frac{x\{\lambda_k^2, \varphi_j\}}{1 - \lambda_k^2 x} - \frac{x\{\mu_k^2, \varphi_j\}}{1 - \mu_k^2 x} \right), \\ \{J(x), \theta_m\} &= J(x) \sum_{k=1}^N \left( \frac{x\{\lambda_k^2, \theta_m\}}{1 - \lambda_k^2 x} - \frac{x\{\mu_k^2, \theta_m\}}{1 - \mu_k^2 x} \right), \end{aligned}$$

and the result follows from the comparison with the results of Lemma 3 and Lemma 4.

**Corollary 5.** *The mapping  $\chi_N$  is a local diffeomorphism.*

*Proof.* Let us prove that the tangent map of  $\chi$  is invertible. Assume that there exist  $(\alpha_j)_{1 \leq j \leq 2N}$  and  $(\beta_j)_{1 \leq j \leq 2N}$  so that

$$\sum_{j=1}^N \alpha_j dI_j + \sum_{j=1}^N \alpha_{N+j} dL_j + \sum_{j=1}^N \beta_j d\varphi_j + \sum_{j=1}^N \beta_{N+j} d\theta_j = 0.$$

Since, by Remark 1 and Proposition 4,  $\{I_j, I_k\} = 0$ ,  $\{L_j, I_k\} = 0$ ,  $\{\theta_j, I_k\} = 0$  and  $\{\varphi_j, I_k\} = -\delta_{jk}$ , by applying the above identity to  $X_{I_k}$ , we get  $\beta_k = 0$  for  $k = 1, \dots, N$ . Doing the same with  $X_{L_k}$ , we get  $\beta_k = 0$  for  $k = N+1, \dots, 2N$ . Applying this identity to  $X_{\varphi_k}$  and then to  $X_{\theta_k}$ , we get  $\alpha_k = 0$ ,  $k = 1, \dots, 2N$ , this completes the proof.  $\square$

**3.4. The surjectivity of the mapping  $\chi_N$ .** In view of the inverse formula of Proposition 17, the surjectivity of  $\chi_N$  is equivalent to the fact that, for every  $(I, L, \varphi, \theta)$  in  $\Omega_N \times \mathbb{T}^{2N}$ , if  $u$  is the right hand side of (26), then  $\chi_N(u) = (I, L, \varphi, \theta)$ . Though the formulae are explicit, this fact is far from trivial and will lead to heavy calculations. Therefore we shall use another approach. Indeed we already know from Corollary 5 that  $\chi_N$  is an open mapping. Since  $\Omega_N \times \mathbb{T}^{2N}$  is connected, it suffices to prove that  $\chi_N$  is proper hence closed to obtain that it is onto. Let us take a sequence  $(I^{(p)}, L^{(p)}, \varphi^{(p)}, \theta^{(p)})$  in  $\Omega_N \times \mathbb{T}^{2N}$  which converges to  $(I, L, \varphi, \theta) \in \Omega_N \times \mathbb{T}^{2N}$ , and such that, for every  $p$ , there exists  $u_p \in \mathcal{M}(N)_{\text{gen}}$  such that

$$\chi_N(u_p) = (I^{(p)}, L^{(p)}, \varphi^{(p)}, \theta^{(p)}).$$

Since

$$\|u_p\|_{H^{1/2}}^2 = \sum_{j=1}^N (\lambda_j^{(p)})^2 = \frac{1}{4} \sum_{j=1}^N (I^{(p)})^2$$

$(u_p)$  is a bounded sequence in  $H_+^{1/2}$ . Up to extracting a subsequence, we may assume that  $(u_p)_{p \in \mathbb{N}}$  converges weakly to some  $u$  in  $H_+^{1/2}$ . At this stage we can appeal to Proposition 2 and conclude that the convergence of  $u_p$  to  $u$  is strong and that

$$2\lambda_j^2(u) = I_j, \quad j = 1, \dots, N, \quad 2\mu_m^2(u) = L_m, \quad m = 1, \dots, N,$$

with  $\lambda_j(u) = 0$  if  $j > N$ ,  $\mu_m(u) = 0$  if  $m > N$ . Therefore  $u \in \mathcal{M}(N)_{\text{gen}}$ . This completes the proof of the surjectivity of  $\chi_N$ .

**3.5. The remaining commutation identities.** At this stage we proved that  $\chi_N$  is a global diffeomorphism. We are going to show that it is symplectic. In view of Proposition 4, it suffices to prove that the Poisson brackets of  $\{\varphi_j, \varphi_k\}$ ,  $\{\theta_k, \theta_{k'}\}$  and  $\{\varphi_j, \theta_\ell\}$  cancel. We first remark that, thanks to the first commutations properties and to the Jacobi identity, these brackets are functions of the actions  $(I, L)$  only. Indeed, applying

$$\{f, \{\varphi_j, \varphi_k\}\} + \{\varphi_j, \{\varphi_k, f\}\} + \{\varphi_k, \{f, \varphi_j\}\} = 0$$

to  $f = I_\ell$  and  $f = L_m$ , we obtain, in view of Proposition 4,

$$\{I_\ell, \{\varphi_j, \varphi_k\}\} = \{L_m, \{\varphi_j, \varphi_k\}\} = 0.$$

Writing  $\{\varphi_j, \varphi_k\} = g(I, L, \varphi, \theta)$ , we infer, by Remark 1 and Proposition 4,

$$\frac{\partial g}{\partial \varphi_\ell} = \frac{\partial g}{\partial \theta_m} = 0.$$

The same holds for  $\{\theta_k, \theta_{k'}\}$  and  $\{\varphi_j, \theta_\ell\}$ .

We now prove the remaining commutation laws by first establishing the following result. Recall that  $J_n(u) := (H_u^n(1)|1)$ .

**Lemma 5.** *One has  $\{J_3, J_1\} = -\frac{i}{2}J_1^2$ .*

*Proof.* From the definition of  $J_1$ , one has  $J_1(u) = (u|1)$  so that  $dJ_1(u)(h) = (h|1)$ . On the other hand,  $J_3(u) = (H_u^2(u)|1)$  so that

$$dJ_3(u)(h) = (H_h H_u^2(1) + H_u H_h H_u(1) + H_u^2 H_h(1)|1) = 2(h|H_u^2(1)) + (u^2|h).$$

As  $dJ_3(u)(h) = 4\text{Im}(h|X_{\text{Re}J_3}) + 4i\text{Im}(h|X_{\text{Im}J_3})$ , it implies that

$$\begin{aligned} X_{\text{Re}J_3} &= -\frac{i}{2}H_u^2(1) - \frac{i}{4}u^2 \\ X_{\text{Im}J_3} &= \frac{1}{2}H_u^2(1) - \frac{1}{4}u^2. \end{aligned}$$

Thus, one obtains

$$\{J_3, J_1\} = dJ_1(X_{\text{Re}J_3}) + idJ_1(X_{\text{Im}J_3}) = -\frac{i}{2}J_1^2 \text{ and the lemma is proved.}$$

□

As a corollary, we get the following commutation laws.

**Corollary 6.** *For any  $j, k$ ,  $\{\varphi_j, \varphi_k\} = 0$ .*

*Proof.* From the definitions of  $J_1$  and  $J_3$ , we have  $J_1 = \sum_j \lambda_j \nu_j^2 e^{-i\varphi_j}$  and  $J_3 = \sum_k \lambda_k^3 \nu_k^2 e^{-i\varphi_k}$  so that

$$\{J_3, J_1\} = \sum_{j,k} e^{-i(\varphi_j + \varphi_k)} [-i\{\lambda_k^3 \nu_k^2, \varphi_j\} \lambda_j \nu_j^2 + i\{\lambda_j \nu_j^2, \varphi_k\} \lambda_k^3 \nu_k^2 - \{\varphi_j, \varphi_k\} \lambda_j \nu_j^2 \lambda_k^3 \nu_k^2].$$



On the other hand, by Lemma 5, one also has

$$\{J_3, J_1\} = -\frac{i}{2}J_1^2 = -\frac{i}{2} \sum_{j,k} \lambda_j \lambda_k \nu_j^2 \nu_k^2 e^{-i(\varphi_j + \varphi_k)} .$$

As the commutators  $\{\varphi_j, \varphi_k\}$ ,  $\{\lambda_k \nu_k^2, \varphi_j\}$  and  $\{\lambda_k^3 \nu_k^3, \varphi_j\}$  only depend on the actions  $(I, L)$ , we can identify the Fourier coefficients of the function  $\{J_3, J_1\}$  as a trigonometric polynomial in the angle variables. We focus on the Fourier coefficient for  $j \neq k$ . Since  $\{\lambda_k, \varphi_j\} = 0$ , one gets

$$(37) \quad (\lambda_k^2 - \lambda_j^2) \left[ -\{\varphi_j, \varphi_k\} + i \left( \frac{\{\nu_j^2, \varphi_k\}}{\nu_j^2} - \frac{\{\nu_k^2, \varphi_j\}}{\nu_k^2} \right) \right] = -i .$$

Taking the real part of both sides, we conclude

$$\{\varphi_j, \varphi_k\} = 0 .$$

□

We now compute the commutation laws between the  $\varphi_j$ 's and the  $\theta_k$ 's. We shall make use of the functionals  $N_{2n+1}(u) = (zu|H_u^{2n}(1))$ . Recall that the operator  $K_u^2$  has the  $\mu_k^2$ 's as eigenvalues with associated eigenfunctions  $g_k = (H_u^2 - \mu_k^2 I)^{-1}u$ , with  $\|g_k\|^2 = b_k = \sum_j \frac{\lambda_j^2 \nu_j^2}{(\lambda_j^2 - \mu_k^2)^2}$ . Hence, by Formula (25),

$$P_u(zu) = \sum_k \frac{1}{b_k} S(g_k) = \sum_k \frac{\mu_k e^{i\theta_k}}{b_k} h_k$$

where  $h_k = (H_u^2 - \mu_k^2 I)^{-1}P_u(1)$ . Hence we have

$$N_{2n+1}(u) = \sum_k \frac{\mu_k e^{i\theta_k}}{b_k} (h_k | H_u^{2n}(1)) = \sum_k \frac{\mu_k e^{i\theta_k}}{b_k} P_n(\mu_k)$$

where  $P_n(\mu) = \sum_j \frac{\lambda_j^{2n} \nu_j^2}{\lambda_j^2 - \mu^2}$ . We first compute the commutator of  $N_3$  with  $J_1$ .

**Lemma 6.** *One has  $\{N_3, J_1\} = 0$ .*

*Proof.* As  $N_3(u) = (zu|H_u^2(1))$ , one has

$$\begin{aligned} dN_3(u)(h) &= (zh|H_u^2(1)) + (zu|H_u(h) + H_h(u)) \\ &= 2(h|H_u(zu)) + (zu^2|h). \end{aligned}$$

So, one gets

$$\begin{aligned} X_{\text{Re}N_3} &= -\frac{i}{2}H_u(zu) - \frac{i}{4}zu^2 , \\ X_{\text{Im}N_3} &= \frac{1}{2}H_u(zu) - \frac{1}{4}zu^2 . \end{aligned}$$

It implies that

$$\{N_3, J_1\} = dJ_1 \cdot X_{\text{Re}N_3} + idJ_1 \cdot X_{\text{Im}N_3} = -\frac{i}{2}(zu^2|1) = 0.$$

□

As a corollary, one gets

**Corollary 7.** *For any  $k$  and  $j$ ,  $\{\theta_k, \varphi_j\} = 0$ .*

*Proof.* The proof follows the same lines as before. One writes that  $\{N_3, J_1\} = 0$ . One has

$$\begin{aligned} 0 &= \{N_3, J_1\} = \sum_{j,k} \left\{ \frac{1}{b_k} \mu_k e^{i\theta_k}, \lambda_j \nu_j^2 e^{-i\varphi_j} \right\} \\ &= \sum_{j,k} e^{i(\theta_k - \varphi_j)} \left[ -i \left( \left\{ \frac{\mu_k}{b_k}, \varphi_j \right\} + \{ \lambda_j \nu_j^2, \theta_k \} \right) + \frac{\mu_k}{b_k} \lambda_j \nu_j^2 \{ \theta_k, \varphi_j \} \right]. \end{aligned}$$

By cancelling the real part of the Fourier coefficient, one gets the result. □

By computing the commutator of  $N_3$  and  $N_5$ , one gets as well that  $\{\theta_k, \theta_{k'}\} = 0$ . Let us give the proof for completeness. As  $N_5(u) = (zu|H_u^4(1))$ , we have

$$\begin{aligned} dN_5(u)(h) &= (zh|H_u^4(1)) + (zu|H_h H_u^3(1) + H_u H_h H_u^2(1) + \\ &\quad + H_u^2 H_h H_u(1) + H_u^3(h)) \\ &= (h|H_u(zH_u^2(u)) + H_u(u)H_u(zu) + H_u^3(zu)) + \\ &\quad + (zuH_u^3(1) + H_u^2(zu)u|h). \end{aligned}$$

So, using the expression of  $X_{\text{Re}N_3}$  and of  $X_{\text{Im}N_3}$ , we get

$$\begin{aligned} \{N_3, N_5\} &= dN_5(u)(X_{\text{Re}N_3}) + idN_5(u)(X_{\text{Im}N_3}) \\ &= -\frac{i}{2}(zu^2|H_u(zH_u^2(u)) + H_u(u)H_u(zu) + H_u^3(zu)) \\ &\quad + i(zuH_u^2(u) + uH_u^2(zu)|H_u(zu)) \\ &= -\frac{i}{2} [(zH_u^2(u) + H_u^2(zu)|H_u(zu^2)) + (zu^2|H_u(u)H_u(zu))] \\ &\quad + i(zuH_u^2(u) + uH_u^2(zu)|H_u(zu)). \end{aligned}$$

Applying the formulae

$$(zf|g) = (z\Pi(f)|\Pi(g)) + (\Pi(\bar{g})|\Pi(\bar{z}\bar{f})) , \quad H_{H_u(a)}(b) = H_u(ab) ,$$

we have

$$\begin{aligned} (zu^2|H_u(u)H_u(zu)) &= (zu\overline{H_u(u)}|\bar{u}H_u(zu)) \\ &= (zH_u^2(u)|H_u(zu^2)) + (H_u^2(zu)|H_u(zu^2)), \\ (uH_u^2(zu)|H_u(zu)) &= (H_u^2(zu)|\bar{u}H_u(zu)) = (H_u^2(zu)|H_u(zu^2)), \\ (zuH_u^2(u)|H_u(zu)) &= (zH_u^2(u)|H_u(zu^2)) , \end{aligned}$$

so that eventually

$$\{N_3, N_5\} = 0.$$

On the other hand, we have, as

$$\begin{aligned} N_3(u) &= \sum_{\ell} \frac{\mu_{\ell}}{b_{\ell}} e^{i\theta_{\ell}} \quad \text{and} \quad N_5(u) = \sum_k \frac{\mu_k^2 + J_2}{b_k} \mu_k e^{i\theta_k}, \\ 0 = \{N_3, N_5\} &= \sum_{\ell, k} e^{i(\theta_{\ell} + \theta_k)} \left[ i \left\{ \frac{\mu_{\ell}}{b_{\ell}}, \theta_k \right\} \frac{\mu_k^2 + J_2}{b_k} \mu_k \right. \\ &\quad \left. - i \left\{ \frac{\mu_k}{b_k} (\mu_k^2 + J_2), \theta_{\ell} \right\} \frac{\mu_{\ell}}{b_{\ell}} - \frac{\mu_{\ell} \mu_k}{b_{\ell} b_k} (\mu_k^2 + J_2) \{ \theta_{\ell}, \theta_k \} \right] \end{aligned}$$

Now, as before, one can cancel the real part of the Fourier coefficients to obtain

$$\{ \theta_{\ell}, \theta_k \} \frac{\mu_{\ell} \mu_k}{b_{\ell} b_k} (\mu_k^2 - \mu_{\ell}^2) = 0$$

and hence,  $\{ \theta_{\ell}, \theta_k \} = 0$ .

We have therefore proved all the commutation relations between our action angle variables. This proves that  $\chi_N$  is a symplectomorphism and completes the proof of Theorem 1.1.

**3.6. The explicit solution of the cubic Szegö equation.** We first prove Corollary 1.

*Proof.* Let us compute

$$\Delta_4 = \text{Tr}(H_u^4) - \text{Tr}(K_u^4) = \text{Tr}(H_u^4) - \text{Tr}((H_u^2 - (\cdot|u)u)^2)$$

in terms of  $J_2$  and  $J_4$ . We get  $\Delta_4 = 2J_4 - J_2^2$ . On the other hand, we already pointed out that  $2J_4 - J_2^2 = \|u\|_{L^4}^4$ . Since the cubic Szegö equation on  $\mathcal{M}(N)$  is the Hamiltonian system associated to the functional  $E(u) = \|u\|_{L^4}^4$  and to the symplectic form  $\omega$ , and since  $\chi_N$  is a symplectomorphism, we obtain that the cubic Szegö equation is equivalent to the Hamiltonian system associated to

$$E(I, L, \varphi, \theta) = \frac{1}{4} \sum_{j=1}^N (I_j^2 - L_j^2).$$

As the new coordinates are symplectic, we obtain that the cubic Szegö equation is equivalent to the system

$$\begin{cases} \dot{I}_j = 0, \quad \dot{L}_m = 0 \\ \dot{\varphi}_j = \frac{1}{2} I_j, \quad \dot{\theta}_m = -\frac{1}{2} L_m \end{cases}$$

□

**Remark 2.** Notice that the above system is explicitly solvable, and therefore that we reduced the cubic Szegö equation to a spectral analysis of the Hankel operator associated to the Cauchy datum  $u_0$ . In [2], section 4.1, we observed that the cubic Szegö equation on  $\mathcal{M}(N)$  could

be written as a system of  $2N$  ordinary differential equations in the variables given by the poles and the residues of the rational function  $u$ . Therefore the above corollary provides an explicit resolution of this system.

#### 4. EXTENSION TO THE INFINITE DIMENSION

In this section, we prove Theorem 1.2. We begin with proving the genericity of the set  $H_{+, \text{gen}}^{1/2}$ .

**Lemma 7.** *The set  $H_{+, \text{gen}}^{1/2}$  is a dense  $G_\delta$  subset of  $H_+^{1/2}$ .*

*Proof.* Let us consider the set  $\mathcal{U}_N$  which consists of functions  $u \in H_+^{1/2}$  such that the first  $N$  eigenvalues of  $H_u^2$  are simple, and such that, for any  $j \in \{1, \dots, N\}$ ,  $\nu_j := \|P_j(1)\| \neq 0$ . This set is obviously open in  $H_+^{1/2}$ . It is also dense in  $H_{+, \text{gen}}^{1/2}$  since any element  $u$  in  $H_{+, \text{gen}}^{1/2}$  may be approximated by an element in  $\mathcal{M}(N')$ ,  $N' > N$ , which can be itself approximated by an element in  $\mathcal{M}(N')_{\text{gen}} \subset \mathcal{U}_N$ , since  $N' \geq N$ . Eventually,  $H_{+, \text{gen}}^{1/2}$  is the intersection of the  $\mathcal{U}_N$ 's which are open and dense, hence  $H_{+, \text{gen}}^{1/2}$  is a dense  $G_\delta$  set.  $\square$

We can now begin the proof of Theorem 1.2. First of all, it is clear that, because of the simplicity assumption on the eigenvalues  $\lambda_j^2$  and  $\mu_m^2$ , each function  $\zeta_j$  and  $\gamma_m$  is continuous. Let  $(u_n)$  in  $H_{+, \text{gen}}^{1/2}$  be a sequence so that  $u_n$  converges to some  $u$  in the topology of  $H_+^{1/2}$ . Since  $H_{u_n}$  converges to  $H_u$  in the Hilbert-Schmidt norm, the  $\ell^2$  norm of  $(\lambda_j(u_n))$  tends to the  $\ell^2$  norm of  $(\lambda_j(u))$  in  $\ell^2$ . As  $K_{u_n}$  tends  $K_u$  in the Hilbert-Schmidt norm as well, the  $\ell^2$  norm of  $(\gamma_j(u_n))$  tends to the  $\ell^2$  norm of  $(\gamma_j(u))$ . This implies that  $\chi(u_n)$  tends to  $\chi(u)$  in  $\ell^2 \times \ell^2$ .

We now show that  $\chi$  is a homeomorphism. Let us first prove that  $\chi$  is onto. Let  $((\zeta_j), (\gamma_m)) \in \Xi$ . As  $\chi_N$  is onto on  $\mathcal{M}(N)_{\text{gen}}$ , for any  $N \geq 1$ , to  $(|\zeta_j|, |\gamma_j|, \varphi_j, \theta_j)_{1 \leq j \leq N}$  corresponds a unique  $u_N \in \mathcal{M}(N)_{\text{gen}}$ . Since

$$\|u_N\|_{H^{1/2}}^2 = \text{Tr}(H_{u_N}^2) = \sum_{j=1}^N \lambda_j^2 \rightarrow \sum_{j=1}^{\infty} \lambda_j^2,$$

$(u_N)$  is bounded in  $H^{1/2}$  and there exists a subsequence, still denoted by  $(u_N)$ , which converges weakly to some  $u \in H_+^{1/2}$ . Appealing to Proposition 2, we infer that  $u$  is the strong limit of  $(u_N)$  in  $H^{1/2}$  and that  $u$  belongs to  $H_{+, \text{gen}}^{1/2}$  with  $\chi(u) = ((\zeta_j), (\gamma_m))$ .

Let us prove that  $\chi$  is one-to-one. Again we use the formula (17),

$$u(z) = (u|(I - \bar{z}S)^{-1}P_u(1)).$$

Arguing as in Section 3, it is easy to check that Formula (26) may be extended here so that  $u$  is uniquely determined from the data in  $\Xi$ , as shown by the following result.

**Proposition 5.** *If  $\chi(u) = ((\zeta_j)_{j \geq 1}, (\gamma_m)_{m \geq 1})$ , then*

$$(38) \quad u(z) = X.(I - zA)^{-1}Y$$

where

$$\begin{aligned} X &:= (\nu_j \zeta_j)_{j \geq 1} \\ Y &:= (\nu_k)_{k \geq 1}^T, \end{aligned}$$

$A := (A_{j,k})_{j,k \geq 1}$  is given by

$$A_{j,k} = \sum_{\ell=1}^{\infty} \frac{\nu_j \nu_k \zeta_k \gamma_\ell}{b_\ell (|\zeta_j|^2 - |\gamma_\ell|^2) (|\zeta_k|^2 - |\gamma_\ell|^2)},$$

and

$$\begin{aligned} \nu_j &= \frac{1}{|\zeta_j|} \frac{\prod_{k=1}^{\infty} (|\zeta_j|^2 - |\gamma_k|^2)^{1/2}}{\prod_{k \neq j} (|\zeta_j|^2 - |\zeta_k|^2)^{1/2}}, \\ b_\ell &= \sum_{j=1}^{\infty} \frac{|\zeta_j|^2 \nu_j^2}{(|\zeta_j|^2 - |\gamma_\ell|^2)^2} = \frac{1}{|\zeta_\ell|^2 - |\gamma_\ell|^2} \prod_{k \neq \ell} \frac{|\gamma_\ell|^2 - |\gamma_k|^2}{|\gamma_\ell|^2 - |\zeta_k|^2}. \end{aligned}$$

*Proof.* Since it is very similar to the proof of Proposition 3, we only indicate the new features. Denote by  $\mathcal{R}_u$  the closure of the range of  $H_u$ . The main difference relies on the spectral theory of  $K_u$  on  $\mathcal{R}_u$ . Indeed, if  $u \in H_{+, \text{gen}}^{1/2}$ , it may happen that  $K_u$  has a kernel in  $\mathcal{R}_u$ , which is equivalent, as we noticed in Subsection 3.1, to the existence of  $g_0 \in \mathcal{R}_u$  such that  $H_u g_0 = 1$ . In this case, an orthogonal basis of the Hilbert space  $\mathcal{R}_u$  is given by the sequence  $(g_m)_{m \geq 0}$ , where  $g_m, m \geq 1$ , is given by the formula (24), and  $g_0$  is as above. However it turns out that the existence of  $g_0$  does not affect the formulae in Proposition 5. Indeed, since  $g_0 \in \mathcal{R}_u$  and  $K_u g_0 = 0 = H_u S g_0$ , we infer  $S g_0 = 0$ , hence, with the notation of Proposition 3, the expression of  $S(\tilde{e}_k)$  is still

$$S(\tilde{e}_k) = \sum_{\ell=1}^{\infty} \frac{\lambda_k \nu_k e^{i\varphi_k}}{b_\ell (\lambda_k^2 - \mu_\ell^2)} \mu_\ell e^{i\theta_\ell} h_\ell, \quad b_\ell := \|g_\ell\|^2,$$

and the expression of

$$A_{j,k} = (\tilde{e}_j | S(\tilde{e}_k))$$

then follows for every  $j, k \geq 1$ .  $\square$

It remains to check that  $\chi^{-1}$  is continuous on  $\Xi$ , that is if  $\chi(u_p)$  tends to  $\chi(u)$  then  $u_p$  tends to  $u$  in  $H^{1/2}$ . First, as  $\chi(u_p)$  converges, the sequence  $(u_p)$  is bounded in  $H^{1/2}$  and hence, admits a convergent subsequence which weakly converges to some  $v$ . Appealing again to Proposition 2, we conclude that  $u_p$  converges strongly to  $v$ . As  $\chi$  is continuous and one-to-one, we have  $u = v$ .

Finally, the evolution formulae of  $\zeta_j$  and of  $\gamma_m$  for the cubic Szegő equation (1) are immediate consequences of similar formulae for  $u \in \mathcal{M}(N)$  derived in Corollary 1, combined with the approximation of  $u$

by elements  $u_N$  in  $\mathcal{M}(N)_{\text{gen}}$ , and the continuity of the flow map of (1) on  $H_+^{1/2}$ , as proved in Theorem 2.1 of [2]. This completes the proof.

### 5. INVERSE SPECTRAL PROBLEMS FOR HANKEL OPERATORS

As a byproduct of the existence of the diffeomorphism  $\chi_N$  and of the homeomorphism  $\chi$ , we first prove Corollary 2.

*Proof.* Denote by  $\mathcal{F} : u \in L_+^2 \mapsto c = (\hat{u}(n))_{n \geq 0} \in \ell^2(\mathbb{N})$  the Fourier transform. Notice that  $\mathcal{F}$  realizes an isomorphism from  $H_+^{1/2}$  onto  $\mathbf{h}^{1/2}$ . Moreover, it is easy to check that

$$\mathcal{F}^{-1} \Gamma_c \Gamma_c^* \mathcal{F} = H_u^2, \quad \mathcal{F}^{-1} \tilde{\Gamma}_c \tilde{\Gamma}_c^* \mathcal{F} = K_u^2.$$

Therefore, the set of sequences  $c \in \mathbf{h}^{1/2}$  such that  $\Gamma_c$  has rank  $N$  and admits  $\lambda_j$ ,  $1 \leq j \leq N$ , as simple singular values, and such that  $\tilde{\Gamma}_c$  has rank  $N$  and admits  $\mu_j$ ,  $1 \leq j \leq N$ , as simple singular values, is sent by  $\mathcal{F}^{-1}$  onto

$$\chi_N^{-1}((I_1, \dots, I_N, L_1, \dots, L_N) \times \mathbb{T}^{2N}),$$

with

$$I_j := 2\lambda_j^2, \quad L_m := 2\mu_m^2.$$

The same argument applies in the infinite dimensional case. This completes the proof.  $\square$

Restricting to the case of selfadjoint Hankel operators will give us the proof of Corollary 3 as follows.

*Proof.* Via the Fourier transformation  $\mathcal{F}$ ,

$$L_{+,r}^2 = \{h \in L_+^2 : \forall n \in \mathbb{N}, \hat{h}(n) \in \mathbb{R}\}.$$

identifies to  $\ell_{\mathbb{R}}^2(\mathbb{N})$ , and the operators  $H_u, K_u$  with  $u \in H_+^{1/2} \cap L_{+,r}^2$  respectively identify to  $\Gamma_c, \tilde{\Gamma}_c$  with  $c = \mathcal{F}u$ . Moreover, for every  $(I, L) \in \Omega_N$ , one easily checks that

$$\mathbf{T}(I, L) \cap L_{+,r}^2 = \chi_N^{-1}((I, L) \times \{0, \pi\}^{2N}),$$

and, if  $u$  belongs to this set, the non zero eigenvalues of  $H_u$  (resp.  $K_u$ ) on  $L_{+,r}^2$  are

$$\zeta_1 = \lambda_1 e^{-i\varphi_1}, \dots, \zeta_n = \lambda_n e^{-i\varphi_n} \text{ (resp. } \gamma_1 = \mu_1 e^{-i\theta_1}, \dots, \gamma_N = \mu_N e^{-i\theta_N}).$$

Indeed, on the one hand  $\chi_N^{-1}((I, L) \times \{0, \pi\}^{2N}) \subset \mathbf{T}(I, L) \cap L_{+,r}^2$  by Proposition 3. On the other hand, if  $u \in \mathbf{T}(I, L) \cap L_{+,r}^2$ , the operator  $H_u$  is selfadjoint on  $L_{+,r}^2$ , hence has real eigenvalues  $\zeta_1, \dots, \zeta_N$  with  $|\zeta_j| = \lambda_j$ . The corresponding normalized eigenvectors  $\tilde{e}_j$  in  $L_{+,r}^2$  satisfy

$$H_u(\tilde{e}_j) = \zeta_j \tilde{e}_j,$$

therefore either  $\tilde{e}_j = \pm e_j$  and  $\varphi_j = \arg(1|\tilde{e}_j|^2) = 0$  if  $\zeta_j = \lambda_j$ , or  $\tilde{e}_j = \pm i e_j$  and  $\varphi_j = \arg[-(1|\tilde{e}_j|^2)] = \pi$  if  $\zeta_j = -\lambda_j$ . The same holds for

$K_u$ . The same argument applies in the infinite dimensional case. This completes the proof.  $\square$

**Remark 3.** Notice that, in addition to Corollaries 2 and 3, the solutions  $c$  are given by

$$c_n = X A^n Y ,$$

with the notation of Proposition 3 in the finite rank case, and Proposition 5 in the infinite rank case.

## 6. STABILITY OF INVARIANT TORI

In this section, we prove Theorem 1.3, which we state again for the convenience of the reader.

**Theorem 6.1.** For  $n = 1, \dots, 2N$ , define

$$(39) \quad j_{2n} = \sum_{j=1}^N 2^{-n} I_j^n \left( 1 - \frac{L_j}{I_j} \right) \prod_{k \neq j} \left( \frac{L_k - I_j}{I_k - I_j} \right) .$$

Then  $\mathbf{T}(I_1, \dots, I_N, L_1, \dots, L_N)$  is the set of the solutions in  $H_+^{1/2}$  of the minimization problem

$$\inf \{ M(u) : J_{2n}(u) = j_{2n} , n = 1, \dots, 2N \} .$$

Consequently,  $\mathbf{T} := \mathbf{T}(I_1, \dots, I_N, L_1, \dots, L_N)$  is stable under the evolution of (1), in the sense that, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, if

$$\inf_{v \in \mathbf{T}} \|u_0 - v\|_{H^{1/2}} \leq \delta ,$$

then the solution  $u$  of (1) with  $u(0) = u_0$  satisfies

$$\sup_{t \in \mathbb{R}} \inf_{v \in \mathbf{T}} \|u(t) - v\|_{H^{1/2}} \leq \varepsilon .$$

*Proof.* First of all, notice that Formula (39) expresses the common value of  $J_{2n}(u)$  as  $u \in \mathbf{T}(I_1, \dots, I_N, L_1, \dots, L_N)$ , in view of formulae (27) and

$$J_{2n} = \sum_{j=1}^N \lambda_j^{2n} \nu_j^2$$

with  $I_j = 2\lambda_j^2, L_m = 2\mu_m^2$ .

Let us assume that the Lagrangian torus

$$\mathbf{T} := \mathbf{T}(I_1, \dots, I_N, L_1, \dots, L_N)$$

is the set of solutions in  $H_+^{1/2}$  of the minimization problem

$$(40) \quad \inf \{ M(u) : J_{2n}(u) = j_{2n}, n = 1, \dots, 2N \} := m$$

where the  $j_{2n}$ 's are given by formula (39) and let us prove that it implies the stability.

Let  $u_0^{(n)}$  so that  $\inf_{v \in \mathbf{T}} \|u_0^{(n)} - v\|_{H^{1/2}}$  tends to zero as  $n$  goes to infinity. We are going to show that the solutions  $u^{(n)}$  of the cubic Szegö equation with  $u^{(n)}(0) = u_0^{(n)}$  are such that  $\sup_{t \in \mathbb{R}} \inf_{v \in \mathbf{T}} \|u^{(n)}(t) - v\|_{H^{1/2}}$  tends as well to zero as  $n$  goes to infinity. As the functionals  $u \mapsto J_{2k}(u)$  are invariant under the cubic Szegö flow and are continuous for the weak topology of  $H^{1/2}$  we get that  $J_{2k}(u^{(n)}(t)) = J_{2k}(u_0^{(n)})$  tends to  $j_{2k}$ . Similarly, since  $M(u)$  is a conservation law,  $u^{(n)}$  is bounded in  $H^{1/2}$  and  $M(u^{(n)})$  tends to  $m$ . Moreover, given any sequence  $(t_n)$  of real numbers, the sequence  $(u^{(n)}(t_n))$  has a subsequence which converges weakly to some  $u \in H_+^{1/2}$ . By the weak continuity of the  $J_{2k}$  and the weak semi-continuity of  $M$ ,  $J_{2k}(u) = j_{2k}$  and  $M(u) \leq m$ . Hence, since  $\mathbf{T}$  is the solution of the minimization problem,  $M(u) = m$ ,  $u_n(t_n)$  converges strongly to  $u$  and  $u$  belongs to  $\mathbf{T}$ . This gives the stability.

It remains to prove that the set of minimizers is  $\mathbf{T}$ . Recall that  $H_u^{2k}(1)$ ,  $k = 1, \dots, N$  are linearly independent if and only if the Gram determinant

$$\det(J_{2(n+m)}(u))_{1 \leq n, m \leq 2N}$$

is non-zero. By the choice of the sequence  $\{j_{2n}\}_{1 \leq n \leq 2N}$ , there exists  $u \in \mathcal{M}(N)_{\text{gen}}$  so that  $J_{2n}(u) = j_{2n}$ ,  $1 \leq n \leq 2N$  — any  $u \in \chi_N^{-1}((I_1, \dots, I_N, L_1, \dots, L_N) \times \mathbb{T}^{2N})$  is convenient. Hence, the determinant

$$\det(j_{2(n+m)})_{1 \leq n, m \leq 2N}$$

is different from zero. Since  $H_u$  is one to one on its range, it follows that if  $u$  satisfies  $J_{2n}(u) = j_{2n}$ ,  $1 \leq n \leq 2N$  then  $u, H_u^2(u), \dots, H_u^{2(N-1)}(u)$  are independent. As a first step, the following proposition implies that the set of functions  $u$  with  $J_{2n}(u) = j_{2n}$  with  $M(u)$  minimal is a subset of  $\mathcal{M}(N)$ .

**Proposition 6.** *Let  $u \in H_+^{1/2}$  and  $N \geq 1$  so that  $u, H_u^2(u), \dots, H_u^{2(N-1)}(u)$  are independent. Then the following inequality holds*

$$M(u) \geq \frac{\det((J_{2(k+\ell+1)}(u))_{0 \leq k, \ell \leq N-1}, (J_{2(k+N+1)}(u))_{0 \leq k \leq N-1})}{\det(J_{2(k+\ell+1)}(u))_{0 \leq k, \ell \leq N-1}}$$

with equality if and only if  $u \in \mathcal{M}(N)$ .

*Proof.* This statement is a direct consequence of the following lemma with  $A = H_u^2$  and  $e = u$ .  $\square$

**Lemma 8.** *Let  $A$  be a trace class positive self-adjoint operator defined on a Hilbert space  $\mathcal{H}$  and let  $e \in \mathcal{H}$ ,  $N \geq 1$ . Assume that  $A(e), A^2(e), \dots, A^N(e)$  are independent. Then,*

$$\text{Tr}(A) \geq \frac{\det((A^{k+\ell}(e), e)_{\substack{0 \leq k \leq N-1 \\ 0 \leq \ell \leq N-2}}, (A^{k+N}(e), e)_{0 \leq k \leq N-1})}{\det((A^{k+\ell}(e), e)_{0 \leq k, \ell \leq N-1})}$$



with equality if and only if the range of  $A$  is  $N$  dimensional and  $e$  belongs to the range of  $A$ .

*Proof.* Denote by  $V$  the space spanned by  $e, A(e), \dots, A^{N-1}e$ . Let  $P$  be the orthogonal projector from  $\mathcal{H}$  to  $V$ . Let  $\tilde{A} = PAP$  then  $\tilde{A}$  is positive self adjoint and  $\text{Tr}(A) \geq \text{Tr}(\tilde{A})$ . In fact, one has

$$\text{Tr}(\tilde{A}) = \text{Tr}(P^2A) = \text{Tr}(PA)$$

so that

$$\text{Tr}(A) - \text{Tr}(\tilde{A}) = \text{Tr}((I-P)A) = \text{Tr}((I-P)^2A) = \text{Tr}((I-P)A(I-P)) \geq 0.$$

By definition,  $\tilde{A}$  is at most of range  $N$  so that by Cayley-Hamilton, there exist  $\sigma_1 = \text{Tr}(\tilde{A}), \sigma_2, \dots, \sigma_N$  so that

$$(\tilde{A})^N = \sum_{j=1}^N (-1)^{j-1} \sigma_j (\tilde{A})^{N-j}.$$

In particular,

$$(\tilde{A})^N(e) = \sum_{j=1}^N (-1)^{j-1} \sigma_j (\tilde{A})^{N-j}(e)$$

so that

$$PA^N(e) = \sum_{j=1}^N (-1)^{j-1} \sigma_j A^{N-j}(e)$$

and taking the scalar product with  $A^k(e)$ ,  $0 \leq k \leq N-1$ , we get

$$(A^{N+k}(e), e) = \sum_{j=1}^N (-1)^{j-1} \sigma_j (A^{N-j+k}(e), e).$$

Solving the corresponding system in  $(\sigma_1, \dots, \sigma_N)$ , we get that  $\sigma_1 = \text{Tr}(\tilde{A})$  coincides with the right hand side of the inequality. Hence, inequality of lemma 8 is proved. Furthermore, there is equality if and only if

$$\text{Tr}((I-P)A(I-P)) = 0.$$

This is equivalent, since  $A$  is positive, to  $(I-P)A(I-P) = 0$  which, in turn is equivalent to  $A(I-P) = 0$ . Indeed, let  $w \in \text{Im}(I-P)$  so that  $(I-P)Aw = 0$  then  $((I-P)Aw, w) = 0 = (Aw, w)$  so that  $Aw = 0$ . In particular, the range of  $A$  is a subspace of  $V$ . On the other hand, by assumption the range of  $A$  is at least  $N$  dimensional, we obtain that the range of  $A$  is exactly  $V$ . In particular, it implies that  $e$  belongs to the range of  $A$ . Conversely, if the range of  $A$  is  $N$  dimensional and if  $e$  belongs to the range of  $A$ , then  $V$  is a subspace of the range of  $A$  and is  $N$  dimensional, hence  $V$  is the range of  $A$ . In particular,  $(I-P)A = 0$  so that  $\text{Tr}(\tilde{A}) = \text{Tr}(A)$ .  $\square$

We now show that Proposition 6 implies the theorem, namely that  $\mathbf{T}$  is the solution of the minimization problem. It remains to prove that, if  $u \in \mathcal{M}(N)$  satisfies  $J_{2n}(u) = j_{2n}$  for  $n = 1, \dots, 2N$ , then  $u \in \mathbf{T}$ . Let  $u$  be such a function. Since  $\det(J_{2(n+m)}(u))_{1 \leq n, m \leq N} = \det(j_{2(n+m)})_{1 \leq n, m \leq N} \neq 0$ , we already know that  $H_u^2$  has  $N$  simple positive eigenvalues  $\tilde{\lambda}_1^2 > \dots > \tilde{\lambda}_N^2$  and its corresponding normalization constants  $\tilde{\nu}_1, \dots, \tilde{\nu}_N$  are all  $> 0$ . Let us prove that

$$\tilde{\lambda}_j = \lambda_j, \quad \tilde{\nu}_j = \nu_j, \quad j = 1, \dots, N$$

where  $\lambda_1, \dots, \lambda_N, \nu_1, \dots, \nu_N$  correspond to any element  $u_0 \in \mathbf{T}$ . The assumption  $J_{2n}(u) = j_{2n}$  for  $n = 1, \dots, 2N$  reads

$$\sum_{j=1}^N \tilde{\lambda}_j^{2n} \tilde{\nu}_j^2 = \sum_{j=1}^N \lambda_j^{2n} \nu_j^2, \quad n = 1, \dots, 2N,$$

or, for every polynomial  $P$  of degree  $\leq 2N$  such that  $P(0) = 0$ ,

$$\sum_{j=1}^N P(\tilde{\lambda}_j^2) \tilde{\nu}_j^2 = \sum_{j=1}^N P(\lambda_j^2) \nu_j^2.$$

Assume that for some  $j_0$ ,  $\tilde{\lambda}_{j_0}$  is different from all the  $\lambda_j$ 's. Then we can select a polynomial  $P$  of degree  $2N$  such that  $P(\lambda_j^2) = 0$  for every  $j$ ,  $P(\tilde{\lambda}_j^2) = 0$  for every  $j \neq j_0$  and  $P(0) = 0$ , but  $P(\tilde{\lambda}_{j_0}^2) \neq 0$ . Plugging these informations into the above identity, we get  $\tilde{\nu}_{j_0} = 0$ , a contradiction. This implies  $\tilde{\lambda}_j = \lambda_j$  for every  $j$ , and finally, by solving a Van der Monde system,  $\tilde{\nu}_j = \nu_j$  for every  $j$ .  $\square$

**Remark 4.** *There is an analogous result of Theorem 1.3 in the infinite dimensional case, though it is easier. Indeed, given two sequences  $I = (I_j)_{j \geq 1}$ ,  $L = (L_m)_{m \geq 1}$  of numbers such that*

$$I_1 > L_1 > I_2 > L_2 > \dots > 0, \quad \sum_{j=1}^{\infty} I_j < \infty,$$

denote by  $\mathbf{T}(I, L)$  the infinite dimensional torus of those  $u \in H_{+, \text{gen}}^{1/2}$  such that  $\chi(u) = ((\zeta_j)_{j \geq 1}, (\gamma_m)_{m \geq 1})$  with  $I_j = 2|\zeta_j|^2$  and  $L_m = 2|\gamma_m|^2$  for all  $j, m$ . First of all, we observe that, for every  $n \geq 1$ ,  $J_{2n}$  has a constant value on  $\mathbf{T}(I, L)$  given by

$$j_{2n} = \sum_{j=1}^{\infty} 2^{-n} I_j^n \left(1 - \frac{L_j}{I_j}\right) \prod_{k \neq j} \left(\frac{L_k - I_j}{I_k - I_j}\right).$$

Then we claim that  $\mathbf{T}(I, L)$  is precisely the solution of the minimization problem

$$\inf\{M(u) : J_{2n}(u) = j_{2n}, \quad n \geq 1\}.$$

Indeed, if  $u \in H_+^{1/2}$  is such that  $J_{2n}(u) = j_{2n}$  for every  $n \geq 1$ , we conclude that

$$\forall x \notin \left\{ \frac{1}{\lambda_j^2} \right\}, \quad J(x)(u) = \prod_{j=1}^{\infty} \frac{1 - \mu_j^2 x}{1 - \lambda_j^2 x},$$

where

$$\lambda_j^2 := \frac{1}{2}I_j, \quad \mu_j^2 := \frac{1}{2}L_j.$$

From formula (19), we infer

$$(41) \quad \prod_{j=1}^{\infty} \frac{1 - \mu_j^2(u)x}{1 - \lambda_j^2(u)x} = \prod_{j=1}^{\infty} \frac{1 - \mu_j^2 x}{1 - \lambda_j^2 x}.$$

Consequently, the sequence  $(\lambda_j^2)$  is a subsequence of the sequence  $(\lambda_j^2(u))$ , and the sequence  $(\mu_j^2)$  is a subsequence of the sequence  $(\mu_j^2(u))$ . We deduce

$$M(u) = \text{Tr}(K_u^2) = \sum_{m=1}^{\infty} \mu_m^2(u) \geq \sum_{m=1}^{\infty} \mu_m^2,$$

with equality if and only if the sequences  $(\mu_j^2)$  and  $(\mu_j^2(u))$  coincide. In that case, in view of (41), we conclude that the sequences  $(\lambda_j^2)$  and  $(\lambda_j^2(u))$  coincide too, and finally that  $u \in \mathbf{T}(I, L)$ . The stability of  $\mathbf{T}(I, L)$  through the evolution of (1) therefore follows by the same compactness arguments as in the proof of Theorem 1.3.

## 7. INSTABILITY OF TRAVELING WAVES

In contrast with the preceding section, we now establish instability of traveling waves which are non minimal.

**Theorem 7.1.** *The following traveling waves of the cubic Szegő equation are orbitally unstable :*

$$\varphi(z) = \alpha \prod_{j=1}^N \frac{z - \bar{p}_j}{1 - p_j z}, \quad \alpha \neq 0, N \geq 1, \quad 0 \leq |p_j| < 1,$$

$$\varphi(z) = \alpha \frac{z^\ell}{1 - p^N z^N}, \quad \alpha \neq 0, \quad N \geq 2, N - 1 \geq \ell \geq 0, \quad 0 < |p| < 1.$$

*Proof.* We first deal with traveling waves with non zero velocity,

$$\varphi(z) = \frac{z^\ell}{1 - p^N z^N}, \quad N \geq 2, N - 1 \geq \ell \geq 0, \quad 0 < |p| < 1,$$

where the constant  $\alpha \neq 0$  has been made 1 for simplicity, in view of the invariances of the equation. Our strategy is to approximate  $\varphi$  by a family  $(u_0^\varepsilon)$  in  $\mathcal{M}(N)_{\text{gen}}$  such that the family of corresponding solutions  $(u^\varepsilon)$  do not satisfy

$$(42) \quad \sup_{t \in \mathbb{R}} \inf_{(\alpha, \beta) \in \mathbb{T}^2} \|u^\varepsilon(t) - \varphi_{\alpha, \beta}\|_{H^{1/2}} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where  $\varphi_{\alpha,\beta}$  denotes the current point of the orbit of  $\varphi$  through the action of  $\mathbb{T}^2$ ,

$$\varphi_{\alpha,\beta}(z) = e^{i\alpha}\varphi(e^{i\beta}z) .$$

Specifically, if (42) holds, then, for  $\varepsilon$  small enough,  $u^\varepsilon(t)$  belongs to a compact subset of  $\mathcal{M}(N)$ , and consequently every continuous function  $f$  on  $\mathcal{M}(N)$  which vanishes on every  $\varphi_{\alpha,\beta}$  satisfies

$$\sup_{t \in \mathbb{R}} |f(u^\varepsilon(t))| \xrightarrow{\varepsilon \rightarrow 0} 0 .$$

We shall choose for  $f$ , the function  $\sigma$  defined by

$$u(z) = \frac{A(z)}{1 - \sigma(u)z + z^2 R(z)} ,$$

where  $A, R$  are polynomial functions. Notice that  $\sigma$  vanishes onto the orbit of  $\varphi$  since  $N \geq 2$ . We now compute  $\sigma(u^\varepsilon(t))$  by means of the explicit inverse formula for  $\chi_N$  given in Proposition 3. This yields

$$\sigma(u^\varepsilon(t)) = \text{tr}(\Gamma) = \sum_{1 \leq j, \ell \leq N} \frac{\lambda_j \nu_j^2 \mu_\ell}{b_\ell (\lambda_j^2 - \mu_\ell^2)^2} e^{-i(\varphi_j + \theta_\ell)} .$$

Notice that, in the above formula, all the quantities depend on  $\varepsilon$ , but only the angles  $\varphi_j, \theta_\ell$  depend on  $t$ . Moreover, from Corollary 1, we know that they depend linearly on  $t$ , with velocities

$$\frac{d}{dt}(\varphi_j + \theta_\ell) = \lambda_j^2 - \mu_\ell^2 .$$

We claim that we may assume that all these velocities are pairwise distinct. Indeed, using the diffeomorphism  $\chi_N$  of Theorem 1.1, this just comes from the fact that, on the open set

$$\Omega_N = \{I_1 > L_1 > I_2 > L_2 > \dots > I_N > L_N > 0\}$$

of  $\mathbb{R}^{2N}$ , the quantities  $I_j - L_\ell$  are generically pairwise distinct. Consequently,

$$\frac{1}{T} \int_0^T |\sigma(u^\varepsilon(t))|^2 dt \xrightarrow{T \rightarrow \infty} \sum_{1 \leq j, \ell \leq N} \frac{\lambda_j^2 \nu_j^4 \mu_\ell^2}{b_\ell^2 (\lambda_j^2 - \mu_\ell^2)^4} .$$

We now estimate the right hand side of the above identity from below as  $\varepsilon$  tends to 0. An elementary spectral study of  $H_\varphi^2$  and of  $K_\varphi^2$  shows that their eigenvectors are

$$\varphi_j = \frac{z^j}{1 - p^N z^N} , \quad j = 0, 1, \dots, N-1,$$

and that their eigenvalues belong to the pair

$$\left\{ \frac{|p|^{2N}}{(1 - |p|^{2N})^2}, \frac{1}{(1 - |p|^{2N})^2} \right\}$$

hence are bounded from above and below. From the continuity deduced from the min max formula, we infer that  $\lambda_j^2, \mu_\ell^2$  are also bounded from

above and below as  $\varepsilon$  tends to 0. Therefore, for some fixed positive constant  $\delta$ ,

$$\sum_{1 \leq j, \ell \leq N} \frac{\lambda_j^2 \nu_j^4 \mu_\ell^2}{b_\ell^2 (\lambda_j^2 - \mu_\ell^2)^4} \geq \delta \left( \sum_{j=1}^N \nu_j^4 \right) \left( \sum_{\ell=1}^N \frac{1}{b_\ell^2} \right) \geq \frac{\delta}{N^2} \left( \sum_{j=1}^N \nu_j^2 \right)^2 \left( \sum_{\ell=1}^N \frac{1}{b_\ell} \right)^2.$$

Moreover,

$$\sum_{\ell=1}^N \frac{1}{b_\ell} = \|u_0^\varepsilon\|^2 \xrightarrow{\varepsilon \rightarrow 0} \|\varphi\|^2, \quad \sum_{j=1}^N \nu_j^2 = \|P_{u_0^\varepsilon}(1)\|^2 \xrightarrow{\varepsilon \rightarrow 0} \|P_\varphi(1)\|^2 = 1 - |p|^{2N},$$

as we proved in [2], Proposition 1. We conclude that

$$\liminf_{\varepsilon \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\sigma(u^\varepsilon(t))|^2 dt > 0,$$

which contradicts

$$\sup_{t \in \mathbb{R}} |\sigma(u^\varepsilon(t))| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Hence  $\varphi$  is orbitally unstable.

We now deal with stationary waves, which are Blaschke products

$$\varphi(z) = \prod_{j=1}^{N-1} \frac{z - \bar{p}_j}{1 - p_j z}$$

with  $N \geq 2, 0 \leq |p_j| < 1$ . Once again, we want to prove that there exists a sequence  $u_0^\varepsilon$  in  $H_+^{1/2}$  such that

$$\|u_0^\varepsilon - \varphi\|_{H^{1/2}} \xrightarrow{\varepsilon \rightarrow 0} 0$$

but the solution  $u^\varepsilon$  of the cubic Szegö equation with Cauchy datum  $u_0^\varepsilon$  satisfies

$$(43) \quad \liminf_{\varepsilon \rightarrow 0} \sup_{t \in \mathbb{R}} \inf_{\alpha \in \mathbb{T}} \|u^\varepsilon(t) - e^{i\alpha} \varphi\|_{H^{1/2}} > 0.$$

Introduce the quantity

$$q := (1|\varphi) = (-1)^{N-1} p_1 \dots p_{N-1}.$$

We claim that we may assume that  $q \in \mathbb{R}_+$ . Indeed, by using invariance of the cubic Szegö equation through multiplication by complex numbers of modulus 1 and by rotations of the circle, property (43) for  $\varphi$  and the sequence  $(u_0^\varepsilon)$  is equivalent to property (43) for

$$\tilde{\varphi}(z) = e^{i\beta} \varphi(e^{i\gamma} z)$$

and

$$\tilde{u}_0^\varepsilon(z) = e^{i\beta} u_0^\varepsilon(e^{i\gamma} z).$$

If we choose  $\beta = -(N-1)\gamma$ , we observe that

$$\tilde{\varphi}(z) = \prod_{j=1}^{N-1} \frac{z - \bar{p}'_j}{1 - p'_j z}, \quad p'_j := e^{i\gamma} p_j,$$

so that  $\tilde{q} = e^{i(N-1)\gamma} q$ . Hence a convenient choice of  $\gamma$  ensures  $\tilde{q} \geq 0$ . We therefore assume from now on that  $q \geq 0$ .

We now introduce

$$u_0^\varepsilon = \varphi + \varepsilon.$$

Let us first determine the spectrum of  $H_{u_0^\varepsilon}^2$  on the vector space

$$\langle 1 \rangle := \text{span}(H_{u_0^\varepsilon}^n(1), n \geq 0).$$

We have

$$H_{u_0^\varepsilon} = H_\varphi + \varepsilon H_1$$

which is identically 0 on  $\ker H_\varphi$ . On the range of  $H_\varphi$ , we have

$$H_{u_0^\varepsilon}^2 = H_\varphi^2 + \varepsilon(H_\varphi H_1 + H_1 H_\varphi) + \varepsilon^2 H_1^2 = I + R_\varepsilon$$

where  $R_\varepsilon$  is the rank two operator defined by

$$R_\varepsilon(h) = \varepsilon((h|1)\varphi + (h|\varphi)) + \varepsilon^2(h|1).$$

Notice that we used the identity  $H_\varphi^2 = 1$  on the range of  $H_\varphi$ , which holds since  $\varphi$  is an inner function. We observe that  $R_\varepsilon$  stabilizes  $\text{span}(1, \varphi)$ , which is therefore  $\langle 1 \rangle$ , and that its matrix in the basis  $(1, \varphi)$  reads

$$M_\varepsilon = \begin{pmatrix} \varepsilon q + \varepsilon^2 & \varepsilon + \varepsilon^2 q \\ \varepsilon & \varepsilon q \end{pmatrix}$$

Consequently, the eigenvalues of  $H_{u_0^\varepsilon}^2$  on  $\text{span}(1, \varphi)$  are the roots  $r_\pm$  of the equation

$$(r-1)^2 - (2\varepsilon q + \varepsilon^2)(r-1) - \varepsilon^2(1-q^2) = 0$$

which are given by

$$r_\pm = 1 + \varepsilon \left( q + \frac{\varepsilon}{2} \pm (1 + \varepsilon q + \frac{\varepsilon^2}{4})^{1/2} \right) = 1 + \varepsilon(q \pm 1) + O(\varepsilon^2).$$

We therefore have

$$H_{u_0^\varepsilon}^4(1) - \sigma_1 H_{u_0^\varepsilon}^2(1) + \sigma_2 = 0,$$

$$\sigma_1 = r_+ + r_- = 2 + 2\varepsilon q + \varepsilon^2, \quad \sigma_2 = r_+ r_- = (1 + \varepsilon q)^2.$$

Applying the Lax pair property described in Formulae (2) and (3), we infer

$$H_{u^\varepsilon}^4(1) - \sigma_1 H_{u^\varepsilon}^2(1) + \sigma_2 = 0$$

for every time  $t$ . Indeed,  $f = H_{u^\varepsilon}^4(1) - \sigma_1 H_{u^\varepsilon}^2(1) + \sigma_2$  satisfies the linear evolution equation

$$\frac{df}{dt} = (B_{u^\varepsilon} + \frac{i}{2} H_{u^\varepsilon}^2) f$$

and  $f(0) = 0$ . Denote by  $w^\varepsilon \in \langle 1 \rangle$  the unique vector such that  $H_{u^\varepsilon}(w^\varepsilon) = 1$ . In view of the above formula, we have

$$w^\varepsilon = \frac{-H_{u^\varepsilon}^3(1) + \sigma_1 H_{u^\varepsilon}(1)}{\sigma_2},$$

We now study the evolution of

$$J_{-1}^\varepsilon(t) = (w^\varepsilon | 1), \quad J_1^\varepsilon(t) = (u^\varepsilon | 1).$$

Again by the Lax pair property, we have

$$i\dot{J}_{-1}^\varepsilon = J_1^\varepsilon, \quad i\dot{J}_1^\varepsilon = \sigma_1 J_1^\varepsilon - \sigma_2 J_{-1}^\varepsilon,$$

which implies that

$$J_1^\varepsilon(t) = \gamma_+ e^{-ir+t} + \gamma_- e^{-ir-t},$$

where  $\gamma_\pm$  are given by initial conditions,

$$\gamma_+ + \gamma_- = q + \varepsilon, \quad \frac{\gamma_+}{r_+} + \frac{\gamma_-}{r_-} = \frac{q}{1 + \varepsilon q}.$$

This leads to

$$\gamma_\pm = \frac{q \pm 1}{2} + O(\varepsilon).$$

We infer, for every  $s > 0$ ,

$$\begin{aligned} \frac{\varepsilon}{s} \int_0^{s/\varepsilon} |J_1^\varepsilon(t)|^2 dt &= \frac{1}{s} \int_0^s (\gamma_+^2 + \gamma_-^2 + 2\operatorname{Re}(\gamma_+ \gamma_- e^{-i(r_+ - r_-)\sigma/\varepsilon})) d\sigma \\ &\xrightarrow{\varepsilon \rightarrow 0} \frac{1 + q^2}{2} - \frac{(1 - q^2) \sin(2s)}{4s} := f(s). \end{aligned}$$

On the other hand, if  $\varphi$  is orbitally stable, we have

$$\sup_{t \in \mathbb{R}} ||J_1^\varepsilon(t)|^2 - q^2| \leq C \sup_{t \in \mathbb{R}} \inf_{\alpha \in \mathbb{T}} \|u^\varepsilon(t) - e^{i\alpha} \varphi\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

which imposes  $f(s) = q^2$  for every  $s$  and contradicts the above formula for  $f$ . We conclude that  $\varphi$  is orbitally unstable. □

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