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Coxeter orbits and Brauer trees II

Olivier Dudas^{*†}

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Abstract

The purpose of this paper is to discuss the validity of the assumptions (W) and (S) stated in [12], about the torsion in the modular ℓ -adic cohomology of Deligne-Lusztig varieties associated with Coxeter elements. We prove that both (W) and (S) hold except for groups of type E_7 or E_8 .

Introduction

Let \mathbf{G} be a quasi-simple algebraic group defined over an algebraic closure of a finite field of characteristic p . Let F be the Frobenius endomorphism of \mathbf{G} associated with a rational \mathbb{F}_q -structure. The finite group $G = \mathbf{G}^F$ of fixed points under F is called a finite reductive group.

Let ℓ be a prime number different from p and Λ be a finite extension of \mathbb{Z}_ℓ . There is strong evidence that the structure of the principal ℓ -block of G is encoded in the cohomology over Λ of some Deligne-Lusztig variety. Precise conjectures have been stated in [5] and [7], and much numerical evidence has been collected. The representation theory of ΛG is highly dependent on the prime number ℓ . In [12], we have studied a special case referred to as the *Coxeter case*. The corresponding primes ℓ are those which divide the cyclotomic polynomial $\Phi_h(q)$ where h is the Coxeter number of W . In that situation, it is to be expected that the cohomology of the Deligne-Lusztig variety $Y(\dot{c})$ associated with a Coxeter element c describes the principal ℓ -block $b\Lambda G$. More precisely,

- *Hiß-Lübeck-Malle conjecture*: the Brauer tree of $b\Lambda G$ (which has a cyclic defect group) can be recovered from the action of G and some power F^δ of F on the cohomology groups $H_c^i(Y(\dot{c}), \overline{\mathbb{Q}}_\ell)$ [16];
- *Geometric version of Broué's conjecture*: the complex $b\mathrm{R}\Gamma_c(Y(\dot{c}), \Lambda)$ induces a derived equivalence between the principal ℓ -blocks of G and the normalizer $N_G(T_c)$ of a torus of type c [5].

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In [12] the author has given a general proof of both of these conjectures, but under some assumptions on the torsion in the cohomology of $Y(\dot{c})$. The weaker assumption concerns only some eigenspaces of the Frobenius:

- (W) For all minimal eigenvalues λ of F^δ , the generalized (λ) -eigenspace of F^δ on $bH_c^*(Y(\dot{c}), \Lambda)$ is torsion-free.

We call here "minimal" the eigenvalues of F^δ on the cohomology group in middle degree. If this assumption holds, then we proved in [12] that the Brauer tree of the principal block has the expected shape. However, a stronger assumption is needed to obtain the planar embedding of the tree and Broué's conjecture:

- (S) The Λ -modules $bH_c^i(Y(\dot{c}), \Lambda)$ are torsion-free.

The purpose of this paper is to discuss these assumptions for the Deligne-Lusztig variety $Y(c)$ and some interesting quotients, and to prove that they are valid in the majority of cases. The main result in this direction is the following:

Theorem. *Let b' be the idempotent associated with the principal ℓ -block of T_c . If the type of G is not E_7 or E_8 , then the Λ -modules $bH_c^i(Y(\dot{c}), \Lambda)b'$ are torsion-free.*

Furthermore, we can also include the groups of type E_7 and E_8 if we assume that we already know the shape of the Brauer tree. These have been now determined by the author and Raphaël Rouquier [13]. This proves a version of [12, Theorem A] where $bH_c^i(Y(\dot{c}), \Lambda)$ is replaced by $bH_c^i(Y(\dot{c}), \Lambda)b'$. Note that it does not change the results in [12] which are implied by this theorem. Note also that the assumption on p in [12, Theorem A] can be dropped, since we no longer use generalised Gelfand-Graev representations to study the torsion (unlike in [11]). In particular, we obtain a significant number of new cases of the geometric version of Broué's conjecture (see Theorem 3.7). We also deduce new planar embeddings of Brauer trees for the groups of type 2G_2 , F_4 and 2F_4 (see Theorem 3.8).

Our proof relies on Lusztig's work on the geometry of Deligne-Lusztig varieties associated with Coxeter elements [17]. Many constructions that are derived from $X(c)$, such as remarkable quotients and smooth compactifications, can be expressed in terms of varieties associated with smaller Coxeter elements. This provides an inductive method for finding the torsion in the cohomology of $X(c)$. A further refinement adapted from [2] is then used to lift the method up to $Y(\dot{c})$ and to show that the torsion part of $bH_c^i(Y(\dot{c}), \Lambda)b'$ is necessarily a cuspidal module. This reduces the problem of finding the torsion to the problem of finding where cuspidal composition factors can occur in the cohomology. We prove that these cannot occur outside the middle degree.

This paper is organized as follows: the first section presents some preliminaries. We have compiled the basic techniques that are used in the modular Deligne-Lusztig theory. In the following section, we use the geometric results of [17] to rephrase the assumption (S) into a more representation-theoretical condition involving cuspidal modules. The last section is devoted to this problem. We prove that the cohomology of $X(c)$ has cuspidal composition factors in the middle degree only whenever the shape of the Brauer tree is known.

1 Preliminaries

In this preliminary section we set up the notation and introduce the main techniques (homological and geometric) that we will use throughout this paper.

1.1 Homological methods

1.1.1. Module categories and usual functors. If \mathcal{A} is an abelian category, we will denote by $C(\mathcal{A})$ the category of cochain complexes, by $K(\mathcal{A})$ the corresponding homotopy category and by $D(\mathcal{A})$ the derived category. We shall use the superscript notation $-$, $+$ and b to denote the full subcategories of bounded above, bounded below or bounded complexes. We will always consider the case where $\mathcal{A} = A\text{-Mod}$ is the module category over some ring A or the full subcategory $A\text{-mod}$ of finitely generated modules. This is actually not a strong restriction, since any small abelian category can be embedded into some module category [18]. Since the categories $A\text{-Mod}$ and $A\text{-mod}$ have enough projective objects, one can define the usual derived bifunctors RHom_A^\bullet and ${}^{\text{L}}\otimes_A$.

Let H be a finite group and ℓ be a prime number. We fix an ℓ -modular system (K, Λ, k) consisting of a finite extension K of the field of ℓ -adic numbers \mathbb{Q}_ℓ , the integral closure Λ of the ring of ℓ -adic integers in K and the residue field k of the local ring Λ . We assume moreover that the field K is big enough for H , so that it contains the e -th roots of unity, where e is the exponent of H . In that case, the algebra KH is split semi-simple.

From now on, we shall focus on the case where $A = \mathcal{O}H$, with \mathcal{O} being any ring among (K, Λ, k) . By studying the modular representation theory of H we mean studying the module categories $\mathcal{O}H\text{-mod}$ for various \mathcal{O} , and also the different connections between them. In this paper, most of the representations will arise in the cohomology of some complexes and we need to know how to pass from one coefficient ring to another. The scalar extension and ℓ -reduction have a derived counterpart: if C is any bounded complex of ΛH -modules we can form $KC = C \otimes_\Lambda K$ and $\overline{C} = kC = C \otimes_\Lambda k$. Since K is a flat Λ -module, the cohomology of the complex KC is exactly the scalar extension of the cohomology of C . However this does not apply to ℓ -reduction, but the obstruction can be related to the torsion.

Theorem 1.1 (Universal coefficient theorem). *Let C be a bounded complex of ΛH -modules. Assume that the terms of C are free over Λ . Then for all $n \geq 1$ and $i \in \mathbb{Z}$, there exists a short exact sequence of ΛH -modules*

$$0 \longrightarrow H^i(C) \otimes_\Lambda \Lambda/\ell^n \Lambda \longrightarrow H^i(C \otimes_\Lambda \Lambda/\ell^n \Lambda) \longrightarrow \text{Tor}_1^\Lambda(H^{i+1}(C), \Lambda/\ell^n \Lambda) \longrightarrow 0.$$

In particular, whenever there is no torsion in $H^\bullet(C)$ (and the terms of C are still assumed to be torsion-free) then the cohomology of \overline{C} is exactly the ℓ -reduction of the cohomology of C .

1.1.2. Composition factors in the cohomology. Let C be a complex of kH -modules and L be a simple kG -module. We denote by P_L the projective cover of L in $kG\text{-mod}$. We can determine the cohomology groups of C in which L occurs as a composition factor using the following standard result.

Lemma 1.2. *Given $i \in \mathbb{Z}$, the following assertions are equivalent:*

- (i) *The i -th cohomology group of the complex $\mathrm{RHom}_{kH}^\bullet(P_L, C)$ is non-zero;*
- (ii) *$\mathrm{Hom}_{K^b(kH)}(P_L, C[i])$ is non-zero;*
- (iii) *L is a composition factor of $H^i(C)$.*

Proof. See for example [11, Section 1.1.2]. □

The formulation (i) is particularly adapted to our framework and will be extensively used in Section 3.

1.1.3. Generalized eigenspaces over Λ . Let M be a finitely generated Λ -module and $f \in \mathrm{End}_\Lambda(M)$. Assume that the eigenvalues of f are in the ring Λ . For $\lambda \in \Lambda$, we have defined in [12, Section 1.2.2] the generalized (λ) -eigenspace $M_{(\lambda)}$ of f on M . Here are the principal properties that we shall use:

- $M_{(\lambda)}$ is a direct summand of M and M is the direct sum of the generalized (λ) -eigenspaces for various $\lambda \in \Lambda$;
- if λ and μ are congruent modulo ℓ then $M_{(\lambda)} = M_{(\mu)}$;
- $(kM)_{(\lambda)} := M_{(\lambda)} \otimes_\Lambda k$ is the usual generalized $\bar{\lambda}$ -eigenspace of \bar{f} ;
- $(KM)_{(\lambda)} := M_{(\lambda)} \otimes_\Lambda K$ is the sum of the usual generalized μ -eigenspaces of f where μ runs over the elements of Λ that are congruent to λ modulo ℓ .

1.2 Geometric methods

To any quasi-projective variety X defined over $\bar{\mathbb{F}}_p$ and acted on by H , one can associate a classical object in the derived category $D^b(\mathcal{O}H\text{-Mod})$, namely the cohomology with compact support of X , denoted by $\mathrm{R}\Gamma_c(X, \mathcal{O})$. It is quasi-isomorphic to a bounded complex of finitely generated $\mathcal{O}H$ -modules that are free over \mathcal{O} . Moreover, the cohomology complex behaves well with respect to scalar extension and ℓ -reduction. We have indeed in $D^b(\mathcal{O}H\text{-Mod})$:

$$\mathrm{R}\Gamma_c(X, \Lambda) \overset{\mathrm{L}}{\otimes}_\Lambda \mathcal{O} \simeq \mathrm{R}\Gamma_c(X, \mathcal{O}).$$

In particular, the universal coefficient theorem (Theorem 1.1) will hold for ℓ -adic cohomology with compact support.

We give here some quasi-isomorphisms we shall use in Sections 2 and 3. The reader will find references or proofs of these properties in [1, Section 3].

Proposition 1.3. *Let X and Y be two quasi-projective varieties acted on by H . Then we have the following isomorphisms in the derived category $D^b(\mathcal{O}H\text{-Mod})$:*

(i) *The Künneth formula:*

$$R\Gamma_c(X \times Y, \mathcal{O}) \simeq R\Gamma_c(X, \mathcal{O}) \overset{L}{\otimes} R\Gamma_c(Y, \mathcal{O}).$$

(ii) *The quotient variety $H \backslash X$ exists. Moreover, if the order of the stabilizer of any point of X is prime to ℓ , then*

$$R\Gamma_c(H \backslash X, \mathcal{O}) \simeq \mathcal{O} \overset{L}{\otimes}_{\mathcal{O}H} R\Gamma_c(X, \mathcal{O}).$$

If N is a finite group acting on X on the right and on Y on the left, we can form the amalgamated product $X \times_N Y$, as the quotient of $X \times Y$ by the diagonal action of N . Assume that the actions of H and N commute and that the order of the stabilizer of any point for the diagonal action of N is prime to ℓ . Then $X \times_N Y$ is an H -variety and we deduce from the above properties that

$$R\Gamma_c(X \times_N Y, \mathcal{O}) \simeq R\Gamma_c(X, \mathcal{O}) \overset{L}{\otimes}_{\mathcal{O}N} R\Gamma_c(Y, \mathcal{O})$$

in the derived category $D^b(\mathcal{O}H\text{-Mod})$.

Proposition 1.4. *Assume that Y is an open subvariety of X , stable under the action of H . Denote by $Z = X \setminus Y$ its complement. Then there exists a distinguished triangle in $D^b(\mathcal{O}H\text{-Mod})$:*

$$R\Gamma_c(Y, \mathcal{O}) \longrightarrow R\Gamma_c(X, \mathcal{O}) \longrightarrow R\Gamma_c(Z, \mathcal{O}) \rightsquigarrow$$

Moreover, if Y is both open and closed, then this triangle splits.

Finally, for a smooth quasi-projective variety, Poincaré-Verdier duality [8] establishes a remarkable relation between the cohomology complexes $R\Gamma_c(X, \mathcal{O})$ and $R\Gamma(X, \mathcal{O})$. We shall only need the weaker version for cohomology groups:

Theorem 1.5 (Poincaré duality). *Let X be a smooth quasi-projective variety of pure dimension d . Then if \mathcal{O} is the field K or k , there exists a non-canonical isomorphism of $\mathcal{O}H$ -modules*

$$H_c^i(X, \mathcal{O})^* = \text{Hom}_{\mathcal{O}}(H_c^i(X, \mathcal{O}), \mathcal{O}) \simeq H^{2d-i}(X, \mathcal{O}).$$

2 General results on the torsion

We present in this section some general results on the torsion in the cohomology of Deligne-Lusztig varieties associated with Coxeter elements. We are motivated by the study of the principal ℓ -block of G in the *Coxeter case*, involving only a specific class of prime numbers ℓ . We will only briefly review the results that we will need about the principal ℓ -block but a fully detailed treatment of the Coxeter case can be found in [12].

The problem of finding the torsion in the cohomology of a given variety is a difficult problem. We shall use here all the features of the Deligne-Lusztig varieties associated with Coxeter elements: smooth compactifications, filtrations and remarkable quotients by some finite groups. These will be the principal

ingredients to prove that the contribution of the principal ℓ -block to the torsion in the cohomology is necessarily a cuspidal module (see Corollary 2.10). This addresses the problem of finding where cuspidal composition factors can occur in the cohomology. We will discuss this general problem in the next section.

2.1 Finite reductive groups

We keep the basic assumptions of the introduction, with some slight modification: \mathbf{G} is a quasi-simple algebraic group, together with an isogeny F , some power of which is a Frobenius endomorphism. In other words, there exists a positive integer δ such that F^δ defines an \mathbb{F}_{q^δ} -structure on \mathbf{G} for a certain power q^δ of the characteristic p (note that q might not be an integer). Given an F -stable algebraic subgroup \mathbf{H} of \mathbf{G} , we will denote by H the finite group of fixed points \mathbf{H}^F .

We fix a Borel subgroup \mathbf{B} containing a maximal torus \mathbf{T} of \mathbf{G} such that both \mathbf{B} and \mathbf{T} are F -stable. We denote by \mathbf{U} (resp. \mathbf{U}^-) the unipotent radical of \mathbf{B} (resp. the opposite Borel subgroup \mathbf{B}^-). These groups define a root system Φ with basis Δ , and a set of positive (resp. negative) roots Φ^+ (resp. Φ^-). Note that the corresponding Weyl group W is endowed with an action of F , compatible with the isomorphism $W \simeq N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Therefore, the image under F of a root is a positive multiple of some other root, which will be denoted by $\phi^{-1}(\alpha)$, defining thus a bijection $\phi: \Phi \rightarrow \Phi$. Since \mathbf{B} is also F -stable, this map preserves Δ and Φ^+ . We will also use the notation $[\Delta/\phi]$ for a set of representatives of the orbits of ϕ on Δ .

2.2 Review of the Coxeter case

Recall that the p' -part of the order of G is a product of cyclotomic polynomials $\Phi_d(q)$ for various divisors d of the degrees of W (some precautions must be taken for Ree and Suzuki groups [4]). Therefore, if ℓ is any prime number different from p , it should divide at least one of these polynomials. Moreover, if we assume that ℓ is prime to W^F , then there is a unique d such that $\ell \mid \Phi_d(q)$.

In this paper we will be interested in the case where d is maximal. Since W is irreducible, it corresponds to the case where $d = h$ is the Coxeter number of the pair (W, F) . Explicit values of h can be found in [12]. For a more precise statement – including the Ree and Suzuki groups – recall that a Coxeter element of the pair (W, F) is a product $c = s_{\beta_1} \cdots s_{\beta_r}$ where $\{\beta_1, \dots, \beta_r\} = [\Delta/\phi]$ is any set of representatives of the orbits of the simple roots under the action of ϕ . Then the *Coxeter case* corresponds to the situation where ℓ is prime to $|W^F|$ and satisfies:

- "non-twisted" cases: ℓ divides $\Phi_h(q)$;
- "twisted" cases (Ree and Suzuki groups): ℓ divides the order of T_c for some Coxeter element c .

Note that these conditions ensure that the class of q in k^\times is a primitive h -th root of unity.

As in Section 1.1, the modular framework will be given by an ℓ -modular system (K, Λ, k) , which we require to be big enough for G . We denote by b an idempotent associated with the principal block of ΛG . With the assumptions made on ℓ , the ℓ -component of \mathbf{T}^{cF} is a Sylow ℓ -subgroup of G and as such is a defect group of b . It will be denoted by T_ℓ .

The structure of the block is closely related to the cohomology of the Deligne-Lusztig varieties associated with c . Fix an F^δ -stable representative \dot{c} of c in $N_G(\mathbf{T})$ and define the varieties Y and X by

$$\begin{array}{c} Y = \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}\dot{c}\mathbf{U}\} \\ \pi_c \downarrow / \mathbf{T}^{cF} \\ X = \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}\dot{c}\mathbf{B}\} \end{array}$$

where π_c denotes the restriction to Y of the canonical projection $\mathbf{G}/\mathbf{U} \rightarrow \mathbf{G}/\mathbf{B}$. They are both quasi-projective varieties endowed with a left action of G by left multiplication. Furthermore, \mathbf{T}^{cF} acts on the right of Y and π_c is isomorphic to the corresponding quotient map, so that it induces a G -equivariant isomorphism of varieties $Y/\mathbf{T}^{cF} \xrightarrow{\sim} X$. Combining the results of [9], [17] and [6] we can parametrize the irreducible characters of the principal ℓ -block:

- the non-unipotent characters of bKG are exactly the θ -isotypic components $H_c^r(Y, K)_\theta$ where θ runs over the F^δ -orbits in $\text{Irr } T_\ell \setminus \{1_{T_\ell}\}$;
- the unipotent characters of bKG are the eigenspaces of F^δ on $H_c^*(X, K)$. Each eigenvalue is congruent modulo ℓ to $q^{j\delta}$ for a unique integer $j \in \{0, \dots, h/\delta - 1\}$. We denote by χ_j the corresponding irreducible character. Moreover, each Harish-Chandra series that intersects the block corresponds to a root of unity $\zeta \in K$ and the characters in this series are arranged as follows:

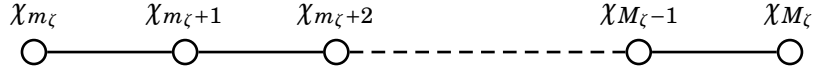
$H_c^r(X, K)$	$H_c^{r+1}(X, K)$	$H_c^{r+2}(X, K)$	\dots	$H_c^{r+M_\zeta-m_\zeta}(X, K)$
χ_{m_ζ}	$\chi_{m_\zeta+1}$	$\chi_{m_\zeta+2}$	\dots	χ_{M_ζ}

Furthermore, the distinction "non-unipotent/unipotent" corresponds to the distinction "exceptional/non-exceptional" in the theory of blocks with cyclic defect groups. The connection is actually much deeper: Hiß, Lübeck and Malle have observed in [16] that the cohomology of the Deligne-Lusztig variety X should not only give the characters of the principal ℓ -block, but also its Brauer tree Γ :

Conjecture HLM (Hiß-Lübeck-Malle). *Let Γ^\bullet denote the graph obtained from the Brauer tree of the principal ℓ -block by removing the exceptional node and all edges incident to it. Then the following holds:*

- The connected components of Γ^\bullet are labeled by the Harish-Chandra series, hence by some roots of unity $\zeta \in K$.

(ii) *The connected component corresponding to a root ζ is*



(iii) *The vertices labeled by χ_{m_ζ} are the only nodes connected to the exceptional node.*

The validity of this conjecture has been checked in all cases where the Brauer tree was known, that is for all quasi-simple groups except the groups of type E_7 and E_8 . A general proof has been exposed in [12], but under a precise assumption on the torsion in the cohomology of Y :

(W) For all $\zeta \in K$ corresponding to an Harish-Chandra series in the block, the generalized $(q^{m_\zeta \delta})$ -eigenspace of F^δ on $bH_c^\bullet(Y, \Lambda)$ is torsion-free.

This assumption concerns only the "minimal" eigenvalues, that is the eigenvalues of F^δ on the cohomology of X in middle degree. Unfortunately, we will not be able to prove it in its full generality for the variety Y , but only for the variety X (see Proposition 2.12).

We have investigated in [12] the consequences of a stronger assumption, concerning all the eigenvalues of F^δ :

(S) The Λ -modules $bH_c^i(Y, \Lambda)$ are torsion-free.

Such an assumption is known to be valid for groups with \mathbb{F}_q -rank 1 (since the corresponding Deligne-Lusztig variety is an irreducible affine curve) and for groups of type A_n [2]. The purpose of this paper is to prove that it is actually valid for any quasi-simple group, including E_7 and E_8 if Conjecture (HLM) holds (see Theorem 3.6). As a byproduct, we obtain a proof of the geometric version of Broué's conjecture in the Coxeter case. We also deduce the planar embedding of Brauer trees for groups of type 2G_2 , F_4 and 2F_4 .

2.3 Torsion and cuspidality

Let $I \subset \Delta$ be a ϕ -stable subset of simple roots. We denote by \mathbf{P}_I the standard parabolic subgroup, by \mathbf{U}_I its unipotent radical and by \mathbf{L}_I its standard Levi complement. The corresponding Weyl group is the subgroup W_I of W generated by the simple reflections in I . All these groups are F -stable. One obtains a Coxeter element c_I of (W_I, F) by removing in c the reflections associated with the simple roots which are not in I . Written with the Borel subgroup $\mathbf{B}_I = \mathbf{B} \cap \mathbf{L}_I$ of \mathbf{L}_I , the Deligne-Lusztig variety X_I associated with c_I is by definition

$$X_I = X_{\mathbf{L}_I}(c_I) = \{g\mathbf{B}_I \in \mathbf{L}_I/\mathbf{B}_I \mid g^{-1}F(g) \in \mathbf{B}_I c_I \mathbf{B}_I\}.$$

Recall that the Harish-Chandra induction and restriction functors are defined over any coefficient ring \mathcal{O} among (K, Λ, k) by

$$\begin{aligned}
R_{L_I}^G : \mathcal{O}L_I\text{-mod} &\longrightarrow \mathcal{O}G\text{-mod} \\
N &\longmapsto \mathcal{O}[G/U_I] \otimes_{\mathcal{O}L_I} N
\end{aligned}$$

and

$$\begin{aligned}
{}^*R_{L_I}^G : \mathcal{O}G\text{-mod} &\longrightarrow \mathcal{O}L_I\text{-mod} \\
M &\longmapsto M^{U_I}.
\end{aligned}$$

By the results in [17], the restriction of the cohomology of X can be expressed in terms of the cohomology of smaller Coxeter varieties, leading to an inductive approach for studying the torsion. Given the assumptions made on ℓ in Section 2.2, we can prove the following:

Proposition 2.1. *If I is a proper ϕ -stable subset of Δ , then the groups $H_c^i(X_I, \Lambda)$ are torsion-free.*

Proof. By induction on the cardinality of I , the case $I = \emptyset$ being trivial. Assume that I is non-empty, and let J be any maximal proper ϕ -stable subset of I . By [17, Corollary 2.10] there exists an isomorphism of varieties $(U_J \cap L_I) \backslash X_I \simeq X_J \times G_m$ which yields the following isomorphism of Λ -modules

$${}^*R_{L_J}^{L_I}(H_c^i(X_I, \Lambda)) \simeq H_c^{i-1}(X_J, \Lambda) \oplus H_c^{i-2}(X_J, \Lambda).$$

Consequently, the Harish-Chandra restriction to L_J of $H_c^*(X_I, \Lambda)_{\text{tor}}$ is a torsion submodule of $H_c^*(X_J, \Lambda)$ and hence it is zero by assumption. In other words, any torsion submodule of $H_c^*(X_I, \Lambda)_{\text{tor}}$ is a cuspidal ΛL_I -module.

Let $r_I = |I/\phi|$ be the number of ϕ -orbits in I . Following [17], we can define a smooth compactification \overline{X}_I of X_I . It has a filtration by closed L_I -subvarieties $\overline{X}_I = D_{r_I}(I) \supset D_{r_I-1}(I) \supset \cdots \supset D_0(I) = L_I/B_I$ such that each pair $(D_a(I), D_{a-1}(I))$ leads to the following long exact sequences of kL_I -modules:

$$\cdots \longrightarrow \bigoplus_{\substack{J \subset I \text{ } \phi\text{-stable} \\ |J/\phi|=a}} R_{L_J}^{L_I}(H_c^i(X_J, k)) \longrightarrow H^i(D_a(I), k) \longrightarrow H^i(D_{a-1}(I), k) \longrightarrow \cdots$$

Since kL_I is a semi-simple algebra, we are in the following situation:

- any cuspidal composition factor of a kL_I -module M is actually a direct summand of M . Therefore we can naturally define the cuspidal part M_{cusp} of M as the sum of all cuspidal submodules;
- if $J \neq I$ then an induced module $R_{L_J}^{L_I}(M)$ has a zero cuspidal part;
- by Poincaré duality (see Theorem 1.5) $H^i(\overline{X}_I, k)$ is isomorphic to the k -dual of the kL_I -module $H^{2r_I-i}(\overline{X}_I, k)$.

Consequently, we can argue as in [17] and use the following isomorphisms :

$$H_c^i(X_I, k)_{\text{cusp}} \simeq H^i(\overline{X}_I, k)_{\text{cusp}} \simeq H^{2r_I-i}(\overline{X}_I, k)_{\text{cusp}}^* \simeq H_c^{2r_I-i}(X_I, k)_{\text{cusp}}^*$$

to deduce that $H_c^i(X_I, k)$ has a zero cuspidal part whenever $i > r_I$.

Finally, $H_c^i(X_I, \Lambda)_{\text{tor}} \otimes_{\Lambda} k$ is a cuspidal submodule of $H_c^i(X_I, \Lambda) \otimes_{\Lambda} k$ which is also a submodule of $H_c^i(X_I, k)$ by the universal coefficient theorem (see Theorem 1.1). Therefore it must be zero if $i \neq r_I$ and $H_c^i(X_I, \Lambda)$ is torsion-free. Note that

the cohomology group in middle degree is also torsion-free since X_I is an irreducible affine variety: we have indeed $H_c^{r_I-1}(X_I, k) = 0$ so that $H_c^{r_I}(X_I, \Lambda)_{\text{tor}} = 0$ by the universal coefficient theorem. \square

Remark 2.2. We have actually shown that the cohomology of a Deligne-Lusztig variety associated with a Coxeter element is torsion-free whenever ℓ does not divide the order of the corresponding finite reductive group. One can conjecture that this should hold for any Deligne-Lusztig variety. Note that the assumption on ℓ is crucial, otherwise an induced module can have cuspidal composition factors. For example if ℓ divides $\Phi_h(q)$, then $R_T^G(k) = k[G/B]$ has at least one cuspidal composition factor – denoted by S_0 in [12].

As an immediate consequence of Proposition 2.1 and the isomorphism in [17, Corollary 2.10], we obtain:

Corollary 2.3. *The torsion part of any group $H_c^i(X, \Lambda)$ is a cuspidal ΛG -module.*

We can therefore reduce the problem of finding the torsion in the cohomology of X to the problem of finding where cuspidal composition factors can occur in the kG -modules $H_c^i(X, k)$. This is the general approach that we shall use throughout this paper. It is justified by the fact that if we assume that the modules $H_c^i(X, \Lambda)$ are torsion-free and that Conjecture (HLM) holds, then by [12] cuspidal composition factors can occur only in $H_c^i(X, k)$.

2.4 Reduction: from Y to X

In this section we shall give conditions on X ensuring that the principal part of the cohomology of Y is torsion-free. Unlike Y , the variety X has a smooth compactification (the compactification of Y constructed in [3] is only rationally smooth in general). Therefore we can use duality theorems to study precise concentration properties of cuspidal modules in the cohomology of X (see Sections 2.5 and 3.1). We will not only restrict the cohomology of Y to the principal block b of ΛG but also to the principal block b' of $\Lambda \mathbf{T}^{cF}$. This is not a strong restriction since the complexes $bR\Gamma_c(Y, \Lambda)$ and $bR\Gamma_c(Y, \Lambda)b'$ are conjecturally isomorphic in $K^b(\Lambda G\text{-Mod})$. Note that this is already the case when the coefficient ring is K (see [6, Theorem 5.24]). By Proposition 1.3 we have

$$R\Gamma_c(Y, \Lambda)b' \simeq R\Gamma_c(Y, \Lambda) \otimes_{\Lambda T_{\ell'}} \Lambda \simeq R\Gamma_c(Y/T_{\ell'}, \Lambda)$$

where $T_{\ell'}$ is the ℓ' -component of the abelian group \mathbf{T}^{cF} . The quotient variety $Y/T_{\ell'}$ will be denoted by Y_{ℓ} .

From now on, we shall work exclusively with Y_{ℓ} instead of Y . This will not have any effect on the results we have deduced from (W) and (S) in [12]. We start by proving an analog of [17, Corollary 2.10] for the variety Y_{ℓ} :

Proposition 2.4. *Let I be a maximal proper ϕ -stable subset of Δ . Then there exists an isomorphism of Λ -modules*

$$H_c^i(U_I \setminus Y_{\ell}, \Lambda) \simeq {}^*R_{L_I}^G(H_c^i(Y_{\ell}, \Lambda)) \simeq H_c^{i-1}(X_I, \Lambda) \oplus H_c^{i-2}(X_I, \Lambda).$$

Proof. Recall that the isomorphism $U_I \backslash X \simeq X_I \times \mathbf{G}_m$ constructed in [17] is not G -equivariant. We can check that it is nevertheless V_I -equivariant, where $\mathbf{V}_I = \mathbf{U} \cap \mathbf{L}_I$ is a maximal unipotent subgroup of \mathbf{L}_I . We argue as in [2]: the isomorphism cannot be lifted up to Y_ℓ but only to an abelian covering of this variety. Let Y° be the preimage in Y of a connected component of $U \backslash Y$. It is stable by the action of U but not necessarily by the action of \mathbf{T}^{cF} which permutes transitively the connected components of $U \backslash Y$ (since $U \backslash Y / \mathbf{T}^{cF} \simeq U \backslash X$ is connected). If we denote by H the stabilizer of the component $U \backslash Y^\circ$ in \mathbf{T}^{cF} , we have the following $V_I \times (\mathbf{T}^{cF})^{\text{op}}$ -equivariant isomorphism

$$U_I \backslash Y^\circ \times_H \mathbf{T}^{cF} \simeq U_I \backslash Y. \quad (2.5)$$

By Abhyankar's lemma, we can now lift the Galois covering $U \backslash Y^\circ \rightarrow U \backslash X \simeq (\mathbf{G}_m)^r$ up to a covering $\varpi : (\mathbf{G}_m)^r \rightarrow (\mathbf{G}_m)^r$ with Galois group $\prod \mu_{m_i}$. Details of this construction are given in [2] and [10]. We define the variety \tilde{Y} using the following diagram in which all the squares are cartesian:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\quad} & (\mathbf{G}_m)^r \\ \downarrow /N & & \downarrow \varpi /N \\ U_I \backslash Y^\circ & \xrightarrow{\quad} & U \backslash Y^\circ \\ \downarrow \pi^\circ /H & & \downarrow \pi^\circ /H \\ U_I \backslash X & \xrightarrow{\quad} & U \backslash X \simeq (\mathbf{G}_m)^r \end{array} \quad \begin{array}{c} \nearrow / \prod \mu_{m_i} \\ \\ \end{array} \quad (2.6)$$

Under the isomorphism $U_I \backslash X \simeq X_I \times \mathbf{G}_m$ the map $U_I \backslash X \rightarrow (\mathbf{G}_m)^r$ decomposes into $\pi_I \times \text{id}_{\mathbf{G}_m}$ where $\pi_I : X_I \rightarrow V_I \backslash X_I \simeq (\mathbf{G}_m)^{r-1}$ is the quotient map associated with X_I . In particular, we can form the following fiber product

$$\begin{array}{ccc} \tilde{Y}_I & \xrightarrow{\quad} & (\mathbf{G}_m)^{r-1} \\ \downarrow / \prod_{i \neq j} \mu_{m_i} & & \downarrow / \prod_{i \neq j} \mu_{m_i} \\ X_I & \xrightarrow{\pi_I} & (\mathbf{G}_m)^{r-1} \end{array} \quad (2.7)$$

in order to decompose the variety \tilde{Y} into a product $\tilde{Y} \simeq \tilde{Y}_I \times \mathbf{G}_m$ which is compatible with the group decomposition $\prod \mu_{m_i} \simeq (\prod_{i \neq j} \mu_{m_i}) \times \mu_{m_j}$.

Recall from [2] that the cohomology of \mathbf{G}_m endowed with the action of μ_m can be represented by a complex concentrated in degrees 1 and 2:

$$0 \longrightarrow \Lambda \mu_m \xrightarrow{\zeta-1} \Lambda \mu_m \longrightarrow 0$$

where ζ is any primitive m -th root of 1. Such a complex will be denoted by $Z(\mu_m)[-1]$ so that the cohomology of $Z(\mu_m)$ vanishes outside the degrees 0 and 1. As in [2, Section 3.3.2], we shall write $Z_{\mathbf{T}^{cF}}(\mu_m)$ for $Z(\mu_m) \otimes_{\Lambda \mu_m} \Lambda \mathbf{T}^{cF}$.

Using Proposition 1.3, we can now express the previous constructions in cohomological terms. Equation 2.5 together with Diagram 2.6 leads to

$$\begin{aligned} \mathrm{R}\Gamma_c(U_I \setminus Y, \Lambda) &\simeq \mathrm{R}\Gamma_c(U_I \setminus Y^\circ, \Lambda) \otimes_{\Lambda H}^L \Lambda \mathbf{T}^{cF} \\ &\simeq \mathrm{R}\Gamma_c(\tilde{Y}, \Lambda) \otimes_{\Lambda \prod_{i \neq j} \mu_{m_i}}^L \Lambda \mathbf{T}^{cF} \end{aligned}$$

Using the isomorphism $\tilde{Y} \simeq \tilde{Y}_I \times \mathbf{G}_m$ and the notation that we have introduced, it can be written as

$$\begin{aligned} \mathrm{R}\Gamma_c(U_I \setminus Y, \Lambda) &\simeq \left(\mathrm{R}\Gamma_c(\tilde{Y}_I, \Lambda) \otimes_{\Lambda \prod_{i \neq j} \mu_{m_i}}^L \Lambda \mathbf{T}^{cF} \right) \otimes_{\Lambda \mathbf{T}^{cF}}^L \left(\mathrm{R}\Gamma_c(\mathbf{G}_m, \Lambda) \otimes_{\Lambda \mu_{m_j}}^L \Lambda \mathbf{T}^{cF} \right) \\ &\simeq \left(\mathrm{R}\Gamma_c(\tilde{Y}_I, \Lambda) \otimes_{\Lambda \prod_{i \neq j} \mu_{m_i}}^L \Lambda \mathbf{T}^{cF} \right) \otimes_{\Lambda \mathbf{T}^{cF}}^L Z_{\mathbf{T}^{cF}}(\mu_{m_j})[-1] \end{aligned} \quad (2.8)$$

in the derived category $D^b(\mathrm{Mod}\text{-}\Lambda \mathbf{T}^{cF})$.

In general, the cohomology groups of the complex $Z_{\mathbf{T}^{cF}}(\mu_{m_j})$ are endowed with a non-trivial action of \mathbf{T}^{cF} since the map $\mu_{m_j} \rightarrow \mathbf{T}^{cF}$ is not surjective. One can actually compute explicitly the image of this map: by [2, Proposition 3.5] it is precisely the group $N_c(Y_{c,c_I})$ (see [1, Section 4.4.2] for the definition). Therefore we obtain

$$H^0(Z_{\mathbf{T}^{cF}}(\mu_{m_j})) = H^1(Z_{\mathbf{T}^{cF}}(\mu_{m_j})) \simeq \Lambda[\mathbf{T}^{cF}/N_c(Y_{c,c_I})].$$

By [1, Proposition 4.4], the quotient $\mathbf{T}^{cF}/N_c(Y_{c,c_I})$ is isomorphic to $\mathbf{T}^{c_I F}/N_{c_I}(Y_{c,c_I})$ which has order prime to ℓ . In particular, the image of the map $\mu_{m_j} \rightarrow \mathbf{T}^{cF}$ contains the ℓ -Sylow subgroup of \mathbf{T}^{cF} and the map $\mu_{m_j} \rightarrow \mathbf{T}^{cF}/T_{\ell'}$ is definitely onto. Since $Z_{\mathbf{T}^{cF}}(\mu_{m_j})$ fits into the following distinguished triangle in $D^b(\Lambda \mathbf{T}^{cF}\text{-Mod})$

$$\Lambda[\mathbf{T}^{cF}/N_c(Y_{c,c_I})] \rightarrow Z_{\mathbf{T}^{cF}}(\mu_{m_j}) \rightarrow \Lambda[\mathbf{T}^{cF}/N_c(Y_{c,c_I})][-1] \rightsquigarrow$$

we deduce that the coinvariants under T_ℓ have a relatively simple shape

$$\Lambda \rightarrow Z_{\mathbf{T}^{cF}}(\mu_{m_j}) \otimes_{\Lambda T_{\ell'}}^L \Lambda \rightarrow \Lambda[-1] \rightsquigarrow \quad (2.9)$$

Together with the expression of $\mathrm{R}\Gamma_c(\tilde{Y}_I, \Lambda) \otimes_{\Lambda \prod_{i \neq j} \mu_{m_i}}^L \Lambda \mathbf{T}^{cF}$ given in Formula 2.8, this triangle yields

$$\mathrm{R}\Gamma_c(\tilde{Y}_I, \Lambda) \otimes_{\Lambda \prod_{i \neq j} \mu_{m_i}}^L \Lambda[-1] \rightarrow \mathrm{R}\Gamma_c(U_I \setminus Y_\ell, \Lambda) \rightarrow \mathrm{R}\Gamma_c(\tilde{Y}_I, \Lambda) \otimes_{\Lambda \prod_{i \neq j} \mu_{m_i}}^L \Lambda[-2] \rightsquigarrow$$

and then simply

$$\mathrm{R}\Gamma_c(X_I, \Lambda)[-1] \rightarrow \mathrm{R}\Gamma_c(U_I \setminus Y_\ell, \Lambda) \rightarrow \mathrm{R}\Gamma_c(X_I, \Lambda)[-2] \rightsquigarrow$$

by definition of \tilde{Y}_I .

We claim that the connecting maps $H_c^{i-2}(X_I, \Lambda) \rightarrow H_c^i(X_I, \Lambda)$ coming from the long exact sequence associated with the previous triangle are actually zero. Since the triangle 2.9 splits over K , the scalar extension of these morphisms are

zero. But by Proposition 2.1 the modules $H_c^i(X_I, \Lambda)$ are torsion-free, so that the connecting morphisms are indeed zero over Λ . Consequently, we obtain short exact sequences

$$0 \longrightarrow H_c^{i-1}(X_I, \Lambda) \longrightarrow H_c^i(Y_\ell, \Lambda)^{U_I} \longrightarrow H_c^{i-2}(X_I, \Lambda) \longrightarrow 0$$

which finishes the proof. \square

Corollary 2.10. *The torsion part of any cohomology group $H_c^i(Y_\ell, \Lambda)$ is a cuspidal ΛG -module.*

As a consequence, Assumption (S) holds for Y_ℓ if every cuspidal module in the block occurs in the cohomology group in middle degree only. By construction of Y_ℓ , it is actually sufficient to consider the cohomology of X :

Corollary 2.11. *Assume that one of the following holds:*

- (1) *The cohomology of X over Λ is torsion-free and Conjecture (HLM) holds.*
- (2) *Cuspidal composition factors occur in $H_c^i(X, k)$ for $i = r$ only.*

Then the cohomology of Y_ℓ over Λ is torsion-free.

Proof. We have already mentioned at the end of Section 2.3 that assertions (1) and (2) are equivalent. Denote by $\pi_\ell : Y_\ell \longrightarrow X$ the quotient map by the ℓ -Sylow subgroup of \mathbf{T}^{cF} . The push-forward of the constant sheaf \underline{k}_{Y_ℓ} on Y_ℓ is obtained from successive extensions of constant sheaves \underline{k}_X :

$$(\pi_\ell)_*(\underline{k}_{Y_\ell}) \simeq \begin{array}{c} \underline{k}_X \\ \underline{k}_X \\ \vdots \\ \underline{k}_X \end{array}.$$

In the derived category, the complex $R\Gamma_c(Y_\ell, k) = R\Gamma_c(X, (\pi_\ell)_*(\underline{k}_{Y_\ell}))$ can also be obtained from extensions of $R\Gamma_c(X, k)$'s. In other words, there exists a family of complexes $R\Gamma_c(Y_\ell, k) = C_0, C_1, \dots, C_n = R\Gamma_c(X, k)$ that fit into the following distinguished triangles in $D^b(kG\text{-Mod-}kT_\ell)$:

$$C_{i+1} \longrightarrow C_i \longrightarrow R\Gamma_c(X, k) \rightsquigarrow$$

If we apply the functor $\mathrm{RHom}_{kG}^\bullet(P, -)$ to each of these triangles, we can deduce that the complexes $\mathrm{RHom}_{kG}^\bullet(P, C_i)$ are concentrated in degree r whenever $\mathrm{RHom}_{kG}^\bullet(P, C_n)$ is. Taking P to be the projective cover of any cuspidal kG -module, we conclude that cuspidal composition factors of the cohomology of Y_ℓ can only occur in middle degree. \square

2.5 On the assumption (W) for X

We give here a proof of the analog of the assumption (W) for the Deligne-Lusztig variety X , using the smooth compactification \bar{X} constructed in [9].

Proposition 2.12. *If λ is an eigenvalue of F^δ on $H_c^r(X, K)$, then the Λ -modules $H_c^i(X, \Lambda)_{(\lambda)}$ are all torsion-free.*

Proof. In line with [17], we shall denote by $D = D_{r-1}(\Delta)$ the complement of X in \bar{X} . Let I be a ϕ -stable subset of Δ . By [17, Section 7], the generalized (λ) -eigenspaces of F^δ on $H_c^i(X_I, K)$ are zero whenever $i \neq |I/\phi|$. By Proposition 2.1, this remains true for cohomology groups with coefficients in k . From the long exact sequences

$$\cdots \longrightarrow \bigoplus_{\substack{I \text{ } \phi\text{-stable} \\ |I/\phi|=a}} R_{L_I}^G(H_c^i(X_I, k)) \longrightarrow H_c^i(D_a(\Delta), k) \longrightarrow H_c^i(D_{a-1}(\Delta), k) \longrightarrow \cdots \quad (2.13)$$

we deduce that $H_c^i(D, k)_{(\lambda)}$ is zero if $i \geq r$. Consequently, for all $i > r$ we have

$$H_c^i(X, k)_{(\lambda)} \simeq H^i(\bar{X}, k)_{(\lambda)}.$$

The eigenvalue $\mu = q^{2r} \lambda^{-1}$ is a "maximal" eigenvalue of F^δ and as such does not occur in the cohomology of the varieties X_I for proper subsets I . From the previous long exact sequences we deduce that the generalized $\bar{\mu}$ -eigenspace of F^δ on $H_c^i(D, k)$ is always zero. Since \bar{X} is a smooth variety, we can apply Poincaré duality (see Theorem 1.5) in order to obtain the following isomorphisms:

$$H_c^i(X, k)_{(\lambda)} \simeq H^i(\bar{X}, k)_{(\lambda)} \simeq (H^{2r-i}(\bar{X}, k)_{(\mu)})^* \simeq (H_c^{2r-i}(X, k)_{(\mu)})^*.$$

Since X is an irreducible affine variety, the cohomology of X vanishes outside the degrees $r, \dots, 2r$. This proves that $H_c^i(X, k)_{(\lambda)} = 0$ whenever $i > r$, and the result follows from the universal coefficient theorem. \square

Remark 2.14. One cannot deduce that the assumption (W) holds for Y_ℓ using this result. The method that we used in the proof of Corollary 2.11 does not preserve the generalized eigenspaces of F^δ .

Note however that Bonnafé and Rouquier have constructed in [3] a compactification \bar{Y}_ℓ of Y_ℓ such that $\bar{Y}_\ell \setminus Y_\ell \simeq \bar{X} \setminus X$. This compactification is only rationally smooth in general, but if it happened to be k -smooth, the previous method would extend to Y_ℓ and finish the proof of the conjecture of Hiß-Lübeck-Malle.

3 Cuspidal composition factors in $H_c^\bullet(X, k)$

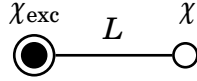
We have reduced the proof of the assumption (S) to showing that the kG -modules $H_c^i(X, k)$ have no cuspidal composition factors unless $i = r$. In other words, the cohomology of $\mathrm{RHom}_{kG}^\bullet(\bar{P}_L, \mathrm{R}\Gamma_c(X, k))$ should vanish outside the degree r whenever L is cuspidal.

Throughout this section and unless otherwise specified, we shall always assume that Conjecture (HLM) holds for (G, F) (which is known to be true except for the groups of type E_7 or E_8). In that case, the above result can be deduced from the characteristic zero case if we can show that the cohomology of

$\mathrm{RHom}_{\Lambda G}^\bullet(P_L, \mathrm{R}\Gamma_c(X, \Lambda))$ is torsion-free. Indeed, the irreducible unipotent components of $[P_L]$ are the characters χ_{m_ζ} 's, and they occur in the cohomology of X in middle degree only. We shall divide the proof according to the depth of χ_{m_ζ} : if χ_{m_ζ} is cuspidal, we prove that the contribution of P_L to the cohomology of X and its compactification \bar{X} are the same. In the case where χ_{m_ζ} is not cuspidal, it is induced from the cohomology of a smaller Coxeter variety and we show that the contribution of P_L can be computed in terms of this cohomology, which we know to be torsion-free.

3.1 Harish-Chandra series of length 1

We start by dealing with the case of cuspidal kG -modules that occur as ℓ -reductions of cuspidal unipotent characters. By Conjecture (HLM), these correspond to subtrees of the Brauer tree of the form



where χ is a cuspidal unipotent character and L is the unique composition factor of the ℓ -reduction of χ . By [17, Proposition 4.3], the character χ occurs only in the cohomology of X in middle degree. The following is a modular analog of this result.

Proposition 3.1. *Assume that Conjecture (HLM) holds for (G, F) . Let χ be a cuspidal character of G occurring in $H_c^r(X, K)$. The ℓ -reduction of χ gives a unique simple cuspidal kG -module and it occurs as a composition factor of $H_c^i(X, k)$ for $i = r$ only.*

Proof. We keep the notation used in the course of the proof of Proposition 2.12: D is the complement of X in the compactification \bar{X} . Recall that the cohomology of D can be computed in terms of induced characters afforded by the cohomology of smaller Coxeter varieties. Since χ is cuspidal, we can therefore use the long exact sequences 2.13 to show that χ does not occur in $H_c^*(D, K)$.

Denote by L the unique composition factor of the ℓ -reduction of χ . By Conjecture (HLM), L does not appear in the ℓ -reduction of any other unipotent character of the block. Therefore, using Proposition 2.1 and the long exact sequences 2.13 again, we deduce that L cannot occur as a composition factor of $H_c^*(D, k)$. In particular, if we denote by $\bar{P}_L \in kG\text{-mod}$ the projective cover of L then

$$\mathrm{Hom}_{kG}(\bar{P}_L, H_c^i(X, k)) \simeq \mathrm{Hom}_{kG}(\bar{P}_L, H_c^i(\bar{X}, k)).$$

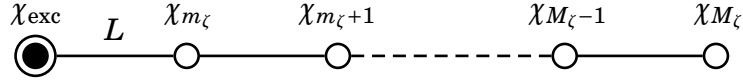
Moreover, since duality preserve cuspidality, the same argument applies for the dual of the cohomology of D . Subsequently, using Poincaré duality, we obtain

$$\mathrm{Hom}_{kG}(\bar{P}_L, H_c^i(\bar{X}, k)) \simeq \mathrm{Hom}_{kG}(\bar{P}_L, H_c^{2r-i}(\bar{X}, k)^*) \simeq \mathrm{Hom}_{kG}(\bar{P}_L, H_c^{2r-i}(X, k)^*).$$

Now the cohomology of the irreducible affine variety X vanishes outside the degrees $r, \dots, 2r$. It follows that $\mathrm{Hom}_{kG}(\bar{P}_L, H_c^i(X, k)) = 0$ whenever $i > r$. \square

3.2 Induced characters

We now turn to the case of cuspidal kG -modules occurring as composition factors of the ℓ -reduction of a non-cuspidal unipotent character. If we assume that Conjecture (HLM) holds, then by the results of [12] the situation corresponds to branches of the following form:



where $m_\zeta < M_\zeta$.

At the level of characters, χ_{m_ζ} is obtained by inducing a cuspidal unipotent character occurring in the cohomology of a smaller Coxeter variety X_I . The projective cover of L can in turn be obtained by a suitable induction using the following result.

Lemma 3.2. *Let I be an F -stable subset of Δ , and J be its complement. Let $\mathbf{V}_I = \mathbf{L}_I \cap \mathbf{U}$ (resp. $\mathbf{V}_J = \mathbf{L}_J \cap \mathbf{U}$) be the maximal unipotent subgroup of \mathbf{L}_I (resp. \mathbf{L}_J) contained in \mathbf{U} . Then there is an isomorphism of varieties, compatible with the actions of F^δ and $V_I \times V_J$*

$$(U_I \cap U_J) \backslash X \simeq X_I \times X_J.$$

Proof. For $\alpha \in \Phi^+$ we denote by \mathbf{U}_α the corresponding one-parameter subgroup of \mathbf{U} and we put $\mathbf{U}_\alpha^\# = \mathbf{U}_\alpha \setminus \{1\}$. By [17, Theorem 2.6], there is a U -equivariant isomorphism of varieties

$$X \simeq \left\{ u \in \mathbf{U} \mid u^{-1}F(u) \in \prod_{\alpha \in I} \mathbf{U}_\alpha^\# \prod_{\beta \in J} \mathbf{U}_\beta^\# \right\}.$$

The method used in [10, Proposition 1.2] to describe the quotient of \mathbf{B} by the group $D(\mathbf{U})^F$ extends to any unipotent normal subgroup of \mathbf{B} normalized by \mathbf{T} instead of $D(\mathbf{U})$ (see [11, Section 2.3.2] for more details). Using the isomorphism $(\mathbf{U}_I \cap \mathbf{U}_J) \backslash \mathbf{U} \simeq \mathbf{V}_I \times \mathbf{V}_J$, we can therefore realize the quotient of the Deligne-Lusztig variety by $U_I \cap U_J$ as:

$$(U_I \cap U_J) \backslash X \simeq \left\{ (\bar{u}, v_1, v_2) \in \mathbf{U} \times \mathbf{V}_I \times \mathbf{V}_J \mid \begin{array}{l} \pi_{\mathbf{U}_I \cap \mathbf{U}_J}(\bar{u}) = v_1^{-1}F(v_1)v_2^{-1}F(v_2) \\ \bar{u} \in \prod_{\alpha \in I} \mathbf{U}_\alpha^\# \prod_{\beta \in J} \mathbf{U}_\beta^\# \end{array} \right\}.$$

This can be rephrased in the $V_I \times V_J$ -equivariant isomorphism $V \backslash X \simeq X_{\mathbf{L}_I}(c_I) \times X_{\mathbf{L}_J}(c_J) \simeq X_I \times X_J$. \square

Recall that a *regular* character V_J is any linear character of V_J , trivial on $D(\mathbf{V}_J)^F$ such that the induced character on the abelian group $V_J/D(\mathbf{V}_J)^F$ is a product of non-trivial characters (see [2, Section 2] for more details). If ψ is such a character, and Λ_ψ is the corresponding one-dimensional ΛV_J -module we define

$$\Gamma_I = \text{Ind}_U^G \text{Res}_U^{V_I \times V_J} (\Lambda V_I \otimes_\Lambda \Lambda_\psi)$$

where the restriction is taken through the map $U \longrightarrow U/U_I \cap U_J \simeq V_I \times V_J$.

Lemma 3.3. *Let I to be the minimal F -stable subset of Δ such that ${}^*R_{L_I}^G(\chi_{m_\zeta}) \neq 0$. Then P_L is a direct summand of Γ_I .*

Proof. Since Γ_I is projective, it is enough to show that χ_{m_ζ} is the only constituent of $[\Gamma_I]$ in the Harish-Chandra series corresponding to ζ (cut by the principal block). To this end, we can compute the contribution of Γ_I to the cohomology of the Coxeter variety using the previous lemma.

$$\begin{aligned} \mathrm{RHom}_{\Lambda G}^\bullet(\Gamma_I, \mathrm{R}\Gamma_c(X, \Lambda)) &\simeq \mathrm{RHom}_{\Lambda U}^\bullet(\mathrm{Res}_U^{V_I \times V_J}(\Lambda V_I \otimes_\Lambda \Lambda_\psi), \mathrm{R}\Gamma_c(X, \Lambda)) \\ &\simeq \mathrm{RHom}_{\Lambda(V_I \times V_J)}^\bullet(\Lambda V_I \otimes_\Lambda \Lambda_\psi, \mathrm{R}\Gamma_c((U_I \cap U_J) \backslash X, \Lambda)) \\ &\simeq \mathrm{R}\Gamma_c(X_I, \Lambda) \otimes_\Lambda \mathrm{RHom}_{\Lambda V_J}^\bullet(\Lambda_\psi, \mathrm{R}\Gamma_c(X_J, \Lambda)). \end{aligned}$$

From [2, Theorem 3.10] we deduce that the complex $\mathrm{RHom}_{\Lambda V_J}^\bullet(\Lambda_\psi, \mathrm{R}\Gamma_c(X_J, \Lambda))$ is quasi-isomorphic to the trivial module shifted in degree $r_J = \dim X_J$ and we obtain finally

$$\mathrm{RHom}_{\Lambda G}^\bullet(\Gamma_I, \mathrm{R}\Gamma_c(X, \Lambda)) \simeq \mathrm{R}\Gamma_c(X_I, \Lambda)[-r_J]. \quad (3.4)$$

Tensoring this quasi-isomorphism with K gives the contribution of the character of Γ_I to the graded character afforded by the cohomology of X . In particular, a character χ in the series associated to ζ and in the principal ℓ -block occurs in Γ_I if and only if the corresponding eigenvalue of F^δ occurs in $H_c^*(X_I, K)$. Since I is chosen to be minimal for ζ , then by [17, Theorem 7.1] this eigenvalue is necessarily $\zeta q^{\delta \dim X_I/2}$ and it occurs in degree $r = \dim X = \dim X_I + \dim X_J$ only. This forces χ to be the unique character of the series occurring in $H_c^*(X, K)$, that is χ_{m_ζ} .

Finally, using the shape of the Brauer tree, any projective indecomposable module that has χ_{m_ζ} as its only constituent in the Harish-Chandra series associated to ζ must be the projective cover of L . \square

Proposition 3.5. *Assume that Conjecture (HLM) holds. Let L be the unique simple cuspidal kG -module that occur in any ℓ -reduction of χ_{m_ζ} . Then L occurs as a composition factor of $H_c^i(X, k)$ for $i = r$ only.*

Proof. Let I be minimal such that ${}^*R_{L_I}^G(\chi_{m_\zeta}) \neq 0$. By 3.4 and Proposition 2.1 the cohomology of $\mathrm{RHom}_{\Lambda G}^\bullet(\Gamma_I, \mathrm{R}\Gamma_c(X, \Lambda))$ is torsion-free. Since P_L is a direct summand of Γ_I the same holds for P_L . Consequently, the cohomology of $\mathrm{RHom}_{\Lambda G}^\bullet(P_L, \mathrm{R}\Gamma_c(X, \Lambda))$ vanishes outside the degree r since it already does over K by [17, Proposition 4.3]. \square

3.3 Main results

We now have all the ingredients for proving that there is no torsion in the cohomology of Y_ℓ . Recall that we have shown that the torsion part of the cohomology is necessarily a cuspidal ΛG -module (see Corollary 2.10). By the universal coefficient theorem, it is therefore sufficient to prove that the module $H_c^i(Y_\ell, k)$ has no cuspidal composition factors if $i > r$. By Corollary 2.11, this property holds whenever it holds for the Deligne-Lusztig variety X . In the framework of

derived categories, we are then reduced to show that for any cuspidal kG -module L lying in the block, the complex

$$\mathrm{RHom}_{\Lambda G}^\bullet(\overline{P}_L, \mathrm{R}\Gamma_c(X, k))$$

is quasi-isomorphic to a complex concentrated in degree r . If L happens to be a composition factor of the ℓ -reduction of a cuspidal unipotent character, then by Proposition 3.1 it cannot occur outside the cohomology in middle degree. Corollary 3.5 deals with the case where L is a composition factor of the ℓ -reduction of an induced character, but we have to assume that Conjecture (HLM) holds.

Theorem 3.6. *Let G be a quasi-simple group. Assume that Conjecture (HLM) holds. Then in the set-up of Section 2.2, the Λ -modules $bH_c^i(Y_\ell, \Lambda)$ are torsion-free.*

We deduce from [12, Theorem 4.12] the geometric version of Broué's conjecture holds for any quasi-simple group except E_7 and E_8 . This extends significantly the previous results of Puig [19] (for $\ell \mid q-1$), Rouquier [20] (for $\ell \mid \phi_h(q)$ and $r=1$) and Bonnafé-Rouquier [2] (for $\ell \mid \phi_h(q)$ and (G, F) of type A_n).

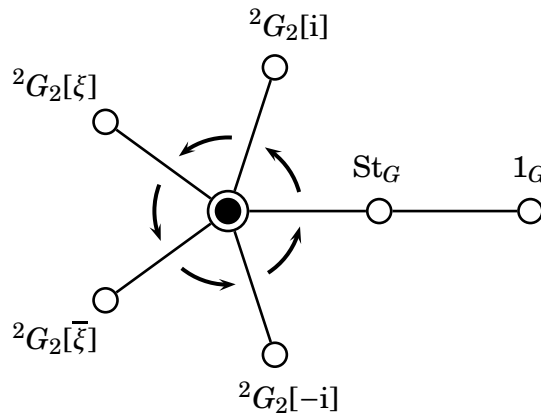
Theorem 3.7. *Let (G, F) be a quasi-simple group different from E_7 and E_8 and c be a Coxeter element of (W, F) . Let ℓ be a prime number not dividing the order of W^F and satisfying one of the two following assumptions, depending on the type of (G, F) :*

- "non-twisted" cases: ℓ divides $\Phi_h(q)$;
- "twisted" cases: ℓ divides the order of T_c .

Then the complex $b\mathrm{R}\Gamma_c(Y(\dot{c}), \Lambda)b'$ induces a splendid and perverse equivalence between the principal ℓ -blocks $b\Lambda G$ and $b'\Lambda N_G(T_c)$

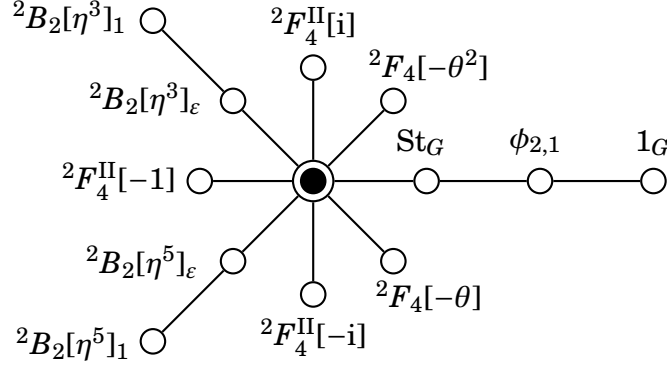
Using [12, Theorem 4.14] we also deduce the planar embedding of the Brauer tree of the principal ℓ -block for groups of type 2G_2 , 2F_4 and F_4 (compare with [14] and [15]).

Theorem 3.8. (i) *Assume that ℓ is odd and divides $q^2 - q\sqrt{3} + 1$. Then the planar embedded Brauer tree of the principal ℓ -block of the Ree group ${}^2G_2(q^2)$ is*



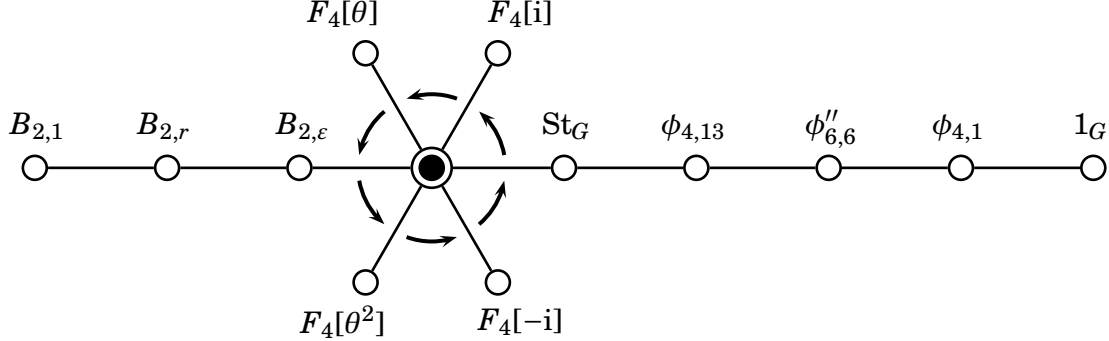
where $i = \xi^3$ and ξ is the unique 12-th root of unity in Λ^\times congruent to q^5 modulo ℓ .

(ii) Assume that ℓ divides $q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$. Then the planar embedded Brauer tree of the principal ℓ -block of the simple group of type ${}^2F_4(q)$ is



where θ (resp. i , resp. η) the unique primitive 3-rd (resp. 4-th, resp. 8-th) root of unity in Λ^\times congruent to q^8 (resp. q^6 , resp. q^3) modulo ℓ .

(iii) Assume that $\ell \neq 2, 3$ and divides $q^4 - q^2 + 1$. Then the planar embedded Brauer tree of the principal ℓ -block of the simple group of type $F_4(q)$ is



where θ (resp. i) is the unique third (resp. fourth) root of unity in Λ^\times congruent to q^4 (resp. q^3) modulo ℓ .

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