Perturbations of C1-diffeomorphisms and dynamics of generic conservative diffeomorphisms of surface

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0 Introduction

When one studies a mechanical system with no dissipation, the motion is governed by some ordinary differential equations which preserve a volume form. As an example, the forced damped pendulum with no friction gives rise to a conservative dynamics on the open annulus (see [Hu]): let θ be the angle between the axis of a rigid pendulum with the vertical and $\dot\theta$ be the angle velocity. The configurations $(\theta, \dot\theta)$ live on the infinite annulus $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ and the motion is governed by the equation

$$\frac{d^2}{dt^2}\theta = -\sin(2\pi\theta) + h(t),$$

where h is the forcing. The volume form $d\theta \wedge d\dot\theta$ is preserved. By integrating the system, one obtains a global flow $(\phi_t)_{t\in\mathbb{R}}$, that is a familly of diffeomorphisms of $\mathbb{A}$ which associates to any initial configuration $(\theta, \dot\theta)$ at time 0 the configuration $\phi_t(\theta, \dot\theta)$ at time t. If the forcing h is T-periodic, the flow satisfies the additional relation $\phi_{t+T} = \phi_t \circ \phi_T$ and the dynamics of the pendulum can be studied through the iterates of the conservative annulus diffeomorphism $f = \phi_T$. Several simple questions may be asked about this dynamics:

- *What are the regions $U \subset \mathbb{A}$ that are* invariant *by f?* Invariant means that $f(U) = U$.
- *Does there exist a dense set of initial data $(\theta, \dot\theta)$ which are periodic?* Periodic means that $f^\tau(\theta, \dot\theta) = (\theta, \dot\theta)$ for some integer $\tau \geq 1$.
- *How do the orbit separate?* More precizely, let us consider two initial data $(\theta, \dot\theta)$ and $(\theta', \dot\theta')$ that are close. How does the distance $d(f^n(\theta, \dot\theta), f^n(\theta, \dot\theta))$ behaves when n increases?

An other example of conservative diffeomorphism is the standard map on the two-torus:

$$(x, y) \mapsto (x + y, y + a\sin(2\pi(x + y))) \mod \mathbb{Z}^2.$$

It is sometimes considered by physicists as a model for chaotic dynamics: the equations defining such a diffeomorphism are simple but we are far from beeing able to give a complete description of its dynamics. However one can hope that *some other systems, arbitrarily close to the original one, could be much easier to be described.* To reach this goal, one has to precise what "arbitrarily close" and "be described" mean: the answer to our problem will depend a lot on these two definitions. The viewpoint we adopt in this text allows to give a rather deep description of the dynamics. However one should not forget that one can choose other definitions which could seem also (more?) relevant and that very few results were obtained in this case.

The setting, Baire genericity. In the following we consider a compact and boundaryless smooth connected surface M endowed with a smooth volume v (which is, after normalization, a probability measure) and we fix a diffeomorphism f on M which preserves v. We are aimed to describe the space of orbits of f and in particular the space of periodic orbits.

Lot of difficulties appear if one chooses an arbitrary diffeomorphism. Our philosophy here will be to forget the dynamics which seem too pathological hoping that the set of diffeomorphisms that we describe is large (at least dense in the space of dynamical systems we are working with). For us, such a set will be large if it is generic in the sense of Baire category.

This notion requires to choose carefully the space of diffeomorphisms, that should be a Baire space: for example for any $k \in \mathbb{N}$, the space Diff^k_v of C^k-diffeomorphisms of M which preserve v. A set of diffeomorphisms is *generic* (or *residual*) if it contains a dense G_δ subset of Diff^k_v, i.e. by Baire theorem if it contains a countable intersection of dense open sets of Diff^k_v (so that the intersection of two generic sets remains generic).
In the sequel, we are interested in exhibiting generic properties of diffeomorphisms: these are properties that are satisfied on a generic set of diffeomorphisms.

An example: generic behavior of periodic orbits. Robinson has proven in [R_1, R_2] the following generic property which extends a previous result of Kupka and Smale to the conservative diffeomorphisms. It is a consequence of Thom's transversality theorem.

Theorem 0.1 (Robinson)**.** *When $k \geq 1$, for any generic diffeomorphism $f \in \mathrm{Diff}^k_v$, and any periodic orbit $p, f(p), \ldots, f^\tau(p) = p$, one of the two following cases occurs (figure 1):*

- *either the orbit of p is* elliptic*: the eigenvalues of $\mathrm{D}_p f^\tau$ are non-real (in particular, this tangent map is conjugate to a rotation);*
- *or p is a* hyperbolic saddle*: the eigenvalues are real and have modulus different from* 1*. In this case, there are some one-dimensional invariant manifolds (one stable $W^s(p)$ and one unstable $W^u(p)$) through p. Points on the stable manifold converge towards the orbit of p in the future, and the same for points on the unstable manifold in the past.*

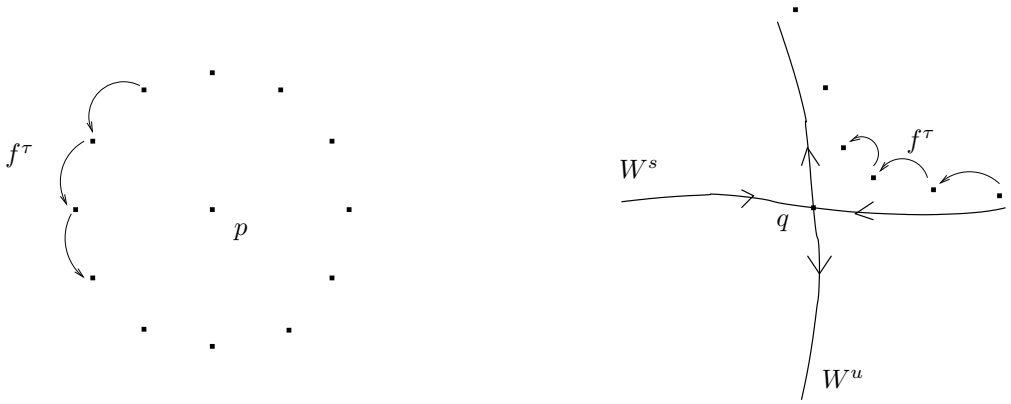


Figure 1: Dynamics near an elliptic point, p, and a saddle point, q.

One wants to say that the dynamics of the return map f^τ near a τ-periodic p "looks like" the dynamics of the tangent map $\mathrm{D}_p f^\tau$. This is the case if p is hyperbolic: D. Grobman and P.

Hartman have shown that if p is hyperbolic, the map f^τ is topologicall conjugate to $\mathrm{D}\, f^\tau$ near p; more precizely, there exist a neighborhood U of p and V of 0 in the tangent space T_pM and a homeomorphism $h\colon U \to V$ such that $h \circ f^\tau = \mathrm{D}_p\, f^\tau \circ h$.

Theorem 0.1 implies in particular:

Corollary 0.2. *When $k \geq 1$, for any generic diffeomorphism $f \in \mathrm{Diff}^k_v$, and any $\tau \in \mathbb{N} \setminus \{0\}$, the set of periodic points of period τ is finite.*

In the next part we will give other examples of generic properties. They are often obtained in the same way:

- One first proves a *perturbation result*, which is in general the difficult part. In the previous example, one shows that for every integers $\tau, k \geq 1$, any C^k-diffeomorphism f can be perturbed in the space Diff^k_v as a diffeomorphism g whose periodic orbits of period τ are elliptic or hyperbolic.
- One then uses Baire theorem for getting the genericity. An example of this standard argument is given at section I.3.1.

The last two parts are devoted to some important perturbation results. In part II, we discuss Pugh's closing lemma that allows to create periodic orbits and Hayashi's closing lemma that allows to glue two half orbits together; the perturbations in these two cases are local. In part III, we state a connecting lemma for pseudo-orbits obtained with M.-C. Arnaud and C. Bonatti through global perturbations and explain the main ideas of its proof.

I Overview of genericity results on the dynamics of C^1 conservative surface diffeomorphisms

This part is a survey of the properties satisfied by the C^1-generic conservative diffeomorphisms of compact surfaces

I.1 Discussions on the space Diff^k_v

In general, the generic properties depend strongly on the choice of the space Diff^k_v. We will here illustrate this on an example and explain why we will focus on the C^1-topology.

I.1.1 Some generic properties in different spaces. One of the first result was given by Oxtoby and Ulam [OU], in the C^0 topology.

Theorem I.1 (Oxtoby-Ulam)**.** *For any generic homeomorphism $f \in \mathrm{Diff}^0_v$, the invariant measure v is ergodic.*

Ergodicity means that for the measure v, the system can not be decomposed: any invariant Borel set A has either measure 0 or 1. By Birkhoff's ergodic theorem, *the orbit of v-almost every point is equidistributed in M.*
The C^0 topology also seems very weak: one can show that since for any generic diffeomorphism, once there exists a periodic point of some period p, then the set of p-periodic points is uncountable. (In particular corollary 0.2 does not hold for Diff^0_v.)

In high topologies, one gets Kolmogorov-Arnold-Moser theory. One of the finest forms is given by Herman (see [Mo, section II.4.c], [He, chapitre IV] or [Y]).

Theorem I.2 (Herman)**.** *There exists a non-empty open subset $\mathcal{U}$ of Diff^k_v, with $k \geq 4$ and for any diffeomorphism $f \in \mathcal{U}$, there exists a smooth closed disk $D \subset M$ which is periodic by f: the disks D, $f(D)$,..., $f^{\tau-1}(D)$ are disjoint and $f^\tau(D) = D$.*

Theses disks are obtained as neighborhoods of the elliptic periodic orbits. The dynamics in this case is very different from the generic dynamics in Diff^0_v since the existence of invariant domains breaks down the ergodicity of v: the orbit of any point of D can not leave the set $D \cup f(D) \cup \cdots \cup f^{\tau-1}(D)$.

Remark I.3. *We should notice that by a result of Zehnder [Z_2] for each $k \geq 1$, the C^∞-diffeomorphisms are dense in Diff^k_v. Therefore, for any $1 \leq k < \ell$, any property that is generic in Diff^ℓ_v will be dense in Diff^k_v. This result is not known is this generality in higher dimensions for conservative diffeomorphisms.*

I.1.2 An elementary perturbation lemma. The reason why theorem I.1 is true is that perturbations in Diff^0_v are very flexible: for any homeomorphism $f \in \mathrm{Diff}^0_v$ and any point $x \in M$, one can perturb f in order to modify the image of $f(x)$. More precisely, if y is close to $f(x)$, one chooses a small path γ that joints $f(x)$ to y. Pushing along γ, one can modify f as homeomorphism g so that $g(x) = y$. The homeomorphisms f and g will coincide outside a small neighborhood of γ. Hence, the C^0-norm of the perturbation is about equal to the distance between x and y.

In the space Diff^1_v, the C^1-norm of the perturbation also should be small (for example smaller than $\varepsilon > 0$) and one has to perturb f on a larger domain (in a ball of radius about $\varepsilon^{-1}\,\mathrm{d}(x,y)$). This can been seen easily from the mean value theorem: let x, y be two points and φ be a perturbation of the identity which satisfies $\varphi(x) = y$ and such that $\|\,\mathrm{D}\,\varphi - \mathrm{Id}\,\| \leq \varepsilon$; then, if a point z is not perturbed by φ (i.e. $\varphi(z) = z$), we get

$$\|y - x\| = \|(\varphi(x) - x) - (\varphi(z) - z)\| \leq \varepsilon.\|z - x\|.$$

As a consequence, when ε is small, the perturbation domain has a large radius and lot of the orbits of f will be modified.

This remark will be at the root of all the genericity results that will be presented below (a more precise statement will be given at section I.1.2). It explains the difficulty of the perturbations in Diff^1_v. In higher topologies, the situation becomes much more complicate since in Diff^k_v, the radius of the perturbation domain should be at least $(\varepsilon^{-1}\,\mathrm{d}(x,y))^{\frac{1}{k}}$.

This justifies why we will now work in Diff^1_v: we need a space of *diffeomorphisms* where the *elementary perturbations* don't have a too large support.

I.2 The closing and connecting lemmas

From the elementary perturbation lemma, one derivates more sophisticated perturbation lemmas.

I.2.1 Pugh's closing lemma. The first result was shown by Pugh [Pu_1, Pu_2, PR, A_1]. It allows to create by perturbation some periodic orbit once the dynamics is recurrent. More precisely, one considers the points z whose orbit is *non-wandering*: for any neighborhood U of z, there is a forward iterate $f^n(U)$ of U (with $n \geq 1$) which intersects U.

The local perturbation result is the following (see also figure 2):

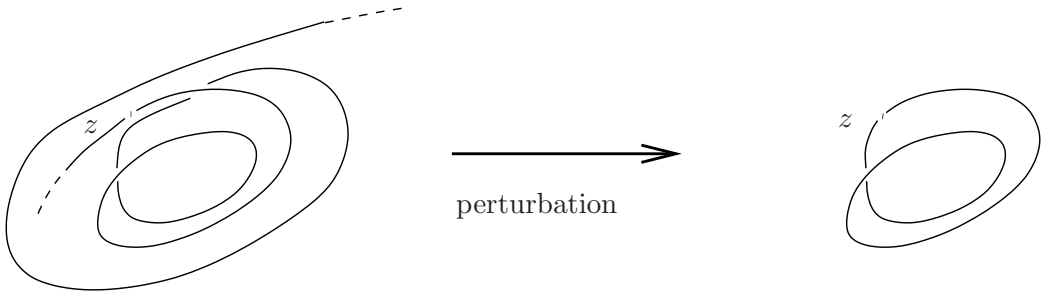


Figure 2: Perturbation given by the closing lemma.

Theorem I.4 (Closing lemma, Pugh)**.** *Let f be a C^1-diffeomorphism in Diff^1_v and $z \in M$ a non-wandering point. Then, there exists a C^1-small perturbation $g \in \mathrm{Diff}^1_v$ of f such that z is a periodic orbit of f.*

I.2.2 Hayashi's connecting lemma. We have seen that the closing lemma allows to connect an orbit to itself. About 30 years later, Hayashi [Ha, WX2, A_2] proved a second local perturbation lemma and showed how to connect an orbit to another one (see figure 3).

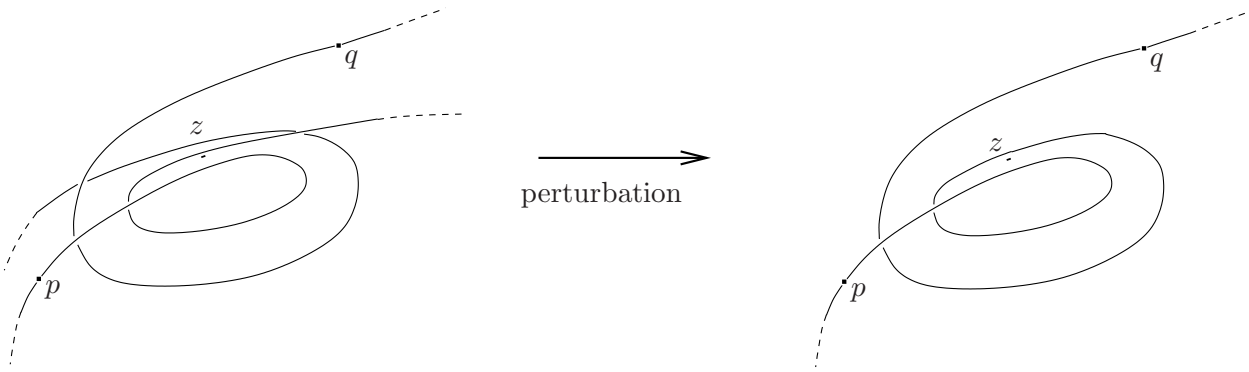


Figure 3: Perturbation given by the connecting lemma.

Theorem I.5 (Connecting lemma, Hayashi)**.** *Let f be a C^1 diffeomorphism in Diff^1_v and $p, q, z \in M$ three points such that:*

- *both the accumulation sets of the forward orbit of p and of the backward orbit of q contain the point z;*
- *the point z is not periodic.*

Then, there exists a C^1-small perturbation $g \in \mathrm{Diff}^1_v$ of f and an integer $n \geq 1$ such that $g^n(p) = q$.

The second assumption in the connecting lemma is purely technical (and maybe not essential). Some stronger versions of this result are given below.

Remark I.6. *In the closing and connecting lemmas, the perturbations are local: there exists an integer $N \geq 1$ such that the support of the perturbation is contained in an arbitrarily small neighborhood of the segment of orbit $\{z, f(z), \dots, f^{N-1}(z)\}$.*

I.2.3 The connecting lemma for pseudo-orbits. One can restate Hayashi's connecting lemma in the following form. It is possible to connect p to q by perturbation provided that these points are almost on the same orbit: at some place (close to z) one allows a small jump between the forward orbit of p and the backward orbit of q. The connecting lemma for pseudo-orbits, proved in [BC, ABC] allows to deal with any number of jumps (see figure 4).

For any $\varepsilon > 0$, we say that a sequence $(z_0, \dots, z_\ell)$ is a ε-pseudo-orbit of f if for any $k \in \{0, \dots, \ell - 1\}$, we have $\mathrm{d}(f(z_k), z_{k+1}) < \varepsilon$. In other terms, this sequence is an orbit with small errors, bounded by ε, at each iterations.

Theorem I.7 (Connecting lemma for pseudo-orbits, Bonatti-Crovisier, Arnaud-B-C)**.** *Let f be a C^1-diffeomorphism in Diff^1_v such that for each $\tau \geq 1$, the set periodic points with period τ of f is finite. Let $p, q \in M$ be two points such that for each $\varepsilon > 0$, there exists a ε-pseudo-orbit $(p = z_0, z_1, \dots, z_{\ell-1}, z_\ell = q)$ which joints p to q.*

Then, there exists a C^1-small perturbation $g \in \mathrm{Diff}^1_v$ of f and an integer $n \geq 1$ such that $g^n(p) = q$.

By corollary 0.2, the first assumption of theorem I.7 is generic in Diff^1_v (here again, this technical assumption is probably not essential). We will see below that the second one is always satisfied (we assumed that M is connected).

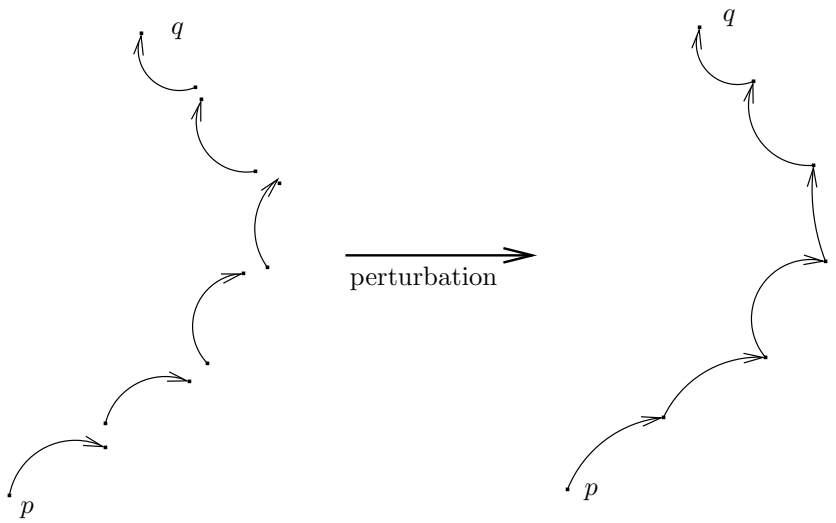


Figure 4: Perturbation given by the connecting lemma for pseudo-orbits.

I.3 C^1-generic properties

We now give some consequences of these perturbation lemmas.

I.3.1 Density of periodic points. The first consequence shows the important role played by the periodic orbits in C^1-dynamics. For any diffeomorphism f and any non-empty open set U, using the fact that f preserves a smooth finite measure, we get that a forward iterate $f^n(U)$ of U intersects U. This shows that any point in M is non-wandering.

Now, Pugh's closing lemma implies:

Theorem I.8 (Pugh)**.** *For any generic diffeomorphism $f \in \mathrm{Diff}^1_v$, the periodic points of f are dense in M.*

We recall that according to Robinson's theorem 0.1, each periodic orbit is either elliptic or hyperbolic.

The proof follows from the closing lemma by a classical argument. We give it as an example of Baire theory.

Sketch of the proof of theorem I.8. Let (U_n) be a countable basis of neighborhoods in M. We fix n and we have to show that any generic diffeomorphism has a periodic point in U_n.

By the implicit function theorem, the diffeomorphisms which have a hyperbolic or elliptic periodic point in U_n form an open set $\mathcal{U}_n$. It now sufficient to show that for each n, the set $\mathcal{U}_n$ is dense: the countable intersection $\bigcap \mathcal{U}_n$ then will be a dense G_δ set, hence generic.

Let us consider any diffeomorphism f_0. Since any point in U_n is non-wandering, by a small perturbation, one can create in U_n a periodic point p for a diffeomorphism f_1 close to f_0. For a new perturbation f_2 (given by a transversality argument extracted from the proof of theorem 0.1) the point p will be elliptic or hyperbolic, as required. □

I.3.2 Study of the elliptic orbits. According to the last result, it should be interesting to study separately the two kinds of periodic orbits that appear in generic dynamics. The elliptic behavior was studied intensively but mostly in higher differentiability. The following result is not specific to the C^1-topology.

Theorem I.9 (Zehnder[Z_1])**.** *Let f be a generic diffeomorphism in Diff^1_v. Any elliptic periodic orbit is accumulated by hyperbolic saddles. Hence, the hyperbolic periodic points are dense in M.*

We give a different proof from Zehnder's original one (which used approximations by smooth diffeomorphisms). We need a useful perturbation lemma by Franks which allows to change the tangent map along the periodic orbits.

Theorem I.10 (Franks lemma, [F_2, BDP])**.** *Let q, $f(q)$,..., $f^\tau(q) = q$ be a periodic orbit of a conservative diffeomorphism f. Then, any C^0-perturbation of the sequence of linear tangent maps $(\mathrm{D}_q f, \mathrm{D}_{f(q)} f, \ldots, \mathrm{D}_{f^{\tau-1}(q)} f)$, that preserves the volume forms v, can be realized as the sequence of tangent maps $(\mathrm{D}_q g, \mathrm{D}_{g(q)} g, \ldots, \mathrm{D}_{g^{\tau-1}(q)} g)$ associated to a conservative diffeomorphism g that is C^1-close to f and preserves the orbit $\{q, f(q), \ldots, f^{\tau-1}(q)\}$.*

Sketch of the proof of theorem I.9. Let us consider some elliptic periodic point p. We explain how to create by perturbation a hyperbolic saddle q close to p. Theorem I.9 then follows from a standard Baire argument.

We first recall that the derivative $\mathrm{D}_p f^\tau$ along the orbit of p is conjugate to a rotation. By a transversality argument, one may perturb the dynamics so that the angle of this rotation is irrational. The dynamics in a neighborhood of the orbit of p looks like the dynamics of an irrational rotation. Hence, by a small perturbation one creates in a neighborhood of the orbit of p a second periodic point q whose period τ' is an arbitrarily large multiple $k\tau$ of τ: this argument uses Pugh's closing lemma for dynamics C^1-close to the rotation.

Note that the derivative along the orbit of q is close to the rotation $(\mathrm{D}_p f^\tau)^k$. Since the period $k\tau$ of q can be chosen arbitrarily large, one can perturb the derivative at points q, $f^\tau(q), \ldots, f^{k-1}\tau(q)$ (by compositing by small rotations) in order to obtain a sequence of linear maps whose product is equal to the identity. By using Franks lemma (theorem I.10), one gets a C^1-small perturbation of f whose derivative along the orbit of q is the identity. By a new arbitrarily C^1-small perturbation, the orbit of q can be made of saddle type. □

I.3.3 Study of the hyperbolic saddles. We have seen that the hyperbolic periodic orbits are dense in M. The next result shows the existence of *homoclinic intersections*, i.e. of transverse intersections between the stable and unstable manifolds of these periodic orbits. This is important since a transverse intersection between the invariant manifolds of a saddle periodic orbit implies the existence of non-trivial *hyperbolic sets*: a compact and invariant set is *hyperbolic* if the tangent bundle over K splits as two one-dimensional bundles, one is uniformly contracted and the other one is uniformly expanded. The periodic saddle are the simplest examples of hyperbolic sets but Smale has shown that the homoclinic intersections imply the existence of larger hyperbolic sets, that are Cantor sets, and called *horseshoes*. The dynamics on horseshoes is very rich, but has been well described.

Theorem I.11 (Takens [T])**.** *For any generic diffeomorphism f and any periodic point p, the transverse intersection points between the invariant manifolds $W^s(p)$ and $W^u(p)$ of p are dense in $W^s(p)$ and $W^u(p)$.*

The proof now is easy from the connecting lemma.

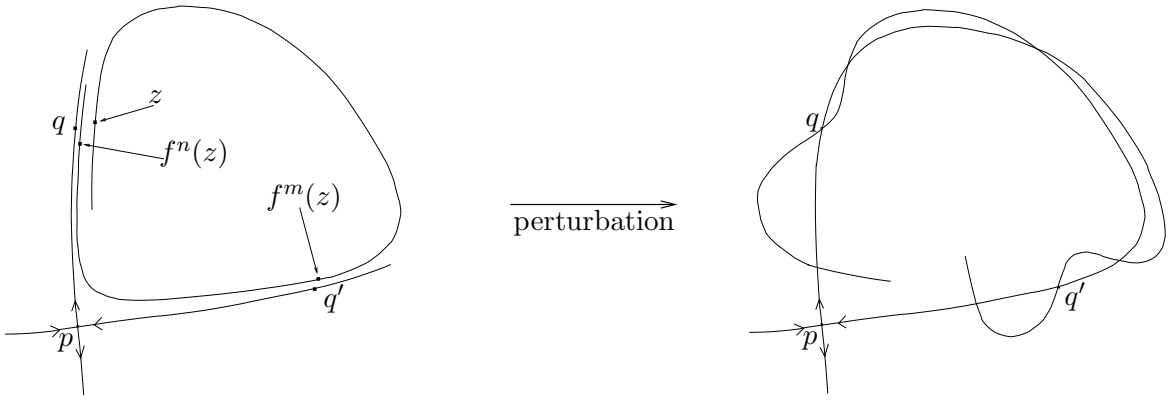


Figure 5: Creation of a homoclinic point by perturbation.

Sketch of the proof of theorem I.11. Let p be a saddle of some diffeomorphism f. We show how to build one intersection between the stable and unstable manifolds.

Let us choose some point q on the unstable manifold of p. Since q is non-wandering, it is accumulated by a sequence (z_k) and by a sequence of forward iterates $(f^{n_k}(z_k))$ of the points z_k. We note that each sequence $(z_k, f(z_k), \dots, f^{n_k}(z_k))$ should come close to the stable manifold of p before visiting q. Hence, there is a sequence $(f^{m_k}(z_k))$ with $1 < m_k < n_k$ which converges to a point q' on the stable manifold of p.

Now, the connecting lemma [1] (used twice) allows to connect the unstable manifold at q and the stable manifold at q' to a same segment of orbit $(z_k, f(z_k), \dots, f^{m_k}(z_k))$ (see figure 5). One gets an intersection at q between the stable and unstable manifolds of p. By a new perturbation, this intersection becomes transverse. □

I.3.4 Global dynamics. Up to here, we proved that the periodic points, and saddles, are dense in M. One could imagine however that M has some invariant domains so that each periodic orbit stay in a small region of M. We will see that this picture is wrong generically.

We state before a property of conservative diffeomorphism which justifies the role of pseudo-orbits:

Proposition I.12. *For any diffeomorphism $f \in \mathrm{Diff}^1_v$ and any $\varepsilon > 0$, any two points $p, q \in M$ can always be jointed by an ε-pseudo-orbit.*

Proof. We recall Poincaré recurrence theorem: almost any point $x \in M$ is *recurrent*: the accumulation set of the forward orbit of x contains x. In particular, the recurrent points are dense in M.

Let us fix some $\varepsilon > 0$ and denote by $X \subset M$ be the set of points that can be attained from p by a ε-pseudo-orbit. We choose $x \in X$. There exists a ε-pseudo-orbit from p to x. It is possible to change a little bit the point x so that x is a recurrent point.

1. In fact, we use here a variation on the connecting lemma given later in the text. This variation asserts than one chooses first the place where the perturbation is realized and then the orbits that should be connected. This will be stated precisely at section II.3.5.

Hence, one can consider a return $f^\ell(x)$ of the orbit of x which is close to x. This shows that any point y in the ball $\mathrm{B}(x, \varepsilon/2)$ can be jointed from x by a ε-pseudo-orbit. By concatening the pseudo-orbit from p to x with the pseudo-orbit from x to y, one deduces that y may be attained from p by a ε-pseudo-orbit. Thus, $\mathrm{B}(x, \varepsilon/2)$ is contained in X, for any point $x \in X$. Since M is connected, one deduces that $X = M$ and in particular $q \in X$. □

As a consequence of the connecting lemma for pseudo-orbits, one gets the following global result:

Theorem I.13 (Bonatti-Crovisier)**.** *Any generic diffeomorphism $f \in \mathrm{Diff}^1_v$ is transitive: there is a G_δ and dense subset $\mathcal{G} \subset M$ such that the forward orbits of any $x \in \mathcal{G}$ is dense in M.*

Proof. Let (U_n) be a countable basis of neighborhoods of M. Using proposition I.12 and the connecting lemma for pseudo-orbits, we get that for each open set U_n, the set $\mathcal{G}_n \subset M$ of points whose forward orbit meets U_n is dense in M. It is also open. Hence, $\mathcal{G} = \bigcap \mathcal{G}_n$ is a dense G_δ subset of M whose points have a forward orbit dense in M. □

An other consequence of theorem I.7, is that for a generic diffeomorphism f and any pair of periodic saddles p and q, the invariant manifolds $W^s(f^i(p))$ and $W^u(f^j(q))$ of some iterates of p and q intersect transversally. One says that f possesses a unique *homoclinic class*.

This property implies the existence of many other periodic points. For example, there exists a periodic orbit (whose period is very large) which shadows the orbit of p during a large number of iterates and then shadows the orbit of q. Since the periodic saddles are dense in M (by theorem I.9), we get the announced property:

Corollary I.14. *For any generic diffeomorphism in Diff^1_v and any ε, there exists a periodic orbit which is ε-dense in M (i.e. which meets any ball of radius ε in M).*

I.3.5 Repartition of periodic orbits. It is possible to improve the density theorem I.8 by describing the repartition of the periodic orbits. From the measure theory, this is the ergodic closing lemma due to Mañé [Man].

Theorem I.15 (Ergodic closing lemma, Mañé)**.** *Let f be a generic diffeomorphism in Diff^1_v. Then, any invariant probability measure[2] μ of f is the weak limit of periodic measures (μ_n) whose supports converge towards the support of μ.*

From the topological point of view, one may wonder what are the regions of M that are shadowed by a single periodic orbit. This requires another global perturbation lemma that we won't detail here (see [C]).

We say that an invariant compact set K is *chain transitive* if for any points $p, q \in K$ and any $\varepsilon > 0$, there is a ε-pseudo-orbit contained in K that joints p to q. The result is the following:

Theorem I.16 (Crovisier)**.** *For any generic diffeomorphism in Diff^1_v and any invariant compact set K, the set K is the Hausdorff limit of a sequence of periodic orbits[3] if and only if it is chain-transitive.*

2. We do not assume here that the measure μ is ergodic. Hence, a consequence of this result is that in the convex and compact set $\mathcal{M}$ of invariant measures, the ergodic measures (these are the extremal points of $\mathcal{M}$) are dense.

3. This means that for any $\delta > 0$, there exists a periodic orbit contained in the δ-neighborhood of K and which crosses all the δ-balls centered at points of K: at scale δ, the periodic orbit $\mathcal{O}$ and the set K can not be distinguished.

As a consequence, we get a weak shadowing property: pseudo-orbits whose jumps are very small "can not be distinguished" from "true orbits".

Corollary I.17. *For any generic diffeomorphism $f \in \mathrm{Diff}^1_v$, for any $\delta > 0$, there exists $\varepsilon > 0$ such that any ε-pseudo-orbit $PO = \{z_0, \dots, z_m\}$ of f is δ-close to a genuine segment of orbit $O = \{z, f(z), \dots, f^n(z)\}$ of f: this means that PO is contained in the δ-neighborhood of O and that O is contained in the δ-neighborhood of PO.*

I.4 Hyperbolic versus elliptic dynamics

Whereas the hyperbolic saddles of a generic diffeomorphism always exists and are dense in M, the same is not true for the elliptic points.

I.4.1 Hyperbolic behavior: Anosov diffeomorphisms. An important example of conservative surface dynamics are the *Anosov diffeomorphisms*: these are the diffeomorphisms such that the whole manifold M is a hyperbolic set.

For example, any linear automorphism in $SL(2, \mathbb{Z})$ acts on the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and preserves the canonical Haar measure. When the modulus of the eigenvalues is different from 1, one gets an Anosov diffeomorphism.

The general Anosov diffeomorphisms are important because they form an open subset of Diff^k_v. Their dynamics are well understood since Franks has shown [F$_1$] that any Anosov surface diffeomorphism is conjugate by a homeomorphism to a linear Anosov automorphism of $\mathbb{T}^2$.

I.4.2 The dynamics far away from the Anosov diffeomorphisms. It is clear that Anosov diffeomorphisms don't have any elliptic periodic orbit. For the other diffeomorphisms, Newhouse showed [N] that the situation is completely different.

Theorem I.18 (Newhouse)**.** *For any generic diffeomorphism in Diff^1_v, which is not Anosov, the elliptic periodic points are dense in M.*

In particular, if M is not the torus, we get the same conclusion for any generic diffeomorphism in Diff^1_v. Combining the technics of [BDP] and [BC], one can improve this result and show that for any generic diffeomorphism which is not Anosov, for any $\varepsilon > 0$, there exists an elliptic periodic orbit which is ε-dense in M.

We note also that if f is an Anosov diffeomorphism, any diffeomorphism g that is C^1-close to f is transitive (f is said *robustly transitive*). This property characterizes the Anosov diffeomorphisms.

Proposition I.19. *For any diffeomorphism $f \in \mathrm{Diff}^1_v$ which is not Anosov, there is a C^1-small perturbation g which is not transitive.*

Proof. Using remark I.3, one can approach f by a smooth diffeomorphism $\bar{f}$ which has an elliptic periodic orbit. For a new perturbation g, the assumptions of KAM theorem I.2 are satisfied and g is not transitive. □

I.4.3 The Lyapunov exponents. The dichotomy hyperbolic/elliptic can be also detected from the Lyapunov exponents of the diffeomorphism: Oseledets theorem asserts that for any

diffeomorphism $f \in \mathrm{Diff}^1_v$ and at v-almost any point, the *upper Lyapunov exponent* exists:

$$\lambda^+(f,x) = \lim_{n\to+\infty} \frac{1}{n} \log \| \mathrm{D}_x f^n \|.$$

This quantity, which is always non-negative, describes how the infinitesimal dynamics along the orbits of x is stretched.

Bochi proved [B] the following property.

Theorem I.20 (Bochi)**.** *For any generic diffeomorphism $f \in \mathrm{Diff}^1_v$ two cases can occur:*
- *either f is Anosov, (and $\lambda^+(f,x)$ is strictly positive at any point x where it is defined);*
- *or $\lambda^+(f,x) = 0$ for v-almost every point x.*

The proof is very interesting since it combines perturbation technics of the derivative along some segments of orbits and a control of the measure of the points which exhibit the required behavior.

I.5 Questions

We conclude this survey with some open problems.

1. *Is any generic diffeomorphism topologically mixing?*
   A diffeomorphism f is *topologically mixing* if for any non-empty open sets U and V, there is an integer $n_0 \geq 1$ such that $f^n(U)$ intersects V once $n \geq n_0$. It is a stronger property than the transitivity. In fact, theorem I.7 implies that for a generic diffeomorphism, the manifold splits into a finite number of pieces Λ, $f(\Lambda),\ldots,$ $f^\tau(\Lambda) = \Lambda$ that are cyclically permuted by the dynamics. The return map f^τ on Λ is topologically mixing. The problem here is to decide whether there is only one piece ($\tau = 1$).
2. *Is any generic diffeomorphism ergodic?*
   A weaker problem would be: is there a full measure set of point whose orbits are dense?
3. *Is there a closing lemma in higher differentiability?*
   About perturbation lemmas on surfaces and in higher differentiability, there exist some partial results that adopt a topological approach, see [R_2, Mat, P, O, FL].

II Local perturbations in C^1-dynamics

In this part we explain the main ideas for closing or connecting orbits by perturbation. Most of the technical details that are skipped here can be read in [PR, A_1, WX1, BC].

II.1 The elementary perturbation lemma

II.1.1 Statement. The basic result that allows perturbations is the elementary perturbation lemma introduced at section I.1.2. Since the perturbation is local, we state it in $\mathbb{R}^2$ without loosing in generality (see figure 6).

Proposition II.1 (Elementary perturbation lemma)**.** *For any neighborhood $\mathcal{U}$ of* Id *in the space of conservative C^1-diffeomorphisms of $\mathbb{R}^2$, there exists $\eta > 0$ with the following property:*
If $x, y \in \mathbb{R}^2$ are close enough to each other there is $\varphi \in \mathcal{U}$ which sends x onto y and coincides with Id *outside the ball* $\mathrm{B}\left(\frac{x+y}{2}, \frac{1+\eta}{2}\,\mathrm{d}(x,y)\right)$.

One gets the perturbation g of f announced at section I.1.2, as a composition $\varphi \circ f$.

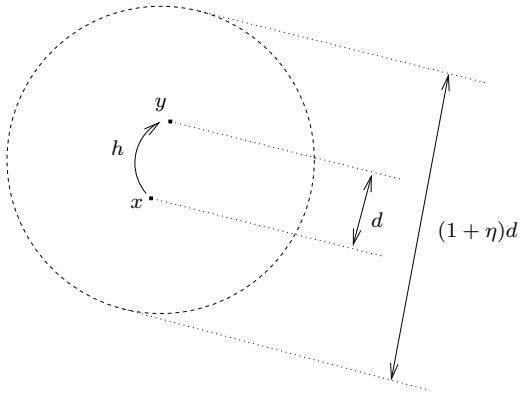


Figure 6: An elementary perturbation.

II.1.2 Why this result is not sufficient to our purpose? One can hope that this result will be sufficient to get the closing lemma: if one considers a segment of orbit $(p, f(p), \dots, f^n(p))$ and if p and $f^n(p)$ are close enough one may want to introduce a perturbation φ given by proposition II.1 which sends $f^n(p)$ on p. However the point p will not be periodic for the diffeomorphism $g = \varphi \circ f$ in general. The reason is that the segment of orbit $(p, f(p), \dots, f^n(p))$ could have many intermediate returns in the support of the perturbation φ and could be broken by the perturbation (figure 7).

II.1.3 Proof of the closing lemma when $\eta = \frac{1}{2}$. When $\eta \leq \frac{1}{2}$ (but in this case the C^1-size of the perturbation is not so small) one can deduce the closing lemma directly from the elementary perturbation lemma. The main difficulty is to select two close iterates that could be jointed to each other by a perturbation (figure 8).

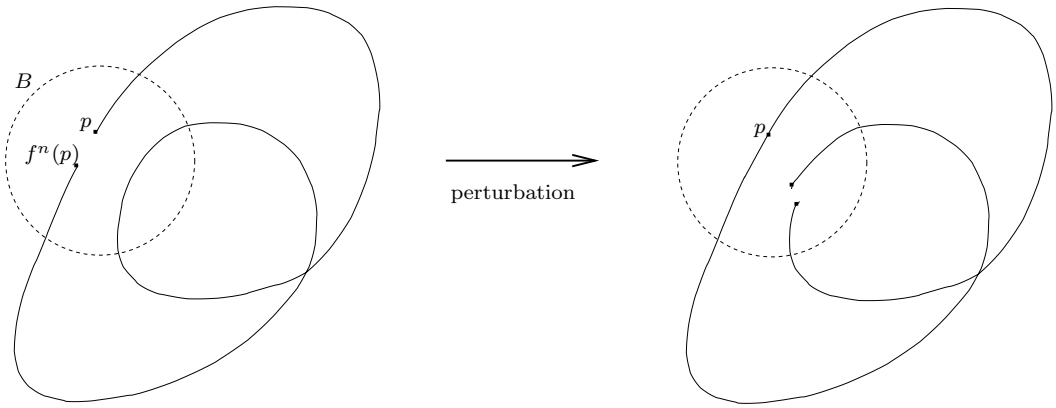


Figure 7: Orbit broken by the perturbation.

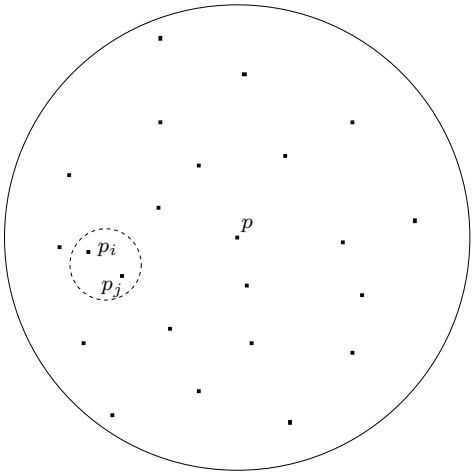


Figure 8: Selection of returns in B.

Let us consider a point p whose orbit is recurrent [4]: the forward orbit of p has some returns arbitrarily close to p. We fix also a small ball $B = \mathrm{B}(p, r)$ centered at p, that will contain the support of the perturbation. We choose a time $n \geq 1$ such that $f^n(p)$ and p are close enough.

Let us denote by $p_0 = p, p_1, \ldots, p_{s-1}, p_s = f^n(p)$ the points of $\{p, f(p), \ldots, f^n(p)\}$ that belong to the interior of B. Among the pairs (p_i, p_j) with $i < j$, we choose one which minimizes the quantity

$$D_{i,j} = \frac{\mathrm{d}(p_i, p_j)}{\mathrm{d}\left(\frac{p_i+p_j}{2}, \partial B\right)}.$$

This plays the role of a "hyperbolic distance in the ball B" between the points p_i. Note that since the distance between x and $f^n(x)$ has been chosen arbitrarily small in comparison to the radius r of B, the minimum of the $D_{i,j}$ can be assumed arbitrarily small.

4. The assumption in the closing lemma is that p is non-wandering. In order to simplify the proof, our assumption here is a little bit stronger but the argument is the same.

An easy estimate gives the following claim.

Claim II.2. *If D_{i_0,j_0} is small enough and realizes the minimum over all the $D_{i,j}$, the ball $\mathrm{B}\left(\frac{p_{i_0}+p_{j_0}}{2}, \frac{1+\eta}{2}\,\mathrm{d}(p_{i_0},p_{j_0})\right)$ is contained in B and does not intersect any other point $p_0,\dots,p_s$.*

In particular, one now can realize the perturbation φ which sends p_{j_0} onto p_{i_0}. The segment of orbit $(p_{i_0}, f(p_{i_0}),\dots, f^{-1}(p_{j_0}))$ is not perturbed by the perturbation and the point p_{i_0} is now periodic. Hence, we get a periodic point close to the initial point p. A small perturbation (one conjugates by a translation) can move the periodic point p_{i_0} onto p, as required.

II.1.4 Composition of perturbations. In the general case, a unique elementary perturbation φ is not sufficient and we will realize several perturbations at different places. We will now see that this is allowed, since the size of the perturbation one obtains do not increase with the number of independent elementary perturbations that have been realized.

If $\mathcal{U}$ is a neighborhood of a diffeomorphism f, the *support* of a perturbation $g \in \mathcal{U}$ is the set of points x where $f(x)$ and $g(x)$ differ. If $g_1 = \varphi_1 \circ f$ and $g_2 = \varphi_2 \circ f$ are two perturbations with distinct supports, the *composition of the perturbations* is the diffeomorphism $g = \varphi_1 \circ \varphi_2 \circ f = \varphi_2 \circ \varphi_1 \circ f$. By shrinking the neighborhood $\mathcal{U}$ if necessary, one can choose it with the *composition property*:

$$g_1, g_2 \in \mathcal{U} \;\Rightarrow\; g \in \mathcal{U}.$$

Hence, one may compose an arbitrary number of elementary perturbations without leaving the neighborhood $\mathcal{U}$ of f.

II.2 The closing lemma

In order to prove the closing lemma, we would like to realize a small perturbation given by proposition II.1 with a constant $\eta \leq \frac{1}{2}$. As we explained in section I.1.2 this is not possible. The idea of Pugh was to spread the perturbation in the time and to obtain it as a composition of several elementary perturbation. This idea allows roughly to divide the (a priori large) constant η, given by the size of the perturbations that are allowed, by the time we consider to spread the perturbation.

II.2.1 Pugh's perturbation lemma.

Theorem II.3. *Let $f \in \mathrm{Diff}^1_v$ be a diffeomorphism and let us consider a point z which is not periodic. Then, for any neighborhood $\mathcal{U}$ of f in Diff^1_v, there exists an integer N and a Riemannian metric d' which have the following property:*

If x, y are two points contained in a small ball $S = \mathrm{B}_{\mathrm{d}'}(z,\delta)$ centered at z, for the metric d', there is a perturbation $g \in \mathcal{U}$ of f which sends x onto $f^N(y)$ by g^N.

The support of the perturbation g is contained in the union $\hat{S} \cup f(\hat{S}) \cup \cdots \cup f^{N-1}(\hat{S})$ of the ball $\widehat{S} = \mathrm{B}_{\mathrm{d}'}(z, \frac{3}{2}\delta)$ with its $N-1$ first iterates.

Note that since z is not periodic, the ball $\widehat{S}$ and its $N-1$ first iterates are disjoint. With this result we "recover a constant $\eta = \frac{1}{2}$": the perturbation around x and y occurs in the ball of diameter less than $1 + \frac{1}{2}$ times the diameter of the ball S which contains x and y. The price to pay is that we also perturb in the $N-1$ iterates of the ball $\widehat{S}$ (figure 9). However, the argument given at section II.1.3 remains and theorem II.3 implies the closing lemma.

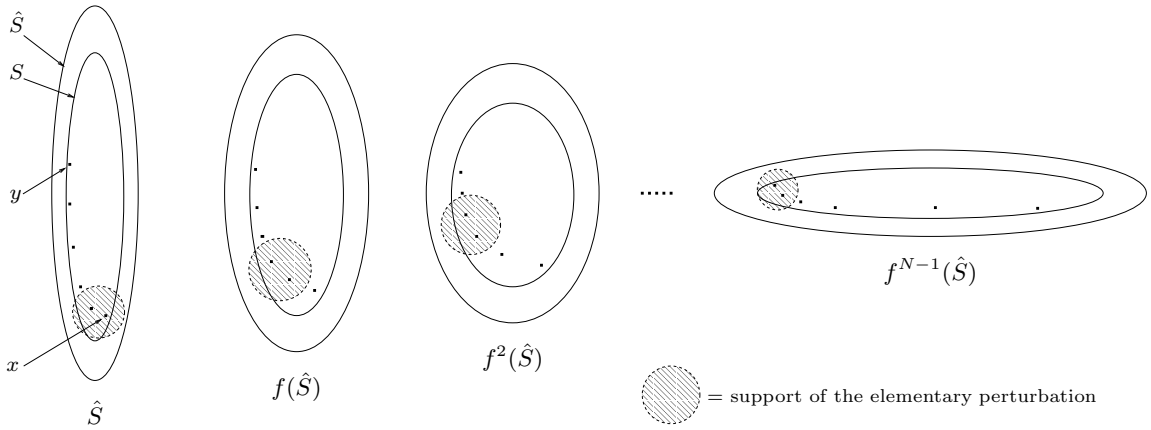


Figure 9: Perturbations in theorem II.3.

Remark II.4. *Note that all the sets $\widehat{S}$, $f(\widehat{S})$,..., $f^{N-1}(\widehat{S})$ have roughly the shape of ellipsis. An important improvement (used in the connecting lemma below) is that in each ellipsis $f^k(\widehat{S})$ with $k \in \{0, , \dots, N-1\}$, the support of the perturbation g is contained in a ball which is small with respect to the smaller axis of $f^k(\widehat{S})$.*

II.2.2 Proof of theorem II.3 when f is conformal. In order to explain how to spread the perturbation in the time and get theorem II.3, we first consider the case where f is "conformal" (i.e. the image of a small Euclidean ball is roughly a small euclidian ball). One considers the (large) constant η given by the elementary perturbation lemma, a small ball $S = \mathrm{B}(z, \delta)$ around z (for the standard metric) and two points $x, y \in S$.

We choose $N = 4(1+\eta)$ and we divide the segment between x and y by a sequence $(\zeta_0 = x, \dots, \zeta_N = y)$ of points at distance $\mathrm{d}(x,y)/N$ from each other.

For each $i \in \{0, \dots, N-1\}$, the image of the ball $\widehat{S} = \mathrm{B}(z, \frac{3}{2}\delta)$ by f^i is roughly a ball, by assumption. The two points $f^i(\zeta_i), f^i(\zeta_{i+1})$ are contained in this ball. Moreover, by our assumption, the relative distance between these two points in comparison to their distance to the boundary of $f^i(\widehat{S})$ is small: it is close to the relative distance between ζ_i and ζ_{i+1} in comparison to their distance to the boundary of $\hat{S}$. So that the elementary perturbation lemma gives a perturbation $g_i = \varphi_i \circ f$ in $\mathcal{U}$ with support in $f^i(\widehat{S})$ (figure 10) such that $g_i(f^i(\zeta_i)) = f^{i+1}(\zeta_{i+1})$.

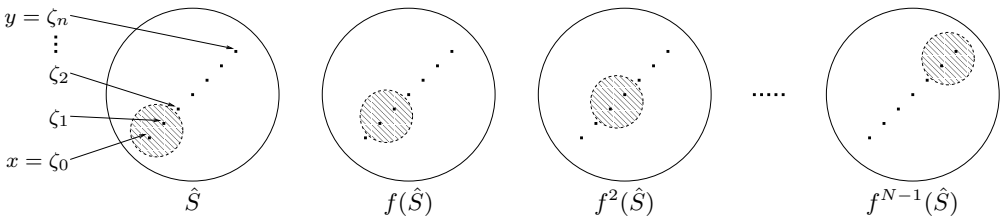


Figure 10: Perturbation when f is conformal.

All the perturbations g_i have by construction disjoint supports so that the composed pertur-

bation $g = (\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_{N-1}) \circ f$ belongs to $\mathcal{U}$ (by the composition property of section II.1.4) and satisfies $g^N(\zeta_0) = f^N(\zeta_N)$, as announced.

II.2.3 Proof of theorem II.3 when f is not conformal. The difficulty when f is not conformal is that the image of a ball is no more a ball (after several iterations, it could be a ellipsis with a large eccentricity). Hence, the two points $f^i(\zeta_i), f^i(\zeta_{i+1})$ could be at a small distance from the boundary of the ellipsis $f^i(\widehat{S})$ (in comparison to their relative distance). This is the case in particular if the segment that joints them follows the direction of the largest axis of the ellipsis. Therefore, the support of perturbation given by proposition II.1 which sends $f^i(\zeta_i)$ on $f^i(\zeta_{i+1})$ is no more contained in $f^i(\widehat{S})$. On the contrary, if one assumes that the segment between these points follows the direction of the small axis of the ellipsis, the perturbation can be realized inside $f^i(\widehat{S})$ as in the conformal case. On figure 11, the right part shows the case where the points are in a bad position so that the elementary perturbation can not be realized; a good position is pictured on the left part of the figure.

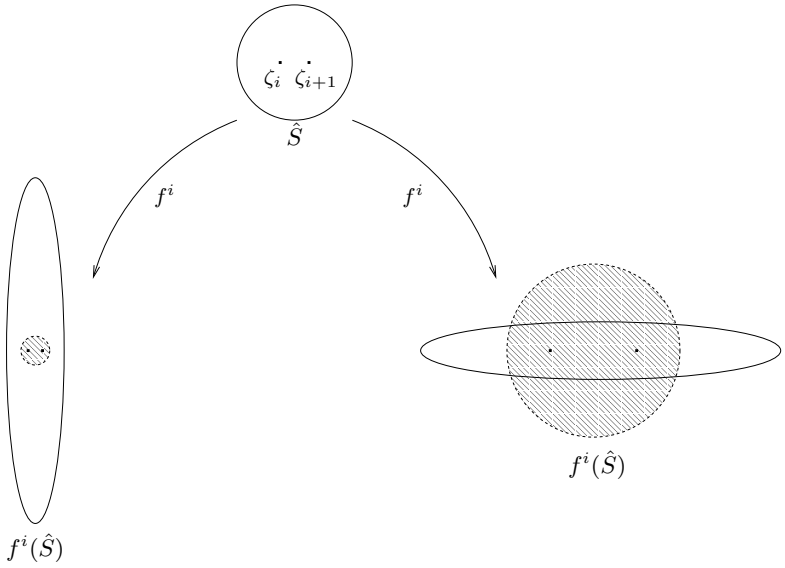


Figure 11: Difficulty when f is not conformal.

In order to bypass this problem, we introduce a different metric d', so that S and $\widehat{S}$ are chosen as balls for d' but as ellipsis for the initial metric. The perturbation is not realized at each time i (hence the integer N could be larger than in the previous case, depending on the behavior of the derivatives of f along the orbit of z) and the points (ζ_i) are not chosen along a segment. What is important is that there are times $0 \leq n_0 < n_1 < \cdots < n_i < \cdots \leq N$ such that the segment between the points $f^{n_i}(\zeta_i)$ and $f^{n_i}(\zeta_{i+1})$ follows roughly the direction of the small axis of the ellipsis $f^{n_i}(\widehat{S})$. At these times, one realizes the elementary perturbations. Figure 9 gives an idea of the way the perturbations are chosen. We refer the reader to [WX1] for a good detailed presentation of this proof.

Remark II.5. *One can choose N large enough and the sequences (ζ_i) and (n_i) carefully so that*

at each time n_i, the distance between $f^{n_i}(\zeta_i)$ and $f^{n_i}(\zeta_{i+1})$ is small in comparison to the small axis of $f^{n_i}(S)$. This implies the remark II.4.

II.3 The connecting lemma

II.3.1 Why the connecting lemma is more difficult than the closing lemma? In the proof of the closing lemma (section II.1.3) we had to select two returns $f^{i_0}(p)$ and $f^{j_0}(p)$ in a ball B, with the property that they should be far enough from the other intermediate iterates. When one tries to connect one orbit $\{f^n(p)\}$ to another one $\{f^{-m}(q)\}$, one may also selects two points in B among the returns $\{f^{n_\ell}(p)\} \cup \{f^{-m_k}(q)\}$ of the two orbits. However if the two selected returns that we get belong both to the first or both to the second orbit, a perturbation will produce a periodic orbit that crosses B but not an orbit that joints the two points p and q. Hence, one should require that the two selected returns does belong to different orbits. In general, it is not possible to find a pair of points which has this additional property (figure 12).

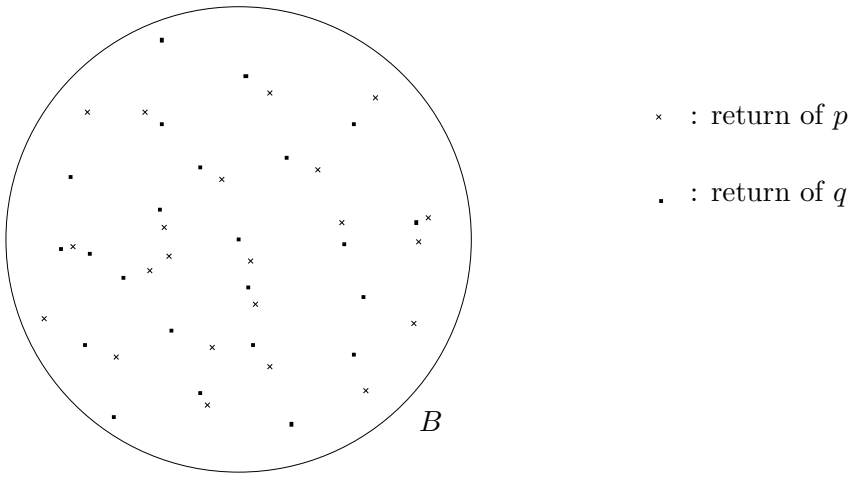


Figure 12: A wrong selection of returns for the connecting lemma.

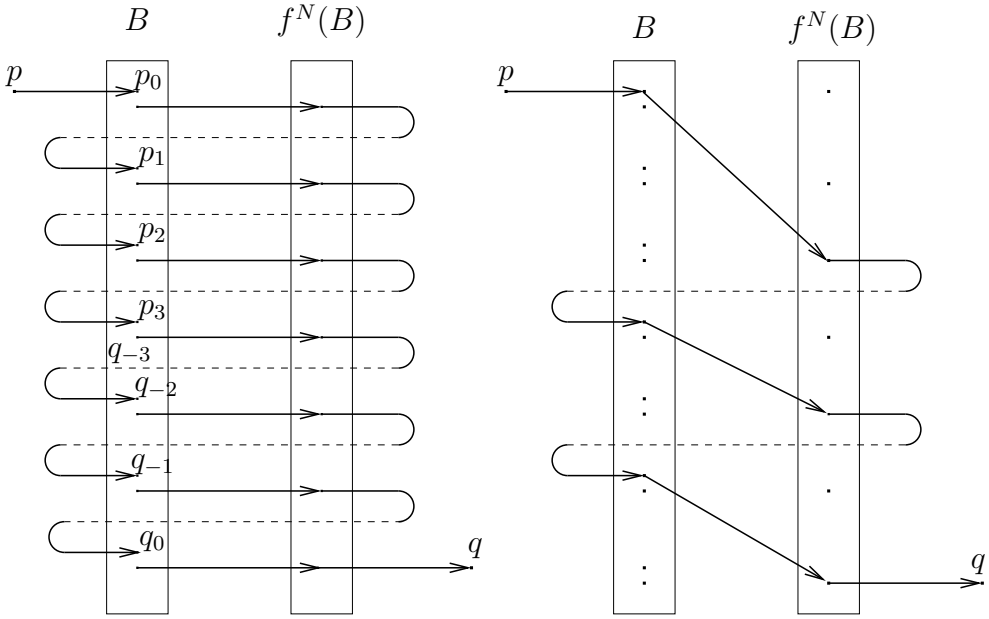
II.3.2 Hayashi's strategy. One idea of Hayashi is to clean up the cloud of returns of the two orbits in B by forgetting some of them. If two returns of the same orbit are close enough (for example two iterates $f^{n_1}(p)$ and $f^{n_2}(p)$ of p), they prevent us from using the argument of section II.1.3 as we explained at II.3.1. In this case, one will consider that these two points are the same ($f^{n_1}(p) = f^{n_2}(p)$) and forget the intermediate returns in between. One should also hope that a small perturbation could move $f^{n_1}(p)$ on $f^{n_2}(p)$ so that the assumption is fulfilled. Hence, Hayashi's strategy consists in selecting a large number of pairs of returns and not only one pair as in the closing lemma. We then realize, for each of these pairs, a perturbation given by Pugh's theorem II.3 in order to close the orbits. One difficulty is to guarantee that all these perturbations have disjoint supports.

More precisely, let us consider the first returns p_0, p_1, ..., p_r of the forward orbit of p in B, ordered chronologically and the last returns q_{-s}, ..., q_{-1}, q_0 of the backward orbit of q in B, also ordered chronologically. Recall that the two orbits accumulate on a same point z. It is

thus possible to choose the ball B centered at z so that one can assume moreover that the last returns $f^{n(p)}(p) = p_r$ and $f^{-n(q)}(q) = q_{-s}$ are very close to the center of the ball B.

In the chronological sequence $(p_0, \dots, p_r, q_{-s}, \dots, q_0)$ we extract a subsequence of the form $(x_0, y_0, x_1, y_1, \dots, x_\ell, y_\ell)$ so that (using Pugh's theorem II.3) for each i one can perturb f as a diffeomorphism $g_i = \psi_i \circ f$ which satisfies $g_i^N(x_i) = f^N(y_i)$. The support of this perturbation is contained in a small ball $\widehat{S}_i$ contained in B, and in the $N-1$ first iterates of $\hat{S}_i$. Moreover, the supports of the different perturbations (g_i) should be pairwise disjoint.

If moreover one has, $x_0 = p_0$, $y_\ell = q_0$ and if for each i, the point x_{i+1} is the first return of the orbit of y_i to the ball B, then, by composing all these perturbations g_i of f, one gets a diffeomorphism $g = \psi_0 \circ \cdots \circ \psi_{N-1} \circ f \in \mathcal{U}$ which sends by some forward iteration the point p on q. After perturbation the segment of orbit from p to q is shorter than the initial pseudo-orbit $(p, f(p), \dots, f^{n(p)}(p), f^{-n(q)+1}(q), \dots, f^{-1}(q), q)$ (see figure 13).



Before perturbation. After perturbation.

Figure 13: Combinatorics of the perturbations realized by the connecting lemma.

The main difficulty is to choose the subsequence $(x_0, y_0, x_1, y_1, \dots, x_\ell, y_\ell)$. It is not built directly: we first introduce (section II.3.3) an intermediary sequence $(x'_0, y'_0, x'_1, y'_1, \dots, x'_{\ell'}, y'_{\ell'})$ by cleaning up the points in the regions where there are too much accumulations. We then select a second time (section II.3.4) so that the perturbations associated to each pair (x_i, y_i) have disjoint supports.

II.3.3 Tiled cubes: first selection. In order to select the points, it is more convenient to replace the euclidian ball B by a square [5] (for a metric d' given by theorem II.3). Viewed with

5. We one replaces a ball by a square, one should specify the orientation of the axes. One chooses the axis in the directions of the axis of the ellipsis B. In other terms, the square will be viewed in the initial metric as a rectangle and not as a parallelogram.

the initial metric, the set B is a rectangle. We then tile it as pictured on figure 14. This tiling allows us to decide when too much points of the cloud of returns accumulate in a region of the cube B.

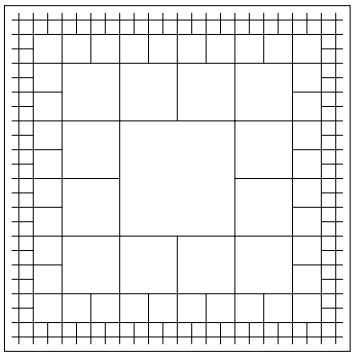


Figure 14: A tiled cube.

We select a first subsequence $(x'_0, y'_0, x'_1, y'_1, \dots, x'_{\ell'}, y'_{\ell'})$ from $(p_0, \dots, p_r, q_{-s}, \dots, q_0)$ so that each tile of the square B contains at most one pair (x'_i, y'_i). This can be done by induction: once (x'_i, y'_i) has been defined, one chooses x'_{i+1} as the first return of y'_i to B. The point y'_{i+1} is then the last point in $\{p_0, \dots, p_r, q_{-s}, \dots, q_0\}$ which belong to the tile of B which contains x'_{i+1}. The only assumption used here is that p_r and q_{-s} are close enough to z so that they belong to the same central tile of B. The picture after this first selection is represented at figure 15.

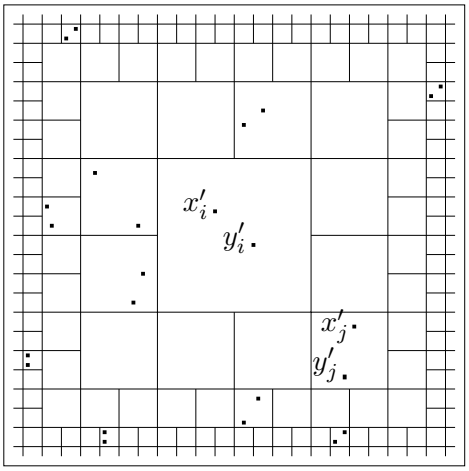


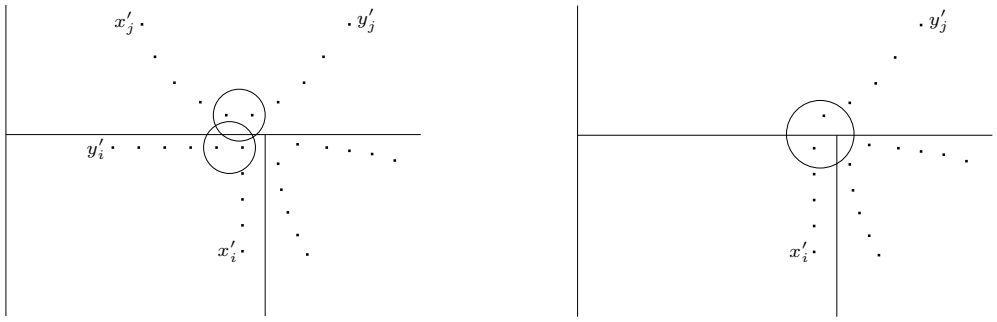
Figure 15: The first selection.

This first selection is not sufficient for the connecting lemma. Indeed, if one applies Pugh's theorem II.3 to define some perturbations g'_i such that $(g'_i)^N(x_i) = f^N(y_i)$, the supports of these perturbations may overlap: if x_i and y_i belong to a tile T of B, the support of the perturbation will be contained in the rectangle $\widehat{T}$ (obtained from T by an homothety of ratio $3/2$) and in the $N-1$ first iterates of $\widehat{T}$. The problem here appears if one has to perturb in two adjacent tiles, since the two perturbations will conflict. Hence, we can not compose in

general these perturbations g'_i and we before need to select a second sequence from the cloud $\{x'_0, y'_0, x'_1, y'_1, \dots, x'_{\ell'}, y'_{\ell'}\}$.

II.3.4 Shortcuts: second selection. In order to explain how to handle the conflicts described before, let us consider the case where the support of two perturbations g'_i and g'_j defined above overlap (they may not overlap in the cube B but in the image $f^k(B)$ of B for some $k \in \{0, \dots, N-1\}$). This means that the points $\{x'_i, y'_i\}$ (in a tile T_i) and the points $\{x'_j, y'_j\}$ (in a tile T_j) have their images by f^k close. We also will assume that $i < j$.

The idea to solve the conflict is two replace the two perturbations which send x'_i and on $f^N(y'_i)$ and x'_j on $f^N(y'_j)$ respectively, by a single perturbation which sends x'_i on $f^N(y'_j)$ (figure 16). After this construction, we erase the intermediary points $\{x'_k, y'_k\}$ with $k \in \{i+1, \dots, j-1\}$ from the sequence $(x'_0, y'_0, x'_1, y'_1, \dots, x'_{\ell'}, y'_{\ell'})$ and get a new pseudo-orbit from p to q. In other words, we realized a shortcut in the pseudo-orbit that connects p to q.



Conflict. New perturbation.

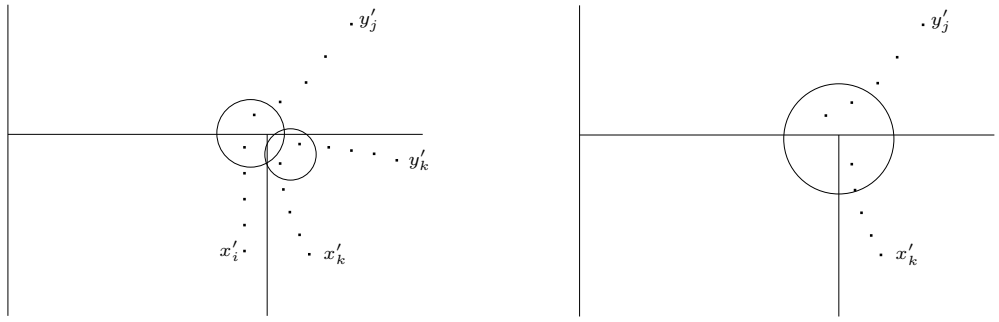
Figure 16: A shortcut.

Recall that the supports of the perturbations g'_i and g'_j should be very small in comparison to the iterates of the tiles T_i and T_j (remark II.4) hence, we get that the tiles T_i and T_j should be adjacent and that the support of the new perturbation remains small in comparison to these tiles.

We then continue in this way in order to solve all the conflicts. We note that each time one solve a conflict, one gets a new perturbation whose support is a little bit larger. Hence, it could meet the support of another perturbation g'_k and one should solve a new conflict. One may wonder if the number of conflicts one should consider starting from an initial pair (x'_i, y'_j) can be arbitrarily large so that the support of the final perturbation may become huge in comparison to the size of the initial tile T_i.

This is not the case: we control a priori the size of all the perturbations so that all the conflicts that can occur happen with tiles that are adjacent to the initial tile T_i. Since the geometry of the tiling is bounded, the number of tiles that are adjacent to T_i is bounded (by 12) and we know that we will have to solve at most 12 conflicts. If one chooses (in Pugh's theorem) the supports of the perturbations (g'_k) very small in comparison to the size of the corresponding tiles as it is allowed by remark II.4, the perturbation that we will get after solving 12 conflicts will remains small in comparison to the initial tile T_i. Hence, its support can not reach any new tile (the other tiles are not tangent to T_i and consequently are far from the

support of the perturbation we obtained, see figure 17). After solving all the conflicts, we get



New conflict. No more conflict.

Figure 17: The number of conflicts remains bounded.

a new pseudo-orbit, a new sequence $(x_0, y_0, x_1, y_1, \ldots, x_\ell, y_\ell)$ and a collection of perturbations (g_i) whose supports are pairwise disjoint and such that, for each i, we have $g_i^N(x_i) = y_i$.

II.3.5 Conclusion. The proof gives a better statement than theorem I.5. The tiled cube B and the integer N are built independently from the orbits of the points p and q. The only assumption we used was that p and q have two returns $f^{n(p)}(p)$ and $f^{-n(q)}(q)$ in a same tile of B (for example the central tile of the cube).

Theorem II.6 (Connecting lemma, 2^{nd} version)**.** *Let f be a C^1-diffeomorphism and $\mathcal{U}$ a neighborhood of f in* Diff^1_v*. For any point z that is not periodic, there is an integer N such that for any neighborhood U of z, there exists a smaller neighborhood V which has the following property:*

If p and q are two points outside $U \cup f(U) \cup \cdots \cup f^N(U)$ that have some iterates $f^{n(p)}(p)$ and $f^{-n(q)}(q)$ inside V, then, there is a perturbation $g \in \mathcal{U}$ of f with support in $U \cup f(U) \cup \cdots \cup f^{N-1}(U)$ and an integer $n \geq 1$ which satisfy $g^n(p) = q$.

This is this statement that was used to get theorem I.11. We prove it by choosing the square B inside U; the neighborhood V is the central tile of B.

III Global perturbations in conservative dynamics

In this part, we explain the proof of the connecting lemma for pseudo-orbits.

Theorem III.1. *Let $f \in \mathrm{Diff}^1_v$ be a diffeomorphism such that for any $\tau \geq 1$, the periodic orbits of period less than τ are finite. Then, for any $p, q \in M$, there exists a C^1-small perturbation $g \in \mathrm{Diff}^1_v$ of f and an integer $n \geq 1$ such that $g^n(p) = q$.*

The proof we give here is due to Arnaud-Bonatti-Crovisier but is slightly different from the arguments in [BC, ABC].

III.1 Introduction

It is clear that one can not hope to connect a pseudo-orbit into an orbit by performing only a local perturbation: we shall use several perturbations, given by Hayashi's connecting lemma. Provided they have disjoint supports, the composed perturbation remains small as it is explained at section II.1.4.

The proof will use two main ingredients:

1. we will consider (section III.2) a generalization of the tiled cubes given by the proof of the local connecting lemma (see section II.3.3): we call them *perturbation domains*;
2. using strongly the existence of an invariant probability measure with full support (here, a smooth volume), we build (section III.3), for each pair of points x, y that are not periodic, some perturbation domains and a pseudo-orbit that joints x to y and whose jumps are contained in the tiles of the perturbation domains.

The final argument (section III.4) consists in perturbing in each perturbation domain.

III.2 The perturbation domains

One can revisit the proof of Hayashi's connecting lemma and obtain a more general statement. Let us recall the main ideas. We consider a diffeomorphism $f \in \mathrm{Diff}^1_v$ and a neighborhood $\mathcal{U}$ of f. To any point $z \in M$, which is not periodic, one can associates:

– an integer $N \geq 1$,

– a small tiled cube B, as pictured on figure 14 and disjoint from its $N-1$ first iterates,

such that the following property is satisfied:

> *Let p be a point whose forward orbit meets the central tile T_c of the tiling of B, and q a point whose backward orbit meets T_c. Then, there exists a perturbation $g \in \mathcal{U}$ of f with support in $B \cup f(B) \cup \cdots \cup f^{N-1}(B)$ and an integer $m \geq 1$ such that $g^m(p) = q$.*

We also recall that the existence of the cube B was given by Pugh's perturbation lemma (theorem II.3, see also section II.3.3).

We now explain how to generalize the assumptions of Hayashi's connecting lemma.

III.2.1 More jumps. The connecting lemma allows to connect an orbit to another one provided that the concatenation of these two segments of orbit is a pseudo-orbit whose unique jump is moreover contained in the cental tile T_c of B. However, the central tile T_c may be replaced by any other tile of B, and the proof of the connecting lemma, as explained at section II.3, can also deal with pseudo-orbits (z_k) with any number of jumps. The only assumption that should

be required is that "*the jumps of the pseudo-orbit are contained in the tiles of B*". This means that for each k, if $f(z_k) \neq z_{k+1}$ then $f(z_k)$ and z_{k+1} are contained in a same tile. (Note that for proving the connecting lemma, we already considered this kind of pseudo-orbit: in the first step of the proof we cleaned up the cloud of returns of the two orbits and obtained, after a first selection, a pseudo-orbit whose jumps are contained in the tiles of the cube B.)

Sometimes it is useful to allow also jumps that are slightly larger than the tiles of B: we introduce the *enlarged tiles $\hat{T}$ of B* as the squares, having the same centers and the same axis as the tiles T of B, but with sizes $1 + \frac{1}{10}$ times larger. Then, we will say that a pseudo-orbit (z_k) *respects the tiling of B* if for any k such that $f(z_k) \neq z_{k+1}$, there exists an enlarged tile $\hat{T}$ which contains both $f(z_k)$ and z_{k+1}.

We now claim that the following property is satisfied by the cubes B given by the connecting lemma (figure 18):

(P) *For any pseudo-orbit $(z_0, \dots, z_n)$ which respects the tiling of B, there exists a perturbation $g \in \mathcal{U}$ of f with support in $B \cup f(B) \cup \cdots \cup f^{N-1}(B)$ and an integer $m \in \{1, \dots, n\}$ such that $g^m(z_0) = z_n$.*

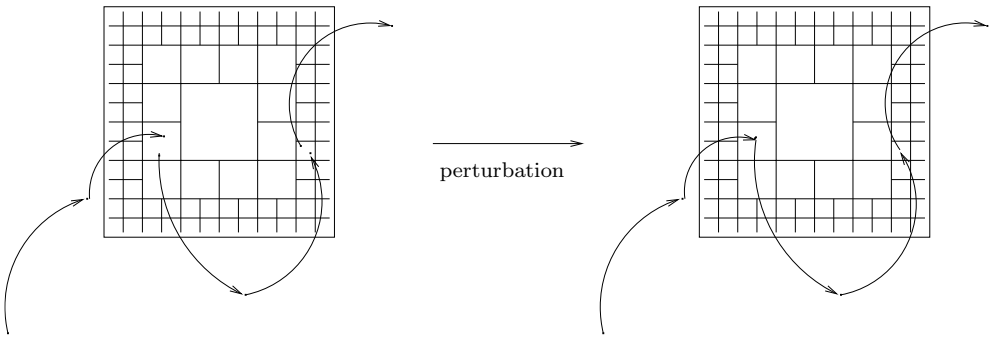


Figure 18: Perturbation in a tiled cube (property (P)).

Property (P) is obtained easily from the arguments of section II.3: by a first selection, one can assume that each enlarged tile of B is associated at most to one jump of the pseudo-orbit. One then removes all the jumps, using the elementary perturbation lemma (proposition II.1) for each enlarged tile that contains a jump. One has to solve the conflicts if necessary: the crucial argument is that the number of conflicts that are related to a jump is a priori bounded by 12: This comes from the following facts:

– if two enlarged tiles $\hat{T}$ and $\hat{T}'$ intersect, then the tiles T and T' are adjacent,
– the number of tiles T' adjacent to a tile T is bounded by 12.

III.2.2 Tiled domains. The cube B which supports the perturbations is obtained from the metric d' given by Pugh's theorem II.3. More precisely, there is a local chart $\psi_z \colon U_z \to \mathbb{R}^2$ defined on a small neighborhood U_z of z such that Pugh's metric d' is the pull back by ψ_z of the standard Euclidian metric on $\mathbb{R}^2$. In the proof of the connecting lemma, we did not use however the fact that B was a cube; we just needed the bounded geometry of the tiling: the image B_0 of B by ψ_z is a tiled open set of $\mathbb{R}^2$ which has the following properties:

(T1) The tiles of B_0 are squares of $\mathbb{R}^2$ that are oriented along the canonical axis of $\mathbb{R}^2$.

(T2) Let us associate to any tile T_0, the enlarged cube $\hat{\hat{T}}_0$ with the same center but obtained from T_0 by scaling by a ratio $1 + \frac{1}{5}$. Then, any two tiles T_0 and T_0' are adjacent once the enlarged cubes $\hat{\hat{T}}_0$, $\hat{\hat{T}}_0'$ intersect.

(T3) There exits a uniform constant (here 12) that bounds the number of tiles T_0' that are adjacent to a tile T_0 of B_0.

Any tiled open set $B \subset U_z$ which satisfies these two properties will be called a *tiled open set for the chart* (ψ_z, U_z) (figure 19).

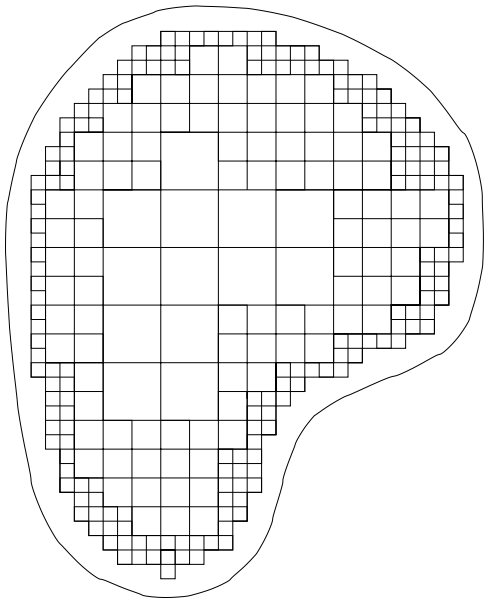


Figure 19: A tiled open set.

One ends this paragraph by showing how to tile any open set (figure 20):

Proposition III.2. *Any open set $\tilde{B}$ of $\mathbb{R}^2$ admits a tiling by squares which satisfies properties (T1), (T2) and (T3).*

Proof. We introduce the standard tilings $\mathcal{T}_n$ of $\mathbb{R}$ by squares of size 2^{-n}. Let U be any open subset of $\mathbb{R}^2$. The announced tiling $\mathcal{T}$ is built from the standard tiling $\mathcal{T}_n$ by deciding inductively what are the tiles of $\mathcal{T}_n$ that belong to $\mathcal{T}$: a tile $T \in \mathcal{T}_n$ will belong to $\mathcal{T}$ if

- T and all the tiles in $\mathcal{T}_n$ that are adjacent to T are contained in U,
- T is not contained in the sub-domain of U tiled at the previous steps by tiles of lower tiling $\mathcal{T}_m$, $m < n$.

□

III.2.3 An improved connecting lemma We mention two other technical improvements: up to here, we have assumed that the point z where we defined the tiled cube or open set is not periodic. This was necessary, since we need to spread the perturbations in the time during N iterates. However, the construction of Pugh's metric d' around z, or equivalently of the chart

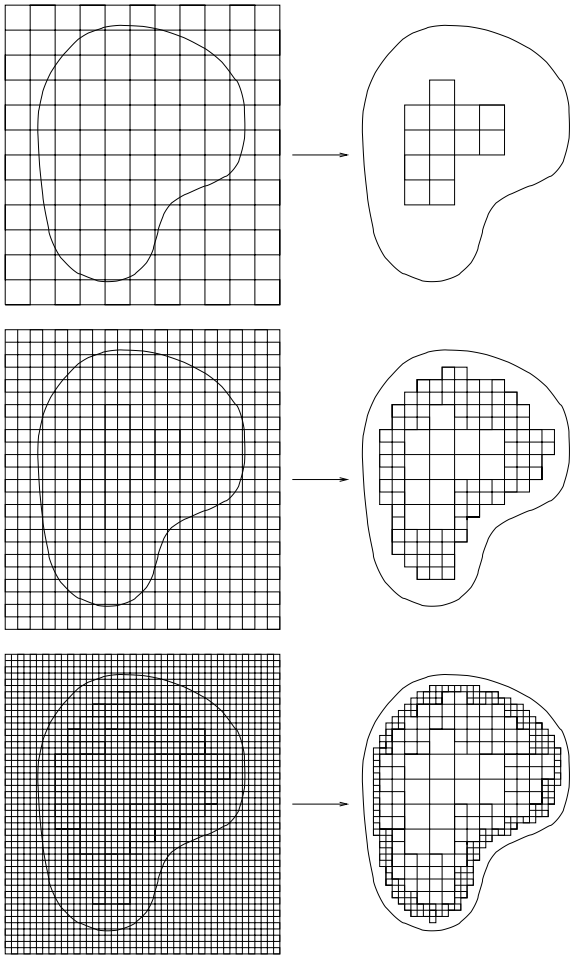


Figure 20: How to tile an open set.

$\psi_z \colon U_z \to \mathbb{R}^2$ does not require that z is non-periodic: we only have to consider the sequence of derivatives $\mathrm{D}_z f^n$ at z in order to analyze what are the directions mostly contracted or expanded by the dynamics. Then, any tiled cube (or tiled open set) contained in U_z and disjoint from its $N-1$ first iterates will satisfy property (P). This remark is important since it allows by compactness to cover M by finitely many charts ψ_z. Hence, it is now clear that the integer N depends only on $(f,\mathcal{U})$ and not on the choice of the point z.

We sum up all the previous remarks by saying that the connecting lemma asserts the existence of tiled open sets which satisfy condition (P): these sets will be called *perturbation domain*.

Theorem III.3 (Connecting lemma, 3rd version)**.** *For any diffeomorphism f and any neighborhood $\mathcal{U}$ of f in Diff^1_v, there exists an integer $N \geq 1$ and, at any point $z \in M$, there is a chart $\psi_z \colon U_z \to \mathbb{R}^2$ such that any tiled open set $B \subset U_z$ for the chart (ψ_z, U_z), that is disjoint from its $N-1$ first iterates, satisfies property (P).*

III.2.4 Additional remark One can state theorem III.3 without using tilings. This uses the idea of the proof of the closing lemma: once Pugh's metric d' has been given in a chart U_z, one

considers the "hyperbolic distance" of B defined for pair of points $z, z' \in B$ as

$$\mathrm{d}^B(z,z') = \frac{\mathrm{d}'(z,z')}{\inf(\mathrm{d}'(z,\partial B), \mathrm{d}'(z',\partial B))}.$$

One extends it to the pair of points in M: if z or z' does not belong to B, then $\mathrm{d}^B(z,z') = \infty$ unless $z = z'$ (then $\mathrm{d}^B(z,z') = 0$). If B is tiled as it is described in the proof of proposition III.2 and if $\mathrm{d}^B(z,z')$ is smaller than $\frac{1}{10}$, then z and z' are contained in a same enlarged tile of B or are equal. This implies the following statement of the connecting lemma:

Theorem III.4 (Connecting lemma, 4th version)**.** *For any pair $(f,\mathcal{U})$, there exists an integer $N \geq 1$ and, at any point $z \in M$, there exists an open neighborhood U_z endowed with a metric d'_z such that any open set $B \subset U_z$ that is disjoint from its $N-1$ first iterates satisfies the following property:*

if two points $p, q \in M$ may be jointed by a pseudo-orbit of f whose jumps are smaller than $\frac{1}{10}$ for the hyperbolic distance d^B of B associated to d', then there exists a perturbation $g \in \mathcal{U}$ with support in $B \cup f(B) \cup \cdots \cup f^{N-1}(B)$ and an integer $m \geq 1$ such that $g^m(p) = q$.

III.3 Choice of the perturbation domains

In order to prove theorem III.1, we consider a neighborhood $\mathcal{U}$ of f (which has the composition property, section II.1.4) and we apply the connecting lemma (theorem III.3) so that we get

- an integer N,
- a finite covering of M by charts $\psi_z \colon U_z \to \mathbb{R}^2$.

We consider two points $p, q \in M$ that one wants to connect by a same orbit. One difficulty is that the perturbation domains given by theorem III.3 should be disjoint from their $N-1$ first iterates so that they cannot contain any periodic point of small period. This motivates the following argument.

We denote by Σ_{2N} the set of points that are periodic of period less or equal to $2N$. It is finite by assumption. It is always possible to assume that p and q are not periodic (and not in Σ_{2N}): if this is not the case, one chooses p' and q' not periodic and close to p and q (using the fact that the set of periodic points is only countable), one then realizes the perturbation as explained in the section below and connects p' to q'; one ends by a little conjugacy of the perturbed map in order to move p' on p and q' on q.

III.3.1 Connection by segments of orbits and paths. In a first step, we build a "geometrical pseudo-orbit" that joints the points p to q: the jumps are obtained as translations along small paths which are pairwise disjoint.

Proposition III.5. *There are a sequence of points $(x_0, \ldots, x_s)$ in M, a sequence of integers $(n_0, \ldots, n_{s-1})$, and a sequence of paths $(\gamma_0, \ldots, \gamma_{s-1})$ in M such that:*

- *we have $x_0 = p$ and $x_s = q$;*
- *for each $k \in \{0, \ldots, s-1\}$, the points $f^{n_k}(x_k)$ and x_{k+1} are the endpoints of the path γ_k;*
- *each path γ_k is small and contained in some domain U_z;*
- *all the paths $f^i(\gamma_k)$ with $i \in \{0, \ldots, N-1\}$ and $k \in \{0, \ldots, s-1\}$ are disjoint.*

In particular the points p and q are connected by a pseudo-orbit of the form

$$(p = x_0, f(x_0), \dots, f^{n_0-1}(x_0), x_1, f(x_1), \dots, f^{n_1-1}(x_1), \dots, x_{s-1}, f^{n_{s-1}-1}(x_{s-1}), x_s = q).$$

Proof. The proof is very similar to the proof of proposition I.12: since p and q are not periodic, one chooses a large compact set K which contains in its interior p and q and which is disjoint from the compact set Σ_{2N}. One may also assume that the interior of K is connected.

There is a pseudo-orbit from p to q whose jumps are contained in the interior of K and arbitrarily small (in particular each jump is contained in a domain U_z). Hence, for any two consecutive points x and y of the pseudo-orbit, there is a simple path σ that connects $f(x)$ to y. One can choose σ small and contained in the interior of K so that σ is disjoint from its $2N$ first iterates and contained in some open set U_z. We get a family of path (σ_k). This gives all the required properties but the last one (a path σ may have some iterate $f^\ell(\sigma)$ with $\ell \in \{0, \dots, N\}$ that intersects another path σ').

We now suppress the intersections between the paths. This is done inductively: one considers the smallest i such that one iterate of σ_i intersects one iterate of some other path σ_j. One considers also the largest possible j. One then modifies the pseudo-orbit and the sequence (σ_k): we suppress the intersections between σ_i and σ_j, we do not create any new intersection and we do not use anymore the paths σ_k with $k \in \{i+1, \dots, j-1\}$. Hence, after a finite number of modifications, we have removed all the intersections and we get the announced sequences.

Let us explain how to handle with the intersections (we will assume for instance that $f^\ell(\sigma_i)$ intersect σ_j): the path σ_i connects the point $f^{n_i}(x_i)$ to x_{i+1} and the path σ_j connects the two points $f^{n_i}(x_j)$ and x_{j+1}. One considers the path $\sigma'_i \subset \sigma_i$ which connects $f^{n_i}(x_i)$ to the first intersection point x'_{i+1} of σ_i with $f^{-\ell}(\sigma_j)$. We introduce also the path $\tilde{\sigma}_j \subset \sigma_j$ which connects $f^\ell(x'_{i+1})$ to x_{j+1}. Note that $f^\ell(\sigma'_i)$ and $\tilde{\sigma}_j$ only intersect at their endpoint $f^\ell(x'_{i+1})$.
One now considers a return $f^{n'_{i+1}}(x'_{i+1})$ of $f^\ell(x'_{i+1})$ close to $f^\ell(x'_{i+1})$ (by Poincaré recurrence theorem, changing a little bit the point x'_{i+1} if necessary). One may also assume that $f^{n'_{i+1}}(x'_{i+1})$ does not belong to $f^\ell(\sigma'_i)$ (perturbing a little bit σ'_i again). It is then possible to modify $\tilde{\sigma}_j$ in a neighborhood of $f^\ell(x'_{i+1})$ so that it connects $f^{n'_{i+1}}(x'_{i+1})$ to x_{j+1} and is disjoint from $f^\ell(\sigma'_i)$: we get a path σ'_j (see figure 21).

By this construction, there is no more intersection between the iterates of σ'_i and σ'_j. There is also a segment of orbit that connects one endpoint of σ'_i to an endpoint of σ'_j. Hence, one can forget the intermediary paths σ_k with $k \in \{i+1, \dots, j-1\}$. □

III.3.2 Construction of perturbation domains. In the second step, we here build:

1. A finite number of perturbation domains $(B_s)_{s\in S}$. One requires that all the sets $f^i(B_s)$, with $s \in S$, and $i \in \{0, \dots, N-1\}$ are pairwise disjoint.
2. A pseudo-orbit $(z_0, \dots, z_n)$ between p and q which respects the tiling of the perturbation domains B_k.

For this, one first considers the sequences (x_k), (n_k) and (γ_k) introduced at the previous section and that connect p to q. One thicks each path γ_k as a small open set B_k. As it is the case for the paths γ_k, one can assume that each B_k is contained in some open set U_z and all the sets $f^i(B_k)$ with $i \in \{0, \dots, N-1\}$ and $s \in S$ are pairwise disjoint.
By proposition III.2, one can tile each B_k as a tiled domain of the chart (ψ_z, U_z) and by theorem III.3, it becomes a perturbation domain.

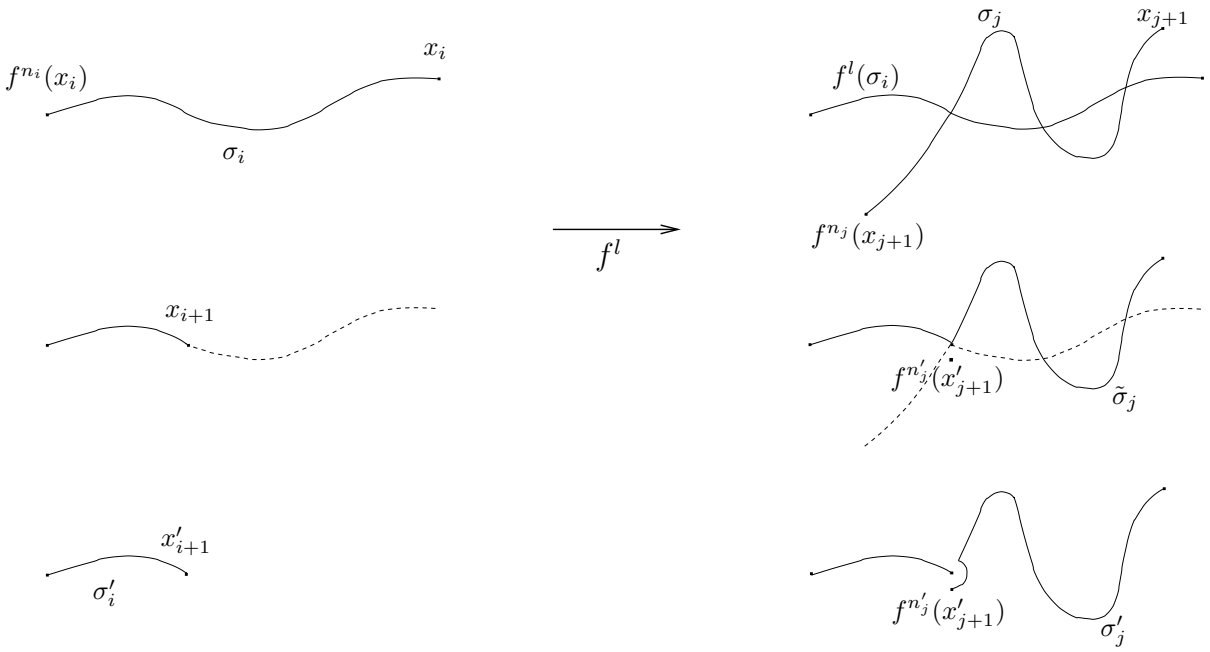


Figure 21: Modification of the paths σ_i and σ_j.

It remains to define the pseudo-orbit $(z_0, \ldots, z_n)$. Note that the pseudo-orbit

$$(p = x_0, f(x_0), \ldots, f^{n_0-1}(x_0), x_1, \ldots, x_s = q)$$

has its jumps contained in the domains B_k but maybe not in the tiles of the B_k. Therefore, one will define a longer pseudo-orbit by introducing between each pair of points $(f^{n_i}(x_i), x_{i+1})$ a segment of pseudo-orbit which has its jumps in the enlarged tiles of the domains (B_k).
The argument is again the same as in the proof of proposition I.12: since γ_k is connected, one can build a pseudo-orbit from $f^{n_i}(x_i)$ to x_{i+1} (using the recurrence of almost every point of M by f) and having jumps arbitrarily small and all contained in any neighborhood of γ_k. In particular, if the size of the jumps and the distance from each jump to γ_k are small with respect to the tiles of B_k that meet γ_k, then, the pseudo-orbit respect the tiling of B_k. This gives the pseudo-orbit (z_k).

III.4 Conclusion: proof of theorem III.1

We have built some domains (B_k) and a pseudo-orbit (z_i) from p to q at section III.3.2.
Using the definition of the perturbation domains (theorem III.3), one may perturb in each B_k in order to remove the jumps inside (property (P)). Since the perturbation domains are disjoint, one can perturb independently in each of them, the final perturbation will stay in $\mathcal{U}$ by the composition property.
Hence, one considers each perturbation domains one after the other and eliminates in this way all the jumps of the pseudo-orbit (z_i) (figure 22). We get at the end for the perturbed map a genuine orbit that connects p to q as required.

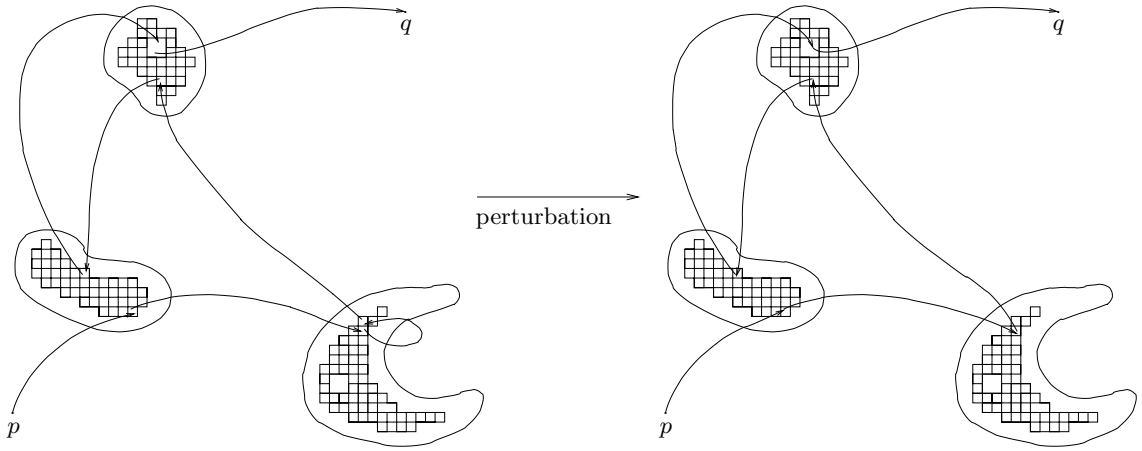


Figure 22: Final perturbation.

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