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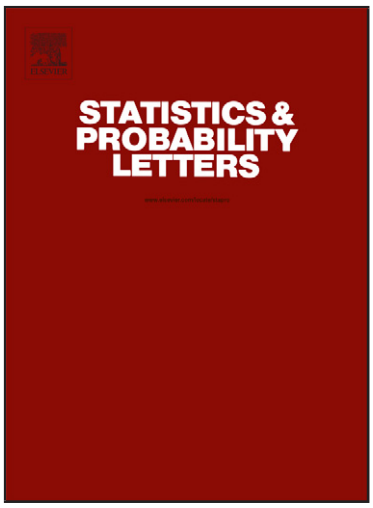

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SHIGA-WATANABE's TIME INVERSION PROPERTY FOR SELF-SIMILAR DIFFUSION PROCESSES

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Summary : Assume (X_t, P^0) is $\frac{1}{2}$-self-similar, rotation invariant diffusion on R^d, d$\geq$ 2, starting at 0 and assume $\{0\}$ is a polar set. We will show, using the corresponding well-known result for the radial process, that Shiga-Watanabe's time inversion property holds for (X_t, P^0). The generalization for an α-self-similar, rotation invariant diffusion, $\alpha > 0$, is also given.

Key words :Time inversion, self-similar, Bessel process, diffusion, rotation invariant, skew product, radial process, spherical process

AMS Mathematics Subject Classification (2000): 60G18, 60J60, 60J25

0.*Introduction. Theorem*

The following time inversion property is well known for Brownian motion in R^d, d$\geq$ 1, and Bessel diffusions on [0, ∞), starting at 0 (see Shiga, Watanabe (1973) and Watanabe (1975):

(X_t) has the same finite dimensional distributions as $(tX_{1/t})$ under P^0. (1)

Graversen, Vuolle-Apiala (2000) show that (1) also holds for symmetrized Bessel processes on R, starting at 0, in the case of the index $\nu \in$(-1,0). In this case (X_t) can both hit 0 and can be started at 0. Symmetrized Bessel processes form the class of one dimensional rotation invariant $\frac{1}{2}$-self similar diffusions (see the definition below). If the index $\nu \leq$-1 then 0 is an exit boundary point. That is, (X_t) can hit 0 but cannot be started there. If $\nu \geq 0$ then 0 is an entrance boundary point, that is, (X_t) can be started there and it will never come back. Thus in this case (X_t) in fact lives either on [0, ∞) or on (-∞, 0] and $\{0\}$ is a polar set. Obviously, (1) is valid. Result (1) has been generalized for α-self-similar diffusions on [0, ∞) and for symmetric α-self-similar diffusions on R,

$\alpha > 0$, in Graversen, Vuolle-Apiala (2000). The corresponding generalization of (1) is then

(X_t) has the same finite dimensional distributions as $(t^{2\alpha}X_{1/t})$ under P^0.(2)

In this note we will show that (2) holds for all rotation invariant, d-dimensional, α-self-similar diffusions (that is, strong Markov processes with continuous paths), $d \geq 2$, $\alpha > 0$, for which $\{0\}$ is a polar set. Our main tool is a skew product representation for rotation invariant diffusions starting at 0; see Ito, Mc Kean Jr. (1974), p. 274-276.

Let (X_t, P^x) be a rotation invariant (RI) α-self-similar (α-ss) diffusion on R^d, $d \geq 2$, $\alpha > 0$, such that $\{0\}$ is a polar set. By α-self-similarity we mean that

(X_t) under P^x has the same finite dimensional distributions as $(a^{-\alpha}X_{at})$ under $P^{a^\alpha x}$ for all $x \in R^d$, $a > 0$ (3)

and by (RI) that

(X_t) under P^x has the same finite dimensional distributions as $(T^{-1}(X_t))$ under $P^{T(x)}$ for all $T \in O(d)$. (4)

Self-similar diffusions on R or on $[0, \infty)$ are defined similarly. Brownian motion fullfills both (3) and (4). See more about self-similar Markov processes in Graversen, Vuolle-Apiala (1986), Lamperti (1972) and Vuolle-Apiala, Graversen (1986). According to Graversen, Vuolle-Apiala (1986) and Vuolle-Apiala (2003), when $X_0 \neq 0$ the diffusion processes which fullfill (3) and (4) can be represented as skew products

$$[|X_t|, \theta_{A_t}], \tag{5}$$

where $A_t = \lambda \int_0^t |X_s|^{-\frac{1}{\alpha}} ds$ for some $\lambda > 0$, the radial part $(|X_t|)$ is an α-ss diffusion on $(0,\infty)$, and (θ_t) is a spherical Brownian motion on S^{d-1} independent of $(|X_t|)$.

REMARK: As showed in Graversen, Vuolle-Apiala (1986), (5) is valid for all strong Markov processes with cadlag paths fullfilling (3) and (4). However, as showed by a counterexample by J. Bertoin, W. Werner (1996), the independence between $(|X_t|)$ and (θ_t) is not necessarily true if the paths are only right continuous. There is an error in the proof of Proposition 2.4, p.19-20 in Graversen, Vuolle-Apiala (1986). It was shown in Vuolle-Apiala (2003), Lemma 2.1, that $(|X_t|)$ and (θ_t) are independent in the case of continuous paths.

We want to prove the following:

THEOREM: Let (X_t, P^0) be an (RI) α-ss diffusion on R^d, $\alpha > 0$, $d \geq 2$, starting at 0, having $\{0\}$ as a polar set. Then the time inversion property (2) is valid.

1. *The Proof of the Theorem*

The proof will be based on

PROPOSITION: Let (r_t) be an α-ss diffusion on $[0, \infty)$, $\alpha > 0$, such that 0 is an entrance, non-exit boundary point. Then the skew product

$$[r_t, \theta_{\lambda \int_0^t r_s^{-\frac{1}{\alpha}} ds}], \; t > 0, \; r_0 > 0, \tag{6}$$

where (θ_t, Q^θ) is a spherical Brownian motion on S^{d-1} independent of (r_t), such that $Q^\theta(\theta_0 = 0) = 1 \; \forall \theta \in S^{d-1}$,

can be completed to be an α-ss diffusion on R^d by defining

$$[r_t, \nu_{\lambda \int_1^t r_s^{-\frac{1}{\alpha}} ds}], \; t > 0, \text{ when } r_0 = 0, \tag{7}$$

where (ν_t, Q) is an independent, spherical Brownian motion defined for $-\infty < t < +\infty$ and the law of ν_0 is the uniform spherical distribution $m(d\theta)$.

REMARK 1: Because of a uniquenness result of RI measures on S^{d-1} there is at most one way to complete (6) to be RI on the R^d.

REMARK 2: It is obvious that (ν_t, Q) in fact is a stationary process and ν_t is uniformly distributed for all $t \in R$ (see Kuznetsov, 1973).

We have

$$Q\{\nu(t_1) \in d\theta_1, \ldots, \nu(t_n) \in d\theta_n\} = m(d\theta) Q^{\theta_1}(\theta(t_2 - t_1) \in d\theta_2) \ldots Q^{\theta_{n-1}}(\theta(t_n - t_{n-1}) \in d\theta_n) \tag{8}$$

for $-\infty < t_1 < \ldots$ ¡$t_n < +\infty$ (for typographical reasons we use in formula (8) notations $\nu(t_i)$ and $\theta(t_i)$ instead of ν_{t_i} and θ_{t_i})

In order to prove Proposition we need

LEMMA 1: $(\nu_{t_1}, \ldots, \nu_{t_n})$ under Q has the same distribution as $(\nu_{-t_1}, \ldots, \nu_{-t_n})$.

PROOF: Follows immediately from (8) and the fact that (θ_t, Q^θ) has a symmetric density with respect to the uniform measure $m(d\theta)$ on S^{d-1} (see Vuolle-Apiala, Graversen (1986), Lemma 3, p.329).□

In the proof of Proposition we will use the result of Ito-McKean (1974), p.275, which says that the skew product (6) can be completed to be a diffusion

(which obviously is RI) on the whole R^d having the skew product (7) when $r_0 = 0$ iff $A_{0+} = \infty$ a.s. P^0. Here we need

LEMMA 2: Let (r_t) be an α-ss diffusion on $[0,\infty)$ such that 0 is an entrance, non-exit boundary point. Then

$$P^0\{ \int_0^\epsilon r_s^{-\frac{1}{\alpha}} ds = \infty\} = 1 \quad \forall \epsilon > 0.$$

PROOF of LEMMA 2: The property (2) is valid for (r_t) which implies

$$P^0\{ \int_0^\epsilon r_s^{-\frac{1}{\alpha}} ds = \infty\} = P^0\{ \int_0^\epsilon (s^{2\alpha} r_{1/s})^{-\frac{1}{\alpha}} ds = \infty\} = P^0\{ \int_0^\epsilon s^{-2} r_{1/s}^{-\frac{1}{\alpha}} ds = \infty\}.$$

Substituting $1/s = h$ gives this equal to

$$P^0\{\int_\infty^{1/\epsilon} (- r_h^{-\frac{1}{\alpha}}) dh = \infty\} = P^0\{ \int_{1/\epsilon}^\infty r_h^{-\frac{1}{\alpha}} dh = \infty\}.$$

Using the Markov property we obtain

$$P^0\{\int_{1/\epsilon}^\infty r_h^{-\frac{1}{\alpha}} dh = \infty\} = E^0\{P^{r_{1/\epsilon}}\{ \int_0^\infty r_h^{-\frac{1}{\alpha}} dh = \infty\}\}.$$

Because 0 is an entrance, non-exit boundary point, $r_{1/\epsilon} > 0$ a.s. (P^0). Now, according to Lamperti (1972),

$$P^r\{ \int_0^\infty r_h^{-\frac{1}{\alpha}} dh = \infty\} = 1 \quad \text{for all } r > 0$$

and thus

$$E^0\{P^{r_{1/\epsilon}}\{ \int_0^\infty r_h^{-\frac{1}{\alpha}} dh = \infty\}\} = 1$$

which implies

$$P^0\{ \int_0^\epsilon r_h^{-\frac{1}{\alpha}} dh = \infty\} = 1 \ . \ \square$$

PROOF OF PROPOSITION: It only remains to prove that the skew product

$$[r_t, \nu_{\lambda \int_1^t r_s^{-\frac{1}{\alpha}} ds}] \ , r_0 = 0, \tag{7}$$

fullfills the α-self-similarity condition (3) under P^0. Let $I_1, \ldots, I_n$ be Borel subsets of $[0,\infty)$ and $J_1, \ldots, J_n$ Borel subsets of S^{d-1}. We will show

$$P^0\{r_{t_1} \in I_1, \ldots, r_{t_n} \in I_n, \ \nu_{\lambda \int_1^{t_1} r_s^{-\frac{1}{\alpha}} ds} \in J_1, \ldots, \ \nu_{\lambda \int_1^{t_n} r_s^{-\frac{1}{\alpha}} ds} \in J_n\} =$$

$$P^0\{a^{-\alpha} r_{at_1} \in I_1, \ldots, a^{-\alpha} r_{at_n} \in I_n, \ \nu_{\lambda \int_1^{at_1} r_s^{-\frac{1}{\alpha}} ds} \in J_1, \ldots, \ \nu_{\lambda \int_1^{at_n} r_s^{-\frac{1}{\alpha}} ds} \in J_n\} (*)$$

for all $t > 0$. For simplicity, assume $n = 2$, the general case is analogeous.

Now the right hand side of (*) for n = 2 is equal to

$$P^0\{a^{-\alpha}r_{at_1} \in I_1,\, a^{-\alpha}r_{at_2} \in I_2,\ \nu_{\lambda\int_{1/a}^{t_1}(a^{-\alpha}\,r_{as})^{-\frac{1}{\alpha}}ds} \in J_1, \nu_{\lambda\int_{1/a}^{t_2}(a^{-\alpha}\,r_{as})^{-\frac{1}{\alpha}}ds} \in J_2\}=$$

$$P^0\{r_{t_1} \in I_1,\, r_{t_2} \in I_2,\ \nu_{\lambda\int_{1/a}^{t_1} r_s^{-\frac{1}{\alpha}}ds} \in J_1, \nu_{\lambda\int_{1/a}^{t_2} r_s^{-\frac{1}{\alpha}}ds} \in J_2\}$$

because (r_t) fullfills (3) and because of independence between $(r_.)$ and $(\nu_.)$.

This is further equal to

$$P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2,\ \nu_{\lambda\int_{1/a}^{1} r_s^{-\frac{1}{\alpha}}ds + \lambda\int_1^{t_1} r_s^{-\frac{1}{\alpha}}ds} \in J_1, \nu_{\lambda\int_{1/a}^{1} r_s^{-\frac{1}{\alpha}}ds + \lambda\int_1^{t_2} r_s^{-\frac{1}{\alpha}}ds} \in J_2\}=$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P^0\{r_{t_1} \in I_1,\, r_{t_2} \in I_2,\, \lambda\int_{1/a}^{1} r_s^{-\frac{1}{\alpha}}ds \in du, \lambda\int_1^{t_1} r_s^{-\frac{1}{\alpha}}ds \in dv,$$

$$\lambda\int_1^{t_2} r_s^{-\frac{1}{\alpha}}ds \in dw, \nu_{u+v} \in J_1,\, \nu_{u+w} \in J_2\}=$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P^0\{r_{t_1} \in I_1,\, r_{t_2} \in I_2,\, \lambda\int_{1/a}^{1} r_s^{-\frac{1}{\alpha}}ds \in du,\, \lambda\int_1^{t_1} r_s^{-\frac{1}{\alpha}}ds \in dv,$$

$$\lambda\int_1^{t_2} r_s^{-\frac{1}{\alpha}}ds \in dw\}\, Q(\nu_{u+v} \in J_1,\, \nu_{u+w} \in J_2)=$$

because of independence between $(r_.)$ and $(\nu_.)$. Now $(\nu_.)$ is a stationary process and thus this is equal to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P^0\{r_{t_1} \in I_1,\, r_{t_2} \in I_2,\, \lambda\int_{1/a}^{1} r_s^{-\frac{1}{\alpha}}ds \in du,\, \lambda\int_1^{t_1} r_s^{-\frac{1}{\alpha}}ds \in dv,$$

$$\lambda\int_1^{t_2} r_s^{-\frac{1}{\alpha}}ds \in dw\}\, Q(\nu_v \in J_1,\, \nu_w \in J_2) =$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2,\ \lambda\int_1^{t_1} r_s^{-\frac{1}{\alpha}}ds \in dv, \lambda\int_1^{t_2} r_s^{-\frac{1}{\alpha}}ds \in dw\}\, Q(\nu_v \in J_1, \nu_w \in J_2)=$$

$$P^0\{r_{t_1} \in I_1,\, r_{t_2} \in I_2,\ \nu_{\lambda\int_1^{t_1} r_s^{-\frac{1}{\alpha}}ds} \in J_1, \nu_{\lambda\int_1^{t_2} r_s^{-\frac{1}{\alpha}}ds} \in J_2\}. \qquad \square$$

Now we can prove Theorem:

PROOF OF THEOREM: (X_t) has according to Graversen, Vuolle-Apiala (1986), Vuolle-Apiala (2003) and Proposition a skew product representation

$$[r_t, \theta_{\lambda\int_0^t r_s^{-\frac{1}{\alpha}}ds}] \ \text{ as } X_0 \neq 0 \tag{6}$$

and

$$[r_t, \nu_{\lambda\int_1^t r_s^{-\frac{1}{\alpha}}ds}] \ \text{as}\, X_0=0, \tag{7}$$

where (r_t) is the radial process, (θ_t, Q^θ) is an independent spherical Brownian motion such that $Q^\theta(\theta_0 = \theta) = 1$ and (ν_h, Q) is an independent, stationary, spherical Brownian motion defined for $-\infty < h < +\infty$ and the law of (ν_h) is the uniform spherical distribution for all $h \in R$. To show (2) let us consider the distribution of $\{t_1^{2\alpha} X_{1/t_1}, \ldots, t_n^{2\alpha} X_{1/t_n}\}$ under P^0. Assume for simplicity $n = 2$, $\alpha = \frac{1}{2}$, the general case is analogeous.

Let I_i and J_i, $i = 1, 2$, be Borel subsets of $(0, \infty)$ and S^{d-1},respectively. Now

$$P^0\{t_1 X_{1/t_1} \in (I_1, J_1), t_2 X_{1/t_2} \in (I_2, J_2)\} =$$

$$P^0\{t_1 r_{1/t_1} \in I_1, t_2 r_{1/t_2} \in I_2, \ \nu_{\lambda \int_1^{1/t_1} r_s^{-2} ds} \in J_1, \nu_{\lambda \int_1^{1/t_2} r_s^{-2} ds} \in J_2\} =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P^0\{t_1 r_{1/t_1} \in I_1, t_2 r_{1/t_2} \in I_2, \nu_u \in J_1, \nu_v \in J_2, \lambda \int_1^{1/t_1} r_s^{-2} ds \in du,$$

$$\lambda \int_1^{1/t_2} r_s^{-2} ds \in dv\} =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P^0\{t_1 r_{1/t_1} \in I_1, t_2 r_{1/t_2} \in I_2, \lambda \int_1^{1/t_1} r_s^{-2} ds \in du, \lambda \int_1^{1/t_2} r_s^{-2} ds \in dv\} Q(\nu_u \in J_1, \nu_v \in J_2) =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P^0\{t_1 r_{1/t_1} \in I_1, t_2 r_{1/t_2} \in I_2, -\lambda \int_1^{t_1} (s r_{1/s})^{-2} ds \in du, -\lambda \int_1^{t_2} (s r_{1/s})^{-2} ds \in dv\} Q(\nu_u \in J_1, \nu_v \in J_2).$$

Because (1) is true for (r_t) this is equal to

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, -\lambda \int_1^{t_1} r_s^{-2} ds \in du, -\lambda \int_1^{t_2} r_s^{-2} ds \in dv\} Q(\nu_u \in J_1, \nu_v \in J_2) =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \lambda \int_1^{t_1} r_s^{-2} ds \in du, \lambda \int_1^{t_2} r_s^{-2} ds \in dv\} Q(\nu_{-u} \in J_1, \nu_{-v} \in J_2).$$

Using Lemma 1 we get this equal to

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \lambda \int_1^{t_1} r_s^{-2} ds \in du, \lambda \int_1^{t_2} r_s^{-2} ds \in dv\} Q(\nu_u \in J_1, \nu_v \in J_2) =$$

$$P^0\{r_{t_1} \in I_1, r_{t_2} \in I_2, \ \nu_{\lambda \int_1^{t_1} r_s^{-2} ds} \in J_1, \nu_{\lambda \int_1^{t_2} r_s^{-2} ds} \in J_2\} =$$

$$P^0\{X_{t_1} \in (I_1, J_1), X_{t_2} \in (I_2, J_2)\}.$$

□

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