A FOURIER APPROACH FOR THE LEVEL CROSSINGS OF SHOT NOISE PROCESSES WITH JUMPS

HERMINE BIERMÉ AND AGNÈS DESOLNEUX

Abstract. We use here a change of variables formula in the framework of functions of bounded variation to derive an explicit formula for the Fourier transform of the level crossings function of shot noise processes with jumps. We illustrate the result on some examples and give some applications. In particular, it allows us to study the asymptotic behavior of the mean number of level crossings as the intensity of the Poisson point process of the shot noise process goes to infinity.

In this paper, we will consider a *shot noise process* which is a real-valued random process given by

$$X(t) = \sum_{i\in\mathbb{Z}} \beta_i g(t-\tau_i), \quad t \in \mathbb{R}, \tag{1}$$

where g is a given (deterministic) measurable function (it will be called the *kernel function* of the shot noise), the $\{\tau_i\}_{i\in\mathbb{Z}}$ are the points of a homogeneous Poisson point process on the line of intensity $\lambda > 0$, and the $\{\beta_i\}_{i\in\mathbb{Z}}$ are independent copies of a random variable β (called the *impulse*), independent of $\{\tau_i\}_{i\in\mathbb{Z}}$.

Such a process has many applications (see [16] and the references therein for instance) and it is a well-known and studied mathematical model (see [10], [15], [5] for some of its properties).

We will be interested here in the level crossings of such a process. Usually, the mean number of level crossings of a stochastic process is computed thanks to a Rice's formula (see [12] or [1]) that requires some regularity conditions on the joint probability density of X and of its derivative. This joint probability density is generally not easy to obtain and to study. Its existence is also sometimes a question in itself. This is why, instead of working directly on the mean number of level crossings, we will work on the Fourier transform of the function that maps each level α to the mean number of crossings of the level α per unit length. Thanks to a change of variables formula, we will be able to relate this Fourier transform to the characteristic function of the shot noise process (which, unlike the probability density, always exists and is explicit).

1. General result

In [3], we have studied the level crossings of the shot noise process X when the kernel function g is smooth on $\mathbb{R}$. We will consider here the case where g is a piecewise smooth function, that is not necessarily continuous. We first introduce some definitions and notations.

Let I be an open interval of $\mathbb{R}$ and $k \geq 0$ be an integer. A function $f : I \to \mathbb{R}$ is said piecewise C^k on I if there exists a finite set of points of discontinuity of f on I, denoted by

$$S_f = \{s_1, s_2, \ldots, s_m\}, \text{ with } m \geq 1 \text{ and } s_1 < \ldots < s_m,$$

and called the jump set of f on I, such that f is of class C^k at any point s of I such that $s \notin S_f$. We moreover assume that f admits finite left and right limits at each point of S_f. For a point $s \in I$, we will denote

$$f(s+) = \lim_{t\to s, t>s} f(t) \quad \text{and} \quad f(s-) = \lim_{t\to s, t<s} f(t)$$

the respective right and left limits of f at s. Notice that when $s \notin S_f$, we simply have $f(s+) = f(s-) = f(s)$. We will also use the following notations:

$$\forall s \in I, \;\; f^*(s) := \max\{f(s+), f(s-)\}, \quad f_*(s) := \min\{f(s+), f(s-)\}, \tag{2}$$

2000 *Mathematics Subject Classification.* Primary: 60G17, 60E10, 26A45; Secondary: 60G10, 60F05.

Key words and phrases. shot noise, crossings, stationary process, Poisson process, characteristic function, functions of bounded variation, change of variables formula.

$$\text{and} \quad \Delta f(s) := f(s+) - f(s-). \tag{3}$$

Finally, when $k \geq 1$, we will denote by $f', \ldots, f^{(k)}$ the functions that are defined at all points $s \notin S_f$ by the usual derivatives $f'(s), \ldots, f^{(k)}(s)$.

In the sequel, we will need assumptions on the kernel function g and on the impulse β of the shot noise process X defined by (1). These assumptions are grouped together into the following condition denoted by (C):

$$\text{(C)}: \ \mathbb{E}(|\beta|) < \infty, \ \ g \text{ is piecewise } C^2 \text{ on } \mathbb{R} \ \text{ and } \ g, g', g'' \in L^1(\mathbb{R}).$$

We will also denote the jump set of g on $\mathbb{R}$ by

$$S_g = \{t_1, t_2, \ldots, t_n\}, \text{ with } t_1 < \ldots < t_n.$$

As consequence of (C), g has finite total variation on $\mathbb{R}$, which means that

$$TV(g, \mathbb{R}) = \sup_P \sum_{k=1}^{n_P} |g(a_k) - g(a_{k-1})| = \int_{\mathbb{R}} |g'(s)|\, ds + \sum_{j=1}^{n} |\Delta g(t_j)| < \infty,$$

where the supremum is taken over all partitions $P = \{a_0, \ldots, a_{n_P}\}$ of $\mathbb{R}$ with $n_P \geq 1$ and $a_0 < \ldots < a_{n_P}$.

Finally, we assume that the points $\{\tau_i\}$ of the Poisson point process are indexed by $\mathbb{Z}$ in such a way that for any $k \in \mathbb{N}$, one has $0 < \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$ and $0 > \tau_{-1} > \tau_{-2} > \ldots > \tau_{-k} > \ldots$.

1.1. **Piecewise regularity of the shot noise process.** The shot noise process "inherits" the regularity of the kernel function g. More precisely, we have the following result.

Theorem 1. *Assume that the condition (C) holds, then the shot noise process X defined by* (1) *is a stationary process which is almost surely piecewise C^1 on any interval (a, b) of $\mathbb{R}$. The jump set of X on (a, b) is*

$$S_X \cap (a, b) \ \ \textit{where} \ \ S_X = \bigcup_{i \in \mathbb{Z}} (\tau_i + S_g)$$

and

$$\forall t \notin S_X, \quad X'(t) = \sum_{i \in \mathbb{Z}} \beta_i g'(t - \tau_i).$$

Proof. Note that, since $\mathbb{E}(|\beta|) < \infty$ and $g \in L^1(\mathbb{R})$, for any $t \in \mathbb{R}$, the random variable $X(t)$ is well-defined and $\mathbb{E}(X(t)) = \lambda \mathbb{E}(\beta) \int_{\mathbb{R}} g(s) ds$, according to [16]. Moreover, since the Poisson point process is homogeneous, X is a stationary process.

Let us first remark that since the jump set S_g of g contains exactly n points, we can write g as the sum of $2n$ piecewise C^2 functions on $\mathbb{R}$, each of them having only one discontinuity point and having the same regularity properties as g. Therefore we may and will assume that g has only one discontinuity point and we write $S_g = \{t_1\}$. Let $i_0 \in \mathbb{Z}$. We set $I_{i_0} := [t_1 + \tau_{i_0}, t_1 + \tau_{i_0+1}]$, Then, for any $t \in \mathbb{R}$,

$$X(t) = \sum_{i > i_0+1} \beta_i g(t - \tau_i) + \sum_{i < i_0} \beta_i g(t - \tau_i) + \beta_{i_0} g(t - \tau_{i_0}) + \beta_{i_0+1} g(t - \tau_{i_0+1}).$$

The function $t \mapsto g(t - s)$ is C^2 on I_{i_0} for any $s < \tau_{i_0}$ such that almost surely, for any $i < i_0$, the function $t \mapsto g(t - \tau_i)$ is C^2 on I_{i_0} with $g((t_1 + \tau_{i_0} - \tau_i)+) = g((t_1 + \tau_{i_0} - \tau_i)-)$. Moreover

$$\mathbb{E}\left(\sum_{i < i_0} |\beta_i| \sup_{t \in I_{i_0}} |g'(t - \tau_i)| \,\Big|\, \tau_{i_0}, \tau_{i_0+1} \right) \ = \ \lambda \mathbb{E}(|\beta|) \int_{-\infty}^{0} \sup_{t \in I_{i_0}} |g'(t - s - \tau_{i_0})| ds,$$

using the fact that $\{\tau_i - \tau_{i_0}; i < i_0\}$ is a homogeneous Poisson point process with intensity λ on $]-\infty, 0[$, and independent of τ_{i_0}, τ_{i_0+1}. Now, for any $t \in I_{i_0}$ and $s < 0$,

$$g'(t - s - \tau_{i_0}) = \int_{t_1 + \tau_{i_0}}^{t} g''(u - s - \tau_{i_0}) du + g'(t_1 - s),$$

such that by Fubini-Tonelli Theorem,

$$\int_{-\infty}^{0} \sup_{t\in I_{i_0}} |g'(t-s-\tau_{i_0})|ds \quad \leq \quad (\tau_{i_0+1}-\tau_{i_0}) \int_{\mathbb{R}} |g''(s)|ds + \int_{\mathbb{R}} |g'(s)|ds.$$

Then

$$\mathbb{E}\left(\sum_{i<i_0} |\beta_i| \sup_{t\in I_{i_0}} |g'(t-\tau_i)|\right) \leq \lambda \mathbb{E}(|\beta|)\left(\frac{1}{\lambda}\int_{\mathbb{R}} |g''(s)|ds + \int_{\mathbb{R}} |g'(s)|ds\right) < +\infty,$$

so that a.s. the series $t \mapsto \sum_{i<i_0} \beta_i g'(t-\tau_i)$ is uniformly convergent on I_{i_0}. Therefore, a.s. the series $t \mapsto \sum_{i<i_0} \beta_i g(t-\tau_i)$ is continuously differentiable on I_{i_0} with $\left(\sum_{i<i_0} \beta_i g(t-\tau_i)\right)' = \sum_{i<i_0} \beta_i g'(t-\tau_i)$ and

$$\sum_{i<i_0} \beta_i g((t_1+\tau_{i_0}-\tau_i)+) = \sum_{i<i_0} \beta_i g((t_1+\tau_{i_0}-\tau_i)-).$$

The same proof applies for $\sum_{i>i_0+1} \beta_i g(t-\tau_i)$. To conclude it is sufficient to remark that for $i \in \{i_0, i_0+1\}$, the function $t \mapsto g(t-\tau_i)$ is continuously differentiable in the interior of I_{i_0}. Moreover $g((t_1+\tau_{i_0}-\tau_{i_0+1})+) = g((t_1+\tau_{i_0}-\tau_{i_0+1})-)$, $g((t_1+\tau_{i_0}-\tau_{i_0})+) = g(t_1+)$ and $g((t_1+\tau_{i_0}-\tau_{i_0})-) = g(t_1-)$. Finally, a.s. X is continuously differentiable in the interior of I_{i_0} with

$$X'(t) = \sum_i \beta_i g'(t-\tau_i),$$

and $X((t_1+\tau_{i_0})+) - X((t_1+\tau_{i_0})-) = \beta_{i_0}\left(g(t_1+) - g(t_1-)\right)$. This ends the proof of the theorem since $\mathbb{R} = \cup_{i_0\in\mathbb{Z}} I_{i_0}$. □

Remark 1. *Theorem 1 implies that, under the condition (C), the shot noise process X has almost surely a finite total variation on any interval (a,b) of $\mathbb{R}$. By stationarity we can focus on what happens on the interval $(0,1)$. Then X has almost surely a finite number of points of discontinuity in $(0,1)$ and its total variation on $(0,1)$ is given by*

$$TV(X,(0,1)) = \int_0^1 |X'(t)|dt + \sum_{j=1}^{n} \sum_{\tau_i\in(-t_j,1-t_j)} |\beta_i||\Delta g(t_j)|.$$

1.2. **Level crossings.** We start this section with a general definition and a result on the level crossings of a piecewise smooth function.

Let f be a piecewise C^1 function on an interval (a,b) of $\mathbb{R}$. We can define its level crossings on (a,b) by considering, for any level $\alpha \in \mathbb{R}$,

$$N_f(\alpha,(a,b)) = \#\{s\in(a,b)\ ;\ f_*(s) \leq \alpha \leq f^*(s)\} \in \mathbb{N}\cup\{\infty\},$$

where the notation $\#\{\cdot\}$ stands for the number of elements of the set $\{\cdot\}$ and using the notations defined by (2). The number $N_f(\alpha,(a,b))$ may be infinite. This is for instance what happens if there exists a sub-interval of (a,b) on which f is constant equal to α (the value α is then called a critical value of f). But, according to Morse-Sard Theorem ([13] p.10), the set of these critical values has Lebesgue measure zero. Let us also mention that a weak variant of Morse-Sard Theorem for Lipschitz functions can be found in [8], p.112.

Then, a change of variables formula for piecewise C^1 functions is obtained in the following proposition.

Proposition 1. *Let $a,b\in\mathbb{R}$ with $a<b$ and f be a piecewise C^1 function on (a,b). Then, for any bounded continuous function h defined on $\mathbb{R}$,*

$$\int_{\mathbb{R}} h(\alpha)N_f(\alpha,(a,b))\,d\alpha = \int_a^b h(f(s))|f'(s)|\,ds + \sum_{s\in S_f\cap(a,b)} \int_{f_*(s)}^{f^*(s)} h(\alpha)\,d\alpha. \tag{4}$$

Proof. Let us assume that $S_f \cap (a,b) = \{s_j\,;\, 1 \leq j \leq m\}$ with $m \geq 1$ and $s_0 := a < s_1 < \ldots < s_m < s_{m+1} := b$. Then

$$N_f(\alpha,(a,b)) = \sum_{j=0}^{m} \#\{s \in (s_j, s_{j+1})\ ;\ f(s) = \alpha\} + \sum_{j=1}^{m} \mathbf{1}_{[f_*(s_j), f^*(s_j)]}(\alpha). \tag{5}$$

Let h be a bounded continuous function on $\mathbb{R}$. According to the change of variables formula for Lipschitz functions (see [8] p.99), for any $j = 0, \ldots, m$, we have

$$\int_{\mathbb{R}} h(\alpha) \#\{s \in (s_j, s_{j+1})\ ;\ f(s) = \alpha\}\, d\alpha = \int_{s_j}^{s_{j+1}} h(f(s)) |f'(s)|\, ds.$$

Then, we sum up these equalities for $j = 0$ to m, and using (5), this achieves the proof of the proposition. □

Notice that a change of variables formula for functions of bounded variation was also obtained in [6]. But their formula is valid away from the jump set of f, whereas our formula (4) takes explicitly into account the contribution of the jump set. Let us also mention the fact our formula (4) is a kind of extension of Banach's Theorem used by Rychlik in [17] (p.335). The main difference is that our formula involves a function h that is not necessarily constant, but on the other hand, we are restricted to a piecewise C^1 function f (whereas Banach's Theorem is valid for functions of bounded variation).

Let us be now interested in the shot noise process X. For $\alpha \in \mathbb{R}$, let $N_X(\alpha)$ be the random variable that counts the number of crossings of the level α by the process X in the interval $(0,1)$. Using (2), it is defined by

$$N_X(\alpha) = \#\{t \in (0,1)\ ;\ X_*(t) \leq \alpha \leq X^*(t)\}.$$

We will be mainly interested in its expectation, namely in

$$C_X(\alpha) = \mathbb{E}(N_X(\alpha)).$$

The function $\alpha \mapsto C_X(\alpha)$ is called the mean level crossings function and we compute its Fourier transform in the following theorem. This result has to be related to the heuristic approach of [2] (in particular their formula (13) that involves the joint density of $X(t)$ and $X'(t)$ in a Rice's formula - but without checking any of the hypotheses for its validity). Our theorem will involve the characteristic function of the shot noise process which is easily computable. Actually, if we denote for all $u, v \in \mathbb{R}$,

$$\psi(u,v) = \mathbb{E}(e^{iuX(0)+ivX'(0)}) \quad \text{and} \quad \widehat{F}(u) = \mathbb{E}(e^{iu\beta}) \tag{6}$$

then it is well-known (see [16] for instance) that

$$\psi(u,v) = \exp\left(\lambda \int_{\mathbb{R}} (\widehat{F}(ug(s) + vg'(s)) - 1)\, ds\right). \tag{7}$$

Theorem 2. *Assume that the condition (C) is satisfied, then the mean level crossings function C_X belongs to $L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} C_X(\alpha)\, d\alpha = \mathbb{E}(TV(X,(0,1))) \leq \lambda \mathbb{E}(|\beta|) TV(g,\mathbb{R}).$$

Moreover, its Fourier transform, denoted by $u \mapsto \widehat{C_X}(u)$ is given for $u \neq 0$ by

$$\widehat{C_X}(u) = \mathbb{E}(e^{iuX(0)}|X'(0)|) + \lambda \mathbb{E}(e^{iuX(0)}) \cdot \frac{1}{iu} \sum_{j=1}^{n} \left(\mathbb{E}(e^{iu(\beta g)^*(t_j)}) - \mathbb{E}(e^{iu(\beta g)_*(t_j)})\right),$$

and for $u = 0$ by

$$\widehat{C_X}(0) = \mathbb{E}(TV(X,(0,1))) = \mathbb{E}(|X'(0)|) + \lambda \mathbb{E}(|\beta|) \sum_{j=1}^{n} |\Delta g(t_j)|.$$

Proof. According to Theorem 1, we can apply Proposition 1 such that, for any bounded continuous functions h defined on $\mathbb{R}$, almost surely

$$\int_{\mathbb{R}} h(\alpha) N_X(\alpha)\, d\alpha = \int_0^1 h(X(t))|X'(t)|\, dt + \sum_{t\in S_X\cap(0,1)} \int_{X_*(t)}^{X^*(t)} h(\alpha)\, d\alpha. \tag{8}$$

Taking $h = 1$, we obtain that

$$\int_{\mathbb{R}} N_X(\alpha)\, d\alpha = \int_0^1 |X'(t)|\, dt + \sum_{t\in S_X\cap(0,1)} |\Delta X(t)| = TV(X,(0,1)).$$

Using the stationarity of X', we have $\mathbb{E}\left(\int_0^1 |X'(t)|dt\right) = \mathbb{E}(|X'(0)|) \leq \lambda\mathbb{E}(|\beta|)\int_{\mathbb{R}} |g'(s)|ds$ and

$$\mathbb{E}\left(\sum_{t\in S_X\cap(0,1)} |\Delta X(t)|\right) = \lambda\mathbb{E}(|\beta|)\sum_{j=1}^n |\Delta g(t_j)|.$$

Therefore,

$$\int_{\mathbb{R}} C_X(\alpha)\, d\alpha \;\leq\; \lambda\mathbb{E}(|\beta|)\left(\int_{\mathbb{R}} |g'(s)|ds + \sum_{j=1}^n |\Delta g(t_j)|\right) \leq \lambda\mathbb{E}(|\beta|)TV(g,\mathbb{R}).$$

Now, taking $h(\alpha) = e^{iu\alpha}$ for some $u \in \mathbb{R}$ with $u \neq 0$ in (8), we get

$$\begin{aligned}\int_{\mathbb{R}} e^{iu\alpha} N_X(\alpha)\, d\alpha &= \int_0^1 e^{iuX(t)}|X'(t)|\, dt + \sum_{t\in S_X\cap(0,1)} \int_{X_*(t)}^{X^*(t)} e^{iu\alpha}\, d\alpha, \\ &= \int_0^1 e^{iuX(t)}|X'(t)|\, dt + \sum_{j=1}^n A(j), \end{aligned} \tag{9}$$

where $A(j) := \frac{1}{iu}\sum_{\tau_i\in(-t_j,1-t_j)} e^{iuX((t_j+\tau_i)+)}\left(e^{iu\max(\beta_i\Delta g(t_j),0)} - e^{iu\min(\beta_i\Delta g(t_j),0)}\right)$. Now, let us write $X((t_j+\tau_i)+) = \sum_{\tau_k\neq\tau_i} \beta_k g(t_j+\tau_i-\tau_k) + \beta_i g(t_j+)$ and

$$B(u,t_j,\beta_i) := \frac{1}{iu}\left(e^{iu(\beta_i g)^*(t_j)} - e^{iu(\beta_i g)_*(t_j)}\right),$$

such that

$$A(j) = \sum_{\tau_i\in(-t_j,1-t_j)} \exp\left(iu\sum_{k\neq i}\beta_k g(t_j+\tau_i-\tau_k)\right) B(u,t_j,\beta_i).$$

Then, for $M > \max_{1\leq j\leq n}|t_j| + 1$, we consider

$$A_M(j) = \sum_{\tau_i\in(-t_j,1-t_j)} \exp\left(iu\sum_{k\neq i;|\tau_k|\leq M}\beta_k g(t_j+\tau_i-\tau_k)\right) B(u,t_j,\beta_i), \text{ such that } A_M(j) \underset{M\to+\infty}{\longrightarrow} A(j) \text{ a.s..}$$

We have that

$$A_M(j) \overset{d}{=} \sum_{i=1}^{N_M} \mathbf{1}_{(-t_j,1-t_j)}(U_i) \exp\left(iu\sum_{k=1,k\neq i}^{N_M}\beta_k g(t_j+U_i-U_k)\right) B(u,t_j,\beta_i),$$

where $(U_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of random variables of uniform law on $[-M,M]$ independent from $(\beta_k)_{k\in\mathbb{N}}$, and N_M is a Poisson random variable of parameter $2\lambda M$, independent from $(U_k)_{k\in\mathbb{N}}$

and from $(\beta_k)_{k\in\mathbb{N}}$. We adopt the convention that $\sum_{i=1}^{0} = 0$. Now, by conditioning, we get

$$\mathbb{E}(A_M(j)) = \sum_{m=0}^{+\infty} \mathbb{E}(A_M(j)|N_M = m)\mathbb{P}[N_M = m],$$

and then computing $\mathbb{E}(A_M(j)|N_M = m)$, for all $m \geq 0$, using the independence of $(U_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$, finally leads to

$$\mathbb{E}(A_M(j)) = \lambda\mathbb{E}(B(u,t_j,\beta)) \int_{-t_j}^{1-t_j} \exp\left(\lambda \int_{-M+x+t_j}^{M+x+t_j} \left(\widehat{F}(ug(s)) - 1\right) ds\right) dx,$$

where $\widehat{F}$ is given by (6). As a consequence, using Lebesgue's Theorem and Formula (7) for $\mathbb{E}(e^{iuX(0)})$, we get

$$\mathbb{E}(A(j)) = \lambda\mathbb{E}(e^{iuX(0)})\mathbb{E}(B(u,t_j,\beta)).$$

Finally, taking the expectation on both sides of Equation (9) and using the stationarity of X, leads to the announced result for $\widehat{C_X}(u)$. □

Proposition 2. *Under the condition (C), if we assume moreover that $\mathbb{E}(\beta^2) < \infty$ and that $g' \in L^2(\mathbb{R})$, then, for $u \neq 0$,*

(10)

$$\widehat{C_X}(u) = \frac{-1}{\pi} \int_0^{+\infty} \frac{1}{v} \left(\frac{\partial\psi}{\partial v}(u,v) - \frac{\partial\psi}{\partial v}(u,-v)\right) dv + \lambda\psi(u,0)\frac{1}{iu}\sum_{j=1}^{n} \left(\mathbb{E}(e^{iu(\beta g)^*(t_j)}) - \mathbb{E}(e^{iu(\beta g)_*(t_j)})\right),$$

where ψ is given by (6).

Proof. Since $g' \in L^1(\mathbb{R})$ and since g has a finite number of discontinuity points with finite left and right limits, it follows that $g \in L^\infty(\mathbb{R})$. Consequently, $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^2(\mathbb{R})$. Therefore, when $\mathbb{E}(\beta^2) < \infty$ and $g' \in L^2(\mathbb{R})$, the characteristic function ψ of $(X(0), X'(0))$ is C^2 on $\mathbb{R}^2$. Then, $\int_0^{+\infty} \frac{1}{v}\left(\frac{\partial\psi}{\partial v}(u,v) - \frac{\partial\psi}{\partial v}(u,-v)\right) dv$ is well-defined and is the Hilbert transform of the function $v \mapsto \frac{\partial\psi}{\partial v}(u,v)$. Moreover, the computations of Theorem 1 in our previous paper [3] yield

$$\mathbb{E}(e^{iuX(0)}|X'(0)|) = \frac{-1}{\pi} \int_0^{+\infty} \frac{1}{v} \left(\frac{\partial\psi}{\partial v}(u,v) - \frac{\partial\psi}{\partial v}(u,-v)\right) dv.$$

This ends the proof of the proposition. □

2. A particular case

The formula for $\widehat{C_X}(u)$ can become simpler in some cases. The first particular case is the one when the kernel g is piecewise constant, since then $X'(0) = 0$ a.s. and thus the term $\mathbb{E}(e^{iuX(0)}|X'(0)|)$ vanishes. Let us also make the following remark.

Remark 2. *When $\beta \geq 0$ a.s., then for any $h_1 < h_2 \in \mathbb{R}$, the function $u \mapsto \frac{1}{iu}\psi(u,0) \cdot (\widehat{F}(uh_2) - \widehat{F}(uh_1))$ is the Fourier transform of the function $\alpha \mapsto \mathbb{P}[\alpha - \beta h_2 \leq X(0) \leq \alpha - \beta h_1]$, where β and $X(0)$ are taken independent.*

We will now give a simpler formula for $\widehat{C_X}(u)$ in the case where g is piecewise non-increasing (meaning that $g' \leq 0$ and thus g is non-increasing on each of the intervals on which it is continuous, but it can have jumps t_j such that $\Delta g(t_j) = g(t_j+) - g(t_j-) > 0$). In that case, we have the following proposition.

Proposition 3. *Assume that condition (C) holds. Assume moreover that $\beta \geq 0$ a.s. and that $g' \leq 0$. Then, for all $u \in \mathbb{R}$,*

$$\widehat{C_X}(u) = 2\lambda\psi(u,0) \cdot \frac{1}{iu} \sum_{t_j:\Delta g(t_j)>0} \left(\widehat{F}(ug(t_j+)) - \widehat{F}(ug(t_j-))\right), \tag{11}$$

where ψ and $\widehat{F}$ are given by (6)*. As a consequence, for almost every $\alpha \in \mathbb{R}$,*

$$C_X(\alpha) = 2\lambda \sum_{t_j:\Delta g(t_j)>0} \mathbb{P}[\alpha - \beta g(t_j+) \leq X(0) \leq \alpha - \beta g(t_j-)], \tag{12}$$

where β and $X(0)$ are taken independent.

Proof. Since $g' \leq 0$, we have $X'(0) \leq 0$ a.s. and consequently

$$\mathbb{E}(e^{iuX(0)}|X'(0)|) = -\mathbb{E}(e^{iuX(0)}X'(0)) = i\frac{\partial \psi}{\partial v}(u,0).$$

Now, since g is piecewise non-increasing and belongs to $L^1(\mathbb{R})$, we have $\lim_{|s|\to+\infty} g(s) = 0$ and thus $\lim_{|s|\to+\infty} \widehat{F}(ug(s)) = 1$. Then, recalling that $\psi(u,v) = \exp\left(\lambda \int_{\mathbb{R}} (\widehat{F}(ug(s) + vg'(s)) - 1)\, ds\right)$, we get

$$\begin{aligned}
\frac{\partial \psi}{\partial v}(u,0) &= \lambda\psi(u,0)\int_{\mathbb{R}} g'(s)\widehat{F}'(ug(s))\, ds \\
&= \lambda\psi(u,0)\cdot\left(\int_{-\infty}^{t_1} g'(s)\widehat{F}'(ug(s))\, ds + \sum_{j=1}^{n-1}\int_{t_j}^{t_{j+1}} g'(s)\widehat{F}'(ug(s))\, ds + \int_{t_n}^{+\infty} g'(s)\widehat{F}'(ug(s))\, ds\right) \\
&= \lambda\psi(u,0)\cdot\frac{1}{u}\left(\widehat{F}(ug(t_1-)) + \sum_{j=1}^{n-1}(\widehat{F}(ug(t_{j+1}-)) - \widehat{F}(ug(t_j+))) - \widehat{F}(ug(t_n+))\right) \\
&= \lambda\psi(u,0)\cdot\frac{1}{u}\sum_{j=1}^{n}(\widehat{F}(ug(t_j-)) - \widehat{F}(ug(t_j+))).
\end{aligned}$$

Consequently, by Theorem 2, we get

$$\begin{aligned}
\widehat{C_X}(u) &= \lambda\psi(u,0)\cdot\frac{i}{u}\sum_{j=1}^{n}(\widehat{F}(ug(t_j-)) - \widehat{F}(ug(t_j+))) \\
&\quad + \lambda\psi(u,0)\cdot\frac{1}{iu}\sum_{t_j:\Delta g(t_j)>0}\left(\widehat{F}(ug(t_j+)) - \widehat{F}(ug(t_j-))\right) + \frac{1}{iu}\sum_{t_j:\Delta g(t_j)<0}\left(\widehat{F}(ug(t_j-)) - \widehat{F}(ug(t_j+))\right) \\
&= 2\lambda\psi(u,0)\cdot\frac{1}{iu}\sum_{t_j:\Delta g(t_j)>0}\left(\widehat{F}(ug(t_j+)) - \widehat{F}(ug(t_j-))\right).
\end{aligned}$$

This achieves the proof of Formula (11) for $\widehat{C_X}(u)$. Formula (12) for $C_X(\alpha)$ for almost every $\alpha \in \mathbb{R}$ follows from Remark 2 and this ends the proof of the proposition. □

A particular case of this proposition is when we make the additional assumption that the function g is positive and that it has only one positive jump at $t_1 = 0$ from the value $g(0-) = 0$ to the value $g(0+) > 0$. Formula (11) then simply becomes

$$\widehat{C_X}(u) = 2\lambda\psi(u,0)\cdot\frac{1}{iu}(\widehat{F}(ug(0+)) - 1). \tag{13}$$

This framework corresponds to the case studied by Hsing in [11], and where he considers up-crossings that are defined in the following way : the point t is an up-crossing of the level α by the process X if it is a point of discontinuity of X and if $X(t-) \leq \alpha$ and $X(t+) > \alpha$. Then, Hsing proves that the expected number of such points in $(0,1)$, denoted by $U_X^{(J)}(\alpha)$, is given by

$$U_X^{(J)}(\alpha) = \lambda\,\mathbb{P}[\alpha - \beta g(0+) < X(0) \leq \alpha], \tag{14}$$

where β and $X(0)$ are taken independent. Note that the left strict inequality comes from the way Hsing defines up-crossings of the level α. Now, let us consider up-crossings of the level α as usually defined (see [7] p.192): the point t is an up-crossing of the level α by the process X if there exists $\varepsilon > 0$ such that $X(s) \leq \alpha$ in $(t-\varepsilon, t)$ and $X(s) \geq \alpha$ in $(t, t+\varepsilon)$; and let us denote by $U_X(\alpha)$ their

expected number in $(0,1)$. In a similar way we can define down-crossings of the level α. Let us remark that, since X is piecewise non-increasing, if moreover X is a.s. not identically equal to α in any interval of $(0,1)$ (this is satisfied for instance when $\mathbb{P}[X(0)=\alpha]=0$), then the crossings of the level α are either up-crossings as defined by Hsing or down-crossings of the level α. Moreover, by stationarity of the process X, the expected number of down-crossings of the level α is equal to $U_X(\alpha)$ and the result of Hsing yields to

$$C_X(\alpha) = 2U_X(\alpha) = 2\lambda\,\mathbb{P}[\alpha - \beta g(0+) < X(0) \leq \alpha]. \tag{15}$$

The result of Hsing given by Equation (14) is stronger than the similar formula given by Equation (12) because his formula is valid for *all* levels α and moreover he needs weaker assumptions on the regularity of g. On the other hand, his proof strongly relies on the fact that g has only one jump and that g is identically 0 before that jump, and thus it can not be generalized to other kernel functions g.

Finally, let us end this section by mentioning that we have studied here the case of a piecewise non-increasing kernel function g, but that, of course, similar results hold for a piecewise non-decreasing kernel.

3. High intensity and Gaussian limit

We assume here that the assumptions of Proposition 2 hold. It is then well-known (see [15, 10] for instance) that, as the intensity λ of the Poisson point process goes to infinity, the normalized process Z_λ defined by

$$t \mapsto Z_\lambda(t) = \frac{X_\lambda(t) - \mathbb{E}(X_\lambda(t))}{\sqrt{\lambda}},$$

where X_λ denotes a shot noise process (as defined by Equation (1)) with intensity λ for the homogeneous Poisson point process, converges to a centered Gaussian process with covariance $R(t) = \mathbb{E}(\beta^2)\int_{\mathbb{R}} g(s)g(s-t)\,ds$.

Now, how does the number of level crossings of Z_λ behave as λ goes to $+\infty$? To answer this, we first determine the asymptotic expansion of the Fourier transform of C_{Z_λ} as $\lambda \to +\infty$.
For $u \in \mathbb{R}$, we have:

$$\begin{aligned}\widehat{C_{Z_\lambda}}(u) &= \frac{1}{\sqrt{\lambda}}\widehat{C_{X_\lambda}}\left(\frac{u}{\sqrt{\lambda}}\right)e^{-iu\mathbb{E}(X_\lambda(t))/\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}}\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}(X_\lambda(0)-\mathbb{E}(X_\lambda(0)))}|X'_\lambda(0)|\right)\\ &\quad + \frac{\lambda}{iu}\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}(X_\lambda(0)-\mathbb{E}(X_\lambda(0)))}\right)\sum_{j=1}^{n}\left(\mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}(\beta g)^*(t_j)}) - \mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}(\beta g)_*(t_j)})\right)\end{aligned}$$

As we have already studied it in Section 4 of [3], the first term of the right-hand side admits a limit as $\lambda \to +\infty$, and more precisely, as $\lambda \to +\infty$,

$$\frac{1}{\sqrt{\lambda}}\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}(X_\lambda(0)-\mathbb{E}(X_\lambda(0)))}|X'_\lambda(0)|\right) = \sqrt{\frac{2m_2}{\pi}}e^{-m_0u^2/2} + o(1),$$

where $m_0 = \mathbb{E}(\beta^2)\int_{\mathbb{R}} g^2(s)\,ds$ and $m_2 = \mathbb{E}(\beta^2)\int_{\mathbb{R}} g'^2(s)\,ds$. For the second term, it is the product of two terms and each of them admits an asymptotic expansion as $\lambda \to +\infty$. Indeed, assuming moreover that $g \in L^3(\mathbb{R})$ and $\mathbb{E}(|\beta|^3) < \infty$, we have

$$\begin{aligned}\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}(X_\lambda(0)-\mathbb{E}(X_\lambda(0)))}\right) &= \exp\left(\lambda\int_{\mathbb{R}}(\widehat{F}(\frac{u}{\sqrt{\lambda}}g(s)) - 1)ds - iu\sqrt{\lambda}\mathbb{E}(\beta)\int_{\mathbb{R}} g(s)\,ds\right)\\ &= \exp\left(-\frac{m_0u^2}{2} - \frac{im_3u^3}{3\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right)\right)\\ &= e^{-m_0u^2/2}\left(1 + \frac{2ium_3}{3m_0\sqrt{\lambda}}\right) + o\left(\frac{1}{\sqrt{\lambda}}\right),\end{aligned}$$

where $m_3 = \mathbb{E}(\beta^3)\int_{\mathbb{R}} g^3(s)\,ds$. And for a given jump number j, $1 \le j \le n$, we have

$$\frac{\sqrt{\lambda}}{iu}\left(\mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}(\beta g)^*(t_j)}) - \mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}(\beta g)_*(t_j)})\right) = \mathbb{E}(|\beta|)|\Delta g(t_j)| + \frac{iu}{2\sqrt{\lambda}}\mathbb{E}(\beta^2)|\Delta g^2(t_j)| + o\left(\frac{1}{\sqrt{\lambda}}\right).$$

Finally, we thus have

$$\begin{aligned}\widehat{C_{Z_\lambda}}(u) &= \sqrt{\lambda}\, e^{-m_0u^2/2}\mathbb{E}(|\beta|)\sum_{j=1}^{n}|\Delta g(t_j)| \\ &+ \left(\sqrt{\frac{2m_2}{\pi}} + \frac{2ium_3}{3m_0}\mathbb{E}(|\beta|)\sum_{j=1}^{n}|\Delta g(t_j)| + \frac{iu}{2}\mathbb{E}(\beta^2)\sum_{j=1}^{n}|\Delta g^2(t_j)|\right)e^{-m_0u^2/2} + o(1).\end{aligned}$$

Let us comment this result. When there are no jumps we obtain that $\widehat{C_{Z_\lambda}}(u)$ converges, as λ goes to infinity, to $\sqrt{\frac{2m_2}{\pi}}e^{-m_0u^2/2}$. This implies that $C_{Z_\lambda}(\alpha)$ weakly converges to $\frac{\sqrt{m_2}}{\pi\sqrt{m_0}}e^{-\alpha^2/2m_0}$, which is the usual Rice's formula for the mean number of level crossings of Gaussian processes (see [7] p.194 for instance). Now, when there are jumps, the behavior of $\widehat{C_{Z_\lambda}}(u)$ is different, since the main term in $\sqrt{\lambda}$ doesn't vanish anymore. More precisely we have that $\frac{1}{\sqrt{\lambda}}\widehat{C_{Z_\lambda}}(u)$ goes to $e^{-m_0u^2/2}\mathbb{E}(|\beta|)\sum_{j=1}^{n}|\Delta g(t_j)|$, which implies that

$$\frac{1}{\sqrt{\lambda}}C_{Z_\lambda}(\alpha) \underset{\lambda\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi m_0}}e^{-\alpha^2/2m_0}\mathbb{E}(|\beta|)\sum_{j=1}^{n}|\Delta g(t_j)| \quad \text{in the sense of weak convergence.}$$

Notice also that taking $u = 0$ in the asymptotic formula for $\widehat{C_{Z_\lambda}}(u)$ gives the asymptotic behavior of the total variation of Z_λ. Indeed, we then have

$$\mathbb{E}(TV(Z_\lambda,(0,1))) = \widehat{C_{Z_\lambda}}(0) = \sqrt{\lambda}\,\mathbb{E}(|\beta|)\sum_{j=1}^{n}|\Delta g(t_j)| + \sqrt{\frac{2m_2}{\pi}} + o(1).$$

This kind of asymptotic behavior has also been studied by B. Galerne in [9] in the framework of random fields of bounded variation.

4. Some examples

4.1. **Step functions.** We start this section with some examples of explicit computations in the case of step functions.

(1) The first simple example of step function is the one where the kernel g is a rectangular function : $g(t) = 1$ for $t \in [0,a]$ with $a > 0$ and 0 otherwise. Notice that this is a very simple framework, that also fits in the results of Hsing [11]. In this particular case X is piecewise constant almost surely.

- If the impulse β is such that $\beta = 1$ a.s. we will prove that

$$C_X(\alpha) = \sum_{k=0}^{+\infty} 2\lambda e^{-\lambda a}\frac{(\lambda a)^k}{k!}\mathbb{1}_{\{k<\alpha<k+1\}} \quad \text{for all } \alpha \in \mathbb{R}\backslash\mathbb{N}. \tag{16}$$

Note that X takes values in $\mathbb{N}$ a.s. such that C_X is constant on any interval $(k,k+1)$ with $k \in \mathbb{N}$. Moreover, $\psi(u,v) = \exp(\lambda a(e^{iu}-1))$, which shows on the one hand that $X(0)$ follows a Poisson distribution with parameter λa such that $C_X(\alpha) = +\infty$ for all $\alpha \in \mathbb{N}$. On the other hand, by Formula (13),

$$\widehat{C_X}(u) = 2\lambda\exp(\lambda a(e^{iu}-1))\frac{e^{iu}-1}{iu}.$$

We recognize here the product of two Fourier transforms: the one of a Poisson distribution and the one of the indicator function of $[0,1]$ from which we deduce (16).

- If the impulse β follows an exponential distribution of parameter $\mu > 0$, then $\widehat{F}(u) = \frac{\mu}{\mu - iu}$ and a simple computation gives $\psi(u,v) = \exp\left(\lambda a \frac{iu}{\mu - iu}\right)$. On the one hand, the law of $X(0)$ can be computed: $P_{X(0)}(dx) = e^{-2a\lambda}\delta_0(dx) + \sum_{k=1}^{+\infty} e^{-a\lambda}\frac{(2a\lambda)^k}{k!} f_{\mu,k}(x)dx$, where $f_{\mu,k}$ is the probability density of the Gamma distribution of parameters μ and k, and δ_0 is the Dirac measure at point 0. Then, $\mathbb{P}[X(0) = 0] > 0$ such that $C_X(0) = +\infty$ and, for all $\alpha > 0$, $\mathbb{P}[X(0) = \alpha] = 0$ such that, according to (15), C_X is continuous on $(0, +\infty)$. On the other hand,

$$\widehat{C_X}(u) = 2\lambda \exp\left(\lambda a \frac{iu}{\mu - iu}\right) \frac{1}{\mu - iu}. \tag{17}$$

We recognize here the Fourier transform of a non-central chi-square distribution, and thus

$$\text{for all } \alpha > 0,\ \ C_X(\alpha) = 2\lambda\mu e^{-a\lambda - \mu\alpha} I_0(2\sqrt{a\lambda\mu\alpha}), \tag{18}$$

where I_0 is the modified Bessel function of the first kind of order 0; it is given by $I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta} d\theta = \sum_{m=0}^{+\infty} \frac{x^{2m}}{4^m (m!)^2}$.

(2) A second example is a "double rectangular" function given by: $g(t) = 1$ if $-1 \le t < 0$; $g(t) = -1$ if $0 \le t < 1$, and $g(t) = 0$ otherwise. Notice that this case does not fit anymore in the framework of Hsing [11]. However X is still piecewise constant almost surely.

- If $\beta = 1$ almost surely, then simple computations show that

$$\widehat{C_X}(u) = 4\lambda \frac{\sin u}{u} \exp(2\lambda(\cos u - 1)).$$

The last term above is the characteristic function of the difference of two independent Poisson random variables of same parameter λ (it is also called a Skellam distribution). Thus, as previously we obtain that $C_X(\alpha) = +\infty$ for all $\alpha \in \mathbb{Z}$ and

$$\forall \alpha \in \mathbb{R}\backslash\mathbb{Z},\ \ C_X(\alpha) = \sum_{k=-\infty}^{+\infty} 4\lambda p_k \mathbf{I}_{\{k<\alpha<k+1\}},\ \text{where } \forall k \in \mathbb{Z}, p_k = e^{-2\lambda} \sum_{n=0}^{+\infty} \frac{\lambda^{k+2n}}{n!(k+n)!}.$$

- If β follows an exponential distribution of parameter μ, we can also explicitly compute

$$\widehat{C_X}(u) = \frac{4\lambda\mu}{\mu^2 + u^2} \exp\left(-2\lambda \frac{u^2}{\mu^2 + u^2}\right).$$

4.2. **Exponential kernel.** In this section, we consider an example that has been widely studied in the literature: the impulse β follows an exponential distribution of parameter $\mu > 0$, and the kernel function g is given by $g(s) = 0$ for $s < 0$ and $g(s) = e^{-s}$ for $s \ge 0$.

A simple computation gives that the joint characteristic function of $X(0)$ and $X'(0)$ is

$$\forall u, v \in \mathbb{R},\ \ \psi(u,v) = \frac{\mu^\lambda}{(\mu - iu + iv)^\lambda}.$$

Then, on the one hand, $X(0)$ follows a Gamma distribution such that $\mathbb{P}(X(0) = \alpha) = 0$ for all $\alpha \in \mathbb{R}$, and C_X is a continuous function on $\mathbb{R}$ according to (15). On the other hand, by Formula (13), we get

$$\widehat{C_X}(u) = \frac{2\lambda\mu^\lambda}{(\mu - iu)^{\lambda+1}}.$$

We recognize here the Fourier transform of an other Gamma probability density. Thus it implies that

$$C_X(\alpha) = \frac{2\lambda\mu^\lambda \alpha^\lambda e^{-\mu\alpha}}{\Gamma(\lambda+1)} \mathbf{I}_{\{\alpha \ge 0\}} \quad \text{for all } \alpha \in \mathbb{R}.$$

In the case where λ is an integer, the explicit formula for the mean number of level crossings $C_X(\alpha)$ was already given in [14] by Orsingher and Battaglia (but they had a completely different approach based on the property that in the particular case of an exponential kernel, the process is Markovian). Finally, let us also mention the work of Borovkov and Last in [4] that gives a version

of Rice's formula for the mean number of level crossings in the framework of piecewise-deterministic Markov processes with jumps.

Acknowledgments

This work has been supported by the French National Research Agency under the grant ANR-09-BLAN-0029-01. We also would like to thank the anonymous reviewer for his valuable comments and suggestions.

References

[1] Azais, J. and Wschebor, M. (2009). *Level sets and extrema of random processes and fields*. John Wiley And Sons Ltd, United Kingdom.

[2] Bar-David, I. and Nemirovsky, A. (1972). Level crossings of nondifferentiable shot processes. *IEEE Transactions on Information Theory* **18,** 27–34.

[3] Biermé, H. and Desolneux, A. (2011). Crossings of smooth shot noise processes. *Technical report* No 2011-16. prépublication du MAP5. (avalaible at http://hal.archives-ouvertes.fr/hal-00589560/fr/).

[4] Borovkov, K. and Last, G. (2008). On level crossings for a general class of piecewise-deterministic Markov processes. *Advances in Applied Probability* **40,** 815–834.

[5] Breton, J. (2010). Regularity of the laws of shot noise series and of related processes. *Journal of Theoretical Probability* **23,** 21–38.

[6] Choski, R. and Fonseca, I. (1997). A change of variables formula for mappings in BV. *Proceedings of the American Mathematical Society* **125,** 2065–2072.

[7] Cramér, H. and Leadbetter, M. R. (2004). *Stationary and related stochastic processes*. Dover Publications Inc., Mineola, NY. Sample function properties and their applications, Reprint of the 1967 original.

[8] Evans, L. and Gariepy, R. (1992). *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press.

[9] Galerne, B. (2010). Stochastic image models and texture synthesis. *PhD thesis*. Ecole Normale Supérieure de Cachan.

[10] Heinrich, L. and Schmidt, V. (1985). Normal convergence of multidimensional shot noise and rates of this convergence. *Advances in Applied Probability* **17,** 709–730.

[11] Hsing, T. (1987). On the intensity of crossings by a shot noise process. *Advances in Applied Probability* **19,** 743–745.

[12] Leadbetter, M. R. (1966). On crossings of levels and curves by a wide class of stochastic processes. *Annals of Mathematical Statistics* **37,** 260–267.

[13] Milnor, J. (1965). *Topology from the differential viewpoint*. Princeton University Press.

[14] Orsingher, E. and Battaglia, F. (1982). Probability distributions and level crossings of shot noise models. *Stochastics* **8,** 45–61.

[15] Papoulis, A. (1971). High density shot noise and Gaussianity. *Journal of Applied Probability* **8,** 118–127.

[16] Rice, J. (1977). On generalized shot noise. *Advances in Applied Probability* **9,** 553–565.

[17] Rychlik, I. (2000). On some reliability applications of Rice's formula for the intensity of level crossings. *Extremes* **3:4,** 331–348.

Hermine BIERMÉ, MAP5 (UMR CNRS 8145) Université Paris Descartes, 45 rue des Saints-Pères, 75006 PARIS France

E-mail address: hermine.bierme@mi.parisdescartes.fr

Agnès Desolneux, MAP5 (UMR CNRS 8145) Université Paris Descartes, 45 rue des Saints-Pères, 75006 PARIS France

E-mail address: agnes.desolneux@mi.parisdescartes.fr