A FOURIER APPROACH FOR THE CROSSINGS OF SHOT NOISE PROCESSES WITH JUMPS

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ABSTRACT. We use here a change of variable formula in the framework of functions of bounded variation to derive an explicit formula for the Fourier transform of the crossing function of shot noise processes with jumps. We illustrate the result on some examples and give some applications. In particular we are then able to study the behavior of the mean number of crossings as the intensity of the Poisson point process of the shot noise goes to infinity.

In this paper, we will consider a *shot noise process* which is a real-valued random process given by

$$X(t) = \sum_{i\in\mathbb{Z}} \beta_i g(t-\tau_i), \quad t \in \mathbb{R} \tag{1}$$

where g is a given (deterministic) measurable function (it will be called the *kernel function* of the shot noise), the $\{\tau_i\}_{i\in\mathbb{Z}}$ are the points of a Poisson point process on the line of intensity $\lambda\, ds$, where $\lambda > 0$, and the $\{\beta_i\}_{i\in\mathbb{Z}}$ are independant copies of a random variable β (called the *impulse*), independent of $\{\tau_i\}$.

Such a process has many applications (see [14] for instance) and it is a well-known and studied mathematical model (see [9], [13], [4] for some of its properties).

We will be interested here in the crossings of such a process. Usually, the mean number of crossings of a stochastic process is computed thanks to a Rice's formula (see [11] or [1]) that requires some regularity conditions on the joint probability density of X and of its derivative. This joint probability density is generally not easy to obtain and to study. Its existence is also sometimes a question. This is why, instead of working directly on the mean number of crossings, we will work on the Fourier transform of the function that maps each level α to the mean number of crossings of the level α per unit length. Thanks to a change of variables formula, we will be able to relate this Fourier transform to the characteristic function of the shot noise process (which, unlike the probability density, always exists and is explicit).

1. General result

In [3], we have studied the crossings of X when the kernel function g is smooth on $\mathbb{R}$. We will consider here the case where g is a piecewise smooth function, that is not necessarily continuous.

More precisely, we assume that $g \in L^1(\mathbb{R})$, and that it is piecewise $\mathcal{C}^2$ on $\mathbb{R}$, with a finite number of points of discontinuity denoted by

$$S_g = \{t_1, t_2, \dots, t_n\}, \text{ with } t_1 < \dots < t_n,$$

and called the jump set of g. We moreover assume that g admits finite left and right limits at each point of discontinuity.

In the following we denote g', respectively g'', the function which is defined at all points $s \notin S_g$ by the usual derivative $g'(s)$, respectively $g''(s)$. We also assume that $g' \in L^1(\mathbb{R})$. As a consequence, g has a finite total variation on $\mathbb{R}$, which means that

$$TV(g,\mathbb{R}) = \sup \sum_{k\in K} |g(s_k) - g(s_{k-1})| = \int_{\mathbb{R}} |g'(s)|\, ds + \sum_{j=1}^{n} |g(t_j^+) - g(t_j^-)| < +\infty,$$

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where the supremum is taken over any subdivision $\{s_k\,;\,k \in K\}$ of $\mathbb{R}$ and $g(t_j^+) = \lim_{s\to t_j, s>t_j} g(s)$, $g(t_j^-) = \lim_{s\to t, s<t_j} g(s)$ are the respective right and left limits of g at a discontinuity point t_j. This implies that $g \in BV(\mathbb{R})$, the set of functions of bounded variation on $\mathbb{R}$. We follow the notations and definitions used in the framework of BV functions (see [7]). According to our assumptions the weak derivative of g is the Radon measure given by

$$Dg = g'ds + \sum_{j=1}^{n}(g(t_j^+) - g(t_j^-))\delta_{t_j}.$$

1.1. **Piecewise regularity of the shot noise process.** The shot noise process "inherits" the regularity of the kernel function g. More precisely, we have the following result.

Theorem 1. *Let $\beta \in L^1(\Omega)$. Let g be a piecewise $\mathcal{C}^2$ function with $\#S_g < \infty$ and such that $g, g', g'' \in L^1(\mathbb{R})$, where g' and g'' are the absolutely continuous part of Dg and D^2g. Then the shot noise process X defined by* (1) *is an integrable stationary process which is almost surely piecewise C^1 on $\mathbb{R}$ with*

$$S_X = \bigcup_{i\in\mathbb{Z}} (\tau_i + S_g)$$

and

$$\forall t \notin S_X,\ X'(t) = \sum_i \beta_i g'(t-\tau_i).$$

Proof. Note that according to Section 2 of [3], since $\beta \in L^1(\Omega)$ and $g \in L^1(\mathbb{R})$, for any $t \in \mathbb{R}$, the random variable $X(t)$ is well defined and integrable with $\mathbb{E}(X(t)) = \lambda\mathbb{E}(\beta)\int_{\mathbb{R}} g(s)ds$. Moreover X is a stationary process since the intensity of the Poisson point process is given by λds.

Let us remark that since $\#S_g = n$ we can write g as the sum of $2n$ piecewise $\mathcal{C}^2$ functions on $\mathbb{R}$, each of them having only one discontinuity point and having the same regularity properties as g. Therefore we may and will assume that $\#S_g = 1$ and write $S_g = \{t_1\}$. Let τ_{i_0}, $i_0 \in \mathbb{Z}$, be a fixed point of the Poisson point process. We write $I_{i_0} := [t_1 + \tau_{i_0}, t_1 + \tau_{i_0+1}]$, Then, for any $t \in \mathbb{R}$,

$$X(t) = \sum_{i>i_0+1} \beta_i g(t-\tau_i) + \sum_{i<i_0} \beta_i g(t-\tau_i) + \beta_{i_0} g(t-\tau_{i_0}) + \beta_{i_0+1} g(t-\tau_{i_0+1}).$$

The function $t \mapsto g(t-s)$ is $\mathcal{C}^2$ on I_{i_0} for any $s < \tau_{i_0}$ such that almost surely, for any $i < i_0$, the function $t \mapsto g(t-\tau_i)$ is $\mathcal{C}^2$ on I_{i_0} with $g((t_1 + \tau_{i_0} - \tau_i)^+) = g((t_1 + \tau_{i_0} - \tau_i)^-)$. Moreover

$$\mathbb{E}\left(\sum_{i<i_0} |\beta_i| \sup_{t\in I_{i_0}} |g'(t-\tau_i)| \,\Big|\, \tau_{i_0}, \tau_{i_0+1}\right) \quad = \quad \lambda\mathbb{E}(|\beta|)\int_{-\infty}^{0} \sup_{t\in I_{i_0}} |g'(t-s-\tau_{i_0})| ds,$$

using the fact that $\{\tau_i - \tau_{i_0}; i < i_0\}$ is a Poisson point process with intensity $\lambda \mathbf{1}_{]-\infty,0[}(s)ds$ independant of τ_{i_0}, τ_{i_0+1}. But for any $t \in I_{i_0}$ and $s < 0$,

$$g'(t-s-\tau_{i_0}) = \int_{t_1+\tau_{i_0}}^{t} g''(u-s-\tau_{i_0})du + g'(t_1 - s),$$

such that by Fubini Tonnelli,

$$\int_{-\infty}^{0} \sup_{t\in I_{i_0}} |g'(t-s-\tau_{i_0})| ds \quad \le \quad (\tau_{i_0+1} - \tau_{i_0}) \int_{\mathbb{R}} |g''(s)|ds + \int_{\mathbb{R}} |g'(s)|ds.$$

Then

$$\mathbb{E}\left(\sum_{i<i_0} |\beta_i| \sup_{t\in I_{i_0}} |g'(t-\tau_i)|\right) \le \lambda\mathbb{E}(|\beta|)\left(\frac{1}{\lambda}\int_{\mathbb{R}} |g''(s)|ds + \int_{\mathbb{R}} |g'(s)|ds\right) < +\infty,$$

so that the series $t \mapsto \sum_{i<i_0} \beta_i g'(t-\tau_i)$ is uniformly convergent on I_{i_0}. Therefore, almost surely the series $t \mapsto \sum_{i<i_0} \beta_i g(t-\tau_i)$ is continuously differentiable on I_{i_0} with $\left(\sum_{i<i_0} \beta_i g(t-\tau_i)\right)' = \sum_{i<i_0} \beta_i g'(t-\tau_i)$

and $\sum_{i<i_0} \beta_i g((t_1+\tau_{i_0}-\tau_i)^+) = \sum_{i<i_0} \beta_i g((t_1+\tau_{i_0}-\tau_i)^-)$.
The same proof applies for $\sum_{i>i_0+1} \beta_i g(t-\tau_i)$. To conclude it is sufficient to remark that for $i \in \{i_0, i_0+1\}$, the function $t \mapsto g(t-\tau_i)$ is continuously differentiable in the interior of I_{i_0}. Moreover $g((t_1+\tau_{i_0}-\tau_{i_0+1})^+) = g((t_1+\tau_{i_0}-\tau_{i_0+1})^-)$ and $g((t_1+\tau_{i_0}-\tau_{i_0})^+) = g(t_1^+)$ and $g((t_1+\tau_{i_0}-\tau_{i_0})^-) = g(t_1^-)$.
Finally, a.s. X is continuously differentiable in the interior of I_{i_0} with

$$X'(t) = \sum_i \beta_i g'(t-\tau_i),$$

and $X((t_1+\tau_{i_0})^+) - X((t_1+\tau_{i_0})^-) = \beta_{i_0}\left(g(t_1^+) - g(t_1^-)\right)$.

□

Then, under the above conditions, the shot noise process X is a BV function on any interval (a,b). By stationarity we can focus on what happens on $(0,1)$. Then X has a.s. a finite number of points of discontinuity on $(0,1)$ and the weak derivative of X is given by

$$DX = X'dt + \sum_{j=1}^{n} \sum_{\tau_i \in (-t_j, 1-t_j)} \beta_i (g(t_j^+) - g(t_j^-))\delta_{t_j+\tau_i}.$$

Moreover its total variation on $(0,1)$ is given by

$$TV(X,(0,1)) = \int_0^1 |X'(t)|dt + \sum_{j=1}^{n} \sum_{\tau_i \in (-t_j, 1-t_j)} |\beta_i||g(t_j^+) - g(t_j^-)|.$$

1.2. **Crossings.** We will be interested in the crossings of the shot noise process X. We first start by a general definition and a result on the crossings of a piecewise smooth function.

When f is a piecewise $\mathcal{C}^1$ function on an interval (a,b) of $\mathbb{R}$ with a finite number of discontinuity points, we can define its crossings on (a,b) by considering for any level $\alpha \in \mathbb{R}$,

$$N_f(\alpha,(a,b)) = \#\{s \in (a,b)\ ;\ \min(f(s^-), f(s^+)) \le \alpha \le \max(f(s^-), f(s^+))\}.$$

Then, a change of variables formula for BV functions yields the following result (see [5]).

Proposition 1. *Let $a, b \in \mathbb{R}$ with $a < b$ and f be a piecewise $\mathcal{C}^1$ function on (a,b). Then, for any bounded continuous function h defined on $\mathbb{R}$,*

$$\int_{\mathbb{R}} h(\alpha) N_f(\alpha,(a,b))\, d\alpha = \int_a^b h(f(s))|f'(s)|\, ds + \sum_{s \in S_f \cap (a,b)} \int_{\min(f(s^+),f(s^-))}^{\max(f(s^+),f(s^-))} h(\alpha)\, d\alpha, \tag{2}$$

Proof. Let us assume that $S_f \cap (a,b) = \{s_j\ ;\ 1 \le j \le m\}$ with $m \ge 1$ and $a := s_0 < s_1 < \ldots < s_m < b := s_{m+1}$. Then

$$N_f(\alpha,(a,b)) = \sum_{j=0}^{m} \#\{s \in (s_j, s_{j+1})\ ;\ f(s) = \alpha\} + \sum_{j=1}^{m} \mathbf{1}_{[\min(f(s_j^-),f(s_j^+)),\max(f(s_j^-),f(s_j^+))]}(\alpha).$$

Let h be a bounded continuous function on $\mathbb{R}$. According to the change of variables formula for Lipschitz functions (see [7] p.99), for any $j = 0, \ldots, m$,

$$\int_{\mathbb{R}} h(\alpha) \#\{s \in (s_j, s_{j+1})\ ;\ f(s) = \alpha\}\, d\alpha = \int_{s_j}^{s_{j+1}} h(f(s))|f'(s)|\, ds,$$

and the result follows using Chasles formula. □

Back to the shot noise process X, for $\alpha \in \mathbb{R}$, let $N_X(\alpha)$ be the random variable that counts the number of crossings of the level α by the process X in the interval $(0,1)$. It is defined by

$$N_X(\alpha) = \#\{t \in (0,1)\ ;\ \min(X(t^+), X(t^-)) \le \alpha \le \max(X(t^+), X(t^-))\}.$$

We will be mainly interested in its expectation, namely in

$$C_X(\alpha) = \mathbb{E}(N_X(\alpha)).$$

The following theorem computes these mean numbers of crossings from a Fourier transform point of view. It has to be related to the heuristic approach of [2] (in particular their Formula (13) that involves the joint density of $X(t)$ and $X'(t)$ in a Rice's formula - but without checking any of the hypotheses for its validity).

Theorem 2. *Under the assumptions of Theorem 1, the mean crossing function C_X is in $L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} C_X(\alpha)\, d\alpha = \mathbb{E}(TV(X,(0,1))) \leq \lambda \mathbb{E}(|\beta|) TV(g,\mathbb{R}).$$

Moreover, its Fourier transform, denoted by $u \mapsto \widehat{C_X}(u)$ is given for $u \neq 0$ by

$$\widehat{C_X}(u) = \mathbb{E}(e^{iuX(0)}|X'(0)|) + \lambda \mathbb{E}(e^{iuX(0)}) \sum_{j=1}^{n} \frac{\mathbb{E}(e^{iu\max(\beta g(t_j^-),\beta g(t_j^+))}) - \mathbb{E}(e^{iu\min(\beta g(t_j^-),\beta g(t_j^+))})}{iu},$$

and for $u=0$ by

$$\widehat{C_X}(0) = \mathbb{E}(TV(X,(0,1))) = \mathbb{E}(|X'(0)|) + \lambda \mathbb{E}(|\beta|) \sum_{j=1}^{n} |g(t_j^+) - g(t_j^-)|.$$

Proof. According to Proposition 1 for any bounded continuous functions h defined on $\mathbb{R}$, almost surely

$$\int_{\mathbb{R}} h(\alpha) N_X(\alpha)\, d\alpha = \int_0^1 h(X(t))|X'(t)|\, dt + \sum_{t \in S_X \cap (0,1)} \int_{\min(X(t^+),X(t^-))}^{\max(X(t^+),X(t^-))} h(\alpha)\, d\alpha, \tag{3}$$

Taking $h = 1$, we obtain that

$$\int_{\mathbb{R}} N_X(\alpha)\, d\alpha = \int_0^1 |X'(t)|\, dt + \sum_{t \in S_X \cap (0,1)} |X(t^+) - X(t^-)| = TV(X,(0,1)).$$

Using the stationarity of X', we have $\mathbb{E}\left(\int_0^1 |X'(t)|\right) = \mathbb{E}(|X'(0)|) \leq \lambda \mathbb{E}(|\beta|) \int_{\mathbb{R}} |g'(s)| ds$ and

$$\mathbb{E}\left(\sum_{t \in S_X \cap (0,1)} |X(t^+) - X(t^-)|\right) = \lambda \mathbb{E}(|\beta|) \sum_{j=1}^{n} |g(t_j^+) - g(t_j^-)|.$$

Therefore,

$$\int_{\mathbb{R}} C_X(\alpha)\, d\alpha \quad \leq \quad \lambda \mathbb{E}(|\beta|) \left(\int_{\mathbb{R}} |g'(s)| ds + \sum_{j=1}^{n} |g(t_j^+) - g(t_j^-)|\right) \leq \lambda \mathbb{E}(|\beta|) TV(g,\mathbb{R}).$$

Now, taking $h(\alpha) = e^{iu\alpha}$ for some $u \in \mathbb{R}$ with $u \neq 0$ in (3), we get

$$\begin{aligned} \int_{\mathbb{R}} e^{iu\alpha} N_X(\alpha)\, d\alpha &= \int_0^1 e^{iuX(t)}|X'(t)|\, dt + \sum_{t \in S_X \cap (0,1)} \int_{\min(X(t^+),X(t^-))}^{\max(X(t^+),X(t^-))} e^{iu\alpha}\, d\alpha, \\ &= \int_0^1 e^{iuX(t)}|X'(t)|\, dt + \sum_{j=1}^{n} A(j), \end{aligned} \tag{4}$$

where $A(j) := \sum_{\tau_i \in (-t_j, 1-t_j)} \frac{e^{iuX((t_j+\tau_i)^+)}}{iu} \left(e^{iu\max(\beta_i(g(t_j^-)-g(t_j^+)),0)} - e^{iu\min(\beta_i(g(t_j^-)-g(t_j^+)),0)}\right)$. Now, let us write $X((t_j+\tau_i)^+) = \sum_{\tau_k \neq \tau_i} g(t_j + \tau_i - \tau_k) + \beta_i g(t_j^+)$ and

$$B(u, t_j, \beta_i) := \frac{e^{iu\max(\beta_i g(t_j^-),\beta_i g(t_j^+))} - e^{iu\min(\beta_i g(t_j^-),\beta_i g(t_j^+))}}{iu}$$

such that

$$A(j) = \sum_{\tau_i \in (-t_j, 1-t_j)} e^{iu\sum_{k\neq i} \beta_k g(t_j+\tau_i-\tau_k)} B(u, t_j, \beta_i).$$

Then, for $M > 0$ consider

$$A_M(j) = \sum_{\tau_i \in (-t_j, 1-t_j), |\tau_i|\leq M} e^{iu\sum_{k\neq i; |\tau_k|\leq M} \beta_k g(t_j+\tau_i-\tau_k)} B(u, t_j, \beta_i) \text{ such that } A_M(j) \underset{M\to+\infty}{\longrightarrow} A(j) \text{ a.s.}$$

Then

$$A_M(j) \stackrel{d}{=} \sum_{i=1}^{N_M} \mathbf{1}_{(-t_j, 1-t_j)}(U_i) e^{iu\sum_{k\neq i} \beta_k g(t_j+U_i-U_k)} B(u, t_j, \beta_i),$$

where $(U_k)_{k\in\mathbb{N}}$ is an iid sequence of uniform law on $[-M, M]$ independent from $(\beta_k)_{k\in\mathbb{N}}$ and N_M is a Poisson random variable of intensity $2\lambda M$ independent from $(U_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ and the convention is $\sum_{i=1}^{0} = 0$. Now, classical computations lead to

$$\mathbb{E}(A_M(j)) = \lambda \mathbb{E}(B(u, t_j, \beta)) \int_{-t_j}^{1-t_j} \exp\left(\lambda \int_{-M+x+t_j}^{M+x+t_j} \left(\widehat{F}(ug(s)) - 1\right) ds\right) dx,$$

where $\widehat{F}(u) = \mathbb{E}(e^{iu\beta})$. Then using Lebesgue's Theorem

$$\mathbb{E}(A(j)) = \lambda \mathbb{E}(B(u, t_j, \beta)) \mathbb{E}(e^{iuX(0)}).$$

Finally, taking the expectation on both sides of Equation (4) and using the stationarity of X, leads to the announced result for $\widehat{C_X}(u)$. □

The Fourier transform $\widehat{C_X}$ may be rewritten in terms of the characteristic function of the shot noise process which is easily computable. Actually, if we denote for all $u, v \in \mathbb{R}$,

$$\psi(u, v) = \mathbb{E}(e^{iuX(0)+ivX'(0)}) \text{ and } \widehat{F}(u) = \mathbb{E}(e^{iu\beta}) \tag{5}$$

then it is well-known that

$$\psi(u, v) = \exp\left(\lambda \int_{\mathbb{R}} (\widehat{F}(ug(s) + vg'(s)) - 1)\, ds\right).$$

Proposition 2. *Under the assumptions of Theorem 1, if we assume moreover that $\beta \in L^2(\Omega)$ and that $g' \in L^2(\mathbb{R})$. Then, for $u \neq 0$, $\widehat{C_X}(u)$ is equal to*

(6)
$$\frac{-1}{\pi} \int_0^{+\infty} \frac{1}{v} \left(\frac{\partial\psi}{\partial v}(u, v) - \frac{\partial\psi}{\partial v}(u, -v)\right) dv + \lambda\psi(u, 0) \sum_{j=1}^{n} \frac{\mathbb{E}(e^{iu\max(\beta g(t_j^-), \beta g(t_j^+))}) - \mathbb{E}(e^{iu\min(\beta g(t_j^-), \beta g(t_j^+))})}{iu}.$$

Proof. Since $g' \in L^1(\mathbb{R})$ and since g has a finite number of discontinuity points with finite left and right limits, it follows that $g \in L^\infty(\mathbb{R})$. Consequently, $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^2(\mathbb{R})$. Therefore, when $\beta \in L^2(\Omega)$ and $g' \in L^2(\mathbb{R})$, the characteristic function ψ of $(X(0), X'(0))$ is $\mathcal{C}^2$ on $\mathbb{R}^2$. Then, $\frac{-1}{\pi}\int_0^{+\infty} \frac{1}{v}\left(\frac{\partial\psi}{\partial v}(u, v) - \frac{\partial\psi}{\partial v}(u, -v)\right) dv$ is well defined and is the Hilbert transform of the function $v \mapsto \frac{\partial\psi}{\partial v}(u, v)$. Moreover, the computations of Proposition 12 in our previous paper [3] yields that

$$\mathbb{E}(e^{iuX(0)}|X'(0)|) = \frac{-1}{\pi} \int_0^{+\infty} \frac{1}{v} \left(\frac{\partial\psi}{\partial v}(u, v) - \frac{\partial\psi}{\partial v}(u, -v)\right) dv.$$

□

Let us remark that, when $\beta \geq 0$ a.s., the part involving the jumps in Equation (6) can be written as

$$\sum_{t_j; g(t_j^+) > g(t_j^-)} \frac{\widehat{F}(ug(t_j^+)) - \widehat{F}(ug(t_j^-))}{iu} + \sum_{t_j; g(t_j^+) < g(t_j^-)} \frac{\widehat{F}(ug(t_j^-)) - \widehat{F}(ug(t_j^+))}{iu},$$

where $\widehat{F}$ is defined by (5).

Moreover, remark also that for any $s_1 < s_2$, the function $u \mapsto \psi(u,0)\frac{\widehat{F}(us_2)-\widehat{F}(us_1)}{iu}$ is the Fourier transform of the function $\alpha \mapsto \mathbb{P}[\alpha - \beta s_2 \leq X(0) \leq \alpha - \beta s_1]$.

2. A particular case

The formula for $\widehat{C_X}(u)$ can become simpler in some cases. The first particular case if when the kernel g is piecewise constant, since then $X'(0) = 0$ a.s. and thus the term $\mathbb{E}(e^{iuX(0)}|X'(0)|)$ vanishes. The second particular case is when g is piecewise non-increasing. In that case, we have the following proposition.

Proposition 3. *Assume that $\beta \geq 0$ a.s. and that g is piecewise non-increasing (meaning that $g' \leq 0$ and thus g is non-increasing on each of the intervals on which it is continuous, but it can have positive jumps, defined as discontinuity points t_j such that $g(t_j+) > g(t_j-)$). Then,*

$$\widehat{C_X}(u) = 2\lambda\psi(u,0) \sum_{t_j \, positive\ jump} \frac{\widehat{F}(ug(t_j^+)) - \widehat{F}(ug(t_j^-))}{iu}, \tag{7}$$

and as a consequence,

$$\text{for almost every } \alpha \in \mathbb{R}, \;\; C_X(\alpha) = 2\lambda \sum_{t_j \, positive\ jump} \mathbb{P}[\alpha - \beta g(t_j^+) \leq X(0) \leq \alpha - \beta g(t_j^-)]. \tag{8}$$

Proof. Since $g' \leq 0$, we have $X'(0) \leq 0$ a.s. and consequently $\mathbb{E}(e^{iuX(0)}|X'(0)|) = -\mathbb{E}(e^{iuX(0)}X'(0)) = i\frac{\partial\psi}{\partial v}(u,0)$. Now, since g is piecewise non-increasing and in $L^1(\mathbb{R})$, we have $\lim_{|s|\to+\infty} g(s) = 0$ and thus $\lim_{|s|\to+\infty} \widehat{F}(ug(s)) = 1$. Then, after reordering the term coming from Chasles' Formula,

$$\begin{aligned}
\frac{\partial\psi}{\partial v}(u,0) &= \lambda\psi(u,0)\int_{\mathbb{R}} g'(s)\widehat{F}'(ug(s))\,ds \\
&= \lambda\psi(u,0)\sum_{j=1}^{n} \frac{\widehat{F}(ug(t_j^-)) - \widehat{F}(ug(t_j^+))}{u}.
\end{aligned}$$

Consequently, in the formula for $\widehat{C_X}(u)$, grouping the terms for each jump t_j, we get that all the terms with the negative jumps vanish because in that case $\widehat{F}(ug(t_j^-))-\widehat{F}(ug(t_j^+)) = -\widehat{F}(u\min(g(t_j^+), g(t_j^-)))+\widehat{F}(u\max(g(t_j^+), g(t_j^-)))$ and thus finally we get Formula (7) for $\widehat{C_X}(u)$. The remark at the end of the previous section gives as a consequence Formula (8) for $C_X(\alpha)$ for almost every $\alpha \in \mathbb{R}$. $\square$

A particular case of this is when we make the assumption that the function g is positive and that it has only one positive jump at $t_1 = 0$ from the value $g(0^-) = 0$ to the value $g(0^+) > 0$. The formula above then simply becomes

$$\widehat{C_X}(u) = 2\lambda\psi(u,0)\frac{\widehat{F}(ug(0^+)) - 1}{iu}. \tag{9}$$

This framework corresponds to the case studied by Hsing in [10], and where he proves that if $U_X(\alpha)$ denotes the expected number of up-crossings of the level α by the process X in $[0,1]$, then

$$\forall \alpha \in \mathbb{R}, \;\; U_X(\alpha) = \lambda\,\mathbb{P}[\alpha - \beta g(0^+) < X(0) \leq \alpha]. \tag{10}$$

The result of Hsing given by Equation (10) is stronger than the similar formula given by Equation (8) because his formula is valid for *all* levels α and moreover he needs weaker assumptions on the regularity of g. On the other hand, his proof strongly relies on the fact that g has only one positive jump and that its value is 0 before that jump, and thus it can not be generalized to kernel functions g with other more general shapes. Another point that has to be discussed is the type of crossings that are considered. In his paper, Hsing considers up-crossings that are defined in the following way : the point t is an up-crossing of the level α by the the process X if it is a point of discontinuity of X and if $X(t^-) \leq \alpha$ and $X(t^+) > \alpha$. This explains the left strict inequality in his formula (10).

Notice that we have studied here in this section the case of a piecewise non-increasing kernel function g, but of course similar results hold for a piecewise non-decreasing kernel.

3. High intensity and Gaussian limit

We assume here that the assumptions of Proposition 2 hold. It is then well-known (see [13, 9] for instance) that as the intensity λ of the Poisson point process goes to infinity, then the normalized process Z_λ defined by

$$t \mapsto Z_\lambda(t) = \frac{X_\lambda(t) - \mathbb{E}(X_\lambda(t))}{\sqrt{\lambda}},$$

where X_λ denotes a shot noise process (as defined by Equation (1)) with intensity λ for the Poisson point process, converges to a centered Gaussian process with covariance $R(t) = \mathbb{E}(\beta^2)\int_{\mathbb{R}} g(s)g(s-t)\,ds$.

Now, how do the crossings of Z_λ behave as λ goes to $+\infty$? To answer this, we first determine the asymptotic expansion of the Fourier transform of C_{Z_λ} as $\lambda \to +\infty$.
We have :

$$\begin{aligned}\widehat{C_{Z_\lambda}}(u) &= \frac{1}{\sqrt{\lambda}}\widehat{C_{X_\lambda}}\left(\frac{u}{\sqrt{\lambda}}\right)e^{-iu\mathbb{E}(X_\lambda(t))/\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}}\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}(X_\lambda(0)-\mathbb{E}(X_\lambda(0)))}|X_\lambda'(0)|\right)\\ &\quad+\sqrt{\lambda}\,\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}(X_\lambda(0)-\mathbb{E}(X_\lambda(0)))}\right)\sum_{j=1}^{n}\frac{\mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}\max(\beta g(t_j^+),\beta g(t_j^-))}) - \mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}\min(\beta g(t_j^+),\beta g(t_j^-))})}{iu/\sqrt{\lambda}}\end{aligned}$$

As we have already studied in [3], the first term of the right-hand side admits a limit as $\lambda \to +\infty$, and more precisely we have proved that, as $\lambda \to +\infty$,

$$\frac{1}{\sqrt{\lambda}}\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}(X_\lambda(0)-\mathbb{E}(X_\lambda(0)))}|X_\lambda'(0)|\right) = \sqrt{\frac{2m_2}{\pi}}e^{-m_0u^2/2} + o(1),$$

where $m_0 = \int_{\mathbb{R}} g^2(s)\,ds$ and $m_2 = \int_{\mathbb{R}} g'^2(s)\,ds$. For the second term, it is the product of two terms that can be explicitely asymptotically developed as $\lambda \to +\infty$. Indeed we have

$$\begin{aligned}\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}}(X_\lambda(0)-\mathbb{E}(X_\lambda(0)))}\right) &= \exp\left(\lambda\int_{\mathbb{R}}(e^{i\frac{u}{\sqrt{\lambda}}g(s)}-1)ds - iu\sqrt{\lambda}\int_{\mathbb{R}}g(s)\,ds\right)\\ &= \exp\left(-\frac{m_0u^2}{2} - \frac{im_3u^3}{3\sqrt{\lambda}} + O(\frac{1}{\lambda})\right) = e^{-m_0u^2/2}\left(1+\frac{2ium_3}{3m_0\sqrt{\lambda}}\right) + O\left(\frac{1}{\lambda}\right),\end{aligned}$$

where $m_3 = \int_{\mathbb{R}} g^3(s)\,ds$. And for a given jump j, $1 \leq j \leq n$, we have

$$\begin{aligned}\frac{\mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}\max(\beta g(t_j^+),\beta g(t_j^-))}) - \mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}\min(\beta g(t_j^+),\beta g(t_j^-))})}{iu/\sqrt{\lambda}} &= \mathbb{E}(|\beta|)|g(t_j^+)-g(t_j^-)|\\ &\quad+\frac{iu}{2\sqrt{\lambda}}\mathbb{E}(\beta^2)|g^2(t_j^+)-g^2(t_j^-)| + O\left(\frac{1}{\lambda}\right).\end{aligned}$$

Finally, we thus have

$$\begin{aligned}\widehat{C_{Z_\lambda}}(u) &= \sqrt{\lambda}\,e^{-m_0u^2/2}\mathbb{E}(|\beta|)\sum_{j=1}^{n}|g(t_j^+)-g(t_j^-)|\\ &\quad+\left(\sqrt{\frac{2m_2}{\pi}} + \frac{2ium_3}{3m_0}\mathbb{E}(|\beta|)\sum_{j=1}^{n}|g(t_j^+)-g(t_j^-)| + \frac{iu}{2}\mathbb{E}(\beta^2)\sum_{j=1}^{n}|g^2(t_j^+)-g^2(t_j^-)|\right)e^{-m_0u^2/2} + o(1).\end{aligned}$$

Let us comment a bit this result. When there are no jumps we obtain that $\widehat{C_{Z_\lambda}}(u)$ converges, as λ goes to infinity, to $\sqrt{\frac{2m_2}{\pi}}e^{-m_0u^2/2}$. This implies that $C_{Z_\lambda}(\alpha)$ weakly converges to $\frac{\sqrt{m_2}}{\pi\sqrt{m_0}}e^{-\alpha^2/2m_0}$, which is the usual Rice's formula for the crossings of Gaussian processes (see [6] for instance). Now, when there are jumps, the behavior of $\widehat{C_{Z_\lambda}}(u)$ is different, since the main term in $\sqrt{\lambda}$ doesn't vanish anymore. More precisely we have that $\frac{1}{\sqrt{\lambda}}\widehat{C_{Z_\lambda}}(u)$ goes to $e^{-m_0u^2/2}\mathbb{E}(|\beta|)\sum_{j=1}^n|g(t_j^+)-g(t_j^-)|$, which implies that

$$\frac{1}{\sqrt{\lambda}}C_{Z_\lambda}(\alpha) \underset{\lambda\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi m_0}}e^{-\alpha^2/2m_0}\mathbb{E}(|\beta|)\sum_{j=1}^n|g(t_j^+)-g(t_j^-)| \quad \text{in the sense of weak convergence.}$$

Notice also that taking $u=0$ in the asymptotic formula for $\widehat{C_{Z_\lambda}}(u)$ gives the asymptotic behavior of the total variation of Z_λ. Indeed, we then have

$$\mathbb{E}(TV(Z_\lambda,(0,1))) = \widehat{C_{Z_\lambda}}(0) = \sqrt{\lambda}\,\mathbb{E}(|\beta|)\sum_{j=1}^n|g(t_j^+)-g(t_j^-)| + \sqrt{\frac{2m_2}{\pi}} + o(1).$$

This kind of asymptotic behavior has also been studied by B. Galerne in [8] in the framework of random fields of bounded variation.

4. Some examples

4.1. Step functions. We start this section with some examples of explicit computations in the case of step functions.

(1) The first simple example of step function is the one where the kernel g is a rectangular function : $g(t)=1$ for $t\in[0,a]$ with $a>0$ and 0 otherwise. Notice that this is a very simple framework, that also fits in the results of Hsing [10].

- If the impulse β is such that $\beta=1$ a.s., then $\psi(u,0)=\exp(\lambda a(e^{iu}-1))$, which shows that $X(0)$ follows a Poisson distribution with parameter λa. Then, by Formula (9),
$$\widehat{C_X}(u) = 2\lambda\exp(\lambda a(e^{iu}-1))\frac{e^{iu}-1}{iu}.$$
We recognize here the product of two Fourier transforms: the one of a Poisson distribution and the one of the indicator function of $[0,1]$. Thus,
$$C_X(\alpha) = \sum_{k=0}^{+\infty}2\lambda e^{-\lambda a}\frac{(\lambda a)^k}{k!}\mathbb{1}_{\{k<\alpha<k+1\}} \quad \text{for all } \alpha\in\mathbb{R}\backslash\mathbb{N}.$$
(The inequality $k<\alpha<k+1$ comes from the way we have defined the crossings $C_X(\alpha)$, and $C_X(\alpha)=+\infty$ for all $\alpha\in\mathbb{N}$.)
- If the impulse β follows an exponential distribution of parameter $\mu>0$, then $\widehat{F}(u)=\frac{\mu}{\mu-iu}$ and a simple computation gives
$$\widehat{C_X}(u) = 2\lambda\exp\left(\lambda a\frac{iu}{\mu-iu}\right)\frac{1}{\mu-iu}.$$
We recognize here the Fourier transform of a non-central chi-square distribution, and thus
$$C_X(0)=+\infty \text{ and } C_X(\alpha)=2\lambda\mu e^{-a\lambda-\mu\alpha}I_0(2\sqrt{a\lambda\mu\alpha}) \quad \text{for all } \alpha>0,$$
where I_0 is the modified Bessel function of the first kind of order 0; it is given by $I_0(x)=\frac{1}{\pi}\int_0^\pi e^{x\cos\theta}d\theta=\sum_{m=0}^{+\infty}\frac{x^{2m}}{4^m(m!)^2}$. By taking the inverse Fourier transform, we only have that the formula above for $C_X(\alpha)$ holds for almost every $\alpha\in\mathbb{R}$. But $C_X(\alpha)$ can be written as $C_X(\alpha)=U_X(\alpha)+D_X(\alpha)+T_X(\alpha)$, where $U_X(\alpha)$ are the up-crossings (as defined by Hsing), $D_X(\alpha)$ are the down-crossings (defined in a symmetric way) and $T_X(\alpha)=\mathbb{E}(\#\{t\in(0,1)\cap S_X^c; X'(t)=0 \text{ and } X(t)=\alpha\})$. Now, the law of $X(0)$ can be computed: $dP_{X(0)}(x)=e^{-2a\lambda}\delta_0(x)+\sum_{k=1}^{+\infty}e^{-2a\lambda}\frac{(2a\lambda)^k}{k!}f_{\mu,k}(x)dx$, where $f_{\mu,k}$ is the

probability density of the Gamma distribution of parameters μ and k. Then, for all $\alpha > 0$, $\mathbb{P}[X(t) = \alpha] = 0$ and since X is piecewise constant we have that $T_X(\alpha) = 0$. Then, thanks to the result of Hsing in Equation (10), we have that for *all* $\alpha > 0$, $C_X(\alpha) = 2U_X(\alpha) = 2\lambda\,\mathbb{P}[\alpha - \beta < X(0) \leq \alpha]$. This implies that $C_X(\alpha)$ is a continuous function of $\alpha > 0$, and thus the formula holds for every $\alpha > 0$. And for $\alpha = 0$, since $\mathbb{P}[X(t) = 0] > 0$, we have $C_X(0) = +\infty$.

(2) A second example is a "double rectangular" function given by: $g(t) = 1$ if $-1 \leq t < 0$; $g(t) = -1$ if $0 \leq t < 1$, and $g(t) = 0$ otherwise. Notice that this case does not fit anymore in the framework of Hsing.

- If $\beta = 1$ almost surely, then simple computations show that

$$\widehat{C_X}(u) = 4\lambda \frac{\sin u}{u} \exp(2\lambda(\cos u - 1)).$$

The last term above is the characteristic function of the difference of two independent Poisson random variables of same parameter λ (it is also called a Skellam distribution), and thus

$$\forall \alpha \in \mathbb{R}\backslash\mathbb{Z},\;\; C_X(\alpha) = \sum_{k=-\infty}^{+\infty} 4\lambda p_k \mathbf{1}_{\{k<\alpha<k+1\}},\;\text{ where } \forall k \in \mathbb{Z}, p_k = e^{-2\lambda}\sum_{n=0}^{+\infty} \frac{\lambda^{k+2n}}{n!(k+n)!}.$$

(Again, the inequality $k < \alpha < k+1$ comes from the way we have defined the crossings $C_X(\alpha)$, and $C_X(\alpha) = +\infty$ for all $\alpha \in \mathbb{Z}$.)

- If β follows an exponential distribution of parameter μ, we can also explicitely compute

$$\widehat{C_X}(u) = \frac{4\lambda\mu}{\mu^2 + u^2} \exp\left(-2\lambda \frac{u^2}{\mu^2 + u^2}\right).$$

4.2. **Exponential kernel.** In this section, we consider an example that has been widely studied in the literature : the impulse β follows an exponential distribution of parameter $\mu > 0$, and the kernel function g is given by $g(s) = 0$ for $s < 0$ and $g(s) = e^{-s}$ for $s \geq 0$.

A simple computation gives that the joint characteristic function of $X(0)$ and $X'(0)$ is

$$\forall u, v \in \mathbb{R},\;\; \psi(u, v) = \frac{\mu^\lambda}{(\mu - iu + iv)^\lambda}.$$

Then by Formula (9), we get

$$\widehat{C_X}(u) = \frac{2\lambda\mu^\lambda}{(\mu - iu)^{\lambda+1}}.$$

We recognize here the Fourier transform of a Gamma probability density. Thus it implies that

$$C_X(\alpha) = \frac{2\lambda\mu^\lambda \alpha^\lambda e^{-\mu\alpha}}{\Gamma(\lambda + 1)} \mathbf{1}_{\{\alpha \geq 0\}} \quad \text{for all } \alpha \in \mathbb{R}.$$

The fact that the formula holds for *all* $\alpha \in \mathbb{R}$ comes from the continuity of the crossing function. Indeed, using again Hsing result's as in the previous example, and the fact that here the law of $X(0)$ admits a probability density (see [3] for a rigorous proof of this) and that X is stationary and has a.s. no tangencies in $(0, 1)$, it implies that for all $\alpha \in \mathbb{R}$, $C_X(\alpha) = 2U_X(\alpha)$ and thus that the crossing function C_X is continuous.

In the case where λ is an integer, the explicit formula for the crossings $C_X(\alpha)$ was already given in [12] by Orsingher and Battaglia (but they had a completely different approach based on the property that in the particular case of an exponential kernel, the process is Markovian).

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