



HAL
open science

Spectral triples for finitely presented groups, index 1.

Sébastien Palcoux

► **To cite this version:**

| Sébastien Palcoux. Spectral triples for finitely presented groups, index 1.. 2010. hal-00537652v4

HAL Id: hal-00537652

<https://hal.science/hal-00537652v4>

Preprint submitted on 30 Nov 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SPECTRAL TRIPLES FOR FINITELY PRESENTED GROUPS, INDEX 1

SÉBASTIEN PALCOUX

ABSTRACT. Using a Cayley complex (generalizing the Cayley graph) and Clifford algebras, we are able to give, for a large class of finitely presented groups, a uniform construction of spectral triples with D_+ of index 1.

CONTENTS

1. Introduction	1
2. Basic definitions	2
3. Geometric construction	2
4. Clifford algebra	4
5. Dirac operator	5
References	6

Warning 0.1. *This paper is just a first draft, it contains very few proofs. It is possible that some propositions are false, or that some proofs are incomplete or trivially false.*

1. INTRODUCTION

In this paper, we define even θ -summable spectral triples for a large class of finitely presented groups such that D_+ is index 1. We just generalize the unbounded version of the construction of the Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [5]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [4]. We also use topics in geometric group theory as a Cayley complex (generalizing the Cayley graph).

2000 *Mathematics Subject Classification.* Primary 46L87. Secondary 20F65.

Key words and phrases. non-commutative geometry; spectral triple; geometric group theory; Clifford algebra; Cayley graph; Dirac operator; finitely presented groups.

2. BASIC DEFINITIONS

Definition 2.1. A spectral triple (\mathcal{A}, H, D) is given by a unital \star -algebra \mathcal{A} represented on the Hilbert space H , and an unbounded operator D , called the Dirac operator, such that:

- (1) D is self-adjoint.
- (2) $(D^2 + I)^{-1}$ is compact.
- (3) $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$ is dense in \mathcal{A} .

See the article [6] of G. Skandalis, dedicated to A. Connes and spectral triple.

Definition 2.2. A group Γ is finitely presented if it exists a finite generating set S and a finite set of relations R such that $\Gamma = \langle S \mid R \rangle$. We always take S equals to S^{-1} and the identity element $e \notin S$ (see [3] for more details).

3. GEOMETRIC CONSTRUCTION

Definition 3.1. Let Γ_n be the set of irreducible n -blocks, defined by induction:

- $\Gamma_0 = \Gamma$.
- $\Gamma_1 := \{\{g, gs\} \mid g \in \Gamma, s \in S\}$

An $(n+2)$ -block is a finite set a of $(n+1)$ -blocks such that:

$$\forall b \in a, \forall c \in b, \exists ! b' \in a \text{ such that } b \cap b' = \{c\}.$$

Let a, a' be n -blocks then the commutative and associative composition:

$$a.a' := a \Delta a' = (a \cup a') \setminus a \cap a'$$

gives also an n -block if it's non empty (we take $n \neq 0$).

Let $n > 1$, an n -block a'' is called **irreducible** if $\forall a, a'$ n -blocks:

- (1) $a'' = a.a' \Rightarrow \text{card}(a) \text{ or } \text{card}(a') \geq \text{card}(a'')$
- (2) $\forall b \in a'', b$ is an irreducible $(n-1)$ -block.

- Γ_{n+2} is the set of irreducible $(n+2)$ -blocks.

Note that if $b \in \Gamma_n$, we call n the **dimension** of b .

Definition 3.2. An n -block is called **admissible** if it decomposes into irreducibles.

Example 3.3. Let $\mathbb{Z} = \langle s^{\pm 1} \mid \rangle$ then $a = \{e, s^{10}\}$ is an admissible 1-block because $a = \{e, s\}.\{s, s^2\} \dots \{s^9, s^{10}\}$; but, $b = \{\{e, s\}, \{e, s^{-1}\}, \{s^{-1}, s\}\}$ is a non-admissible 2-block, because there is no irreducible 2-block in this case.

Remark 3.4. The graph with vertices Γ_0 and edges Γ_1 is the Cayley graph \mathcal{G} .

Remark 3.5. Let a be an n -block then $a.a = \emptyset$ and if $a = \{b_1, \dots, b_r\}$ then $b_i = b_1.b_2 \dots b_{i-1}.b_{i+1} \dots b_r$ and $b_1.b_2 \dots b_r = \emptyset$.

Remark 3.6. $\Gamma_{n+1} \neq \emptyset$ iff $\exists r > 1; a_1, \dots, a_r \in \Gamma_n$ all distincts with $a_1 \dots a_r = \emptyset$.

Remark 3.7. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\exists N$ such that $\Gamma_N \neq \emptyset$ and $\forall n > N, \Gamma_n = \emptyset$. In fact $N \leq \text{card}(S)$

Examples 3.8. For $\mathbb{F}_r = \langle s_1^{\pm 1}, \dots, s_r^{\pm 1} \mid \rangle$, we have $N = 1$.
For $\mathbb{Z}^r = \langle s_1^{\pm 1}, \dots, s_r^{\pm 1} \mid s_i s_j s_i^{-1} s_j^{-1}, i, j = 1, \dots, r \rangle$, we have $N = r$.
Here an n -block ($n \leq r$) is just an n -dimensional hypercube.

Definition 3.9. We define the action of Γ on Γ_n recursively:

- Γ acts on $\Gamma_0 = \Gamma$ as: $u_g : h \rightarrow g.h$ with $g, h \in \Gamma$.
- Action on Γ_{n+1} : $u_g : a \rightarrow g.a = \{g.b \mid b \in a\}$ with $g \in \Gamma, a \in \Gamma_{n+1}$.

Note that the action is well-defined: $g.\Gamma_n = \Gamma_n, \forall g \in \Gamma$.

Definition 3.10. Let a and b be blocks, then we say that $b \in a$ if $b = a$ or if $b \in a$ or if $\exists c \in a$ such that $b \in c$ (recursive definition).

Definition 3.11. Let $n > 1$ then an n -block c is **connected** if $\forall b \subset c$: ' b is an n -block' $\Rightarrow b = c$.

Definition 3.12. An n -block b is called **maximal** if there is no $(n+1)$ -block c with $b \in c$. We note $\mathbf{\Gamma}_{\max}$ the set of maximal irreducible blocks.

Example 3.13. Let $\Gamma = \mathbb{Z}^2 \star \mathbb{Z} = \langle s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1} \mid s_1 s_2 s_1^{-1} s_2^{-1} \rangle$, then $\{e, s_3\}$ is a maximal 1-block, $\{\{e, s_1\}, \{s_1, s_1 s_2\}, \{s_1 s_2, s_2\}, \{s_2, e\}\}$ is a maximal 2-block.

Definition 3.14. We define the **block length** $\ell(\cdot)$ as follows: let b be a block, then $\ell(b)$ is the minimal number of irreducible blocks decomposing a connected admissible block c with $e \in c$ and, $b \in c$ or $b \cap c \neq \emptyset$.

Definition 3.15. Let b be a block, then a sequence $(c_1, \dots, c_{\ell(b)})$ with $b \in c_1, e \in c_{\ell(b)}, c_i$ irreducible and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic block-path**, from b to e beginning with c_1 .

Definition 3.16. Let Υ_b be the set of irreducible blocks of minimal dimension beginning a geodesic block-path from b to e .

Remark 3.17. In general, Υ_b is not of cardinal one. It is for $CAT(0)$ groups, but not for the Baumslag-Solitar group $BS(1, 2) = \langle a^{\pm 1}, b^{\pm 1} \mid bab^{-1} = a^2 \rangle$.

Remark 3.18. Consider the group Γ and its finite presentation $\langle S \mid R \rangle$, then we can complete the presentation as follows: let T be a finite subset of Γ with $T \cap S = \emptyset, T = T^{-1}$ and $e \notin T$, let $S' = T \cup S$ an amplified generating set and $R' = R \cup \{t = \bar{t} \mid t \in T\}$ where \bar{t} is t considered as a generator.
Then $\Gamma = \langle S' \mid R' \rangle$.

Lemma 3.19. We can choose T such that if we build the blocks with the completed presentation $\langle S' \mid R' \rangle$, then every irreducible blocks are triangular, i.e. $\forall b \in \Gamma_n, \text{card}(b) = n + 1$. We call $\langle S' \mid R' \rangle$ a **triangularized presentation**.

Example 3.20. *The complete triangularization: let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then Γ acts on Γ_{max} (def. 3.9, 3.12); there are only finitely many orbits O_1, \dots, O_r ; choose $b_i \in O_i$; let $E_i = \{g \in \Gamma \mid g \in b_i\}$; let $T_i = \{gh^{-1} \mid g, h \in E_i, gh^{-1} \notin S \cup \{e\}\}$. Then amplifying the generating set with $T = \bigcup T_i$, we obtain obviously a triangularization called the complete triangularization. Note that this process increases the maximal dimension of the blocks. Note that $\text{card}(T)$ is finite because the group is finitely presented.*

4. CLIFFORD ALGEBRA

We first quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

Definition 4.1. *For V a n -dimensional Hilbert space, define the exterior algebra $\Lambda(V)$ equals to $\bigoplus_{k=0}^n \Lambda^k(V)$ with $\Lambda^0(V) = \mathbb{C}\Omega$. We called Ω the vacuum vector. Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$. Note that $\dim(\Lambda^k(V)) = C_n^k$ and $\dim(\Lambda(V)) = 2^n$.*

Definition 4.2. *Let α_v be the creation operator on $\Lambda(V)$ defined by:*

$$\alpha_v(v_1 \wedge \dots \wedge v_r) = v \wedge v_1 \wedge \dots \wedge v_r \text{ and } \alpha_v(\Omega) = v$$

Reminder 4.3. *The dual α_v^* is called the annihilation operator, then:*

$$\alpha_v^*(v_1 \wedge \dots \wedge v_r) = \sum_{i=0}^r (-1)^{i+1} (v, v_i) v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r \text{ and } \alpha_v^*(\Omega) = 0$$

Reminder 4.4. *Let $\gamma_v = \alpha_v + \alpha_v^*$, then $\gamma_v = \gamma_v^*$ and $\gamma_v \gamma_w + \gamma_w \gamma_v = 2(v, w)I$.*

Definition 4.5. *The operators γ_v generate the Clifford algebra $\text{Cliff}(V)$. Note that the operators γ_v are bounded and that $\text{Cliff}(V).\Omega = \Lambda(V)$.*

Remark 4.6. *V admits the orthonormal basis $(v_a)_{a \in I}$.*

We will write γ_a instead of γ_{v_a} , so that $[\gamma_a, \gamma_{a'}]_+ = 2\delta_{a,a'}I$.

Let Γ be a finitely presented group, with a triangularized presentation $\langle S \mid R \rangle$.

Definition 4.7. *For any irreducible block c , let $\Delta_c = \{b \in \bigcup \Gamma_n \mid c \in \Upsilon_b\}$, with Υ_b defined on definition 3.16.*

Remark 4.8. *If $\Delta_c \neq \emptyset$ then $c \in \Delta_c$*

$\bigcup \Gamma_n = \bigcup \Delta_c$ (it's not a partition in general)

If $\Delta_c \cap \Delta_{c'} \neq \emptyset$ with $c \neq c'$ then $\dim(c) = \dim(c')$ and $c.c'$ is connected.

Definition 4.9. *Let $\bigsqcup_{\alpha \in \mathcal{J}} \mathcal{P}_\alpha$ be the **minimal partition** generated by $\bigcup \Delta_c$, ie $\bigsqcup_{\alpha \in \mathcal{J}} \mathcal{P}_\alpha = \bigcup \Gamma_n$, $\forall \alpha \in \mathcal{J}$, $\mathcal{P}_\alpha \neq \emptyset$ and $\exists n > 0$, $\exists c_1, \dots, c_n$ irreducibles such that $\mathcal{P}_\alpha = \Delta_{c_1} \cap \dots \cap \Delta_{c_n}$.*

Lemma 4.10. *For any $\alpha \in \mathcal{J}$, \mathcal{P}_α admits a unique element c_α (resp. c'_α) of minimal dimension m (resp. of maximal dimension M). Denote by I_α the set of blocks of dimension $m + 1$ in \mathcal{P}_α , then \mathcal{P}_α is in one-to-one correspondence with the power set $\mathcal{P}(I_\alpha)$; in particular, the cardinality of \mathcal{P}_α is 2^{M-m} .*

Definition 4.11. *We naturally identify $\ell^2(\mathcal{P}_\alpha)$ with the exterior algebra $\Lambda(\ell^2(I_\alpha))$ on which operates the Clifford algebra $\text{Cliff}(\ell^2(I_\alpha))$ generated by γ_a , $a \in I_\alpha$.*

5. DIRAC OPERATOR

Definition 5.1. *We define the **n -block length** $\ell_n(\cdot)$ as follows: let b be a block, then $\ell_n(b)$ is the minimal number of irreducible blocks decomposing a connected admissible n -dimensional block c with $e \in c$ and, $b \in c$ or $b \cap c \neq \emptyset$.*

Definition 5.2. *Let b be a block, then a sequence $(c_1, \dots, c_{\ell_n(b)})$ with $b \in c_1$, $e \in c_{\ell(b)}$, $c_i \in \Gamma_n$ and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic n -block-path**, from b to e beginning with c_1 .*

Definition 5.3. *For any $\alpha \in \mathcal{J}$, let $n = \dim(c_\alpha) + 1$; for any $a \in I_\alpha$ define $p_a(\alpha)$ the number of geodesic n -block path from c_α to e beginning with a ; let $p(\alpha) = \sum_{a \in I_\alpha} p_a(\alpha)$; let $\lambda_a = \frac{p_a(\alpha)}{p(\alpha)} \ell_n(c_\alpha)$.*

Definition 5.4. *On $\ell^2(\mathcal{P}_\alpha) = \Lambda(\ell^2(I_\alpha))$, define the Dirac operator D_α by:*

$$D_\alpha = \sum_{a \in I_\alpha} \lambda_a \cdot \gamma_a$$

Remark 5.5. $\mathcal{P}_e := \Delta_e = \{e\}$, $\ell^2(\mathcal{P}_e) = \mathbb{C}e_1$, $I_e = \emptyset$ and $D_e = 0$.

Definition 5.6. *Consider then the Hilbert space:*

$$\mathcal{H} = \bigoplus_n \ell^2(\Gamma_n) = \sum_c \ell^2(\Delta_c) = \bigoplus_{\alpha \in \mathcal{J}} \ell^2(\Delta_\alpha) = \bigoplus_{\alpha \in \mathcal{J}} \Lambda(\ell^2(I_\alpha))$$

\mathbb{Z}_2 -graded by the decomposition into even and odd dimensional blocks:

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

Define the unbounded selfadjoint operator $\mathcal{D} = \sum_{\alpha \in \mathcal{J}} D_\alpha$.

Lemma 5.7. $\mathcal{D}^2 = \sum_\alpha \lambda_\alpha^2 \cdot p_\alpha$,
with $\lambda_\alpha^2 = \sum_{a \in I_\alpha} \lambda_a^2$ and p_α the projection on $\ell^2(\Delta_\alpha)$.

Proof. We use the orthonormal decomposition and the Clifford relations. □

Proposition 5.8. $\mathcal{D}_+ : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ is a Fredholm operator of index 1.

Proposition 5.9. $(\mathcal{D}^2 + I)^{-1}$ is compact.

For $t > 0$, the operator $e^{-t\mathcal{D}^2}$ is trass-class.

Definition 5.10. For any $g \in \Gamma$ and for any $s \in S$ define $p_s(g)$ the number of geodesic 1-block path from g to e beginning with $\{g, gs\}$; let $p(g) = \sum_{s \in S} p_s(g)$.

Definition 5.11. Let \mathcal{C} be the class of finitely presented groups $\Gamma = \langle S \mid R \rangle$ such that $\forall g \in \Gamma, \exists K_g \in \mathbb{R}_+$ such that $\forall s \in S$ and $\forall h \in \Gamma$ (with $h, gh \neq e$):

$$\left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| \leq \frac{K_g}{\ell_1(h)}$$

Examples 5.12. The class \mathcal{C} is stable by direct or free product, it contains $\mathbb{Z}^n, \mathbb{F}_n$, the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see [2]).

Proposition 5.13. Let Γ of class \mathcal{C} , $\mathcal{A} = C_r^*(\Gamma)$ and \mathcal{D} as previously then: $\{a \in \mathcal{A} \mid [\mathcal{D}, a] \in B(\mathcal{H})\}$ is dense in \mathcal{A} .

Theorem 5.14. $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even θ -summable spectral triple and \mathcal{D}_+ is index 1. It then gives a non-trivial element for the K -homology of \mathcal{A} .

REFERENCES

- [1] A. Connes, *Noncommutative differential geometry*. Inst. Hautes études Sci. Publ. Math. No. 62 (1985), 257360.
- [2] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.
- [3] P. de la Harpe, *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
- [4] P. Julg, A. Valette, *Fredholm modules associated to Bruhat-Tits buildings*. Mini-conferences on harmonic analysis and operator algebras (Canberra, 1987), 143155, Proc. Centre Math. Anal. Austral. Nat. Univ., 16, Austral. Nat. Univ., Canberra, 1988.
- [5] M. Pimsner, D. Voiculescu, *KK-groups of reduced crossed products by free groups*. J. Operator Theory 8 (1982), no. 1, 131156.
- [6] G. Skandalis *Géométrie non commutative d'après Alain Connes: la notion de triplet spectral*. Gaz. Math. No. 94 (2002), 4451.
- [7] A. Wassermann, *Lecture notes on Atiyah-Singer index theorem*, Lent 2010 course, <http://www.dpmms.cam.ac.uk/~ajw/AS10.pdf>

INSTITUT DE MATHÉMATIQUES DE LUMINY, MARSEILLE, FRANCE.

E-mail address: palcoux@iml.univ-mrs.fr, <http://iml.univ-mrs.fr/~palcoux>