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# SPECTRAL TRIPLES FOR FINITELY PRESENTED GROUPS INDEX 1

SÉBASTIEN PALCOUX

ABSTRACT. Using generalized Cayley graphs and Clifford algebras, we are able to give, for a large class of finitely presented groups, a uniform construction of spectral triples with  $D_+$  of index 1.

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## 1. INTRODUCTION

In this paper, we define even  $\theta$ -summable spectral triples for a large class of finitely presented groups such that  $D_+$  is index 1. We just generalize the unbounded version of the construction of the Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [5]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [4]. We also use topics in geometric group theory as a generalized Cayley graph.

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## 2. BASIC DEFINITIONS

**Definition 2.1.** A spectral triple  $(\mathcal{A}, H, D)$  is given by a unital  $\star$ -algebra  $\mathcal{A}$  represented on the Hilbert space  $H$ , and an unbounded operator  $D$ , called the Dirac operator, such that:

- (1)  $D$  is self-adjoint.
- (2)  $(D^2 + I)^{-1}$  is compact.
- (3)  $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$  is dense in  $\mathcal{A}$ .

See the article [6] of G. Skandalis, dedicated to A. Connes and spectral triple.

**Definition 2.2.** A group  $\Gamma$  is finitely presented if it exists a finite generating set  $S$  and a finite set of relations  $R$  such that  $\Gamma = \langle S \mid R \rangle$ . We always take  $S$  equals to  $S^{-1}$  and the identity element  $e \notin S$  (see [3] for more details).

## 3. GEOMETRIC CONSTRUCTION

**Definition 3.1.** Let  $\Gamma_n$  be the set of irreducible  $n$ -blocks, defined by induction:

- $\Gamma_0 = \Gamma$ .
- $\Gamma_1 := \{\{g, gs\} \mid g \in \Gamma, s \in S\}$

An  $(n+2)$ -block is a finite set  $a$  of  $(n+1)$ -blocks such that:

$$\forall b \in a, \forall c \in b, \exists ! b' \in a \text{ such that } b \cap b' = \{c\}.$$

Let  $a, a'$  be  $n$ -blocks then the commutative and associative composition:

$$a.a' := a \Delta a' = (a \cup a') \setminus a \cap a'$$

gives also an  $n$ -block if it's non empty (we take  $n \neq 0$ ).

Let  $n > 1$ , an  $n$ -block  $a''$  is called **irreducible** if  $\forall a, a'$   $n$ -blocks:

- (1)  $a'' = a.a' \Rightarrow \text{card}(a) \text{ or } \text{card}(a') \geq \text{card}(a'')$
- (2)  $\forall b \in a'', b$  is a irreducible  $(n-1)$ -block.

- $\Gamma_{n+2}$  is the set of irreducible  $(n+2)$ -blocks.

Note that if  $b \in \Gamma_n$ , we call  $n$  the **dimension** of  $b$ .

**Definition 3.2.** An  $n$ -block is called **admissible** if it decomposes into irreducibles.

**Example 3.3.** Let  $\mathbb{Z} = \langle s^{\pm 1} \mid \rangle$  then  $a = \{e, s^{10}\}$  is an admissible 1-block because  $a = \{e, s\}.\{s, s^2\} \dots \{s^9, s^{10}\}$ ; but,  $b = \{\{e, s\}, \{e, s^{-1}\}, \{s^{-1}, s\}\}$  is a non-admissible 2-block, because there is no irreducible 2-block in this case.

**Remark 3.4.** The graph with vertices  $\Gamma_0$  and edges  $\Gamma_1$  is the Cayley graph  $\mathcal{G}$ .

**Remark 3.5.** Let  $a$  be an  $n$ -block then  $a.a = \emptyset$  and if  $a = \{b_1, \dots, b_r\}$  then  $b_i = b_1.b_2 \dots b_{i-1}.b_{i+1} \dots b_r$  and  $b_1.b_2 \dots b_r = \emptyset$ .

**Remark 3.6.**  $\Gamma_{n+1} \neq \emptyset$  iff  $\exists r > 1; a_1, \dots, a_r \in \Gamma_n$  all distincts with  $a_1 \dots a_r = \emptyset$ .

**Remark 3.7.** Let  $\Gamma = \langle S \mid R \rangle$  be a finitely presented group, then  $\exists N$  such that  $\Gamma_N \neq \emptyset$  and  $\forall n > N, \Gamma_n = \emptyset$ . In fact  $N \leq \text{card}(S)$

**Examples 3.8.** For  $\mathbb{F}_r = \langle s_1^{\pm 1}, \dots, s_r^{\pm 1} \mid \rangle$ , we have  $N = 1$ .  
For  $\mathbb{Z}^r = \langle s_1^{\pm 1}, \dots, s_r^{\pm 1} \mid s_i s_j s_i^{-1} s_j^{-1}, i, j = 1, \dots, r \rangle$ , we have  $N = r$ .  
Here an  $n$ -block ( $n \leq r$ ) is just an  $n$ -dimensional hypercube.

**Definition 3.9.** We define the action of  $\Gamma$  on  $\Gamma_n$  recursively:

- $\Gamma$  acts on  $\Gamma_0 = \Gamma$  as:  $u_g : h \rightarrow g.h$  with  $g, h \in \Gamma$ .
- Action on  $\Gamma_{n+1}$ :  $u_g : a \rightarrow g.a = \{g.b \mid b \in a\}$  with  $g \in \Gamma, a \in \Gamma_{n+1}$ .

Note that the action is well-defined:  $g.\Gamma_n = \Gamma_n, \forall g \in \Gamma$ .

**Definition 3.10.** Let  $a$  and  $b$  be blocks, then we say that  $b \in a$  if  $b = a$  or if  $b \in a$  or if  $\exists c \in a$  such that  $b \in c$  (recursive definition).

**Definition 3.11.** Let  $n > 1$  then an  $n$ -block  $c$  is **connected** if  $\forall b \subset c$ :  
' $b$  is an  $n$ -block'  $\Rightarrow b = c$ .

**Definition 3.12.** An  $n$ -block  $b$  is called **maximal** if there is no  $(n+1)$ -block  $c$  with  $b \in c$ . We note  $\mathbf{\Gamma}_{\max}$  the set of maximal irreducible blocks.

**Example 3.13.** Let  $\Gamma = \mathbb{Z}^2 \star \mathbb{Z} = \langle s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1} \mid s_1 s_2 s_1^{-1} s_2^{-1} \rangle$ , then  $\{e, s_3\}$  is a maximal 1-block,  $\{\{e, s_1\}, \{s_1, s_1 s_2\}, \{s_1 s_2, s_2\}, \{s_2, e\}\}$  is a maximal 2-block.

**Definition 3.14.** We define the **block lenght**  $\ell(\cdot)$  as follows: let  $b$  be a block, then  $\ell(b)$  is the minimal number of irreducible blocks decomposing a connected admissible block  $c$  with  $e \in c$  and,  $b \in c$  or  $b \cap c \neq \emptyset$ .

**Definition 3.15.** Let  $b$  be a block, then a sequence  $(c_1, \dots, c_{\ell(b)})$  with  $b \in c_1, e \in c_{\ell(b)}, c_i$  irreducible and  $c_i \cap c_{i+1} \neq \emptyset$  is called a **geodesic block-path**, from  $b$  to  $e$  beginning with  $c_1$ .

**Lemma 3.16.** There is a unique irreducible block  $\beta(b)$  of minimal dimension, beginning a geodesic block-path from  $b$  to  $e$ .

*Proof.* We prove by contradiction: let  $\beta(b)$  and  $\beta'(b)$  be two different such blocks, then they are the same dimension  $n$ . But then there is an admissible connected block  $d$  of dimension  $n+1$ , with  $\beta(b), \beta'(b) \in d$  and  $e \in d$ , such that  $d$  decomposes into strictly less than  $\ell(b)$  irreducible blocks, contradiction.  $\square$

**Remark 3.17.** Consider the group  $\Gamma$  and its finite presentation  $\langle S \mid R \rangle$ , then we can complete the presentation as follows: let  $T$  be a finite subset of  $\Gamma$  with  $T \cap S = \emptyset, T = T^{-1}$  and  $e \notin T$ , let  $S' = T \cup S$  an amplified generating set and  $R' = R \cup \{t = \bar{t} \mid \}$  where  $\bar{t}$  is  $t$  considered as a generator. Then  $\Gamma = \langle S' \mid R' \rangle$ .

**Lemma 3.18.** *We can choose  $T$  such that if we build the blocks with the completed presentation  $\langle S' \mid R' \rangle$ , then every irreducible blocks are triangular, i.e.  $\forall b \in \Gamma_n$ ,  $\text{card}(b) = n + 1$ . We call  $\langle S' \mid R' \rangle$  a **triangularized presentation**.*

**Example 3.19.** *The complete triangularization: let  $\Gamma = \langle S \mid R \rangle$  be a finitely presented group, then  $\Gamma$  acts on  $\Gamma_{\max}$  (def. 3.9, 3.12); there are only finitely many orbits  $O_1, \dots, O_r$ ; choose  $b_i \in O_i$ ; let  $E_i = \{g \in \Gamma \mid g \in b_i\}$ ; let  $T_i = \{gh^{-1} \mid g, h \in E_i, gh^{-1} \notin S \cup \{e\}\}$ . Then amplifying the generating set with  $T = \bigcup T_i$ , we obtain obviously a triangularization called the complete triangularization. Note that this process increases the maximal dimension of the blocks. Note that  $\text{card}(T)$  is finite because the group is finitely presented.*

#### 4. CLIFFORD ALGEBRA

We first quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

**Definition 4.1.** *For  $V$  a  $n$ -dimensional Hilbert space, define the exterior algebra  $\Lambda(V)$  equals to  $\bigoplus_{k=0}^n \Lambda^k(V)$  with  $\Lambda^0(V) = \mathbb{C}\Omega$ . We called  $\Omega$  the vacuum vector. Recall that  $v_1 \wedge v_2 = -v_2 \wedge v_1$  so that  $v \wedge v = 0$ . Note that  $\dim(\Lambda^k(V)) = C_n^k$  and  $\dim(\Lambda(V)) = 2^n$ .*

**Definition 4.2.** *Let  $\alpha_v$  be the creation operator on  $\Lambda(V)$  defined by:*

$$\alpha_v(v_1 \wedge \dots \wedge v_r) = v \wedge v_1 \wedge \dots \wedge v_r \text{ and } \alpha_v(\Omega) = v$$

**Reminder 4.3.** *The dual  $\alpha_v^*$  is called the annihilation operator, then:*

$$\alpha_v^*(v_1 \wedge \dots \wedge v_r) = \sum_{i=0}^r (-1)^{i+1} (v, v_i) v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r \text{ and } \alpha_v^*(\Omega) = 0$$

**Reminder 4.4.** *Let  $\gamma_v = \alpha_v + \alpha_v^*$ , then  $\gamma_v = \gamma_v^*$  and  $\gamma_v \gamma_w + \gamma_w \gamma_v = 2(v, w)I$ .*

**Definition 4.5.** *The operators  $\gamma_v$  generate the Clifford algebra  $\text{Cliff}(V)$ . Note that the operators  $\gamma_v$  are bounded and that  $\text{Cliff}(V).\Omega = \Lambda(V)$ .*

**Remark 4.6.**  *$V$  admits the orthonormal basis  $(v_a)_{a \in I}$ .*

*We will write  $\gamma_a$  instead of  $\gamma_{v_a}$ , so that  $[\gamma_a, \gamma_{a'}]_+ = 2\delta_{a,a'}I$ .*

Let  $\Gamma$  be a finitely presented group, with a triangularized presentation  $\langle S \mid R \rangle$ .

**Definition 4.7.** *For any irreducible block  $c$ , let  $\Delta_c = \{b \in \bigcup \Gamma_n \mid \beta(b) = c\}$ , with  $\beta(b)$  defined on lemma 3.16.*

**Definition 4.8.** *Let  $b, c$  be blocks such that  $b \in c \in \beta(b)$  then we write  $b \propto c$ . In this case, we see that  $\beta(b) = \beta(c)$ , so that  $\propto$  is an order relation.*

**Lemma 4.9.** *For any irreducible block  $c$  with  $\Delta_c \neq \emptyset$ ,  $\Delta_c$  admits a unique minimal element  $c_{min}$  with respect to  $\prec$ . Let  $m$  be the dimension of  $c_{min}$ ; denote by  $I_c$  the set of blocks of dimension  $m+1$  in  $\Delta_c$ ; then  $\Delta_c$  is in one-to-one correspondence with the power set  $\mathcal{P}(I_c)$ ; in particular, the cardinality of  $\Delta_c$  is  $2^{\dim(c)-m}$  (see section 3 of Julg-Valette paper [4]).*

**Definition 4.10.** *We naturally identify  $\ell^2(\Delta_c)$  with the exterior algebra  $\Lambda(\ell^2(I_c))$  on which operates the Clifford algebra  $\text{Cliff}(\ell^2(I_c))$  generated by  $\gamma_a$ ,  $a \in I_c$ .*

## 5. DIRAC OPERATOR

**Definition 5.1.** *We define the  **$n$ -block length**  $\ell_n(\cdot)$  as follows: let  $b$  be a block, then  $\ell_n(b)$  is the minimal number of irreducible blocks decomposing a connected admissible  $n$ -dimensional block  $c$  with  $e \in c$  and,  $b \in c$  or  $b \cap c \neq \emptyset$ .*

**Definition 5.2.** *Let  $b$  be a block, then a sequence  $(c_1, \dots, c_{\ell_n(b)})$  with  $b \in c_1$ ,  $e \in c_{\ell(b)}$ ,  $c_i \in \Gamma_n$  and  $c_i \cap c_{i+1} \neq \emptyset$  is called a **geodesic  $n$ -block-path**, from  $b$  to  $e$  beginning with  $c_1$ .*

**Definition 5.3.** *For any irreducible block  $c$  with  $\Delta_c \neq \emptyset$ , let  $n = \dim(c_{min}) + 1$ ; for any  $a \in I_c$  define  $p_a(c)$  the number of geodesic  $n$ -block path from  $c_{min}$  to  $e$  beginning with  $a$ ; let  $p(c) = \sum_{a \in I_c} p_a(c)$ ; let  $\lambda_a = \frac{p_a(c)}{p(c)} \ell_n(c_{min})$ .*

**Definition 5.4.** *On  $\ell^2(\Delta_c) = \Lambda(\ell^2(I_c))$ , define the Dirac operator  $D_c$  by:*

$$D_c = \sum_{a \in I_c} \lambda_a \cdot \gamma_a$$

**Remark 5.5.**  $\Delta_e = \{e\}$ ,  $\ell^2(\Delta_c) = \mathbb{C}e_1$ ,  $I_e = \emptyset$  and  $D_e = 0$ .

**Definition 5.6.** *Consider then the Hilbert space:*

$$\mathcal{H} = \bigoplus_n \ell^2(\Gamma_n) = \bigoplus_c \ell^2(\Delta_c) = \bigoplus_c \Lambda(\ell^2(I_c))$$

$\mathbb{Z}_2$ -graded by the decomposition into even and odd dimensional blocks:

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

Define the unbounded selfadjoint operators  $\mathcal{D} = \bigoplus_c D_c$ .

**Lemma 5.7.**  $D^2 = \bigoplus_c D_c^2 = \sum_c (\sum_{a \in I_c} \lambda_a^2) \cdot p_c$  with  $p_c$ , projection on  $\ell^2(\Delta_c)$ .

**Proposition 5.8.**  $\mathcal{D}_+ : \mathcal{H}^+ \rightarrow \mathcal{H}^-$  is a Fredholm operator of index 1.

**Proposition 5.9.**  $(\mathcal{D}^2 + I)^{-1}$  is compact.

For  $t > 0$ , the operator  $e^{-t\mathcal{D}^2}$  is trass-class.

**Definition 5.10.** *For any  $g \in \Gamma$  and for any  $s \in S$  define  $p_s(g)$  the number of geodesic 1-block path from  $g$  to  $e$  beginning with  $\{g, gs\}$ ; let  $p(g) = \sum_{s \in S} p_s(g)$ .*

**Definition 5.11.** Let  $\mathcal{C}$  be the class of finitely presented groups  $\Gamma$  admitting a triangularized finite presentation  $\langle S \mid R \rangle$  such that  $\forall g \in \Gamma, \exists K_g \in \mathbb{R}_+$  such that  $\forall s \in S$  and  $\forall h \in \Gamma$  (with  $h, gh \neq e$ ):

$$\left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| \leq \frac{K_g}{\ell_1(h)}$$

**Examples 5.12.** The class  $\mathcal{C}$  is stable by direct or free product, it contains  $\mathbb{Z}^n$ ,  $\mathbb{F}_n$ , the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see [2]).

**Proposition 5.13.** Let  $\Gamma$  of class  $\mathcal{C}$ ,  $\mathcal{A} = C_r^*(\Gamma)$  and  $\mathcal{D}$  as previously then:  $\{a \in \mathcal{A} \mid [\mathcal{D}, a] \in B(\mathcal{H})\}$  is dense in  $\mathcal{A}$ .

**Theorem 5.14.**  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is an even  $\theta$ -summable spectral triple and  $\mathcal{D}_+$  is index 1. It then gives a non-trivial element for the  $K$ -homology of  $\mathcal{A}$ .

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