Spectral triples for finitely presented groups, index 1.

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SPECTRAL TRIPLES FOR FINITELY PRESENTED GROUPS INDEX 1

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Abstract. Using generalized Cayley graphs and Clifford algebras, we are able to give, for a large class of finitely presented groups, a uniform construction of spectral triples with an index 1 Dirac operator.

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1. Introduction

In this paper, we define even θ-summable spectral triples for a large class of finitely presented groups such that the Dirac operator index 1. We just generalize the unbounded version of the construction of the Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [5]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [4]. We also use topics in geometric group theory as a generalized Cayley graph.

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2. Basic definitions

Definition 2.1. *A spectral triple $(\mathcal{A}, H, D)$ is given by a unital $\star$-algebra $\mathcal{A}$ representated on the Hilbert space H, and an unbounded operator D, called the Dirac operator, such that:*

(1) *D is self-adjoint.*
(2) *$(D^2+I)^{-1}$ is compact.*
(3) *$\{a \in \mathcal{A} \mid [D,a] \in B(H)\}$ is dense in $\mathcal{A}$.*

See the article [6] *of G. Skandalis, dedicated to A. Connes and spectral triple.*

Definition 2.2. *A group Γ is finitely presented if it exists a finite generating set S and a finite set of relations R such that $\Gamma = \langle S \mid R \rangle$. We always take S equals to S^{-1} and the identity element $e \notin S$ (see* [3] *for more details).*

3. Geometric construction

Definition 3.1. *Let Γ_n be the set of irreducible n-blocks, defined by induction:*

- $\Gamma_0 = \Gamma$.
- $\Gamma_1 := \{\{g, gs\} \mid g \in \Gamma, s \in S\}$

An $(n+2)$-block is a finite set a of $(n+1)$-blocks such that:

$$\forall b \in a, \forall c \in b, \exists! b' \in a \text{ such that } b \cap b' = \{c\}.$$

Let a, a' be n-blocks then the commutative and associative composition:

$$a.a' := a \triangle a' = (a \cup a') \backslash a \cap a'$$

gives also an n-block if it's non empty (we take $n \neq 0$).
*Let $n > 1$, an n-block a'' is called **irreducible** if $\forall a, a'$ n-blocks:*

(1) $a'' = a.a' \Rightarrow card(a)$ or $card(a') \geq card(a'')$

(2) $\forall b \in a''$, b is a irreducible $(n-1)$-block.

- *Γ_{n+2} is the set of irreducible $(n+2)$-blocks.*

*Note that if $b \in \Gamma_n$, we call n the **dimension** of b.*

Definition 3.2. *An n-block is called **admissible** if it decomposes into irreducibles.*

Example 3.3. *Let $\mathbb{Z} = \langle s^{\pm 1} \mid \rangle$ then $a = \{e, s^{10}\}$ is an admissible 1-block because $a = \{e,s\}.\{s,s^2\}...\{s^9, s^{10}\}$; but, $b = \{\{e,s\}, \{e,s^{-1}\}, \{s^{-1}, s\}\}$ is a non-admissible 2-block, because there is no irreducible 2-block in this case.*

Remark 3.4. *The graph with vertices Γ_0 and edges Γ_1 is the Cayley graph $\mathcal{G}$.*

Remark 3.5. *Let a be an n-block then $a.a = \emptyset$ and if $a = \{b_1, .., b_r\}$ then $b_i = b_1.b_2...b_{i-1}.b_{i+1}...b_r$ and $b_1.b_2...b_r = \emptyset$.*

Remark 3.6. *$\Gamma_{n+1} \neq \emptyset$ iff $\exists r > 1; a_1, ..., a_r \in \Gamma_n$ all distincts with $a_1...a_r = \emptyset$.*

Remark 3.7. *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\exists N$ such that $\Gamma_N \neq \emptyset$ and $\forall n > N$, $\Gamma_n = \emptyset$. In fact $N \leq card(S)$*

Examples 3.8. *For $\mathbb{F}_r = \langle s_1^{\pm 1}, ..., s_r^{\pm 1} \mid \rangle$, we have $N = 1$.*
For $\mathbb{Z}^r = \langle s_1^{\pm 1}, ..., s_r^{\pm 1} \mid s_i s_j s_i^{-1} s_j^{-1}, i, j = 1, ..., r \rangle$, we have $N = r$.
Here an n-block ($n \leq r$) is just an n-dimensional hypercube.

Definition 3.9. *We define the action of Γ on Γ_n recursively:*

- *Γ acts on $\Gamma_0 = \Gamma$ as: $u_g : h \to g.h$ with $g, h \in \Gamma$.*
- *Action on Γ_{n+1}: $u_g : a \to g.a = \{g.b \mid b \in a\}$ with $g \in \Gamma$, $a \in \Gamma_{n+1}$.*

Note that the action is well-defined: $g.\Gamma_n = \Gamma_n$, $\forall g \in \Gamma$.

Definition 3.10. *Let a and b be blocks, then we say that $b \Subset a$ if $b = a$ or if $b \in a$ or if $\exists c \in a$ such that $b \Subset c$ (recursive definition).*

Definition 3.11. *Let $n > 1$ then an n-block c is **connected** if $\forall b \subset c$: 'b is an n-block' $\Rightarrow b = c$.*

Definition 3.12. *An n-block b is called **maximal** if there is no $(n+1)$-block c with $b \in c$. We note $\mathbf{\Gamma_{max}}$ the set of maximal irreducible blocks.*

Example 3.13. *Let $\Gamma = \mathbb{Z}^2 \star \mathbb{Z} = \langle s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1} \mid s_1 s_2 s_1^{-1} s_2^{-1} \rangle$, then $\{e, s_3\}$ is a maximal 1-block, $\{\{e, s_1\}, \{s_1, s_1 s_2\}, \{s_1 s_2, s_2\}, \{s_2, e\}\}$ is a maximal 2-block.*

Definition 3.14. *We define the **block lenght** $\ell(.)$ as follows: let b be a block, then $\ell(b)$ is the minimal number of irreducible blocks decomposing a connected admissible block c with $e \Subset c$ and, $b \Subset c$ or $b \cap c \neq \emptyset$.*

Definition 3.15. *Let b be a block, then a sequence $(c_1, ..., c_{\ell(b)})$ with $b \Subset c_1$, $e \Subset c_{\ell(b)}$, c_i irreducible and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic block-path**, from b to e beginning with c_1.*

Lemma 3.16. *There is a unique irreducible block $\beta(b)$ of minimal dimension, beginning a geodesic block-path from b to e.*

Proof. We prove by contradiction: let $\beta(b)$ and $\beta'(b)$ be two differents such blocks, then they are the same dimension n. But then there is an admissible connected block d of dimension $n+1$, with $\beta(b), \beta'(b) \in d$ and $e \Subset d$, such that d decomposes into strictly less than $\ell(b)$ irreducible blocks, contradiction. □

Remark 3.17. *Consider the group Γ and its finite presentation $\langle S \mid R \rangle$, then we can complete the presentation as follows: let T be a finite subset of Γ with $T \cap S = \emptyset$, $T = T^{-1}$ and $e \notin T$, let $S' = T \cup S$ an amplified generating set and $R' = R \cup \{t = \bar{t} \mid \}$ where $\bar{t}$ is t considered as a generator. Then $\Gamma = \langle S' \mid R' \rangle$.*

Lemma 3.18. *We can choose T such that if we build the blocks with the completed presentation $\langle S' \mid R' \rangle$, then every irreducible blocks are triangular, i.e. $\forall b \in \Gamma_n$, card$(b) = n+1$. We call $\langle S' \mid R' \rangle$ a* ***triangularized presentation****.*

Example 3.19. *The complete triangularization: let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then Γ acts on Γ_{max} (def. 3.9, 3.12); there are only finitely many orbits $O_1, ..., O_r$; choose $b_i \in O_i$; let $E_i = \{g \in \Gamma \mid g \Subset b_i\}$; let $T_i = \{gh^{-1} \mid g, h \in E_i, gh^{-1} \notin S \cup \{e\}\}$. Then amplifying the generating set with $T = \bigcup T_i$, we obtain obviously a triangularization called the complete triangularization. Note that this process increases the maximal dimension of the blocks. Note that card(T) is finite because the group is finitely presented.*

4. Clifford algebra

We first quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

Definition 4.1. *For V a n-dimensional Hilbert space, define the exterior algebra $\Lambda(V)$ equals to $\oplus_{k=0}^{n} \Lambda^k(V)$ with $\Lambda^0(V) = \mathbb{C}\Omega$. We called Ω the vacuum vector. Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$.*
Note that dim$(\Lambda^k(V)) = C_n^k$ and dim$(\Lambda(V)) = 2^n$.

Definition 4.2. *Let α_v be the creation operator on $\Lambda(V)$ defined by:*

$$\alpha_v(v_1 \wedge ... \wedge v_r) = v \wedge v_1 \wedge ... \wedge v_r \text{ and } \alpha_v(\Omega) = v$$

Reminder 4.3. *The dual $\alpha_v^\star$ is called the annihilation operator, then:*

$$\alpha_v^\star(v_1 \wedge ... \wedge v_r) = \textstyle\sum_{i=0}^{r} (-1)^{i+1}(v, v_i) v_1 \wedge ... v_{i-1} \wedge v_{i+1} \wedge ... \wedge v_r \text{ and } \alpha_v^\star(\Omega) = 0$$

Reminder 4.4. *Let $\gamma_v = \alpha_v + \alpha_v^\star$, then $\gamma_v = \gamma_v^\star$ and $\gamma_v\gamma_w + \gamma_w\gamma_v = 2(v,w)I$.*

Definition 4.5. *The operators γ_v generate the Clifford algebra Cliff(V).*
Note that the operators γ_v are bounded and that Cliff$(V).\Omega = \Lambda(V)$.

Remark 4.6. *V admits the orthonormal basis $(v_a)_{a \in I}$.*
We will write γ_a instead of γ_{v_a}, so that $[\gamma_a, \gamma_{a'}]_+ = 2\delta_{a,a'} I$.

Let Γ be a finitely presented group, with a triangularized presentation $\langle S \mid R \rangle$.

Definition 4.7. *For any irreducible block c, let $\Delta_c = \{b \in \bigcup \Gamma_n \mid \beta(b) = c\}$, with $\beta(b)$ defined on lemma 3.16.*

Definition 4.8. *Let b, c be blocks such that $b \Subset c \Subset \beta(b)$ then we write $b \propto c$.*
In this case, we see that $\beta(b) = \beta(c)$, so that $\propto$ is an order relation.

Lemma 4.9. *For any irreducible block c with $\Delta_c \neq \emptyset$, Δ_c admits a unique minimal element c_{min} with respect to $\propto$. Let m be the dimension of c_{min}; denote by I_c the set of blocks of dimension $m+1$ in Δ_c; then Δ_c is in one-to-one correspondence with the power set $\mathcal{P}(I_c)$; in particular, the cardinality of Δ_c is $2^{dim(c)-m}$ (see section 3 of Julg-Valette paper [4]).*

Definition 4.10. *We naturally identify $\ell^2(\Delta_c)$ with the exterior algebra $\Lambda(\ell^2(I_c))$ on which operates the Clifford algebra Cliff$(\ell^2(I_c))$ generated by γ_a, $a \in I_c$.*

5. Dirac operator

Definition 5.1. *We define the **n-block lenght** $\ell_n(.)$ as follows: let b be a block, then $\ell_n(b)$ is the minimal number of irreducible blocks decomposing a connected admissible n-dimensional block c with $e \Subset c$ and, $b \Subset c$ or $b \cap c \neq \emptyset$.*

Definition 5.2. *Let b be a block, then a sequence $(c_1, ..., c_{\ell_n(b)})$ with $b \Subset c_1$, $e \Subset c_{\ell(b)}$, $c_i \in \Gamma_n$ and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic n-block-path**, from b to e beginning with c_1.*

Definition 5.3. *For any irreducible block c with $\Delta_c \neq \emptyset$, let $n = dim(c_{min})+1$; for any $a \in I_c$ define $p_a(c)$ the number of geodesic n-block path from c_{min} to e beginning with a; let $p(c) = \sum_{a \in I_c} p_a(c)$; let $\lambda_a = \frac{p_a(c)}{p(c)} \ell_n(c_{min})$.*

Definition 5.4. *On $\ell^2(\Delta_c) = \Lambda(\ell^2(I_c))$, define the Dirac operator D_c by:*

$$D_c = \sum_{a \in I_c} \lambda_a . \gamma_a$$

Remark 5.5. *$\Delta_e = \{e\}$, $I_e = \emptyset$ and $D_e = 0$.*

Definition 5.6. *Consider then the Hilbert space:*

$$\mathcal{H} = \bigoplus_n \ell^2(\Gamma_n) = \bigoplus_c \ell^2(\Delta_c) = \bigoplus_c \Lambda(\ell^2(I_c))$$

$\mathbb{Z}_2$-graded by the decomposition into even and odd dimensional blocks.
Define the unbounded selfadjoint operator $\mathcal{D} = \bigoplus_c D_c$.

Lemma 5.7. *$\mathcal{D}^2 = \bigoplus_c D_c^2 = \sum_c (\sum_{a \in I_c} \lambda_a^2).p_c$ with p_c, projection on $\ell^2(\Delta_c)$.*

Proposition 5.8. *D is a Fredholm operator of index 1.*

Proposition 5.9. *$(\mathcal{D}^2 + I)^{-1}$ is compact.*
For $t > 0$, the operator $e^{-t\mathcal{D}^2}$ is trass-class.

Definition 5.10. *For any $g \in \Gamma$ and for any $s \in S$ define $p_s(g)$ the number of geodesic 1-block path from g to e beginning with $\{g, gs\}$; let $p(g) = \sum_{s \in S} p_s(g)$.*

Definition 5.11. *Let $\mathcal{C}$ be the class of finitely presented groups Γ admitting a triangularized finite presentation $\langle S \mid R\rangle$ such that $\forall g \in \Gamma$, $\exists K_g \in \mathbb{R}_+$ such that $\forall s \in S$ and $\forall h \in \Gamma$ (with $h, gh \neq e$):*

$$|\frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)}| \leq \frac{K_g}{\ell_1(h)}$$

Examples 5.12. *The class $\mathcal{C}$ is stable by direct or free product, it contains $\mathbb{Z}^n$, $\mathbb{F}_n$, the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see* [2]*).*

Proposition 5.13. *Let Γ of class $\mathcal{C}$, $\mathcal{A} = C_r^\star(\Gamma)$ and $\mathcal{D}$ as previously then: $\{a \in \mathcal{A} \mid [\mathcal{D}, a] \in B(H)\}$ is dense in $\mathcal{A}$.*

Theorem 5.14. *$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even θ-summable spectral triple and $\mathcal{D}$ is index 1. It then gives a non-trivial element for the K-homology of $\mathcal{A}$.*

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