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Some inverse problems around the tokamak *Tore Supra*

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Abstract

We consider two inverse problems related to the tokamak *Tore Supra*. The first one deals with the Cauchy issue of recovering in a two dimensional annular domain boundary magnetic values on the inner boundary, namely the limiter, from available overdetermined data on the outer boundary, for solutions to the Grad-Shafranov equation. Using tools from complex analysis and properties of generalized Hardy spaces, we establish stability and existence properties. Secondly the inverse problem of recovering the shape of the plasma is addressed thanks to tools of shape optimization. Here too, results about existence and optimality are provided. They give rise to a fast algorithm of identification which is applied to several numerical simulations computing good results either for the classical harmonic case or for the data coming from *Tore Supra*.

Keywords: Hardy Spaces, Bounded Extremal Problem, Conjugate Harmonic Function, Inverse Problems, Shape Optimization, Least Square Problems.

AMS classification: Primary: 30H10, 49J20, 65N21; Secondary: 30H05, 42A50, 35N05.

1 Introduction

A very challenging potential application of inverse boundary problems for elliptic equations in doubly connected domains is related to plasma confinement for thermonuclear fusion in a tokamak [11].

First of all the tokamak is a concept invented in the Soviet Union and introduced in the 1950s. The meaning of the word is, after translation from the Russian words, “toroidal chamber with magnetic coils” [39] and simply refers to a magnetic confinement device with toroidal geometry (see Figure 1). Today, tokamaks are the most used devices for the fusion experiments and needless to say represent the most suitable approach for their

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control. In the present paper we focus on the case of the tokamak *Tore Supra* built at the CEA/IRFM Cadarache (France).

In order to control the plasma position by applying feedback control, it is necessary to know its position in a very short time, or in other words in a time which has to be smaller than the sampling frequency of the plasma shape controller (some microseconds). As a consequence, the poloidal flux function has to be fast identified and computed in a very effective way since its knowledge in the domain included between the exterior wall of the tokamak and the boundary of the plasma is sufficient to determine completely the plasma boundary. In so far as the only data available are the ones obtained by measurements uniformly distributed on the entire exterior wall of the tokamak, namely the poloidal flux function and its normal derivative, the identification of the plasma boundary may be viewed as the solution of a free boundary problem.

The computation of the poloidal flux function in the vacuum and the recovery of the plasma shape have already been extensively studied (see for example [11]). Consequently, several numerical codes have been developed and we mention here the code Apollo actually used for the tokamak *Tore Supra* [36] based on an expansion of Taylor and Fourier types for the flux. Others expansions may be found in the literature such as the one making use of toroidal harmonics involving Legendre functions [3]. Naturally, those series expansions are truncated for computations and the coefficients are determined so that they fit to the measurements.

The purpose of this paper is to formulate an alternative and original approach of the resolution of the free boundary problem consisting in combining tools of both Complex Analysis and Shape Optimization. Up to the knowledge of the authors, such an approach has never been carried out before.

Firstly, we are interested in the following inverse problem : assume that the domain under study, denoted by Ω_l , is the vacuum located between the outer boundary Γ_e of the tokamak and the limiter Γ_l (see Section 2.2) in such a way that it refers to a fixed annular domain, or more precisely, to a conformally equivalent doubly-connected domain. The poloidal flux and its normal derivative being given on Γ_e , we want to recover those magnetic data on Γ_l inside the device. This amounts to solve a Cauchy problem from overdetermined data on part of the boundary. This problem is known to be ill-posed since the work of Hadamard. However sufficient conditions on the available data on Γ_e , together with *a priori* hypotheses on the missing data, may be provided for continuity and stability properties to hold. In order to deal with these constraints, a link is established between the equation satisfied by the flux, namely the Grad-Shafranov equation [37], and the conjugate Beltrami equation [8]. As a result of this part, we will be able to provide the flux and its normal derivative everywhere in the domain Ω_l then more particularly on the limiter Γ_l itself. Lastly, the knowledge of the maximum value of the flux on this boundary may be helpful for the initialization of the following part treating with the shape of the plasma.

In a second time, we investigate the inverse problem consisting in recovering the shape of the plasma inside the tokamak, from the knowledge of Cauchy data on the external boundary Γ_e (or on Γ_l if the resolution of the inverse problem of last paragraph has been

performed first). Indeed, if the plasma domain were known, the poloidal flux u inside the tokamak would be determined until the boundary of the plasma Γ_p by an elliptic partial differential equation, namely the homogeneous Grad-Shafranov equation, with Cauchy data on Γ_e . The question of the global existence of a solution for such a system is open and probably quite difficult. The point of view developed in this paper does not need such a result. Indeed, given that the shape of the plasma is a level set of the poloidal flux u , the question of the determination of the shape of the plasma inside the tokamak comes down intrinsically to solve an overdetermined partial differential equation. Noticing that, in general, an overdetermined condition can be interpreted as the first necessary optimality conditions of a Shape Optimization problem (see for instance [2, 27]), we chose to see the shape of the plasma as a minimizer of a shape functional in a given class of admissible domains. Furthermore, we use a standard shape gradient algorithm to compute numerically the boundary Γ_p .

The overview of the article is as follows. Section 2 is devoted to general notations and a description of a model of plasma equilibrium. The equations governing this latter allow to characterize the boundary Γ_p of the plasma as a level line of the flux in the domain bounded by the limiter. A breve description of the geometry of *Tore Supra* is provided too. Our main existence and stability results for the Cauchy problem explained above are stated in Section 3. We introduce some generalized harmonic functions associated with the problem, which in fact belong to generalized Hardy spaces of an annulus. A useful density result in such classes is given which enables the resolution of the Cauchy problem formulated as a bounded extremal one. Finally, in Section 4, we introduce a Shape Optimization problem whose solution is supposed to be the shape of the plasma. We first prove the existence of minimizers for such a problem and state the associated necessary first order optimality conditions. In particular, it is shown that, under a regularity assumption for the optimum, the solution of this problem verifies the system characterizing the shape of the plasma. Some numerical simulations, based on an optimization algorithm and the use of the shape derivative, are included at the end of the paper and permit to recover in a satisfying way the boundary Γ_p of the domain occupied by the plasma.

2 A model of plasma equilibrium

2.1 The Grad-Shafranov equation

We denote by (r, φ, z) the three-dimensional cylindrical coordinates system where r is the radial coordinate, φ is the toroidal angle and z is the height; $\vec{e}_r, \vec{e}_\varphi$ and \vec{e}_z will denote the axis unit vectors. Thus, given a generic vector \vec{A} , its component along the unit vectors will be denoted by A_r, A_φ and A_z , respectively, so as to have

$$\vec{A} = A_r \vec{e}_r + A_\varphi \vec{e}_\varphi + A_z \vec{e}_z.$$

Since the tokamak is an axisymmetric toroidal device, it is possible to assume that all magnetic quantities do not depend on the toroidal angle φ . That involves that a plasma equilibrium may be studied in any cross section (r, z) , named poloidal section (see Figure

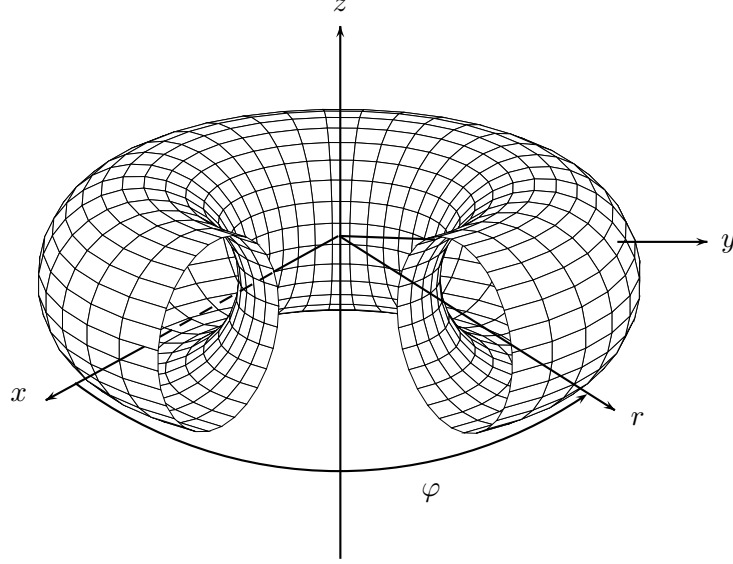


Figure 1: Toroidal geometry of the tokamak Tore Supra.

1). Now writing \vec{B} for the magnetic induction field and u for the poloidal magnetic flux, and denoting by $\nabla \cdot \vec{B} = \frac{\partial B_r}{\partial r} + \frac{\partial B_z}{\partial z}$ and $\nabla u = (\frac{\partial u}{\partial r}, \frac{\partial u}{\partial z})$, it is easy to derive from Maxwell's equations [11], especially from Gauss's law $\nabla \cdot \vec{B} = 0$, the following equations

$$B_r = -\frac{1}{r} \frac{\partial u}{\partial z} \quad \text{and} \quad B_z = \frac{1}{r} \frac{\partial u}{\partial r}. \quad (1)$$

The equilibrium of the plasma is also governed by its own equation

$$\nabla p = \vec{j} \times \vec{B} \quad (2)$$

resulting from the balance between the force of the kinetic pressure p and the one of the magnetic pressure, where \vec{j} stands for the current density field. Then it is a straightforward consequence of (1) and (2) that u satisfies the Grad-Shafranov equation in the whole space [37], so *a fortiori* inside the tokamak,

$$\nabla \cdot \left(\frac{1}{r} \nabla u \right) = j_T \quad (3)$$

for j_T the toroidal component of \vec{j} . Remark that $j_T = 0$ in the vacuum and that in

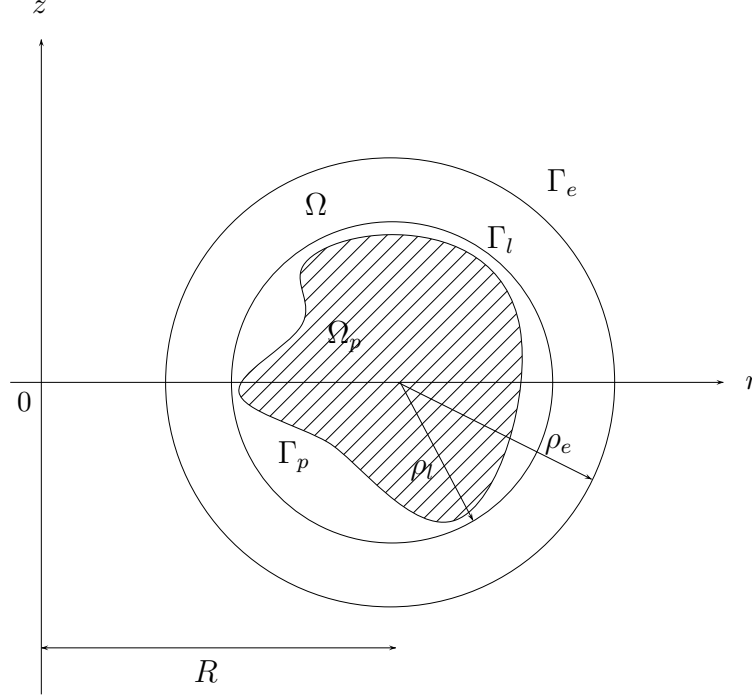


Figure 2: Schematic representation of a poloidal section of Tore Supra.

this case (3) leads to a homogeneous second order elliptic equation in divergence form in two dimensions.

2.2 The geometry of *Tore Supra*

We present here the main features of the tokamak *Tore Supra* built at CEA/IRFM Cadarache in France in a poloidal cross section (r, z) . All numeric quantities for the distances are given in meters.

Here Γ_e stands for the outer boundary of the tokamak. It corresponds to a circle with center $(R, 0) = (2.42, 0)$ and radius $\rho_e = 0.92$. Inside the tokamak, the first material piece encountered is the limiter Γ_l . It corresponds to a closed curve and its goal is to prevent any interaction between the plasma and Γ_e . In a nutshell, the plasma cannot pass through Γ_l . Inside the limiter stands the plasma, referring to the domain Ω_p with a boundary $\partial\Omega_p = \Gamma_p$.

Furthermore we add two notations. In the following we will denote by Ω_v the domain occupied by the vacuum, i.e the one included between Γ_p and Γ_e . Moreover Ω_l will denote

the domain included between Γ_l and Γ_e . Then referring to (3), the equation satisfied by the poloidal magnetic flux u in Ω_v , whence *a fortiori* in Ω_l , is $\nabla \cdot (\frac{1}{r} \nabla u) = 0$.

It should be noticed that in *Tore Supra*, the domain Ω_l is not exactly an annulus as it can be observed on Figure 6. But as explained in [8, Section 6], all properties of functions solutions to (5) below are preserved by composition with conformal maps. Accordingly, the results of the next section will always refer to a annular domain like in Figure 2.

2.3 Definition of the plasma boundary Γ_p

In Section 2.1, a basic condition for the plasma equilibrium was given through (2). This latter implies both following equations

$$\vec{B} \cdot \nabla p = 0 \quad \text{and} \quad \vec{j} \cdot \nabla p = 0.$$

Those equations show that the field lines of the magnetic induction and of the current density lie on isobaric surfaces also called magnetic surfaces. But from (1) we still get

$$\vec{B} \cdot \nabla u = 0,$$

so that the magnetic surfaces correspond to constant poloidal surfaces too. Then the plasma boundary Γ_p is defined as the outermost closed magnetic surface when starting from the center of Γ_e . For the tokamak *Tore Supra*, Γ_p is inside the limiter Γ_l . In fact, two situations can happen for the plasma equilibrium in tokamak devices:

- Γ_p and Γ_l have one common point. In this case, the contact point is unique and the equipotential of u corresponding to Γ_p is tangent to Γ_l . This is the limiter configuration.
- Γ_p and Γ_l have no point in common. In this case, the plasma is strictly included inside the delimiter and qualified as divertor.

Section 3 makes no distinction between both cases unlike Section 4 which focuses only on the limiter case for numerical simulations involving the identification of the plasma boundary in *Tore Supra*. We shall make use of this observation without further notice.

3 How to recover magnetic data in the tokamak *Tore Supra* ?

Let Ω_l be the bounded open domain in \mathbb{R}^2 with C^1 boundary mentioned in Section 2.2 and

$$\sigma(r, z) = \frac{1}{r} \in W^{1,\infty}(\Omega_l; \mathbb{R}) \quad \text{such that} \quad 0 < C_1 \leq \sigma(r, z) \leq C_2 \quad \text{a.e in } \Omega_l \quad (4)$$

for two constants $0 < C_1 < C_2 < +\infty$. The elliptic equation we look at is (3) in the vacuum, i.e

$$\nabla \cdot \left(\frac{1}{r} \nabla u \right) = 0 \quad \text{a.e. in } \Omega_l. \quad (5)$$

Here the differential operator $\nabla \cdot$ and ∇ are understood in the sense of distributions (with respect to the real variables r, z in \mathbb{R}^2). In this section we are dealing with the Cauchy problem described in the introduction. This problem is, as defined by Hadamard, ill-posed for magnetic data u and $\partial_n u$ prescribed on I not only in Sobolev spaces $W^{1-1/p,p}(\Gamma_e)$, $1 < p < +\infty$ but also in $L^p(\Gamma_e)$, where they still make sense [8]. However, well-posedness may be ensured if L^2 -norm constraints is added on Γ_l . It follows that the extrapolation issue from boundary data turned out as a best approximation one on Γ_e , still named bounded extremal problem (BEP).

Note that from now on we restrict ourselves to the particular Hilbertian framework, i.e for $p = 2$, in order to deal with any possible boundary data on Γ_e having finite energy.

Remark 1. When σ is constant, this problem reduces to solving a Cauchy problem for the Laplace operator, that is to say to recover the values of a holomorphic function in a domain of analyticity from part of its boundary values. It is well worth noting that this problem has been extensively studied in Hardy spaces on simply and doubly connected domains (see [6, 7, 23]).

3.1 From Grad-Shafranov equation to conjugate Beltrami equation

To study the Cauchy problem for (5), our approach proceeds via a complex elliptic equation of the first order, namely the conjugate Beltrami equation:

$$\bar{\partial}f = \nu \bar{\partial}f \quad \text{a.e in } \Omega_l, \quad (6)$$

where $\nu \in W^{1,\infty}(\Omega_l; \mathbb{R})$ satisfies

$$\|\nu\|_{L^\infty(\Omega)} \leq \kappa \quad \text{for some } \kappa \in (0, 1) \quad (7)$$

and where we use the standard notations

$$\partial f = \frac{1}{2} \left(\frac{\partial}{\partial r} - i \frac{\partial}{\partial z} \right) f \quad \text{and} \quad \bar{\partial} f = \frac{1}{2} \left(\frac{\partial}{\partial r} + i \frac{\partial}{\partial z} \right) f.$$

From this point, it is easy to verify that (6) decomposes into a system of two real elliptic equations of the second order in divergence form. Indeed, assume first that $f = u + iv$ is a solution to (6) with real valued u and v . Putting this in (6) yields that u satisfies (5) while v satisfies

$$\nabla \cdot \left(\frac{1}{\sigma} \nabla v \right) = 0 \quad \text{a.e. in } \Omega_l \quad (8)$$

with $\sigma = \frac{1-\nu}{1+\nu}$. Moreover, from that definition of σ , we obtain that (7) implies (4). Conversely, assume that u is real valued and satisfies (5). By putting $\nu = \frac{1-\sigma}{1+\sigma}$, a simple computation shows that (6) is equivalent to the system

$$\begin{cases} \frac{\partial v}{\partial r} = -\sigma \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial z} = \sigma \frac{\partial u}{\partial r} \end{cases} \quad \text{a.e. in } \Omega_l. \quad (9)$$

This system admits a solution when Ω is simply connected. Indeed the differential form defined by $W = (W_1, W_2) = (\sigma \frac{\partial u}{\partial r}, -\sigma \frac{\partial u}{\partial z})$ satisfies $\frac{\partial W_1}{\partial r} = \frac{\partial W_2}{\partial z}$. Consequently it is closed and applying Poincaré's Lemma yields the existence of a real valued function v , unique up to an additive constant, such that $f = u + iv$ satisfies (6) with the definition of ν given above. Again, conditions (4) implies (7).

Note that when Ω_l is doubly connected, such like in the situation of Tore Supra, the existence of v as a real single-valued function may only be local. Indeed, from Green's formula applied to u and to any constant function in Ω_l , we get

$$0 = \int_{\Omega_l} \nabla \cdot (\sigma \nabla u) = \int_{\partial \Omega_l} \sigma \frac{\partial u}{\partial n} = \int_{\Gamma_e} \sigma \frac{\partial u}{\partial n} + \int_{\Gamma_l} \sigma \frac{\partial u}{\partial n}.$$

But, since (9) may be rewritten on $\partial \Omega_l$ as $\frac{\partial v}{\partial t} = \sigma \frac{\partial u}{\partial n}$ where $\frac{\partial}{\partial t}$ stands for the tangential partial derivative on $\partial \Omega_l$, it holds that

$$- \int_{\Gamma_e} \sigma \frac{\partial u}{\partial n} = \int_{\Gamma_l} \frac{\partial v}{\partial t}.$$

Hence v is clearly multiplied-valued if $\int_{\Gamma_e} \sigma \frac{\partial u}{\partial n} \neq 0$ and $f = u + iv$ *a fortiori* too. However it is always possible to define u as the real part of a single valued function f in Ω_l . See for example [23] where the holomorphic case is processed by the aid of the logarithmic function (well-known to be multiplied-valued in the unit disk). We only mention, without giving more details, that our situation may be similarly solved by making use of toroidal harmonics [4].

3.2 Generalized Hardy spaces

Assume that $\nu \in W^{1,\infty}(\Omega_l; \mathbb{R})$ verifies (7). When handling boundary data in the fractional Sobolev space $W^{1/2,2}(\partial \Omega_l)$, it is a result from [13] that the Dirichlet problem for (6) admits a unique solution in $W^{1,2}(\Omega_l)$. But as mentioned before, the assumptions on the boundary data are relaxed in a manner that they now belong to the Lebesgue space $L^2(\partial \Omega_l)$. In this case, it is obvious that the solution of the Dirichlet problem for (6) has no more reason to belong to $W^{1,2}(\Omega_l)$ in general. On the other hand the Dirichlet problem will rather have a solution in some generalized Hardy spaces whose definition will follow.

This focuses the attention on the fact that those generalized Hardy spaces are the natural spaces when trying to solve inverse problem linked to (6) for L^2 boundary data (see [20] for the simply connected case). That we only dispose of Dirichlet conditions on Γ_e is an additional motivation to formulate the Cauchy problem as a bounded extremal problem which admits a unique solution in the appropriate Hardy classes (see Section 3.3).

Let us denote by $\mathbb{D}_{R,\rho}$ and $\mathbb{T}_{R,\rho}$ the disk and the circle centered at $(R, 0)$ of radius ρ . Remember that from now Ω_l stands for the annular domain shown in Figure 2. Thus we have $\mathbb{T}_{R,\rho_e} = \Gamma_e$.

Definition 3.1. If $\nu \in W^{1,\infty}(\Omega_l; \mathbb{R})$, we denote by $H_\nu^2(\Omega_l)$ the generalized Hardy space which consists in Lebesgue measurable functions f on Ω_l , solving (6) in the sense of

distributions in Ω_l and satisfying

$$\|f\|_{H_\nu^2(\Omega_l)} := \operatorname{ess\,sup}_{\rho_l < \rho < \rho_e} \|f\|_{L^2(\mathbb{T}_{R,\rho})} < +\infty \quad (10)$$

where

$$\|f\|_{L^2(\mathbb{T}_{R,\rho})} := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(R + \rho e^{i\theta})|^2 d\theta \right)^{1/2}$$

Equipped with the norm defined by (10), $H_\nu^2(\Omega_l)$ is a Hilbert space. Moreover (10) implies $f \in L^2(\Omega_l)$.

We remark immediately that when $\nu = 0$ and $\Omega = \mathbb{D} = \mathbb{D}_{0,1}$, $H_\nu^2(\Omega_l)$ is nothing but the classical $H^2(\Omega_l)$ space of holomorphic functions on the unit disk bounded in norm L^2 on $\mathbb{T} = \mathbb{T}_{0,1}$ (see [19, 24]). Recall just that it consists in the functions $f \in L^2(\mathbb{T})$ which Fourier coefficients of negative order vanish. Most of the properties of generalized Hardy spaces on simply connected domains derive from those of the classical H^2 spaces and still hold for multiply connected domains. Basically, the main idea relies on the connection between functions $f \in H_\nu^2(\Omega_l)$ and functions ω satisfying in the distributional sense, for $\alpha \in L^\infty(\Omega_l)$,

$$\bar{\partial}\omega = \alpha\bar{\omega} \quad \text{on } \Omega_l \quad (11)$$

and such that the condition (10) still holds. That connection has been introduced in [10] and leads to

Proposition 1. *Let $\nu \in W^{1,\infty}(\Omega_l; \mathbb{R})$ satisfy (7) and define $\alpha \in L^\infty(\Omega_l)$ by*

$$\alpha = -\frac{\bar{\partial}\nu}{1-\nu^2} = \bar{\partial} \log \left[\frac{1-\nu}{1+\nu} \right]^{1/2} = \bar{\partial} \log \sigma^{1/2}$$

then $f = u + iv \in H_\nu^2(\Omega_l)$ if and only if $\omega = \frac{f - \nu\bar{f}}{\sqrt{1-\nu^2}} = \sigma^{1/2}u + i\sigma^{-1/2}v$ solves (11) with condition (10).

Proof. The proof is a straightforward computation. Indeed assume $f \in H_\nu^2(\Omega_l)$. Then from $\omega = \frac{f - \nu\bar{f}}{\sqrt{1-\nu^2}}$ we get

$$\bar{\partial}\omega = \frac{\nu\bar{\partial}\nu}{(1-\nu^2)^{3/2}}(f - \nu\bar{f}) + \frac{1}{(1-\nu^2)^{1/2}}(\bar{\partial}f - \nu\bar{\partial}\bar{f} - \bar{f}\bar{\partial}\nu).$$

Insofar as $f \in H_\nu^2(\Omega_l)$, it is true that $\bar{\partial}f - \nu\bar{\partial}\bar{f} = 0$. Hence

$$\begin{aligned} \bar{\partial}\omega &= \frac{\nu\bar{\partial}\nu}{(1-\nu^2)^{3/2}}(f - \nu\bar{f}) - \frac{1}{(1-\nu^2)^{1/2}}\bar{f}\bar{\partial}\nu \\ &= -\frac{\bar{\partial}\nu}{(1-\nu^2)^{3/2}}(\bar{f} - \nu f) = -\frac{\bar{\partial}\nu}{1-\nu^2}\omega, \end{aligned}$$

in other words $\bar{\partial}\omega = \alpha\bar{\omega}$. From the definition of ω , it is obvious that (10) still holds for this latter. By the same way, and noticing that $f = \frac{\omega + \nu\bar{\omega}}{\sqrt{1-\nu^2}}$, the converse is valid. \square

Furthermore solutions of type ω , which satisfy condition (10), can be represented as $\omega = e^s g$ where s is continuous on $\bar{\Omega}_l$ and $g \in H^2(\Omega_l)$ [8, 10]. This remark allows to establish that on simply connected domains, classical and generalized Hardy spaces share similar properties and those latter are naturally exportable to multiply connected domains. We recall in the following result most of them

Proposition 2. *If $\nu \in W^{1,\infty}(\Omega_l; \mathbb{R})$ satisfies (7),*

1. *any function $f \in H_\nu^2(\Omega_l)$ has a non-tangential limit almost everywhere on $\partial\Omega_l = \Gamma_e \cup \Gamma_l$, called the trace of f and denoted by trf , which belongs to $L^2(\partial\Omega_l)$;*
2. *$\|trf\|_{L^2(\partial\Omega_l)}$ defines an equivalent norm on $H_\nu^2(\Omega_l)$;*
3. *$trH_\nu^2(\Omega_l)$ is closed in $L^2(\partial\Omega_l)$;*
4. *if $f \in H_\nu^2(\Omega_l)$, trf cannot vanish on a subset of $\partial\Omega_l$ with positive measure unless $f \equiv 0$;*
5. *each $f \in H_\nu^2(\Omega_l)$ satisfies the maximum principle, i.e $|f|$ cannot assume a relative maximum in Ω_l unless it is constant.*

We refer to [8] for the proofs in simply connected domains.

These properties are necessary to prove the results of next section, i.e. the density of traces of function in $H_\nu^2(\Omega_l)$ in $L^2(I)$ whenever I is a subset of non-full measure of the boundary $\partial\Omega_l = \Gamma_e \cup \Gamma_l$, which are the key point to solve extremal problems with incomplete boundary data.

3.3 Density results and bounded extremal problem

This section is devoted to the resolution of the Cauchy problem formulated in the introduction. The first step which needs to be established is a density result of fundamental importance which asserts that if $I \subset \partial\Omega_l$ has positive measure as well as $J = \partial\Omega_l \setminus I$, then every L^2 -complex function on I can be approximated by the trace of a function in $H_\nu^2(\Omega_l)$. Remember that in the situation under study in this paper, the subset I of the boundary $\partial\Omega_l$ corresponds *a priori* to the whole boundary Γ_e . But in order to ensure density results as well as better algorithm's convergence (see the end of Section 3.3), we restrict from now on the set of available magnetic data to a strict subset, still named I of the outer boundary Γ_e . We mention that the case where I is effectively the whole boundary Γ_e is already under study.

Thus the following result is a natural extension of [8, Theorem 4.5.2.1] for an annular domain.

Theorem 3.2. *Let $I \subset \partial\Omega_l = \Gamma_e \cup \Gamma_l$ be a measurable subset such that $J = \partial\Omega_l \setminus I$ has positive Lebesgue measure. The restrictions to I of traces of H_ν^2 -functions are dense in $L^2(I)$.*

From the magnetic data on I , one can compose the function $F_d = u + iv$ where $v = \int_{\Gamma_e} \sigma \frac{\partial u}{\partial n}$ according to the formulation of (9) on $\partial\Omega_l$. As a consequence of Theorem 3.2, it exists a sequence of functions $(f_n)_{n \geq 0} \in H_\nu^2(\Omega_l)$ which trace on I converges to F_d in $L^2(I)$. Suppose now that $\lim_{n \rightarrow +\infty} \|tr f_n\|_{L^2(J)} \neq +\infty$. Up to extracting a subsequence, we may assume that $(tr f_n)_{n \geq 0}$ is bounded in $L^2(J)$. But $(tr f_n)_{n \geq 0}$ is already bounded in $L^2(I)$ by assumption. Consequently it remains bounded in $L^2(\partial\Omega_l)$ and then in $H_\nu^2(\Omega_l)$ by point 2 of Proposition 2. As $H_\nu^2(\Omega_l)$ is a Hilbert space, there exists a subsequence $(tr f_{n_p})_{p \geq 0}$ converging weakly toward some $f \in H_\nu^2(\Omega_l)$. Now, by restriction, it is obvious that $(tr f_{n_p}|_I)_{p \geq 0}$ converges weakly to $f|_I$ in $L^2(I)$. Since F_d is the strong limit of $(tr f_n)_{n \geq 0}$ in $L^2(I)$, we conclude that $F_d = f$ a.e on I . For this reason F_d can either already be the trace on I of a H_ν^2 -function, or $\|tr f_n\|_{L^2(J)} \rightarrow +\infty$ as $n \rightarrow +\infty$. This behaviour of the approximant on J shows that the Cauchy solution is unstable, which in fact explains that the inverse problem is ill-posed. Note that in general the function F_d does not coincide with the trace of a function belonging to $H_\nu^2(\Omega_l)$ insofar as magnetic data proceeds from sensor subject, like for all mechanical engineering, to roundoff errors.

Therefore, in order to prevent such an unstable behaviour, an upper bound for the L^2 norm of the approximation on J will be add. Thus the Cauchy problem may be expressed as bounded extremal problem corresponding to a problem of approximation of incomplete data in generalized Hardy spaces. In practical terms, define, for $M > 0$ and $\phi \in L^2(J)$,

$$\mathcal{B}_{\Omega_l} = \{g \in tr H_\nu^2(\Omega_l); \|Re g - \phi\|_{L^2(J)} \leq M\}_{|I} \subset L^2(I).$$

In light of this definition, it is now possible to find a unique solution for the minimization problem formulated on the class \mathcal{B}_{Ω_l} .

Theorem 3.3. *Fix $M > 0, \phi \in L^2(J)$. For every function $F_d \in L^2(I)$, there exists a unique solution $g^* \in \mathcal{B}_{\Omega_l}$ such that*

$$\|F_d - g^*\|_{L^2(I)} = \min_{g \in \mathcal{B}_{\Omega_l}} \|F_d - g\|_{L^2(I)}.$$

Moreover, if $F_d \notin \mathcal{B}_{\Omega_l}$, the constraint is saturated, i.e $\|Re g^* - \phi\|_{L^2(J)} = M$.

Proof. Since \mathcal{B}_{Ω_l} is clearly convex and the norm $\|\cdot\|_{L^2(\partial\Omega_l)}$ lower semi-continuous, it is enough to show that \mathcal{B}_{Ω_l} is closed in $L^2(I)$ to ensure the existence and the uniqueness of g^* . Indeed, there is a best approximation on any closed convex subset of a uniformly convex Banach space which is in this case $L^2(I)$ [9, Part 3, Chapter II, 1, Propositions 5 and 8].

Let $(\varphi_k|_I)_{k \geq 0} \in \mathcal{B}_{\Omega_l}$, $\varphi_k|_I \rightarrow \varphi_I$ in $L^2(I)$ as $k \rightarrow \infty$. Put $u_k = Re \varphi_k$. By assumption, $(u_k)_{k \geq 0}$ is bounded in $L^2(\partial\Omega_l)$. Then the application of norm's inequalities contained in [8, Theorem 4.4.2.1] shows that $\|\varphi_k\|_{H_\nu^2(\Omega_l)} \leq c_\nu \|u_k\|_{L^2(\partial\Omega_l)}$. Hence $(\varphi_k)_{k \geq 0}$ is bounded in $L^2(\partial\Omega_l)$ and, up to extracting a subsequence, weakly converges to $\psi \in L^2(\Omega_l)$; necessarily $\psi|_I = \varphi_I$.

Next, $\varphi_k \in tr H_\nu^2(\Omega_l)$, which is weakly closed in $L^2(\partial\Omega_l)$ (see discussion after Theorem 3.2); this implies that $\psi \in tr H_\nu^2(\Omega_l)$. Because $Re \varphi_k = u_k$ satisfies the constraint on J , so does $Re \psi$, whence $\varphi_I \in \mathcal{B}_{\Omega_l}$.

Let us now prove that, if $F_d \notin \mathcal{B}_{\Omega_l}$, then $\|Re g^* - \phi\|_{L^2(J)} = M$. Assume for a contradiction that $\|Re g^* - \phi\|_{L^2(J)} < M$. Since $\|F_d - g^*\|_{L^2(I)} > 0$, by Theorem 3.2, there is a function $h \in trH_\nu^2(\Omega_l)$ such that

$$\|F_d - g^* - h\|_{L^2(I)} < \|F_d - g^*\|_{L^2(I)}.$$

By making use of the triangle inequality,

$$\begin{aligned} \|F_d - g^* - \lambda h\|_{L^2(I)} &= \|\lambda(F_d - g^* - h) + (1 - \lambda)(F_d - g^*)\|_{L^2(I)} \\ &\leq |\lambda| \|F_d - g^* - h\|_{L^2(I)} + |1 - \lambda| \|F_d - g^*\|_{L^2(I)} \\ &< \|F_d - g^*\|_{L^2(I)} \end{aligned}$$

for all λ such that $0 < \lambda < 1$. Now, taking advantage of the assumption, we have that for $\lambda > 0$ sufficiently small $\|Re(g^* + \lambda h) - \phi\|_{L^2(J)} \leq M$. To recap, $g^* + \lambda h \in \mathcal{B}_{\Omega_l}$ and $\|F_d - (g^* + \lambda h)\|_{L^2(I_1)} < \|F_d - g^*\|_{L^2(I_1)}$. This contradicts the optimality of g^* . \square

An explicit formulation of g^* has been established in [1] in the case $\nu = 0$ which is still valid in the present situation. Indeed, denote by \mathcal{P}_ν the orthogonal projection from $L^2(\partial\Omega_l)$ onto $trH_\nu^2(\Omega_l)$ (the operator \mathcal{P}_ν is the natural extension of the classical Riesz projection from $L^2(\mathbb{T})$ onto $trH^2(\mathbb{D})$ [19]). The unique solution of the bounded extremal problem, when $F_d \notin \mathcal{B}_{\Omega_l}$, is given by

$$\mathcal{P}_\nu(\chi_{I_1} g^*) + \lambda \mathcal{P}_\nu(\chi_J Re g^*) = \mathcal{P}_\nu(\chi_{I_1} F_d) + \lambda \mathcal{P}_\nu(\chi_J \phi) \quad (12)$$

where $\lambda > 0$ is the unique (Lagrange-type) parameter such that the constraint on J is saturated, in other words $\|Re g^* - \phi\|_{L^2(J)} = M$ and χ_J is the characteristic function of J . Otherwise, the case where $F_d \in \mathcal{B}_{\Omega_l}$ corresponds to $\lambda = -1$. The behaviour with respect to λ of the error $e(\lambda) = \|F_d - g^*\|_{L^2(I_1)}^2$ and the constraint M is discussed in [1].

Observe that g^* may be computed constructively if a complete family of $trH_\nu^2(\Omega_l)$ is known. Indeed, the operator \mathcal{P}_ν can be expressed thank a conjugation operator which, for every function in $H_\nu^2(\Omega_l)$, associates the trace of $Re f$ to the trace of $Im f$. A formula for this latter may be found in [5]. Thus it is sufficient to expand any function in $L^2(\partial\Omega_l)$ on a basis for which the conjugation operator is easily computed. This method is described in [20] for simply connected domains.

Finally we briefly describe a algorithm for solving numerically the bounded extremal problem. It has been developed in [14] and consists in taking advantage of the fact that initially the magnetic data are available on the entire outer boundary Γ_e . As a first step compute the function F_d thank the magnetic data on Γ_e (as it is explained just after Theorem 3.2) and split the measurement set into two disjoint parts $\Gamma_e = I_1 \cup I_2$. Secondly, solve the bounded extremal problem with respect to $\chi_{I_1} F_d$ and a constraint $M_1 > 0$. One obtain a solution g_1^* which depends of M_1 . From this latter compute the real number $M_2 = \underset{M_1 > 0}{\text{Argmin}} \|g_1^* - F_d\|_{L^2(I_2)}$. And lastly solve the bounded extremal problem with respect to $\chi_{I_1} F_d$ and the constraint $M_2 > 0$. This cross validation procedure provides good results in the harmonic case, i.e. for the Laplace's equation.

We conclude this section by claiming that the solution g^* of the bounded extremal problem could be a starting point of the next section which focuses on the recovery of the plasma boundary by technics of Shape Optimization. Indeed g^* provides magnetic data, i.e. u and $\partial_n u$, at every point of Ω_l . It is then possible to initiate the free boundary problem with an outer boundary Γ_E located in Ω_l , which is then closer to the potential boundary Γ_p of the plasma. Moreover this new boundary Γ_E can still be Γ_l itself.

An other feature, under the assumption that the point of contact M_0 between the plasma and the limiter is known in advance, is to provide the constant $c = g^*(M_0)$ corresponding to the value of the flux at this point.

4 Recovering the shape of the plasma inside the tokamak *Tore Supra*

4.1 A Shape Optimization frame

We will use in this section the same notations as those summed up on Figure 2. In particular, Γ_e denotes the external boundary of a disk with given radius $R > 0$, representing the external wall of the tokamak *Tore Supra*. We are interested in the determination of the shape of the plasma inside the tokamak. In the following analysis, we will assume that the plasma is composed of one simply connected component, whose boundary is denoted Γ_p . Let us introduce Ω , the domain located between the two closed curves Γ_e and Γ_p . Physically, as it is underlined in Section 2.3, the only interesting case for our study is the limiter one. It means that it is reasonable to consider that one point of the boundary denoted by M_0 , where the plasma meets the toroidal belt limiter Γ_p , is known.

We have seen in Section 2.3 that the boundary Γ_p may be seen as a level set of the solution. In other words, we look for a domain Ω and its boundary $\Gamma_p = \partial\Omega \setminus \Gamma_e$ such that

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, & x \in \Omega \\ u = u_0, & x \in \Gamma_e \\ \frac{\partial u}{\partial n} = u_1, & x \in \Gamma_e \\ \Gamma_p = \{u = u(M_0)\}. \end{cases} \quad (13)$$

where u still denotes the poloidal magnetic flux inside the tokamak, ∇ denotes the *nabla* operator with respect to the variables (r, z) , and σ is a given analytic function such that $0 < C_1 < \sigma < C_2$ in Ω . Indeed, let us recall that in the case of the tokamak *Tore Supra*, one has (see Section 3.2)

$$\sigma(r, z) = \frac{1}{r} \text{ and } \Omega \subset \mathbb{D}_{R,\rho}.$$

Although a refined study has been led in Section 3 to obtain explicit weak solutions to (6), whence to (5), with boundary data u_0 and u_1 belonging to a larger space as usually (the trace of a generalised Hardy space), and since we want to use standard technics of Shape Optimization, we make the classical assumptions on the boundary data, that is $u_0 \in W^{1/2,2}(\Gamma_e)$ and $u_1 \in W^{-1/2,2}(\Gamma_e)$.

In other words, Ω can be seen as the solution, if it exists, of the “free boundary problem”

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, & x \in \Omega \\ u = u_0, & x \in \Gamma_e \\ \frac{\partial u}{\partial n} = u_1, & x \in \Gamma_e \\ u = c & x \in \Gamma_p, \end{cases} \quad (14)$$

where $c > 0$ is a constant chosen so that $M_0 \in \Gamma_p$. Written under this form, this problem looks like a free boundary problem and we could have a temptation to apply Holmgren’s uniqueness or a Cauchy-Kowalewski’s type theorem. Nevertheless and to our knowledge (see for instance [32]), until yet, that is only possible to deduce from these theorems, the existence of a neighborhood of the external boundary Γ_e where is defined a local solution of the three first equations of (14), but the existence of a closed level set Γ_p inside the domain delimited by Γ_e seems to be a difficult question.

Notice that a close family of “free boundary problems” have been studied in the past, referring to overdetermined like problems with additional Bernoulli type boundary condition. However, in these problems, the Bernoulli condition applies to the free boundary and permits in general to ensure some kind of regularity of this latter and its uniqueness. The case presented in this section is quite unusual since the overdetermined condition applies on the fixed boundary Γ_e . About the literature on free boundary problem with a Bernoulli condition, we refer to [21, 25, 26, 28, 29, 30, 31, 33].

Furthermore and with respect to the physical experiments, we will make the following assumptions:

$$u_1 < 0 \text{ and } u_0 < c \text{ a.e. on } \Gamma_e, \quad (15)$$

coming from the physical frame. Let us notice that, thanks to these additional informations, the application of Hopf’s maximum principle yields

$$\min_{\bar{\Omega}} u = \min_{\Gamma_e} u_0 \text{ and } \frac{\partial u}{\partial n}(\operatorname{argmin} u) = u_1(\operatorname{argmin} u) < 0.$$

In the same way, one has also

$$\max_{\bar{\Omega}} u = \max_{\Gamma_p} u = c, \text{ and } \frac{\partial u}{\partial n}|_{\Gamma_p} > 0.$$

For the reasons mentioned before, we decided to see this “free boundary problem” as a Shape Optimization one. Indeed, let us introduce, Ω and $c > 0$ being fixed, the solution u_Ω of the following mixed Dirichlet-Neumann elliptic problem

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, & x \in \Omega \\ \frac{\partial u}{\partial n} = u_1, & x \in \Gamma_e \\ u = c & x \in \Gamma_p, \end{cases} \quad (16)$$

Let us denote by D the closure of $\mathbb{D}_{R,\rho}$, i.e. the compact set which boundary is Γ_e . If the capacity of the compact set $D \setminus \Omega$ is strictly positive and u_1 belongs to $W^{-1/2,2}(\Gamma_e)$, then this problem has obviously a unique solution, by virtue of Lax-Milgram’s Theorem.

Now, a natural idea to solve the overdetermined system (16) consists in introducing a least square type shape functional J through the Shape Optimization problem

$$\begin{cases} \min J(\Omega) = \int_{\Gamma_e} (u - u_0)^2 ds \\ \Omega \in \mathcal{O}_{\text{ad}}, \end{cases} \quad (17)$$

where \mathcal{O}_{ad} denotes the set of admissible shapes that has to be clarified. The natural questions arising here are:

- How to choose \mathcal{O}_{ad} to ensure the existence of solutions for Problem (17)?
- In the case where Problem (17) has a solution Ω^* , can we guarantee that $J(\Omega^*) = 0$?

In the following section, we will bring some partial answers to these questions.

4.2 An existence result

This section is devoted first to the statement of an existence result for the Shape Optimization problem (17) and secondly to the writing of the first order necessary optimality conditions of this problem. We need to define some convergence notions for the elements of \mathcal{O}_{ad} , providing a topology within this class.

Definition 4.1. A sequence of domains $(\Omega_n)_{n \geq 0}$ is said

- converging to Ω for the Hausdorff convergence if

$$\lim_{n \rightarrow +\infty} d_H(D \setminus \Omega_n, D \setminus \Omega) = 0,$$

where $d_H(K_1, K_2) = \max(\rho(K_1, K_2), \rho(K_2, K_1))$, for any $(i, j) \in \{1, 2\}^2$, $\rho(K_i, K_j) = \sup_{x \in K_i} d(x, K_j)$, and $\forall x \in D$, $d(x, K_i) = \inf_{y \in K_i} d(x, y)$.

- converging to Ω in the sense of characteristic functions if for all $p \in [1, +\infty)$,

$$\chi_{\Omega_n} \xrightarrow[n \rightarrow \infty]{} \chi_{\Omega} \text{ in } L_{\text{loc}}^p(\mathbb{R}^2).$$

- converging to Ω in the sense of compacts if

1. $\forall K$ compact subset of D , $K \subset \Omega \Rightarrow \exists n_0 \in \mathbb{N}^*$, $\forall n \geq n_0$, $K \subset \Omega_n$.
2. $\forall K$ compact subset of D , $K \subset D \setminus \overline{\Omega} \Rightarrow \exists n_0 \in \mathbb{N}^*$, $\forall n \geq n_0$, $K \subset D \setminus \overline{\Omega_n}$.

It is in general a hard task to get existence results in Shape Optimization. Not only, many such problems are often ill-posed (see for instance [2, 27]), but if the class of admissible domains is too large, a minimizing sequence of domains may converge to a very irregular domain, for which the solution of the partial differential equation may exist in a very weak sense. For these reasons, a partial solution consists in restricting the class of admissible domains, by assuming some kind of regularity. For that purpose, let us define the notion of ε -cone property, introduced in [15].

Definition 4.2. Let y be a point of \mathbb{R}^2 , ξ a normalized vector and $\varepsilon > 0$. We denote by $C(y, \xi, \varepsilon)$, the unpointed cone

$$C(y, \xi, \varepsilon) = \{z \in \mathbb{R}^2, \langle z - y, \xi \rangle \geq \cos \varepsilon \|z - y\| \text{ and } 0 < \|z - y\| < \varepsilon\}.$$

We say that an open set Ω verifies the ε -cone property if

$$\forall x \in \partial\Omega, \exists \xi_x \in \mathbb{S}^1, \forall y \in \overline{\Omega} \cap B(x, \varepsilon), C(y, \xi_x, \varepsilon) \subset \Omega.$$

Let us make precise the class of admissible domains. For a given $\eta > 0$, we call D_η , the annular compact set $D \setminus B(0, \eta)$. For η and ε , two fixed strictly positive numbers, we define

$$\mathcal{O}_{\text{ad}}^{\varepsilon, \eta} = \{\Omega \text{ open subset of } D_\eta \text{ verifying the } \varepsilon \text{ cone property, } \Gamma_e \subset \partial\Omega, \#(D_\eta \setminus \Omega) \leq 1\},$$

where $\#(D_\eta \setminus \Omega)$ denotes the numbers of connected components of the complementary set of Ω in D_η , and we replace the class \mathcal{O}_{ad} by $\mathcal{O}_{\text{ad}}^{\varepsilon, \eta}$ in Problem (17). One has the following existence result.

Proposition 3. *Let η and ε be two fixed strictly positive numbers. Then, Problem (17) admits at least one solution.*

Proof. Let $(\Omega_n)_{n \geq 0}$ be a minimizing sequence for Problem (17) and let $\Gamma_{p,n}$ be the part of the boundary contained in Ω_n , that is $\partial\Omega_n \setminus \Gamma_e$. Since for any $n \in \mathbb{N}$, Ω_n is included in the compact set D_η , there exists Ω^* such that $(\Omega_n)_{n \geq 0}$ converges to Ω^* for the Hausdorff metric and in the sense of characteristic functions (see [27, Chapter 2]). Let us denote by Γ_p^* its internal boundary. Furthermore, since any Ω_n satisfies an uniform ε cone property, the sequence $(\Omega_n)_{n \geq 0}$ converges to Ω^* in the sense of compacts and necessarily, Ω^* satisfies the uniform ε -cone property [27, Theorem 2.4.10].

To prove the existence result, we need to prove at least the lower semi-continuity of the criterion J in the class $\mathcal{O}_{\text{ad}}^{\varepsilon, \eta}$.

Let us first prove that the sequence $(u_{\Omega_n})_{n \geq 0}$ is bounded in $W^{1,2}(D_\eta)$. Notice that due to the homogeneous Dirichlet boundary condition on the lateral boundary $\Gamma_{p,n}$, we can extend u_{Ω_n} by c outside $\Gamma_{p,n}$. So we can consider that the functions are all defined on the box D_η (or D) and the integrals over $\Gamma_{p,n}$ and over D_η (or D) will be the same. For that purpose, let us multiply Equation (16) by $u_{\Omega_n} - c$ and then integrate by part. From Green's formula, we get

$$C_1 \int_{\Omega_n} |\nabla u_{\Omega_n}|^2 \leq \int_{\Omega_n} \sigma |\nabla u_{\Omega_n}|^2 = \int_{\Gamma_e} \sigma u_1 (u_{\Omega_n} - c),$$

and the right hand side is uniformly bounded with respect to n , due to the fact that the sequence $(J(\Omega_n))_{n \geq 0}$ is convergent and hence bounded. Now, it remains to prove that the sequence (u_{Ω_n}) is bounded in $L^2(\Omega_n)$. This is actually a consequence of the non positivity of u_1 . Indeed, the use of Hopf's Theorem yields immediately that u_{Ω_n} attains its maximum value at $x_m \in \Gamma_e \cup \Gamma_{p,n}$ and that $\frac{\partial u_{\Omega_n}}{\partial n}(x_m) > 0$ and hence, $\max_{\overline{\Omega_n}} u_{\Omega_n} = u_{\Omega_n}|_{\Gamma_{p,n}} = c$. Let us introduce $u_{\eta/2}$, the solution of (16) set in the interior of $D_{\eta/2}$. A comparison between

u_{Ω_n} and $u_{\eta/2}$, using Hopf's maximum principle ensures that for any $n \in \mathbb{N}$, $u_{\eta/2} \leq u_{\Omega_n} \leq c$ a.e. in D . Thus, $\beta = \max(\|u_{\eta/2}\|_{L^2(D)}, \pi\rho^2c)$ is an uniform bound of the $L^2(D)$ norm of u_{Ω_n} , which proves the $W^{1,2}(D)$ boundedness of u_{Ω_n} , and shows at the same time that $(u_{\Omega_n})_{n \geq 0}$ is bounded in $W^{1,2}(D_\eta)$.

As a result, using Banach Alaoglu's and Rellich-Kondrachov's Theorems, there exists a function $u^* \in W^{1,2}(D_\eta)$ such that

$$u_{\Omega_n} \xrightarrow{W^{1,2}(D_{\eta/2})} u^* \text{ and } u_{\Omega_n} \xrightarrow{L^2(D_{\eta/2})} u^* \text{ as } n \rightarrow +\infty.$$

It remains to prove that u^* is the solution of Equation (16) on Ω^* . Let us notice that, once this assertion will be proved, the continuity of the shape functional J will immediately follow, because of the above convergence result. Let us write the variational formulation of (16). For any function φ satisfying $\varphi \in W^{1,2}(D_{\eta/2})$, $\varphi = 0$ on $\Gamma_e \cup \Gamma_p^*$, and for all $n \in \mathbb{N}$, the function u_{Ω_n} verifies

$$\int_D \sigma \nabla u_{\Omega_n} \cdot \nabla \varphi = \int_{\Gamma_e} \sigma u_1 \varphi. \quad (18)$$

It suffices to use the weak H^1 -convergence of u_{Ω_n} to u^* to show that u^* satisfies also (18). To conclude, it remains to prove that $u^* = 0$ on Γ_p^* . This is actually a consequence of the convergence of $(\Omega_n)_{n \geq 0}$ into Ω^* and the fact that Ω^* has a Lipschitz boundary. We refer to Theorem 2.4.10 and Theorem 3.4.7 in [27]. Finally, let us notice that the inclusion $\Gamma_e \subset \partial\Omega^*$ is verified. Indeed, it follows from the stability of the inclusion for the Hausdorff convergence. To be convinced, one way consist in considering that Γ_e is the boundary of an open set Σ that does not contain D , and to replace the constraint $\Gamma_e \subset \partial\Omega$ by $\Sigma \subset \tilde{\Omega}$, where $\tilde{\Omega} = \Omega \cup \Gamma_e \cup \Sigma$. Hence, the stability of the inclusion of open sets for the Hausdorff convergence applies and the conclusion follows. \square

Notice that the existence result do not guarantee that the solution u_{Ω^*} of (16) set in the optimal domain Ω^* satisfies the Cauchy conditions on Γ_e , that is the nonhomogeneous Dirichlet and Neumann boundary conditions on Γ_e .

4.3 First order optimality conditions

Let us now write the first order optimality conditions for Problem (17). For that purpose, we will first compute the shape derivative of the criterion J . The major difficulty in dealing with sets of shapes is that they do not have a vector space structure. In order to be able to define shape derivatives and study the sensitivity of shape functionals, we need to construct such a structure for the shape spaces. In the literature, this is done by considering perturbations of an initial domain; see [2, 27, 34, 38].

Practically speaking, it is preferable to enlarge the class of admissible domains to write the first order optimality conditions, even if there is a risk of loosing the existence result. Indeed, it would be else strongly difficult for instance to take into account the fact that any admissible domain must satisfy an ε -cone property. For this reason, let us define

$$\mathcal{O}_{\text{ad}} = \{\Omega \text{ open subset of } D, \#(D_\eta \setminus \Omega) \leq 1\}. \quad (19)$$

Let $\Omega \in \mathcal{O}_{\text{ad}}$ given. Let us consider a regular vector field $V : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with compact support, which does not meet the fixed boundary Γ_e . For small $t > 0$, we define $\Omega_t = (I + tV)\Omega$, the image of Ω by a perturbation of Identity and $f(t) := J(\Omega_t)$. We recall that the shape derivative of J at Ω with respect to V is

$$f'(0) = \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

We will denote it by $dJ(\Omega; V)$. Let us now compute this shape derivative. The classical formulae yield

$$dJ(\Omega; V) = 2 \int_{\Gamma_e} (u_\Omega - u_0) \dot{u} ds,$$

where \dot{u} denotes the shape derivative of u , and is solution of the system

$$\begin{cases} \nabla \cdot (\sigma \nabla \dot{u}) = 0, & x \in \Omega \\ \frac{\partial \dot{u}}{\partial n} = 0, & x \in \Gamma_e \\ \dot{u} = -\frac{\partial u_\Omega}{\partial n} (V \cdot n) & x \in \Gamma_p. \end{cases} \quad (20)$$

The proof of such an assertion may be found for instance in [22, Section 3.1]. It is more convenient to work with another expression of the shape derivative and to write it as a distribution with support Γ_p . For that purpose, let us introduce the adjoint state p defined formally as the solution of the system

$$\begin{cases} \nabla \cdot (\sigma \nabla p) = 0, & x \in \Omega \\ \sigma \frac{\partial p}{\partial n} = 2(u_\Omega - u_0), & x \in \Gamma_e \\ p = 0 & x \in \Gamma_p. \end{cases} \quad (21)$$

Proposition 4. *Assume that $\partial\Omega$ is Lipschitz (in other words, Γ_p is Lipschitz). Then, the criterion J is differentiable with respect to the shape at Ω and one has for all admissible perturbation $V \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$,*

$$dJ(\Omega; V) = \int_{\Gamma_p} \sigma \frac{\partial p}{\partial n} \frac{\partial u_\Omega}{\partial n} (V \cdot n) ds.$$

As a consequence, if Problem (17) has a solution Ω^ having a internal boundary Γ_i^* which is a Lipschitz closed curve, then Ω^* solves the overdetermined problem (14).*

Proof. The shape differentiability is standard (see [17, 27]) under the assumption that the boundary of Ω is regular enough. Let us now prove the expression of the shape derivative. First multiply equation (20) by p and integrate on Ω . The use of the Green formula yields

$$-\int_{\Omega} \sigma \nabla p \cdot \nabla \dot{u} = -\int_{\Gamma_e} \sigma \frac{\partial \dot{u}}{\partial n} p = 0.$$

Multiplying equation (21) by \dot{u} and integrating by parts yields

$$-\int_{\Omega} \sigma \nabla p \cdot \nabla \dot{u} = -\int_{\Gamma_e \cup \Gamma_p} \sigma \frac{\partial p}{\partial n} \dot{u}.$$

The conclusion follows easily. Now, let us investigate the first order optimality conditions. Since $\Gamma_{i,\star}$ is Lipschitz and surrounds a domain whose interior is nonempty, we deduce that if V belongs to the cone of admissible perturbations, then so does $-V$. Hence,

$$\frac{\partial p}{\partial n} \frac{\partial u_\Omega}{\partial n} = 0 \text{ on } \Gamma_p^\star,$$

and it follows that there exists a subset of Γ_p^\star of nonzero surface measure on which $u_{\Omega^\star} = c$ and $\frac{\partial p}{\partial n} \frac{\partial u_{\Omega^\star}}{\partial n} = 0$. Furthermore, by virtue of Holmgren's theorem, necessarily $\frac{\partial u_{\Omega^\star}}{\partial n} \neq 0$ on such a subset. Hence $\frac{\partial p}{\partial n} = 0$ on this subset, and since one has also $p = 0$ on Γ_p^\star , a second application of Holmgren's theorem yields $p \equiv 0$. In other words, $\frac{\partial u_{\Omega^\star}}{\partial n}|_{\Gamma_e} = u_1$, which is the desired result. \square

4.4 Numerical computations

The method presented here is a gradient method, based on the use of the shape derivative $dJ(\Omega; V)$ computed in Proposition 4. It consists basically in finding a deformation of a given domain Ω guaranteeing the decrease of the criterion $J(\Omega)$. To sum up, the algorithm writes:

- Initial guess $\Omega_0 \in \mathcal{O}_{\text{ad}}$ is given;
- Iteration k : $\Omega_{k+1} = (Id + t_k V_k) \Omega_k$ where V_k is a vector field shrewdly chosen and t_k the time step;

Practically speaking, we are looking for a vector field V_k such that

$$dJ(\Omega_k, V_k) = \int_{\Gamma_p} \nabla J(\Omega_k) \cdot V_k ds < 0,$$

where

$$\nabla J(\Omega_k) = -\sigma \frac{\partial u_{\Omega_k}}{\partial n} \frac{\partial p_k}{\partial n} n,$$

and p_k is the adjoint state, solution of Problem (21) set on the domain Ω_k . One possible choice for V_k is

$$\int_{\Gamma_p} \nabla J(\Omega_k) \cdot V_k = -\|V_k\|_{H^1(\Omega_k)}^2. \quad (22)$$

For a precise description of this method or others like the level set method and examples of applications in the domain of Shape Optimization, we refer for instance to [2, 12, 18, 16, 35]. To ensure (22), let us write V_k as the solution of the problem written under variational form

$$\text{Find } V_k \in (H^1(\Omega_k))^2 \text{ such that } \int_{\Omega_k} (\nabla V_k : \nabla \varphi + V_k \cdot \varphi) ds = - \int_{\Gamma_p} \nabla J(\Omega_k) \cdot \varphi ds, \quad (23)$$

where the test functions φ describe the space of functions $H^1(\Omega_k)^2$ verifying $\varphi = 0$ on Γ_e , and the symbol “:” denotes the tensorial notation used to represent the *doubly contracted tensorial product*, i.e. for A and B , two smooth vector fields of \mathbb{R}^2 ,

$$A : B = \sum_{i,j=1}^2 A_{ij} B_{ij} = \text{tr}(A \otimes B^\top).$$

Notice that (23) is equivalent to the partial differential equation

$$\begin{cases} -\Delta V_k + V_k = 0 & x \in \Omega_k \\ V_k = 0, & x \in \Gamma_e \\ \frac{\partial V_k}{\partial n} = -\nabla J(\Omega_k), & x \in \Gamma_p^k. \end{cases} \quad (24)$$

Written under this form, it is easy to see that this partial differential equation is decoupled.

Let us summarize the gradient algorithm:

1. *Initialization*: Choice of an initial domain Ω_0 and of a real number $\varepsilon > 0$;
2. *Iteration k* :
 - (a) computation of the solutions u_k and p_k of Problems (16) and (21);
 - (b) evaluation of $\nabla J(\Omega_k)$ on Γ_p^k ;
 - (c) computation of the solution V_k of Problem (24);
 - (d) determination of $\Omega_{k+1} = (Id + t_k V_k)\Omega_k$;
3. *stopping criterion*: The algorithm stops when $|J(\Omega_{k+1}) - J(\Omega_k)| < \varepsilon$

An alternative approach, inverting the role of Dirichlet and Neumann conditions in (16), has also been tested. More precisely, the criterion to minimize has been changed to

$$J(\Omega) = \int_{\Gamma_e} \left(\frac{\partial u_\Omega}{\partial n} - u_1 \right)^2 dx,$$

where u_Ω solves

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, & x \in \Omega \\ u = u_0, & x \in \Gamma_e \\ u = c & x \in \Gamma_p. \end{cases} \quad (25)$$

We observed that both approach gave similar results. Before presenting our numerical results on the tokamak *Tore Supra*, let us begin with some numerical tests, in order to verify the efficiency of our method.

A first test case has been done, taking $\sigma = 1$, and using the fact that the solution of the Laplacian operator is known on an annular domain. Indeed, let us assume that

$\Omega = D_2 \setminus D_1$ where D_1 (respectively D_2) denotes the disk centered at the origin of radius $R_1 > 0$ (respectively $R_2 > 0$). Then, the solution of

$$\begin{cases} \Delta u = 0, & x \in \Omega \\ u = 0, & x \in \partial D_1 \\ u = c_0 & x \in \partial D_2 \end{cases} \quad (26)$$

is given by

$$u = c_0 \frac{\ln \sqrt{x^2 + y^2} - \ln R_1}{\ln R_2 - \ln R_1}.$$

Note that the normal derivative of u on ∂D_2 is clearly

$$\frac{\partial u}{\partial n} = \frac{c_0}{R_2(\ln R_2 - \ln R_1)}.$$

A first test can then be made, looking for the internal radius R_1 , assuming that the normal derivative of u on ∂D_2 is constant equal to $c_1 > 0$ and the expected solution is obviously

$$R_1 = R_2 \exp\left(-\frac{c_0}{R_2 c_1}\right).$$

The results for this case are plotted on Figure 3.

A second test case has been built, by choosing an arbitrary domain Ω_c and a function u_0 , computing the solution u of (16), and choosing for u_1 , the quantity $\frac{\partial u}{\partial n}$. Running the program with these definitions of u_0 and u_1 permits to compare the computed domain with the theoretical domain Ω_c . The results for this case are plotted on Figure 4.

Let us now present the simulations of the plasma shape in the tokamak *Tore Supra*. Measures of u_0 and u_1 on the external boundary of the tokamak, made by Physicists from CEA/IRFM, are the starting point of our algorithm. We interpolated these datas thanks to cubic splines to have a complete definition of these functions (see Figure 5). All the simulations presented here are been realized using the software Matlab. With these functions, the algorithm provided the following results plotted on Figure 6. Notice that we did not impose in our algorithm for the plasma to stay inside the limiter, but we fortunately observed it. The point marked by a cross is a control point and the free boundary has to contain this point (it is on the limiter). These conditions being fulfilled, validate our method.

A linked interesting inverse problem consists in finding the value of the constant c introduced in Problem (14) from our knowledge of the point M_0 belonging to the shape of the plasma. Our algorithm can easily perform the value of such a constant. Indeed, solving Problem (14) with an arbitrary constant $c_1 \gg c$ yields a solution u and the associated optimal domain is $\Omega_1 = \{u_0 \leq u \leq c_1\}$. In fact, the true solution is provided

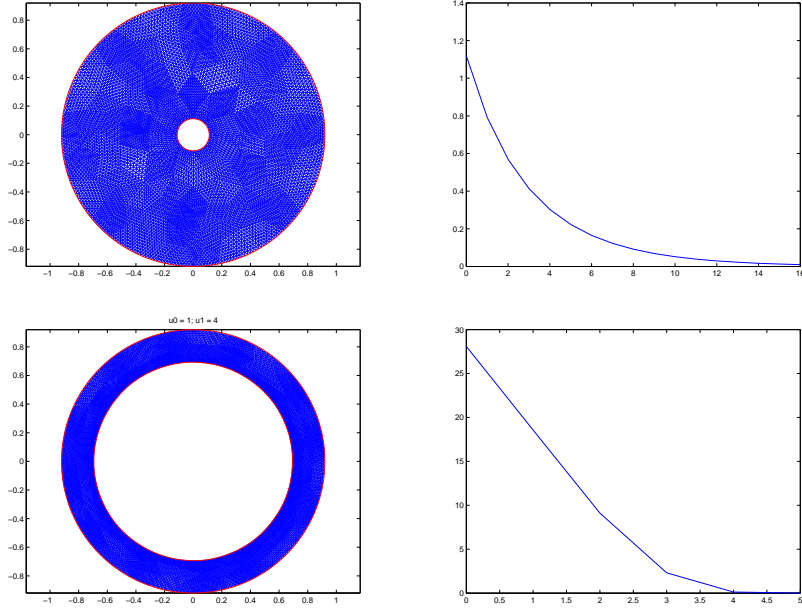


Figure 3: Up: optimum computed (left) and criterion (right) for the test case $c_0 = 2$ and $c_1 = 1$. Hence, the expected value of R_1 is 0.10 which is the case. Down: optimum computed (left) and criterion (right) for the test case $c_0 = 1$ and $c_4 = 1$. Hence, the expected value of R_1 is 0.70 which is the case

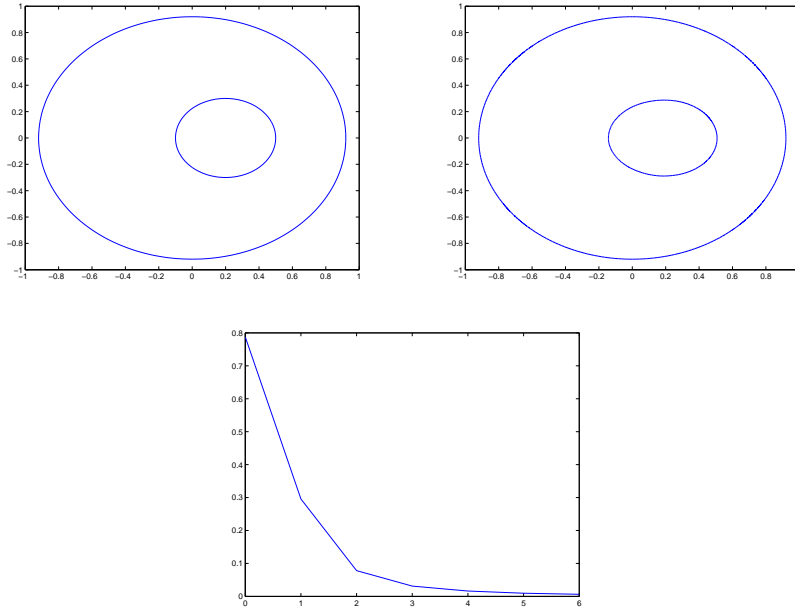


Figure 4: Domain Ω_c expected (left), Domain Ω computed (right) and criterion (down) in the case $u_0 = -0.2x^2 - 0.5y^2$

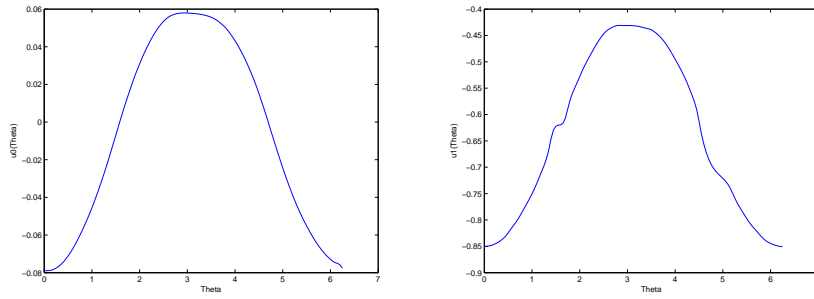


Figure 5: $\theta \mapsto u_0(\theta)$ (left) and $\theta \mapsto u_1(\theta)$ (right)

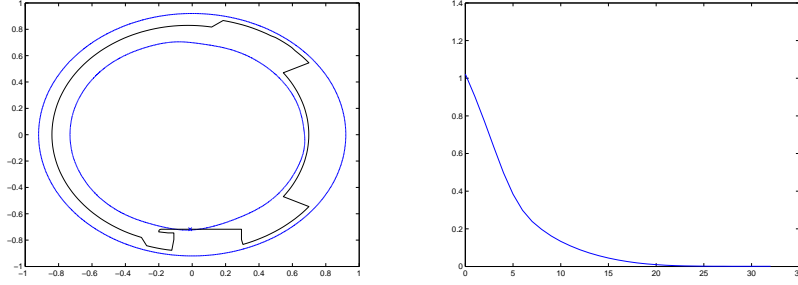


Figure 6: External boundary in blue, toroidal belt limiter in black, computed shape of the plasma in blue (left) and criterion (right)

by $\Omega^\star = \{u_0 \leq u \leq u(M_0)\}$. Indeed, it is very easy to show that, as a consequence of the maximum principle, Ω^\star satisfies (13). On Figure 7 is an example of the solution obtained by running the program with $c_1 = 0.2$, the expected constant being closed to 0.153.

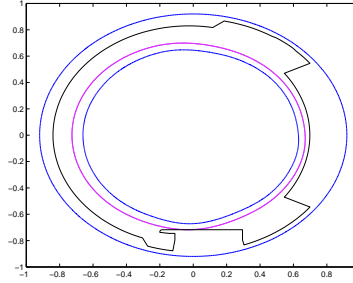


Figure 7: Ω_{c_1} in blue and Ω^\star in purple. The constant c evaluated is 0.1569 whereas the constant c measured by Physicists is 0.1531

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