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On the law of the supremum of Lévy processes

November 17, 2010

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Abstract

We show that the law of the overall supremum $\overline{X}_t = \sup_{s \leq t} X_s$ of a Lévy process X before the deterministic time t is equivalent to the average occupation measure $\mu_t(dx) = \int_0^t \mathbb{P}(X_s \in dx) ds$, whenever 0 is regular for both open halflines $(-\infty, 0)$ and $(0, \infty)$. In this case, $\mathbb{P}(\overline{X}_t \in dx)$ is absolutely continuous for some (and hence for all) $t > 0$, if and only if the resolvent measure of X is absolutely continuous. We also study the cases where 0 is not regular for one of the halflines $(-\infty, 0)$ or $(0, \infty)$. Then we give absolute continuity criterions for the laws of (\overline{X}_t, X_t) , (g_t, \overline{X}_t) and $(g_t, \overline{X}_t, X_t)$, where g_t is the time at which the supremum occurs before t . The proofs of these results use an expression of the joint law $\mathbb{P}(g_t \in ds, X_t \in dx, \overline{X}_t \in dy)$ in terms of the entrance law of the excursion measure of the reflected process at the supremum and that of the reflected process at the infimum. As an application, this law is made (partly) explicit in some particular instances.

KEY WORDS AND PHRASES: Past supremum, equivalent measures, absolute continuity, expected occupation time, excursion measure, reflected process.

MSC 2000 subject classifications: 60 G 51.

1 Introduction

The law of the past supremum $\overline{X}_t = \sup_{s \leq t} X_s$ of Lévy processes before a deterministic time $t > 0$ presents some major interest in stochastic modeling such as queuing and risk theories as it is related to the law of the first passage time T_x above any positive level x through the relation $\mathbb{P}(\overline{X}_t \geq x) = \mathbb{P}(T_x \leq t)$. The importance of knowing features of this law for some domains of application mainly explains the abundance of the literature on this topic. From the works of P. Lévy on Brownian motion [14] to

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the recent developments of A. Kuznetsov [12] for a very large class of stable Lévy processes, an important number of papers have appeared. Most of them concern explicit computations for stable processes and basic features, such as tail behavior of this law, are still unknown in the general case.

The present work is mainly concerned with the study of the nature of the law of the overall supremum \overline{X}_t and more specifically, with the existence of a density for this distribution. In a recent paper, N. Bouleau and L. Denis [3] have proved that the law of \overline{X}_t is absolutely continuous whenever the Lévy measure of X is itself absolutely continuous and satisfies some additional conditions, see Proposition 3 in [3]. This result has raised our interest on the subject and we proposed to determine 'exploitable' necessary and sufficient conditions under which the law of \overline{X}_t is absolutely continuous. Doing so, we also obtained conditions for the absolute continuity of the random vectors (g_t, \overline{X}_t) , (\overline{X}_t, X_t) and $(g_t, \overline{X}_t, X_t)$, where g_t is the time at which the maximum of X occurs on $[0, t]$. The proofs are based on two main ingredients. The first one is the equivalence between the law of X_t in \mathbb{R}_+ and the entrance law of the excursions of the reflected process at its minimum, see Lemma 1. The second argument is an expression of the law of $(g_t, \overline{X}_t, X_t)$ in terms of the entrance laws of the excursions of both reflected processes. From this expression, we may in addition recover the law of $(g_t, \overline{X}_t, X_t)$ for Brownian motion with drift and derive an explicit form of this law for the symmetric Cauchy process. The law of (g_t, \overline{X}_t) , may also be computed in some instances of spectrally negative Lévy processes.

The remainder of this paper is organized as follows. In Section 2, we give some definitions and we recall some basic elements of excursion theory and fluctuation theory for Lévy processes, which are necessary for the proofs. The main results of the paper are stated in Sections 3 and 4. In Section 3, we state continuity properties of the triples $(g_t, \overline{X}_t, X_t)$ and $(g_t^*, \underline{X}_t, X_t)$, whereas Section 4 is devoted to some representations and explicit expressions for the law of $(g_t, \overline{X}_t, X_t)$. Then except for corollaries, proofs of the results are postponed to Section 5.

2 Preliminaries

We denote by \mathcal{D} the space of càdlàg paths $\omega : [0, \infty) \rightarrow \mathbb{R} \cup \{\infty\}$ with lifetime $\zeta(\omega) = \inf\{t \geq 0 : \omega_t = \omega_s, \forall s \geq t\}$, with the usual convention that $\inf\{\emptyset\} = +\infty$. The space \mathcal{D} is equipped with the Skorokhod topology, its Borel σ -algebra \mathcal{F} , and the usual completed filtration $(\mathcal{F}_s, s \geq 0)$ generated by the coordinate process $X = (X_t, t \geq 0)$ on the space \mathcal{D} . We write \overline{X} and \underline{X} for the supremum and infimum processes:

$$\overline{X}_t = \sup\{X_s : 0 \leq s \leq t\} \quad \text{and} \quad \underline{X}_t = \inf\{X_s : 0 \leq s \leq t\}.$$

For $t > 0$, the last passage times by X at its supremum and at its infimum before t are respectively defined by:

$$g_t = \sup\{s \leq t : X_s = \overline{X}_t \text{ or } X_{s-} = \overline{X}_t\} \quad \text{and} \quad g_t^* = \sup\{s \leq t : X_s = \underline{X}_t \text{ or } X_{s-} = \underline{X}_t\}.$$

We also define the first passage time by X in the open halfline $(0, \infty)$ by:

$$\tau_0^+ = \inf\{t \geq 0 : X_t > 0\}.$$

For each $x \in \mathbb{R}$, we denote by \mathbb{P}_x the law on \mathcal{D} of a Lévy process starting from x , and we write $\mathbb{P}_0 = \mathbb{P}$. We assume throughout the sequel that (X, \mathbb{P}) is not a compound Poisson process and that $|X|$ is not a subordinator.

Note that under our assumptions, 0 is always regular for $(-\infty, 0)$ or/and $(0, \infty)$. It is well known that the reflected processes $\overline{X} - X$ and $X - \underline{X}$ are strong Markov processes. Under \mathbb{P} , the state 0 is regular for $(0, \infty)$ (resp. for $(-\infty, 0)$) if and only if it is regular for $\{0\}$, for the reflected process $\overline{X} - X$ (resp. for $X - \underline{X}$). If 0 is regular for $(0, \infty)$, then the local time at 0 of the reflected process $\overline{X} - X$ is the unique continuous, increasing, additive functional L with $L_0 = 0$, a.s., such that the support of the measure dL_t is the set $\{t : \overline{X}_t = X_t\}$ and which is normalized by

$$\mathbb{E} \left(\int_0^\infty e^{-t} dL_t \right) = 1. \quad (2.1)$$

Let G be the set of left end points of the excursions away from 0 of $\overline{X} - X$ and for each $s \in G$, call ϵ^s the excursion which starts at s . Denote by E the set of excursions, i.e. $E = \{\omega \in \mathcal{D} : \omega_t > 0, \text{ for all } 0 < t < \zeta(\omega)\}$ and let \mathcal{E} be the Borel σ -algebra which is the trace of \mathcal{F} on the subset E of \mathcal{D} . The Itô measure n of the excursions away from 0 of the process $\overline{X} - X$ is characterized by the so-called *compensation formula*:

$$\mathbb{E} \left(\sum_{s \in G} F(s, \omega, \epsilon^s) \right) = \mathbb{E} \left(\int_0^\infty dL_s \left(\int F(s, \omega, \epsilon) n(d\epsilon) \right) \right), \quad (2.2)$$

which is valid whenever F is a positive and predictable process, i.e. $\mathcal{P}(\mathcal{F}_s) \otimes \mathcal{E}$ -measurable, where $\mathcal{P}(\mathcal{F}_s)$ is the predictable σ -algebra associated to the filtration (\mathcal{F}_s) . We refer to [2], Chap. IV, [13], Chap. 6 and [9] for more detailed definitions and some constructions of L and n .

If 0 is not regular for $(0, \infty)$, then the set $\{t : (\overline{X} - X)_t = 0\}$ is discrete and following [2] and [13], we define the local time L of $\overline{X} - X$ at 0 by

$$L_t = \sum_{k=0}^{N_t} \mathbf{e}^{(k)}, \quad (2.3)$$

where $N_t = \text{Card}\{s \in (0, t] : \overline{X}_s = X_s\}$, and $\mathbf{e}^{(k)}$, $k = 0, 1, \dots$ is a sequence of independent and exponentially distributed random variables with parameter

$$\gamma = \left(1 - \mathbb{E}(e^{-\tau_0^+}) \right)^{-1}. \quad (2.4)$$

In this case, the measure n of the excursions away from 0 is proportional to the distribution of the process X under the law \mathbb{P} , returned at its first passage time in the positive halfline. More formally, let us define $\epsilon^{\tau_0^+} = (X_{\tau_0^+} - X_{(\tau_0^+ - s)-}, 0 \leq s < \tau_0^+)$, then for any bounded Borel functional K on \mathcal{E} ,

$$\int_{\mathcal{E}} K(\epsilon) n(d\epsilon) = \gamma \mathbb{E}[K(\epsilon^{\tau_0^+})]. \quad (2.5)$$

Define G and ε^s as in the regular case, then from definitions (2.3), (2.5) and an application of the strong Markov property, we may check that the normalization (2.1) and the compensation formula (2.2) are still valid in this case.

The local time at 0 of the reflected process at its infimum $X - \underline{X}$ and the measure of its excursions away from 0 are defined in the same way as for $\bar{X} - X$. They are respectively denoted by L^* and n^* . Then the ladder time processes τ and τ^* , and the ladder height processes H and H^* are the following subordinators:

$$\tau_t = \inf\{s : L_s > t\}, \quad \tau_t^* = \inf\{s : L_s^* > t\}, \quad H_t = X_{\tau_t}, \quad H_t^* = X_{\tau_t^*}, \quad t \geq 0,$$

where $\tau_t = H_t = +\infty$, for $t \geq \zeta(\tau) = \zeta(H)$ and $\tau_t^* = H_t^* = +\infty$, for $t \geq \zeta(\tau^*) = \zeta(H^*)$. The ladder processes (τ, H) and (τ^*, H^*) are (possibly killed) Lévy processes whose characteristic exponents κ and κ^* are given by

$$\mathbb{E}(e^{-\alpha\tau_1 - \beta H_1}) = e^{-\kappa(\alpha, \beta)} \quad \text{and} \quad \mathbb{E}(e^{-\alpha\tau_1^* - \beta H_1^*}) = e^{-\kappa^*(\alpha, \beta)}. \quad (2.6)$$

From (2.1), we derive that $\kappa(1, 0) = \kappa^*(1, 0) = 1$, so that Wiener-Hopf factorization in time which is stated in [2], p. 166 and in [13], p. 166 is normalized as follows

$$\kappa(\alpha, 0)\kappa^*(\alpha, 0) = \alpha, \quad \text{for all } \alpha \geq 0. \quad (2.7)$$

Recall also that the drifts \mathbf{d} and \mathbf{d}^* of the subordinators τ and τ^* satisfy $\mathbf{d} = 0$ (resp. $\mathbf{d}^* = 0$) if and only if 0 is regular for $(-\infty, 0)$, (resp. for $(0, \infty)$) and that:

$$\int_0^t \mathbb{1}_{\{X_s = \bar{X}_s\}} ds = \mathbf{d}L_t \quad \text{and} \quad \int_0^t \mathbb{1}_{\{X_s = \underline{X}_s\}} ds = \mathbf{d}^*L_t^*. \quad (2.8)$$

Suppose that 0 is not regular for $(0, \infty)$ and let \mathbf{e} be an independent exponential time with mean 1, then from (2.1) and (2.8), $\mathbb{P}((X - \underline{X})_{\mathbf{e}} = 0) = \mathbf{d}^*$. From the time reversal property of Lévy processes, $\mathbb{P}((X - \underline{X})_{\mathbf{e}} = 0) = \mathbb{P}(\bar{X}_{\mathbf{e}} = 0) = \mathbb{P}(\tau_0^+ \geq \mathbf{e}) = \gamma^{-1}$, so that $\mathbf{d}^* = \gamma^{-1}$.

We will denote by q_t^* and q_t the entrance laws of the reflected excursions at the maximum and at the minimum, i.e. for $t > 0$,

$$q_t(dx) = n(X_t \in dx, t < \zeta) \quad \text{and} \quad q_t^*(dx) = n^*(X_t \in dx, t < \zeta).$$

They will be considered as measures on $\mathbb{R}_+ = [0, \infty)$. Recall that the law of the lifetime of the reflected excursions is related to the Lévy measure of the ladder time processes, through the equalities:

$$q_t(\mathbb{R}_+) = n(t < \zeta) = \bar{\pi}(t) + a \quad \text{and} \quad q_t^*(\mathbb{R}_+) = n^*(t < \zeta) = \bar{\pi}^*(t) + a^*, \quad (2.9)$$

where $\bar{\pi}(t) = \pi(t, \infty)$ and $\bar{\pi}^*(t) = \pi^*(t, \infty)$ and a, a^* are the killing rates of the subordinators τ and τ^* .

In this paper, we will sometimes write $\mu \ll \nu$, when μ is absolutely continuous with respect to ν . We will say that μ and ν are *equivalent* if $\mu \ll \nu$ and $\nu \ll \mu$. We will denote by λ the Lebesgue measure on \mathbb{R} . A measure which is absolutely continuous with respect to the Lebesgue measure will sometimes be called *absolutely continuous*. A measure which has no atoms will be called *continuous*.

3 Continuity properties of the law of $(g_t, \overline{X}_t, X_t)$

For $t > 0$ and $q > 0$, we will denote respectively by $p_t(dx)$ and $U_q(dx)$ the semigroup and the resolvent measure of X , i.e. for any positive Borel function f ,

$$\mathbb{E}(f(X_t)) = \int_0^\infty f(x)p_t(dx) \quad \text{and} \quad \int_0^\infty f(x)U_q(dx) = \mathbb{E}\left(\int_0^\infty e^{-qt}f(X_t)dt\right).$$

It is clear that for all q and q' the resolvent measures $U_q(dx)$ and $U_{q'}(dx)$ are equivalent. Moreover, each measure U_q is equivalent to the potential measure $U_0(dx) = \int_0^\infty \mathbb{P}(X_t \in dx) dt$. In what follows, when comparing the law of \overline{X}_t to the measures U_q , $q \geq 0$, we will take $U(dx) \stackrel{(\text{def})}{=} U_1(dx)$ as a reference measure. We will say that a Lévy process X is of

- type 1 if 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$,
- type 2 if 0 is not regular for $(-\infty, 0)$,
- type 3 if 0 is not regular for $(0, \infty)$.

Note that since X is not a compound Poisson process, types 1, 2 and 3 define three exhaustive cases. Recall that $\mathbb{R}_+ = [0, \infty)$ and let $\mathcal{B}_{\mathbb{R}_+}$ be the Borel σ -field on \mathbb{R}_+ . For $t > 0$, let μ_t^+ be the restriction to $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ of the average occupation measure of X , on the time interval $[0, t)$, i.e.

$$\int_{[0, \infty)} f(x) \mu_t^+(dx) = \mathbb{E}\left(\int_0^t f(X_s) ds\right),$$

for every positive Borel function f on $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$. Moreover, we will denote by $p_t^+(dx)$ the restriction of the semigroup $p_t(dx)$ to $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$. In particular, we have $\mu_t^+ = \int_0^t p_s^+ ds$. The law of \overline{X}_t will be considered as a measure on $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$. In all the remainder of this article, we assume that the time t is deterministic and finite.

Theorem 1. *For $t > 0$, the law of the past supremum \overline{X}_t can be compared to the occupation measure μ_t^+ as follows.*

1. *If X is of type 1, then for all $t > 0$, the law of \overline{X}_t is equivalent to μ_t^+ .*
2. *If X is of type 2, then for all $t > 0$, the law of \overline{X}_t is equivalent to $p_t^+(dx) + \mu_t^+(dx)$.*
3. *If X is of type 3, then for all $t > 0$, the law of \overline{X}_t has an atom at 0 and its restriction to the open halfline $(0, \infty)$ is equivalent to the restriction of the measure $\mu_t^+(dx)$ to $(0, \infty)$.*

It appears clearly from this theorem that the law of \overline{X}_t is absolutely continuous for all $t > 0$, whenever 0 is regular for $(0, \infty)$ and p_t is absolutely continuous, for all $t > 0$. We will see in Theorem 3 that a stronger result actually holds. Let $U^+(dx)$ be the restriction to $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ of the resolvent measure $U(dx)$. Since μ_t is absolutely continuous with respect to U^+ for all $t > 0$, the law of the past supremum before t can be compared to U^+ as follows.

Corollary 1. *Under the same assumptions as above:*

1. *If X is of type 1, then for any $t > 0$, the law of \overline{X}_t is absolutely continuous with respect to the resolvent measure $U^+(dx)$.*
2. *If X is of type 2, then for any $t > 0$, the law of \overline{X}_t is absolutely continuous with respect to the measure $p_t^+(dx) + U^+(dx)$.*
3. *If X is of type 3, then the same conclusions as in 1. hold for the measures restricted to $(0, \infty)$.*

Under our assumption, the resolvent measure $U^+(dx)$ is always continuous, see Proposition I.15 in [2]. Moreover, the measure $p_t^+(dx)$ is also continuous for all $t > 0$, see Theorem 27.4 in Sato [19]. Hence from Corollary 1, for all $t > 0$, when X is of type 1 or 2, the law of \overline{X}_t is continuous and when it is of type 3, this law has only one atom at 0. This fact has already been observed in [18], Lemma 1.

It is known that for a Lévy process X , the law of X_t may be absolutely continuous for all $t > t_0$, whereas it is continuous singular for $t \in (0, t_0)$, see Theorem 27.23 and Remark 27.24 in [19]. The following theorem shows that when X is of type 1, this phenomenon cannot happen for the law of the supremum, i.e. either absolute continuity of the law of \overline{X}_t holds at any time t or it never holds. We denote by $V(dt, dx)$ the potential measure of the ladder process (τ, H) and by $V(dx)$ the potential measure of the ladder height process H , i.e.

$$V(dt, dx) = \int_0^\infty \mathbb{P}(\tau_s \in dt, H_s \in dx) ds \quad \text{and} \quad V(dx) = \int_0^\infty \mathbb{P}(H_s \in dx) ds.$$

Then let λ^+ be the Lebesgue measure on \mathbb{R}_+ .

Theorem 2. *Suppose that X is of type 1. The following assertions are equivalent:*

1. *The law of \overline{X}_t is absolutely continuous with respect to λ^+ , for some $t > 0$.*
2. *The resolvent measure $U^+(dx)$ is absolutely continuous with respect to λ^+ .*
3. *The resolvent measure $U(dx)$ is absolutely continuous with respect to λ .*
4. *The potential measure $V(dx)$ is absolutely continuous with respect to λ^+ .*

As a consequence, if 1. holds for some $t > 0$, then it holds for all $t > 0$. Moreover assertions 1 – 4 are equivalent to the same assertions formulated for the dual process $-X$. In particular, 1 – 4 hold if and only if the law of $-\underline{X}_t$ is absolutely continuous with respect to λ^+ , for all $t > 0$.

Condition 4 of the above theorem is satisfied whenever the drift coefficient of the subordinator H is positive, see Theorem II.16 and Corollary II.20 in [2]. Let us also mention that necessary and sufficient conditions for $U(dx)$ to be absolutely continuous may be found in Theorem 41.15 of [19], and in Proposition 10, Chap. I of [2]. Formally,

$U \ll \lambda$ if and only if for some $q > 0$ and for all bounded Borel function f , the function $x \mapsto \mathbb{E}_x \left(\int_0^\infty f(X_t) e^{-qt} dt \right)$ is continuous. However, we do not know any necessary and sufficient conditions bearing directly on the characteristic exponent ψ of X . Let us simply recall the following sufficient condition. From Theorem II.16 in [2], if

$$\int_{-\infty}^{\infty} \Re \left(\frac{1}{1 + \psi(x)} \right) dx < \infty, \quad (3.1)$$

then $U(dx) \ll \lambda$, with a bounded density. Therefore, if X is of type 1, then from Theorem 2, condition (3.1) implies that both the laws of \underline{X}_t and \overline{X}_t are absolutely continuous for all $t > 0$.

A famous result from [10] asserts that when X is a symmetric process, condition $U \ll \lambda$ implies that $p_t \ll \lambda$, for all $t > 0$. Then it follows from Theorem 2 that in this particular case, absolute continuity of the law of \overline{X}_t , for all $t > 0$ is equivalent to the absolute continuity of the semigroup p_t , for all $t > 0$.

Theorem 3. *If 0 is regular for $(0, \infty)$, then the following assertions are equivalent:*

1. *The measures p_t^+ are absolutely continuous with respect to λ^+ , for all $t > 0$.*
2. *The measures p_t are absolutely continuous with respect to λ , for all $t > 0$.*
3. *The potential measure $V(dt, dx)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}_+^2 .*

If moreover X is of type 1, then each of the following assertions is equivalent to 1 – 3:

4. *The law of (g_t, \overline{X}_t) is absolutely continuous with respect to the Lebesgue measure on $[0, t] \times \mathbb{R}_+$, for all $t > 0$.*
5. *The law of $(g_t, \overline{X}_t, X_t)$ is absolutely continuous with respect to the Lebesgue measure on $[0, t] \times \mathbb{R}_+ \times \mathbb{R}$, for all $t > 0$.*

When X is of type 1, it is plain that for all $t > 0$, $g_t > 0$ and $\overline{X}_t > 0$, a.s., hence absolute continuity in assertion 1 of Theorem 2 and in assertion 4 of Theorem 3 is actually an equivalence with respect to the Lebesgue measure.

We may wonder if the equivalence between assertions 1 and 2 of Theorem 3 still holds when t is fixed, i.e. when 0 is regular for $(0, \infty)$, does the condition $p_t^+ \ll \lambda^+$, imply that $p_t \ll \lambda$? A counterexample in the case where 0 is not regular for $(-\infty, 0)$ may easily be found. Take for instance, $X_t = Y_t - S_t$, where Y is a compound Poisson process with absolutely continuous Lévy measure and S is a subordinator independent of Y , whose law at time $t > 0$ is continuous singular. Then clearly $p_t^+ \ll \lambda^+$, and there exists a Borel set $A \subset (-\infty, 0)$ such that $\lambda(A) = 0$ and $\mathbb{P}(-S_t \in A) > 0$, so that $p_t(A) > \mathbb{P}(Y_t = 0)\mathbb{P}(S_t \in A) > 0$.

Let Y be a càdlàg stochastic process such that $Y_0 = 0$, a.s. We say that Y is an *elementary process* if there is an increasing sequence (T_n) of nonnegative random

variables such that $T_0 = 0$ and $\lim_{n \rightarrow +\infty} T_n = +\infty$, a.s. and two sequences of finite real-valued random variables $(a_n, n \geq 0)$ and $(b_n, n \geq 0)$ such that $b_0 = 0$ and

$$Y_t = a_n t + b_n \quad \text{if } t \in [T_n, T_{n+1}). \quad (3.2)$$

We say that Y is a *step process* if it is an elementary process with $a_n = 0$, for all n in the above definition.

Proposition 1. *Suppose that 0 is regular for $(0, \infty)$.*

1. *If 0 is regular for $(-\infty, 0)$ and if the law of \overline{X}_t is absolutely continuous for some $t > 0$, then for any step process Y which is independent of X , the law of $\sup_{s \leq t} (X + Y)_s$ is absolutely continuous for all $t > 0$.*
2. *If $p_t^+ \ll \lambda^+$, for all $t > 0$, or if X has unbounded variation and if at least one of the ladder height processes H and H^* has a positive drift, then for any elementary stochastic process Y which is independent of X , the law of $\sup_{s \leq t} (X + Y)_s$ is absolutely continuous for all $t > 0$.*

Sufficient conditions for the absolute continuity of the semigroup may be found in Chapter 5 of [19] and in Section 5 of [11]. In particular if $\Pi(\mathbb{R}) = \infty$ and $\Pi \ll \lambda$, then $p_t \ll \lambda$ for all $t > 0$. Proposition 20 in Bouleau and Denis [3] asserts that under a slight reinforcement of this condition, for any independent càdlàg process Y , the law of $\sup_{s \leq t} (X + Y)_s$ is absolutely continuous provided it has no atom at 0. In the particular case where Y is an elementary process, this result is a consequence of part 2 of Proposition 1.

In view of Theorems 2 and 3, it is natural to look for instances of Lévy processes of type 1 such that the law of \overline{X}_t is absolutely continuous whereas $p_t(dx)$ is not, as well as instances of Lévy processes of type 1 such that the law of \overline{X}_t is not absolutely continuous. The following corollary is inspired from Orey's example [17], see also [19], Exercise 29.12 and Example 41.23.

Corollary 2. *Let X be a Lévy process whose characteristic exponent ψ , i.e. $\mathbb{E}(e^{i\lambda X_t}) = e^{-t\psi(\lambda)}$ is given by:*

$$\psi(\lambda) = \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x \mathbb{I}_{\{|x| < 1\}}) \Pi(dx).$$

Let $\alpha \in (1, 2)$ and c be an integer such that $c > 2/(2 - \alpha)$ and set $a_n = 2^{-c^n}$.

1. *If $\Pi(dx) = \sum_{n=1}^{\infty} a_n^{-\alpha} \delta_{-a_n}(dx)$, then X is of type 1 and for all $t > 0$, the law of \overline{X}_t is absolutely continuous whereas $p_t(dx)$ is continuous singular.*
2. *If $\Pi(dx) = \sum_{n=1}^{\infty} a_n^{-\alpha} (\delta_{-a_n}(dx) + \delta_{a_n}(dx))$, then X is of type 1 and for all $t > 0$, the law \overline{X}_t is not absolutely continuous.*

Proof. We may check that $\int_{(0,1)} x\Pi(dx) = \infty$ in both cases 1 and 2, so that X has unbounded variation and it is of type 1, from Rogozin's criterion, see [2] p. 167.

On the one hand, in part 1, since X has no positive jumps, the ladder height process H is a pure drift, so it follows from Theorem 2 and the remark thereafter that the law of \overline{X}_t is absolutely continuous for all $t > 0$. On the other hand, following [17], we see that $-\log |\psi(\lambda)|$ does not tend to $+\infty$ as $|\lambda| \rightarrow \infty$, so that from the Riemann-Lebesgue Theorem, $p_t(dx)$ is not absolutely continuous. But since $\Pi(dx)$ is discrete with infinite mass, it follows from the Hartman-Wintner Theorem, see Theorem 27.16 in [19], that $p_t(dx)$ is continuous singular.

Then it is proved in Example 41.23 of [19] that the resolvent measure $U(dx)$ of X is not absolutely continuous, so part 2 follows from Theorem 2. \square

4 An expression for the joint law of $(g_t, \overline{X}_t, X_t)$

The following theorem presents a path decomposition of the Lévy process X , over the interval $[0, t]$, at time g_t . Actually, we will essentially focus on its corollaries which provide some representations of the joint law of g_t, \overline{X}_t and X_t at a fixed time t , in terms of the entrance laws (q_s) and (q_s^*) . Besides they will be applied in Section 5 for the proofs of the results of Section 3.

For $\omega \in \mathcal{D}$ and $s \geq 0$, we set $\Delta_s^\pm(\omega) = (\omega_s - \omega_{s-})^\pm$, where $\omega_{0-} = \omega_0$. Then we define the (special) *shift* operator by

$$\theta_s(\omega) = (\omega_{s-} - \omega_{s+u} + \Delta_s^+(\omega), u \geq 0).$$

The *killing* operator and the *return* operator are respectively defined as follows:

$$k_s(\omega) = \begin{cases} \omega_u, & 0 \leq u < s \\ \omega_s, & u \geq s \end{cases}, \quad r_s(\omega) = \begin{cases} \omega_s - \omega_{(s-u)-} - \Delta_s^-(\omega), & 0 \leq u < s \\ \omega_s - \omega_0 - \Delta_s^-(\omega), & u \geq s \end{cases}.$$

We also denote by ω^0 the path which is identically equal to 0.

Theorem 4. *Fix $t > 0$, let f be any bounded Borel function and let F and K be any bounded Borel functionals which are defined on the space \mathcal{D} .*

1. *If X is of type 1, then*

$$\begin{aligned} \mathbb{E}(f(g_t) \cdot F \circ r_{g_t} \cdot K \circ k_{t-g_t} \circ \theta_{g_t}) = \\ \int_0^t f(s) n^*(F \circ k_s, s < \zeta) n(K \circ k_{t-s}, t - s < \zeta) ds. \end{aligned}$$

2. *If X is of type 2, then*

$$\begin{aligned} \mathbb{E}(f(g_t) \cdot F \circ r_{g_t} \cdot K \circ k_{t-g_t} \circ \theta_{g_t}) = \\ \int_0^t f(s) n^*(F \circ k_s, s < \zeta) n(K \circ k_{t-s}, t - s < \zeta) ds \\ + \mathbf{d} f(t) n^*(F \circ k_t, t < \zeta) K(\omega^0). \end{aligned}$$

3. If X is of type 3, then

$$\begin{aligned} \mathbb{E}(f(g_t) \cdot F \circ r_{g_t} \cdot K \circ k_{t-g_t} \circ \theta_{g_t}) = \\ \int_0^t f(s) n^*(F \circ k_s, s < \zeta) n(K \circ k_{t-s}, t-s < \zeta) ds \\ + \mathbf{d}^* f(0) F(\omega^0) n(K \circ k_t, t < \zeta). \end{aligned}$$

Simultaneously to our work, a similar path decomposition has been obtained in [20], when X is of type 1. In the later work, the post- g_t part of $(X_s, 0 \leq s \leq t)$ is expressed in terms of the meander $\mathbb{M}^{(t)} = n(\cdot | t < \zeta) / n(t < \zeta)$, see Theorem 5.1.

By applying Theorem 4 to the joint law of g_t together with the terminal values of the pre- g_t and the post- g_t parts of $(X_s, 0 \leq s \leq t)$, we obtain the following representation for the law of the triple $(g_t, \overline{X}_t, X_t)$. Moreover, when $\lim_{t \rightarrow \infty} X_t = -\infty$, a.s., we define $\overline{X}_\infty = \sup_t X_t$, the overall supremum of X and $g_\infty = \sup\{t : X_t = \overline{X}_\infty\}$, the location of this supremum. Then we obtain the same kind of representation for $(g_\infty, \overline{X}_\infty)$. We emphasize that in the next result, as well as in Corollaries 4 and 5, at least one of the drift coefficients \mathbf{d} and \mathbf{d}^* is zero.

Corollary 3. *The law of $(g_t, \overline{X}_t, X_t)$ fulfills the following representation:*

$$\begin{aligned} \mathbb{P}(g_t \in ds, \overline{X}_t \in dx, \overline{X}_t - X_t \in dy) = & q_s^*(dx) q_{t-s}(dy) \mathbb{I}_{[0,t]}(s) ds \\ & + \mathbf{d} \delta_{\{t\}}(ds) q_t^*(dx) \delta_{\{0\}}(dy) \\ & + \mathbf{d}^* \delta_{\{0\}}(ds) \delta_{\{0\}}(dx) q_t(dy). \end{aligned} \quad (4.1)$$

If moreover $\lim_{t \rightarrow \infty} X_t = -\infty$, a.s., then

$$\mathbb{P}(g_\infty \in ds, \overline{X}_\infty \in dx) = a q_s^*(dx) ds + \mathbf{d}^* a \delta_{\{(0,0)\}}(ds, dx), \quad (4.2)$$

where a is the killing rate of the ladder time process τ .

Proof. Let g and h be two bounded Borel functions on \mathbb{R}_+ and define the functionals K and F on \mathcal{D} by $F(\omega) = g(\omega_\zeta)$ and $K(\omega) = h(\omega_\zeta)$. Then we may check that for $\epsilon \in \mathcal{E}$ and $t < \zeta(\epsilon)$, $F \circ k_t(\epsilon) = g(\epsilon_t)$ and $K \circ k_t(\epsilon) = h(\epsilon_t)$. We also have $F \circ r_{g_t} \circ X = g(\overline{X}_t)$ and $K \circ k_{t-g_t} \circ \theta_{g_t} \circ X = h(\overline{X}_t - X_t)$, so that by applying Theorem 4 to the functionals F and K , we obtain (4.1).

To prove (4.2), we first note that $\lim_{t \rightarrow \infty} (g_t, \overline{X}_t) = (g_\infty, \overline{X}_\infty)$, a.s. Then let f be a bounded and continuous function which is defined on \mathbb{R}_+^2 . We have from (4.1):

$$\mathbb{E}(f(g_t, \overline{X}_t)) = \int_0^t f(s, x) n(t-s < \zeta) q_s^*(dx) ds + \mathbf{d} \int_0^\infty f(t, x) q_t^*(dx) + \mathbf{d}^* n(t < \zeta) f(0, 0).$$

On the one hand, we see from (2.9) that $\lim_{t \rightarrow \infty} n(t < \zeta) = n(\zeta = \infty) = a > 0$. On the other hand, $\lim_{t \rightarrow \infty} n^*(t < \zeta) = 0$, and since the term $\mathbf{d} \int_0^\infty f(t, x) q_t^*(dx)$ is bounded by $C n^*(t < \zeta)$, where $|f(s, x)| \leq C$, for all s, x , it converges to 0 as t tends to ∞ . This allows us to conclude. \square

We derive from Corollary 3 that when X is of type 1, the law of the time g_t is equivalent to the Lebesgue measure, with density $s \mapsto n^*(s < \zeta)n(t - s < \zeta)\mathbb{I}_{[0,t]}(s)$. This corollary illustrates the importance of the entrance laws q_t and q_t^* for the computation of some distributions involved in fluctuation theory. We give below a couple of examples where some explicit forms can be obtained for q_t , q_t^* and the law of $(g_t, \overline{X}_t, X_t)$. When $q_t(dx) \ll \lambda^+$ (resp. $q_t^*(dx) \ll \lambda^+$), we will denote by $q_t(x)$ (resp. $q_t^*(x)$) the density of $q_t(dx)$ (resp. $q_t^*(dx)$).

Example 1: Suppose that X is a Brownian motion with drift, i.e. $X_t = B_t + ct$, where B is the standard Brownian motion and $c \in \mathbb{R}$. We derive for instance from Lemma 1 in Section 5 that

$$q_t(dx) = \frac{x}{\sqrt{\pi t^3}} e^{-(x-c)^2/2t} dx \quad \text{and} \quad q_t^*(dx) = \frac{x}{\sqrt{\pi t^3}} e^{-(x+c)^2/2t} dx.$$

Then expression (4.1) in Corollary 3 allows us to compute the law of the triple $(g_t, \overline{X}_t, X_t)$.

Example 2: Recently, the density of the measure $q_t(dx)$ for the symmetric Cauchy process has been computed in [15]:

$$q_t(x) = q_t^*(x) = \sqrt{2} \frac{\sin\left(\frac{\pi}{8} + \frac{3}{2} \arctan\left(\frac{x}{t}\right)\right)}{(t^2 + x^2)^{3/4}} - \frac{1}{2\pi} \int_0^\infty \frac{y}{(1+y^2)(xy+t)^{3/2}} \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\log(y+s)}{1+s^2} ds\right) dy.$$

As far as we know, this example together with the case of Brownian motion with drift (Example 1) are the only instances of Lévy processes where the measures $q_t(dx)$, $q_t^*(dx)$ and the law of the triplet $(g_t, \overline{X}_t, X_t)$ can be computed explicitly.

Example 3: Recall from (2.9) that $\int_0^\infty q_t(dx) = n(t < \zeta)$ and $\int_0^\infty q_t^*(dx) = n^*(t < \zeta)$, so that we can derive from Theorem 3, all possible marginal laws in the triplet $(g_t, \overline{X}_t, X_t)$. In particular, when X is stable, the ladder time process τ also satisfies the scaling property with index $\rho = \mathbb{P}(X_1 \geq 0)$, so we derive from the normalization $\kappa(1, 0) = 1$ and (2.9) that $n(t < \zeta) = t^{-\rho}/\Gamma(1 - \rho)$. Moreover q_t^* and q_t are absolutely continuous in this case (it can be derived for instance from part 4 of Lemma 1 in the next section). Then a consequence of (4.1) is the following form of the joint law of (g_t, \overline{X}_t) :

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) = \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)} \mathbb{I}_{[0,t]}(s) q_s^*(x) ds dx. \quad (4.3)$$

Note that this computation is implicit in [1], see Corollary 3 and Theorem 5. A more explicit form is given in (4.10), after Proposition 2 in the case where the process has no positive jumps. Note also that when X is stable, the densities q_t and q_t^* satisfy the scaling properties

$$q_t(y) = t^{-\rho-1/\alpha} q_1(t^{-1/\alpha} y) \quad \text{and} \quad q_t^*(x) = t^{\rho-1-1/\alpha} q_1^*(t^{-1/\alpha} x).$$

These properties together with Corollary 3 imply that the three r.v.'s g_t , $\overline{X}_t/g_t^{1/\alpha}$ and $(\overline{X}_t - X_t)/(t - g_t)^{1/\alpha}$ are independent and have for densities

$$\frac{\sin(\pi\rho)}{\pi} s^{\rho-1} (t-s)^{-\rho} \mathbb{1}_{[0,t]}(s), \quad \Gamma(\rho)q_1^*(x) \quad \text{and} \quad \Gamma(1-\rho)q_1(y),$$

respectively. The independence between g_t , $\overline{X}_t/g_t^{1/\alpha}$ and $(\overline{X}_t - X_t)/(t - g_t)^{1/\alpha}$ has recently been proved in Proposition 2.39 of [6].

It is clear that an expression for the law of \overline{X}_t follows directly from Corollary 3 by integrating (4.1) over t and y . However, for convenience in the proofs of Section 5, we write it here in a proper statement. An equivalent version of Corollary 4 may also be found in [8], Lemma 6.

Corollary 4. *The law of \overline{X}_t fulfills the following representation:*

$$\mathbb{P}(\overline{X}_t \in dx) = \int_0^t n(t-s < \zeta) q_s^*(dx) ds + \mathbf{d} q_t^*(dx) + \mathbf{d}^* n(t < \zeta) \delta_{\{0\}}(dx). \quad (4.4)$$

Another remarkable, and later useful, direct consequence of Corollary 3 is the following representation of the semigroup of X in terms of the entrance laws (q_s) and (q_s^*) .

Corollary 5. *Let us denote the measure $q_t(-dx)$ by $\overline{q}_t(dx)$. We extend the measures $\overline{q}_t(dx)$ and $q_t^*(dx)$ to \mathbb{R} by setting $\overline{q}_t(A) = \overline{q}_t(A \cap \mathbb{R}_-)$ and $q_t^*(A) = q_t^*(A \cap \mathbb{R}_+)$, for any Borel set $A \subset \mathbb{R}$. Then we have the following identity between measures on \mathbb{R} :*

$$p_t = \int_0^t \overline{q}_s * q_{t-s}^* ds + \mathbf{d} q_t^* + \mathbf{d}^* \overline{q}_t. \quad (4.5)$$

Now we turn to the particular case where X has no positive jumps. Then, 0 is always regular for $(0, \infty)$. When moreover 0 is regular for $(-\infty, 0)$, since $H_t \equiv t$, it follows from Theorem 2 and the remark thereafter that the law of (g_t, \overline{X}_t) is absolutely continuous. In the next result, we present an explicit form of this law. We set $c = \Phi(1)$, where Φ is the Laplace exponent of the first passage process $T_x = \inf\{t : X_t > x\}$, which in this case is related to the ladder time process by $T_x = \tau_{cx}$.

Proposition 2. *Suppose that the Lévy process X has no positive jumps.*

1. *If 0 is regular for $(-\infty, 0)$, then for $t > 0$, the couple (g_t, \overline{X}_t) has law:*

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) = c x p_s^+(dx) n(t-s < \zeta) s^{-1} \mathbb{1}_{(0,t]}(s) ds \quad (4.6)$$

$$= c n(t-s < \zeta) \mathbb{1}_{(0,t]}(s) \mathbb{P}(\tau_{cx} \in ds) dx. \quad (4.7)$$

In particular, the density of the law of \overline{X}_t is given by the function:

$$x \mapsto \int_0^t c n(t-s < \zeta) \mathbb{P}(\tau_{cx} \in ds).$$

2. If 0 is not regular for $(-\infty, 0)$, then for all $t > 0$,

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) = c\alpha n(t-s < \zeta)s^{-1}\mathbb{1}_{(0,t]}(s)p_s^+(dx)ds + \mathbf{d}c\alpha t^{-1}p_t^+(dx)\delta_{\{t\}}(ds). \quad (4.8)$$

Moreover, we have the following identity between measures on $[0, \infty)^3$:

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx)dt = c\alpha n(t-s < \zeta)\mathbb{1}_{(0,t]}(s)\mathbb{P}(\tau_{cx} \in ds)dxdt + \mathbf{d}c\mathbb{P}(\tau_{cx} \in dt)\delta_{\{t\}}(ds)dx. \quad (4.9)$$

Example 4: Using the series development (14.30), p.88 in [19] for $p_s^+(dx)$, we derive from (4.6) in Proposition 2, the following reinforcement of expression (4.3). When X is stable and spectrally negative, the density of (g_t, \overline{X}_t) is given by:

$$\frac{c}{\pi\Gamma\left(\frac{\alpha-1}{\alpha}\right)(t-s)^{1/\alpha}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(1+n/\alpha)}{n!} \sin\left(\pi n \frac{\alpha-1}{\alpha}\right) s^{-\frac{n+\alpha}{\alpha}} x^n, \quad s \in [0, t], \quad x \geq 0, \quad (4.10)$$

which completes Proposition 1, p.282 in [5].

We end this section with a remark on the existence of a density with respect to the Lebesgue measure, for the law of the local time of general Markov processes. From (4.8), we derive that $\mathbb{P}(\tau_x \geq t)dt = \int_0^x \int_{(0,t]} n^*(t-s < \zeta)\mathbb{P}(\tau_y \in ds)dydt + \mathbf{d}\mathbb{P}(\tau_x \in dt)dt$. Actually, this identity may be generalized to any subordinator S with drift b , killing rate k and Lévy measure ν . Set $\bar{\nu}(t) = \nu(t, \infty) + k$, then the characteristic exponent Φ of S is given by

$$\Phi(\alpha) = \alpha b + \alpha \int_0^{\infty} e^{-\alpha t} \bar{\nu}(t) dt,$$

from which and Fubini Theorem, we derive that for all $x \geq 0$ and $\alpha > 0$:

$$\begin{aligned} \frac{1}{\alpha} \mathbb{E}(1 - e^{-\alpha S_x}) &= \left(b + \int_0^{\infty} e^{-\alpha t} \bar{\nu}(t) dt\right) \frac{\mathbb{E}(1 - e^{-\alpha S_x})}{\Phi(\alpha)}, \\ \int_0^{\infty} e^{-\alpha t} \mathbb{P}(S_x > t) dt &= \left(b + \int_0^{\infty} e^{-\alpha t} \bar{\nu}(t) dt\right) \int_0^{\infty} e^{-\alpha t} \int_0^x \mathbb{P}(S_y \in dt) dy. \end{aligned}$$

Inverting the Laplace transforms on both sides of this identity gives for all $x \geq 0$, the following identity between measures,

$$\mathbb{P}(S_x > t)dt = \int_0^x \int_{(0,t]} \bar{\nu}(t-s) \mathbb{P}(S_y \in ds) dy dt + b \int_0^x \mathbb{P}(S_y \in dt) dy.$$

In particular, if S has no drift coefficient, then the law of $L_t \stackrel{(\text{def})}{=} \inf\{u : S_u > t\}$ has density:

$$\frac{\mathbb{P}(L_t \in dx)}{dx} = \int_{(0,t]} \bar{\nu}(t-s) \mathbb{P}(S_x \in ds).$$

This shows that if $a \in \mathbb{R}$ is a regular state of any real Markov process M such that $\int_0^t \mathbb{1}_{\{M_s=a\}} ds = 0$, a.s. for all t , then the law of the local time of M at level a is

absolutely continuous, for any time $t > 0$. This last result is actually a particular case of [7], where it is proved that for any non creeping Lévy process, the law of the first passage time over $x > 0$ is always absolutely continuous.

5 Proofs and further results

Proof. of Theorem 4. Let \mathbf{e} be an exponential time with parameter $\varepsilon > 0$ which is independent of (X, \mathbb{P}) . Recall the notations of Section 2 and for $\omega \in \mathcal{D}$, define $d_s = \inf\{u > s : \omega_u = 0\}$. From the independence of \mathbf{e} and Fubini Theorem, we have for all bounded function f on \mathbb{R}_+ and for all bounded Borel functionals F and K on \mathcal{D} ,

$$\begin{aligned} \mathbb{E}(f(g_{\mathbf{e}})F \circ r_{g_{\mathbf{e}}}K \circ k_{\mathbf{e}-g_{\mathbf{e}}} \circ \theta_{g_{\mathbf{e}}}) &= \mathbb{E}\left(\int_0^\infty \varepsilon e^{-\varepsilon t} f(g_t)F \circ r_{g_t}K \circ k_{t-g_t} \circ \theta_{g_t} dt\right) \\ &= \mathbb{E}\left(\sum_{s \in G} \varepsilon e^{-\varepsilon s} f(s)F \circ r_s \int_s^{d_s} e^{-\varepsilon(u-s)} K \circ k_{u-s} \circ \theta_s du\right) \\ &\quad + \mathbb{E}\left(\int_0^\infty \varepsilon e^{-\varepsilon t} f(t)F \circ r_t \mathbb{1}_{\{g_t=t\}} dt\right) K(\omega^0). \end{aligned}$$

Recall from Section 2 that ε^s denotes the excursion starting at s . Then

$$\begin{aligned} \mathbb{E}(f(g_{\mathbf{e}})F \circ r_{g_{\mathbf{e}}}K \circ k_{\mathbf{e}-g_{\mathbf{e}}} \circ \theta_{g_{\mathbf{e}}}) &= \mathbb{E}\left(\sum_{s \in G} \varepsilon e^{-\varepsilon s} f(s)F \circ r_s \int_0^{d_s-s} e^{-\varepsilon u} K(\varepsilon^s \circ k_u) du\right) \\ &\quad + \mathbb{E}\left(\int_0^\infty \varepsilon e^{-\varepsilon t} f(t)F \circ r_t \mathbb{1}_{\{X_t=\bar{X}_t\}} dt\right) K(\omega^0). \end{aligned} \quad (5.1)$$

The process

$$(s, \omega, \varepsilon) \mapsto e^{-\varepsilon s} f(s)F \circ r_s(\omega) \int_0^{\zeta(\varepsilon)} e^{-\varepsilon u} K \circ k_u(\varepsilon) du$$

is $\mathcal{P}(\mathcal{F}_s) \otimes \mathcal{E}$ -measurable, so that by applying (2.2) and (2.8) to equality (5.1), we obtain

$$\begin{aligned} \frac{1}{\varepsilon} \mathbb{E}(f(g_{\mathbf{e}})F \circ r_{g_{\mathbf{e}}}K \circ k_{\mathbf{e}-g_{\mathbf{e}}} \circ \theta_{g_{\mathbf{e}}}) &= \mathbb{E}\left(\int_0^\infty dL_s e^{-\varepsilon s} f(s)F \circ r_s\right) n\left(\int_0^\zeta e^{-\varepsilon u} K \circ k_u du\right) \\ &\quad + d \mathbb{E}\left(\int_0^\infty dL_s e^{-\varepsilon s} f(s)F \circ r_s\right) K(\omega^0). \end{aligned} \quad (5.2)$$

From the time reversal property of Lévy processes, see Lemma 2, p.45 in [2], under \mathbb{P} we have $X \circ k_e \stackrel{(d)}{=} X \circ r_e$, so that

$$\mathbb{E}(f(g_{\mathbf{e}})F \circ r_{g_{\mathbf{e}}}K \circ k_{\mathbf{e}-g_{\mathbf{e}}} \circ \theta_{g_{\mathbf{e}}}) = \mathbb{E}(f(\mathbf{e} - g_{\mathbf{e}}^*)K \circ r_{g_{\mathbf{e}}^*}F \circ k_{\mathbf{e}-g_{\mathbf{e}}^*} \circ \theta_{g_{\mathbf{e}}^*}). \quad (5.3)$$

Doing the same calculation as in (5.2) for the reflected process at its minimum $X - \underline{X}$, we get

$$\begin{aligned} & \frac{1}{\varepsilon} \mathbb{E}(f(\mathbf{e} - g_{\mathbf{e}}^*) K \circ r_{g_{\mathbf{e}}^*} F \circ k_{\mathbf{e} - g_{\mathbf{e}}^*} \circ \theta_{g_{\mathbf{e}}^*}) \\ &= \mathbb{E} \left(\int_0^\infty dL_s^* e^{-\varepsilon s} K \circ r_s \right) n^* \left(\int_0^\zeta e^{-\varepsilon u} f(u) F \circ k_u du \right) \\ & \quad + \mathbf{d}^* \mathbb{E} \left(\int_0^\infty dL_s^* e^{-\varepsilon s} K \circ r_s \right) f(0) F(\omega^0). \end{aligned} \quad (5.4)$$

Then we derive from (5.2), (5.3) and (5.4), the following equality

$$\begin{aligned} & \mathbb{E} \left(\int_0^\infty dL_s e^{-\varepsilon s} f(s) F \circ r_s \right) n \left(\int_0^\zeta e^{-\varepsilon u} K \circ k_u du \right) \\ & \quad + \mathbf{d} \mathbb{E} \left(\int_0^\infty dL_s e^{-\varepsilon s} f(s) F \circ r_s \right) K(\omega^0) \\ &= \mathbb{E} \left(\int_0^\infty dL_s^* e^{-\varepsilon s} K \circ r_s \right) n^* \left(\int_0^\zeta e^{-\varepsilon u} f(u) F \circ k_u du \right) \\ & \quad + \mathbf{d}^* \mathbb{E} \left(\int_0^\infty dL_s^* e^{-\varepsilon s} K \circ r_s \right) f(0) F(\omega^0). \end{aligned} \quad (5.5)$$

Then by taking $f \equiv 1$, $F \equiv 1$ and $K \equiv 1$, we derive from (5.2) that

$$\kappa(\varepsilon, 0) = n(1 - e^{-\varepsilon\zeta}) + \varepsilon \mathbf{d}. \quad (5.6)$$

Now suppose that X is of type 1 or 2, so that $\mathbf{d}^* = 0$, from what has been recalled in Section 2. Hence with $K \equiv 1$ in (5.5) and using (5.6), we have

$$\mathbb{E} \left(\int_0^\infty dL_s e^{-\varepsilon s} f(s) F \circ r_s \right) \kappa(\varepsilon, 0) \kappa^*(\varepsilon, 0) = \varepsilon n^* \left(\int_0^\zeta e^{-\varepsilon u} f(u) F \circ k_u du \right). \quad (5.7)$$

But using (2.7) and plunging (5.7) into (5.2) gives

$$\begin{aligned} & \mathbb{E} \left(\int_0^\infty e^{-\varepsilon t} f(g_t) F \circ r_{g_t} K \circ k_{t-g_t} \circ \theta_{g_t} dt \right) = n^* \left(\int_0^\zeta e^{-\varepsilon u} f(u) F \circ k_u du \right) \times \\ & n \left(\int_0^\zeta e^{-\varepsilon u} K \circ k_u du \right) + \mathbf{d} n^* \left(\int_0^\zeta e^{-\varepsilon u} f(u) F \circ k_u du \right) K(\omega^0), \end{aligned}$$

so that part 1 and part 2 of the theorem follow for λ -almost every $t > 0$ by inverting the Laplace transforms in this equality. We easily check that for all $t > 0$, g_t is continuous at time t . Hence for any bounded and continuous functions f , K and F , both sides of these identities are continuous in t , hence they coincide for all $t > 0$. Then we extend this result to any bounded Borel functions f , H and F through a classical density argument. Finally, part 3 is obtained in the same way as parts 1 and 2. \square

Recall that the definition of the ladder height process (H_t) has been given in Section 2. Then define $(\ell_x, x \geq 0)$ as the right continuous inverse of H , i.e.

$$\ell_x = \inf\{t : H_t > x\}.$$

Note that for types 1 and 2, since H is a strictly increasing subordinator, the process $(\ell_x, x \geq 0)$ is continuous, whereas in type 3, since H is a compound Poisson process, then ℓ is a càdlàg jump process. Parts 1 and 2 of the following lemma are reinforcements of Theorems 3 and 5 in [1]. Recall that $V(dt, dx)$ denotes the potential measure of the ladder process (τ, H) .

Lemma 1. *Let X be a Lévy process which is not a compound Poisson process and such that $|X|$ is not a subordinator.*

1. *The following identity between measures holds on \mathbb{R}_+^3 :*

$$u\mathbb{P}(X_t \in dx, \ell_x \in du) dt = t\mathbb{P}(\tau_u \in dt, H_u \in dx) du. \quad (5.8)$$

2. *The following identity between measures holds on \mathbb{R}_+^2 :*

$$\mathbf{d}^* \delta_{\{(0,0)\}}(dt, dx) + q_t^*(dx) dt = V(dt, dx), \quad (5.9)$$

moreover for all $t > 0$, and for all Borel sets $B \in \mathcal{B}_{\mathbb{R}_+}$, we have,

$$q_t^*(B) = t^{-1} \mathbb{E}(\ell(X_t) \mathbb{1}_{\{X_t \in B\}}). \quad (5.10)$$

3. *For all $t > 0$, the measures $q_t^*(dx)$ and $p_t^+(dx)$ are equivalent on \mathbb{R}_+ .*

Proof. When 0 is regular for $(-\infty, 0)$, part 1. is proved in Theorem 3 of [1] and when 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$, part 2. is proved in Theorem 5 of [1]. Although the proofs of parts 1 and 2 follow about the same scheme as in [1], it is necessary to check some details.

First recall the so-called Fristedt identity which is established in all the cases concerned by this lemma, in [13], see Theorem 6.16. For all $\alpha \geq 0$ and $\beta \geq 0$,

$$\kappa(\alpha, \beta) = \exp\left(\int_0^\infty dt \int_{[0, \infty)} (e^{-t} - e^{-\alpha t - \beta x}) t^{-1} \mathbb{P}(X_t \in dx)\right). \quad (5.11)$$

Note that the constant k which appears in this theorem is equal to 1, according to our normalization, see Section 1. Then recall (2.6): $\mathbb{E}(e^{-\alpha\tau_u - \beta H_u}) = e^{-u\kappa(\alpha, \beta)}$. This expression is differentiable, in $\alpha > 0$ and in $u > 0$ and using (5.11), we obtain:

$$\begin{aligned} \mathbb{E}(\tau_u e^{-\alpha\tau_u - \beta H_u}) &= u \mathbb{E}(e^{-\alpha\tau_u - \beta H_u}) \frac{\partial}{\partial \alpha} \kappa(\alpha, \beta) \\ &= -u \int_0^\infty e^{-\alpha t} \mathbb{E}(e^{-\beta X_t} \mathbb{1}_{\{X_t \geq 0\}}) dt \frac{\partial}{\partial u} \mathbb{E}(e^{-\alpha\tau_u - \beta H_u}) \\ &= -u \frac{\partial}{\partial u} \mathbb{E}\left(\int_0^\infty \exp(-\alpha(t + \tau_u) - \beta(\tilde{X}_t + H_u)) \mathbb{1}_{\{\tilde{X}_t \geq 0\}} dt\right), \end{aligned}$$

where \tilde{X} is a copy of X which is independent of (τ_u, H_u) . We may take for instance $\tilde{X} = (X_{\tau_u+t} - X_{\tau_u}, t \geq 0)$, so that it follows from a change of variables and the definition of $(\ell_x, x \geq 0)$,

$$\begin{aligned} \mathbb{E}(\tau_u e^{-\alpha\tau_u - \beta H_u}) &= -u \frac{\partial}{\partial u} \mathbb{E} \left(\int_0^\infty \exp(-\alpha(t + \tau_u) - \beta X_{\tau_u+t}) \mathbb{1}_{\{X_{\tau_u+t} \geq H_u\}} dt \right) \\ &= -u \frac{\partial}{\partial u} \mathbb{E} \left(\int_0^\infty \exp(-\alpha t - \beta X_t) \mathbb{1}_{\{X_t \geq H_u, \tau_u \leq t\}} dt \right) \\ &= -u \frac{\partial}{\partial u} \int_0^\infty dt e^{-\alpha t} \int_{[0, \infty)} e^{-\beta x} \mathbb{P}(X_t \in dx, \ell_x > u), \end{aligned}$$

from which we deduce that

$$\int_{[0, \infty)^2} e^{-\alpha t - \beta x} t \mathbb{P}(\tau_u \in dt, H_u \in dx) du = \int_{[0, \infty)^2} e^{-\alpha t - \beta x} u \mathbb{P}(X_t \in dx, \ell_x \in du) dt,$$

and (5.8) follows by inverting the Laplace transforms.

Let \mathbf{e} be an exponentially distributed random variable with parameter ε , which is independent of X . From identity (6.18), p. 159 in [13], we have

$$\mathbb{E}(\exp(-\beta \bar{X}_{\mathbf{e}})) = \kappa(\varepsilon, 0) \int_{[0, \infty)^2} e^{-\varepsilon t - \beta x} \int_0^\infty \mathbb{P}(\tau_s \in dt, H_s \in dx) ds. \quad (5.12)$$

Suppose that X is of type 1 or 2. By taking the Laplace transforms in x and t of identity (4.4) in Corollary 4, we obtain

$$\mathbb{E}(\exp(-\beta \bar{X}_{\mathbf{e}})) = (\varepsilon \mathbf{d} + n(1 - e^{-\varepsilon \zeta})) n^* \left(\int_0^\zeta e^{-\varepsilon s} e^{-\beta \varepsilon s} ds \right), \quad (5.13)$$

and by comparing (5.6), (5.12) and (5.13), it follows

$$n^* \left(\int_0^\zeta e^{-\varepsilon s} e^{-\beta \varepsilon s} ds \right) = \int_{[0, \infty)^2} e^{-\varepsilon t - \beta x} \int_0^\infty \mathbb{P}(\tau_s \in dt, H_s \in dx) ds. \quad (5.14)$$

Then we derive part 2 from (5.14) and (5.8). If X is of type 3, then taking the Laplace transforms in x and t of identity (4.4), gives

$$\mathbb{E}(\exp(-\beta \bar{X}_{\mathbf{e}})) = n(1 - e^{-\varepsilon \zeta}) \left(\mathbf{d}^* + n^* \left(\int_0^\zeta e^{-\varepsilon t} e^{-\beta \varepsilon t} dt \right) \right), \quad (5.15)$$

so that by comparing (5.6), (5.12) and (5.15), we obtain

$$\mathbf{d}^* + n^* \left(\int_0^\zeta e^{-\varepsilon t} e^{-\beta \varepsilon t} dt \right) = \int_{[0, \infty)^2} e^{-\varepsilon t - \beta x} \int_0^\infty \mathbb{P}(\tau_s \in dt, H_s \in dx) ds, \quad (5.16)$$

and part 2 follows from (5.16) and (5.8) in this case.

Then we show the third assertion. First note that q_t^* is absolutely continuous with respect to p_t^+ for all $t > 0$, since from (5.10) we have for any Borel set $B \subset \mathbb{R}_+$ such that $\mathbb{P}(X_t \in B) = 0$,

$$q_t^*(B) = t^{-1} \mathbb{E}(\ell(X_t) \mathbb{1}_{\{X_t \in B\}}) = 0.$$

Conversely, take a Borel set $B \subset \mathbb{R}_+$, such that $\mathbb{P}(X_t \in B) > 0$. Then since $\mathbb{P}(X_t = 0) = 0$, there exists $y > 0$ such that $\mathbb{P}(X_t \in B, X_t > y) > 0$. As the right continuous inverse of a subordinator, (ℓ_x) is nondecreasing and we have for all $x > 0$, $\mathbb{P}(\ell_x > 0) = 1$. Therefore the result follows from the inequality:

$$0 < \mathbb{E}(\ell_y \mathbb{I}_{\{X_t \in B, X_t > y\}}) \leq \mathbb{E}(\ell(X_t) \mathbb{I}_{\{X_t \in B\}}),$$

together with identity (5.10). \square

Recall from Section 2 that π is the Lévy measure of the ladder time process τ and that $\bar{\pi}(t) = \pi(t, \infty)$.

Lemma 2. *Under the assumption of Lemma 1, for all $t > 0$, the following measures of \mathbb{R}_+ :*

$$\int_0^t \bar{\pi}(t-s) q_s^*(dx) ds \quad \text{and} \quad \int_0^t q_s^*(dx) ds$$

are equivalent.

Proof. For all Borel set $B \subset \mathbb{R}_+$, we have

$$\bar{\pi}(t) \int_0^t q_s^*(B) ds \leq \int_0^t \bar{\pi}(t-s) q_s^*(B) ds,$$

hence $\int_0^t q_s^*(dx) ds$ is absolutely continuous with respect to $\int_0^t \bar{\pi}(t-s) q_s^*(dx) ds$. Moreover, for all $\varepsilon \in (0, t)$ and all Borel set $B \subset \mathbb{R}_+$, we may write

$$\begin{aligned} \int_0^t \bar{\pi}(t-s) q_s^*(B) ds &\leq \\ \bar{\pi}(\varepsilon) \int_0^t q_s^*(B) ds + \int_{t-\varepsilon}^t \bar{\pi}(t-s) q_s^*(B) ds &< \infty. \end{aligned}$$

Hence if $\int_0^t q_s^*(B) ds = 0$, then for all $\varepsilon \in (0, t)$,

$$\int_0^t \bar{\pi}(t-s) q_s^*(B) ds \leq \int_{t-\varepsilon}^t \bar{\pi}(t-s) q_s^*(B) ds < \infty.$$

The finiteness of the right hand side of the above inequality can be derived from Corollary 4. Hence this term tends to 0 as ε tends to 0, so that the equivalence between the measures $\int_0^t \bar{\pi}(t-s) q_s^*(dx) ds$ and $\int_0^t q_s^*(dx) ds$ is proved. \square

Now we are ready to prove all the results of Section 3.

Proof of Theorem 1. When X is of type 1 or 2, it follows from Corollary 4, part 3 of Lemma 1, relation (2.9) and Lemma 2. When X is of type 3, the arguments are the same, except that one has to take account of the fact that the law of \bar{X}_t has an atom at 0, as it is specified in Corollary 4. \square

Proof of Theorem 2. We first prove that part 1 implies that for all $t > 0$, the law of \overline{X}_t is absolutely continuous. To that aim, observe that

$$\overline{X}_{2t} = \max \left\{ \overline{X}_t, X_t + \sup_{0 \leq s \leq t} (X_{t+s} - X_t) \right\} = \max \left\{ \overline{X}_t, X_t + \sup_{0 \leq s \leq t} X_s^{(1)} \right\},$$

where $X^{(1)}$ is an independent copy of X . From this independence and the above expression, we easily deduce that if the law of \overline{X}_t is absolutely continuous, then so is this of \overline{X}_{2t} . Therefore, from Theorem 1, the measure μ_{2t}^+ is absolutely continuous. This clearly implies that for all $s \in (0, 2t]$, the measure μ_s^+ is absolutely continuous. Applying Theorem 1 again, it follows that the law of \overline{X}_s is absolutely continuous, for all $s \in (0, 2t]$. Then we show the desired result by reiterating this argument.

Let us assume that part 1 holds. Then for all $t > 0$, the law of \overline{X}_t is absolutely continuous. Therefore the resolvent measure $U(dx)$ is absolutely continuous. Indeed, let \mathbf{e} be an independent exponentially distributed random time with parameter 1, then the law of $\overline{X}_{\mathbf{e}}$ admits a density, hence the law of $X_{\mathbf{e}} = X_{\mathbf{e}} - \overline{X}_{\mathbf{e}} + \overline{X}_{\mathbf{e}}$ also admits a density, since the random variables $X_{\mathbf{e}} - \overline{X}_{\mathbf{e}}$ and $\overline{X}_{\mathbf{e}}$ are independent, see Chap. VI in [2]. Since the law of $X_{\mathbf{e}}$ is precisely the measure $U(dx)$, we have proved that part 1 implies part 3. Then part 3 clearly implies part 2 and from Corollary 1, part 2 implies part 1.

First observe that $V(dx)$ is absolutely continuous if and only if $\int_0^t q_s^*(dx) ds$ is absolutely continuous, for all $t > 0$. Indeed, from part 2 of Lemma 1, we have $V(dx) = \int_0^\infty q_s^*(dx) ds$, hence if $V(dx)$ is absolutely continuous, then so are the measures $\int_0^t q_s^*(dx) ds$, for all $t > 0$. Conversely assume that the measures $\int_0^t q_s^*(dx) ds$ are absolutely continuous for all $t > 0$. Let A be a Borel set of \mathbb{R}_+ such that $\lambda_+(A) = 0$. From the assumption, $q_s^*(A) = 0$, for λ -almost every $s > 0$, hence $V(A) = \int_0^\infty q_s^*(A) ds = 0$, so that $V(dx)$ is absolutely continuous. Then from Lemma 2 and Corollary 4, for each t , the law of \overline{X}_t is equivalent to the measure $\int_0^t q_s^*(dx) ds$. Therefore part 4 is equivalent to part 1, from the argument of the beginning of this proof. \square

Proof of Theorem 3. If $p_t^+ \ll \lambda^+$ for all $t > 0$, then from part 3 of Lemma 1, $q_t^* \ll \lambda^+$, for all $t > 0$. Suppose moreover that 0 is regular for $(0, \infty)$ and let A be a Borel subset of \mathbb{R} , such that $\lambda(A) = 0$. Then from Corollary 5 and Fubini Theorem, we have

$$p_t(A) = \int_0^t ds q_s^* * \bar{q}_{t-s}(A) + \mathbf{d} q_t^*(A),$$

where \bar{q}_s and q_s^* are extended on \mathbb{R} , as in this corollary. But from the assumptions, for all $0 < s < t$, $q_s^* * \bar{q}_{t-s}(A) = 0$ and $q_t^*(A) = 0$, hence $p_t(A) = 0$, for all $t > 0$ and p_t is absolutely continuous, for all $t > 0$. So part 1 implies part 2 and the converse is obvious.

Then it readily follows from part 3 of Lemma 1 and identity (5.9) that part 1 implies part 3 (recall that $\mathbf{d}^* = 0$ in the present case). Now suppose that $V(dt, dx)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}_+^2 . Then we derive from identity (5.9) that the measures $q_t^*(dx)$ are absolutely continuous for λ -almost

every $t > 0$. From Corollary 5, it means that p_t is absolutely continuous for λ -almost every $t > 0$. But if the semigroup p_t is absolutely continuous for some t , then p_s is absolutely continuous for all $s \geq t$. Hence p_t is actually absolutely continuous, for all $t > 0$ and part 3 implies part 2.

Then suppose that X is of type 1 and recall that $\mathbf{d} = \mathbf{d}^* = 0$ in this case. From Corollary 3 and part 2 of Lemma 1, we have:

$$\mathbb{P}(g_t \in ds, \bar{X}_t \in dx) = n(t - s < \zeta)V(ds, dx). \quad (5.17)$$

Since $n(t - s < \zeta) > 0$, for all $s \in [0, t]$, we easily derive from identity (5.17) that part 3 and part 4 are equivalent.

Let us denote by p_t^- the restriction of p_t to \mathbb{R}_- . If part 2 is satisfied, then p_t^+ and p_t^- are absolutely continuous for all $t > 0$. Then from part 3 of Lemma 1 applied to X and its dual process $-X$, it follows that q_t and q_t^* are absolutely continuous for all $t > 0$, so that from Corollary 3, the triple (g_t, \bar{X}_t, X_t) is absolutely continuous for all $t > 0$, hence part 2 implies part 5. Then part 5 clearly implies part 4. \square

Proof of Proposition 1. In this proof, it suffices to assume that Y is a deterministic process, i.e. (T_n) , (a_n) and (b_n) in (3.2) are deterministic sequences.

In order to prove part 1, let us first assume that $a_n = 0$, for all n . Then recall that from Theorem 2, the law of \bar{X}_t is absolutely continuous, for all $t > 0$. Fix $t > 0$ and let n be such that $t \in [T_n, T_{n+1})$. Set $Z_k = Y_{T_k} + \sup_{T_k \leq s < T_{k+1}} X_s$ and $Z = Y_{T_n} + \sup_{T_n \leq s < t} X_s$, then we have

$$\sup_{s \leq t} X_s + Y_s = \max\{Z_1, Z_2, \dots, Z_{n-1}, Z\}. \quad (5.18)$$

But we can write

$$Z_k = Y_{T_k} + X_{T_k} + \sup_{s \leq T_{k+1} - T_k} X_s^{(k)} \quad \text{and} \quad Z = Y_{T_n} + X_{T_n} + \sup_{s \leq t - T_n} X_s^{(n)}, \quad (5.19)$$

where $X^{(k)}$, $k = 1, \dots, n$ are copies of X such that $X, X^{(k)}$, $k = 1, \dots, n$ are independent. From Theorem 2, the laws of $\sup_{s \leq T_{k+1} - T_k} X_s^{(k)}$ and $\sup_{s \leq t - T_n} X_s^{(n)}$ are absolutely continuous. From the representation (5.19) and the independence hypothesis, we derive that the laws of Z_1, Z_2, \dots, Z_{n-1} and Z are absolutely continuous. Since the maximum of any finite sequence of absolutely continuous random variables is itself absolutely continuous, we conclude that the law of $\sup_{s \leq t} X_s + Y_s$ is absolutely continuous and the first part is proved.

Now we assume that (a_n) is any deterministic sequence. Then we have (5.18) with

$$Z_k = b_k + X_{T_k} + \sup_{s \leq T_{k+1} - T_k} X_s^{(k)} + a_k s \quad \text{and} \quad Z = b_n + X_{T_n} + \sup_{s \leq t - T_n} X_s^{(n)} + a_n s, \quad (5.20)$$

where $X^{(k)}$, $k = 1, \dots, n$ are as above. If $p_t^+ \ll \lambda^+$ for all t , then this property also holds for the process X with any drift a , i.e. $X_t + at$, so from Theorem 1 the laws of $\sup_{s \leq T_{k+1} - T_k} X_s^{(k)} + a_k s$ and $\sup_{s \leq t - T_n} X_s^{(n)} + a_n s$ are absolutely continuous and we conclude that the law of $\sup_{s \leq t} X_s + Y_s$ is absolutely continuous, in the same way as for the first part.

Finally, if X has unbounded variations, then it is of type 1. If moreover for instance the ladder height process at the supremum H has a positive drift, then from Theorem 2 and the remark thereafter, the law of \overline{X}_t is absolutely continuous for all $t > 0$. Since X has unbounded variations, it follows from (iv) p. 64 in [9] that for any $a \in \mathbb{R}$, the ladder height process at the supremum of the drifted Lévy process $X_t + at$ also has a positive drift, and since $X_t + at$ is also of type 1, the law of $\sup_{s \leq t} X_s + as$ is absolutely continuous. Then from Theorem 2, the laws of $\sup_{s \leq T_{k+1} - T_k} X_s^{(k)} + a_k s$ and $\sup_{s \leq t - T_n} X_s^{(n)} + a_n s$ are absolutely continuous and again we conclude that the law of $\sup_{s \leq t} X_s + Y_s$ is absolutely continuous, in the same way as for the first part. \square

Proof of Proposition 2: Recall that under the assumption of this proposition, we have $\mathbf{d}^* = 0$. So, we derive from Corollary 3, by integrating identity (4.1) over y and from part 2 of Lemma 1, that

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) = s^{-1} n(t-s < \zeta) \mathbb{E}(\ell(x) \mathbb{1}_{\{X_s \in dx\}}) \mathbb{1}_{(0,t]}(s) ds + \mathbf{d}\delta_{\{t\}}(ds) t^{-1} \mathbb{E}(\ell(x) \mathbb{1}_{\{X_s \in dx\}}).$$

Since X has no positive jumps, then \overline{X}_t continuous. Moreover, it is an increasing additive functional of the reflected process $\overline{X}_t - X_t$, such that

$$\mathbb{E} \left(\int_0^\infty e^{-t} d\overline{X}_t \right) = \Phi(1)^{-1},$$

where Φ is the Laplace exponent of the subordinator $T_x = \inf\{t : X_t > x\}$. Hence we have $L_t = c\overline{X}_t$, with $c = \Phi(1)$. Then it follows from the definition of H and ℓ , that

$$H_u = c^{-1}u, \text{ on } H_u < \infty, \text{ and } \ell_x = cx \text{ on } \ell_x < \infty.$$

Besides, from part 1 of Lemma 1, we have by integrating (5.8) over $u \in [0, \infty)$,

$$cxp_t^+(dx) dt = ct\mathbb{P}(\tau_{cx} \in dt) dx, \tag{5.21}$$

as measures on $[0, \infty)^2$. This ends the proof of the proposition. \square

Finally note that identity (5.21) may also be derived from Corollary VII.3, p.190 in [2] or from Theorem 3 in [1]. The constant c appearing in our expression is due to the choice of the normalization of the local time in (2.1).

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