Hyperstructures associated to a connected graph

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Abstract

We are searching to associate, in a natural way, a hyperstructure to a connected graph showing that in many relevant cases it is a join-space (under the convention in [3]).

1. Let H be a nonempty set and $\mathcal{P}^*(H) = \{A \mid A \subseteq H, A \neq \emptyset\}$ (where $\subseteq$ stands for the non-strict inclusion). A *hyperoperation* on H is a function $(x,y) \longrightarrow xoy$ from $H \times H$ to $\mathcal{P}^*(H)$. We will denote $A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}$ for $A, B \in \mathcal{P}^*(H)$.

A hyperoperation o is called *associative* if

$$(\forall)x,y,z \in H : (x \circ y) \circ z = x \circ (y \circ z).$$

In this case (H,o) will be referred to as a *semi-hypergroup*. The hyperoperation o is called *reproducible*, if $xoH = Hox = H$ for each $x \in$H.

The semi-hypergroup $(H, \circ)$ with " o " reproducible is called a *hypergroup*. In this case we introduce for each $a, b \in H$ $a/b = \{c \in H \mid a \in b \circ c\}$.

A commutative hypergroup (H,o) which satisfies:

$$(J) \qquad (\forall)(a,b,c,d) \in H^4, a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset$$

is called a *join-space* (as in [3]).

If in addition the join-space (H,o) has the property (see [1], pag. 71):

$$x \circ x = \{x\} = x/x \text{ for all } x \in H$$

$(H, \circ)$ is said to be a *geometrical join-space*. Otherwise it is called a non-geometrical join-space.

2. Let G=(V,E) be a tree (from now on we suppose *card* $(V) \geq 2$).

(a) $\forall x \in V, x \circ x = \{x\}$;

(b) $\forall x, y \in V, x \neq y, x \circ y = [xy] = \{z \in V \mid z$ is a vertex on the elementary path that joins x and y $\}$. Note that because G is a tree, (b) is consistent and

non-contradictory (G tree $\Longrightarrow \forall x, y \in V, x \neq y$, there exists exactly one path joining x and y).

Theorem 1. *Let G=(V,E) be a tree and (V,o) be its associated hyperstructure by the method above. Then (V,o) is a non-geometrical join-space.*

Proof. We notice for the beginning that o is commutative, $x \circ x = x/x = \{x\}$ for each $x \in V$, and $\{x, y\} \subseteq x \circ y$, for any $x, y \in V$ with $x \neq y$.

To prove the associativity we start by showing that:

$$(\forall) x, y, z \in V : (x \circ y) \circ z \subseteq x \circ (y \circ z). \tag{1}$$

If $x = y$ then $(x \circ y) \circ z = x \circ z, x \circ (y \circ z) = x \circ (x \circ z)$ and since $x \circ z \subseteq x \circ (x \circ z)$, (1) follows.

If $x \neq y$, denote by $x = \alpha_1, \alpha_2, ..., \alpha_n = y$ the path $[xy]$ that joins x and y, and by $y = \beta_1, \beta_2, ..., \beta_k = z$ the path [yz]. Two situations occur:

i) $x \circ y \cap y \circ z = \{y\}$; then one can easily see that

$$(x \circ y) \circ z = x \circ y \cup y \circ z = \{\alpha_1, \alpha_2, ..., \alpha_n = y = \beta_1, \beta_2, ..., \beta_k = z\}$$

and also

$$x \circ (y \circ z) = x \circ \{\beta_1, \beta_2, ..., \beta_k = z\} = \{\alpha_1, \alpha_2, ..., \alpha_n = y = \beta_1, \beta_2, ..., \beta_k = z\};$$

so we have equality.

ii) $x \circ y$, $y \circ z$ have common points different from y. As G is a tree, there exists $p \in \{1, ..., n\}, q \in \{1, ..., k\}$ such that $q = \max\{i \mid \beta_i \in x \circ y\}$ $\alpha_p = \beta_q$ and

$$x \circ y = \{\alpha_1 = x, \alpha_2, ..., \alpha_{p+1} = \beta_{q-1}, ...\alpha_p = \beta_q, \alpha_n = y = \beta_1\}$$

$$y \circ z = \{\beta_1 = y = \alpha_n, \beta_2 = \alpha_{n-1}, ..., \beta_q = \alpha_p, \beta_{q+1}, ..., \beta_k = z\}.$$

As a consequence of this,
$x \circ z = \{x = \alpha_1, \alpha_2, ..., \alpha_p = \beta_q, \beta_{q+1}, ..., \beta_k = z\}$(because G has no cycles).
Finally one obtains

$$(x \circ y) \circ z = x \circ (y \circ z) = \{\alpha_1, ..., \alpha_n, \beta_{q+1}, ..., \beta_k\}.$$

So far we proved (1), therefore we can write:

$$(x \circ y) \circ z \subseteq x \circ (y \circ z) = (y \circ z) \circ x \subseteq y \circ (z \circ x) = (z \circ x) \circ y \subseteq z \circ (x \circ y) = (x \circ y) \circ z.$$

Now all the signs $\subseteq$ can be replaced by $=$ in this last relation; and therefore, the associativity of o is proved.

We noticed at the beginning that $(\forall) x, y \in V : y \in x \circ y$. So $(\forall) y \in V : y \in x \circ y \subseteq x \circ V$ where from $(\forall) y \in V : y \in x \circ V = V \circ x$. Hence o is reproducible.

To finish the proof we must show that (J) holds. Let (a,b,c,d)$\in V^4$ such that $a/b \cap c/d \neq \emptyset$. This means $(\exists)\alpha \in V, \alpha \in a/b \cap c/d$ (hence $a \in b \circ \alpha, c \in d \circ \alpha$). We have two cases to analyze:

i) $b \circ \alpha \cap d \circ \alpha = \{\alpha\}$. As $a \circ d \cap b \circ c$ contains at least a,c,α (J) follows.

ii) $b \circ \alpha \cap d \circ \alpha \neq \{\alpha\}$. Let $b \circ \alpha = \{\alpha_1 = b, ..., \alpha_i = a, ..., \alpha_n = \alpha\}, d \circ \alpha = \{\beta_1 = d, ..., \beta_j = c, ..., \beta_m = \alpha\}$ and $\beta_{j_0} = \max\{k \mid \beta_k \in b \circ \alpha\}, \alpha_{i_0} = \beta_{j_0} = e$.

There are two possibilities for a: $a \in b \circ e$ or $a \in e \circ \alpha$and likewise for c:$c \in d \circ e$ or $c \in e \circ \alpha$. As we will see, in all cases (J) is valid:

a) $a \in b \circ e, c \in e \circ \alpha; b \circ c \cap a \circ d$ contains a, e.

b) $a \in b \circ e, c \in d \circ e$; we find a,c,e are in are in $boc \cap a \circ d$.

c) $\{a \in e \circ \alpha, c \in e \circ \alpha\}$ or $\{a \in b \circ e, c \in d \circ e\}$; then $e \in b \circ c \cap a \circ d$.

The proof is now complete.

Remark 1. Denote by H_G the join-space (V,o) associated to a tree G=(V,E) like above. If Γ_t is the set of isomorphism classes of trees and $\mathcal{G}$ the set of isomorphism classes of join-spaces, then the function $f : \Gamma \longrightarrow \mathcal{G}$ defined by $f(\overline{G}) = \overline{H_G}$ is injective (here $\overline{x}$ is the class of x). Indeed, the condition $x \circ y = \{x, y\}$ in H_G is equivalent to (x, y) being an edge in the tree G and so we can construct in only one way the graph G starting from H_G.

3. Let G=(V,E) be a graph. We say that G is *2-connected* if $card(V) > 2$, G is connected and by removing a vertex one obtains a connected graph too. A *block* in G is a maximal 2-connected subgraph of G.

Let G=(V,E) be a connected graph and denote by $b_1, \ldots, b_k$ its blocks. Construct the graph $G^* = (V^*, E^*)$ where $V^* = V \cup \{b_1, \ldots, b_k\}$ and $(x, y) \in E^*$ if and only if one of the following cases occurs:

i) $x \in V$, $y = b_i$ (for some $i \in \{1, \ldots, k\}$ and x is a vertex of b_i in G (or vice versa $y \in V$, $x = b_i$.)

ii) $(x, y) \in E$ and there is no elementary cycle passing through x and y.

Example 1. Let G as below (fig. 4), $b_1 = \{6, 7, 8, 9\}, b_2 = \{2, 3, 4, 5\}$. Then G^* is as in fig. 5.

Remark 2. G^* is a tree. Indeed, G connected implies G^* connected, and if in G^* there exists a cycle it contains at least two blocks b_i and b_j of G as points. Then in G^* are two paths from b_i to b_j and the same will be in G. But then b_i and b_j are contained in the same block of G, contradiction.

Let G=(V,E) be a connected graph and G^* its block-tree. We denote by H_G the join space H_{G^*} and we will shall say that H_G is the join-space associated to G.

Remark 3. i) If G is a tree then G is its own block-tree and so the definition above extends the one we gave in **2**.

ii) If we denote by Γ the set of isomorphism classes of graphs then $f^* : \Gamma \to \mathcal{G}$ defined by $f^*(\bar{G}) = \bar{H}_G$ is (unlike in the case of f) not injective.

iii) When G is 2-connected its block-tree is a star-shaped graph (there exists a point $b \in V$ such that any edge in G^* is of a form (b,a) or $(a,b), a \in V$) and the join -space associated to it

$$(V^*, \circ) : V^* = V \cup \{b\}, x \circ y = \{x, y, b\} \text{ for any } x, y \in V, x \neq y$$

gives no information about the graph-structure of G. In fact H_G characterizes the tree-structure of a graph; so it is natural to ask if there exists another hyperstructure construction for 2-connected graphs. An answer is given in Section 5.

4. In Section **3** we associated a join-space to a connected graph using its path-structure. A question arises: given a join-space $(J, \circ)$ with $card(J) < \infty$, under which conditions upon J there exists a graph G (in fact a tree) such that $J = H_G$?

In this section we suppose J is a finite set having at least two elements and $\circ$ is a hyperoperation on J. Let us consider the following conditions:

(A_1) $\circ$ is associative;
(A_2) $\circ$ is commutative;
(A_3) $x \circ J = J \circ x = J, \ \forall x \in J$;
(A_4) $x \circ x = x, \ \forall x \in J$.

Proposition 1. *If $(A_1) - (A_4)$ and (J) are valid for $(J, \circ)$, then for any $x, y, z \in J$ such that $z \in x \circ y$ and $y \in x \circ z$ one has y=z.*

Proof. Indeed, $z \in x \circ y \Rightarrow x \in z/y$ and $y \in x/y$ and $y \in x \circ z \Rightarrow x \in y/z$. Then $z/y \cap y/z \neq \emptyset \Rightarrow y \circ y \cap z \circ z \neq \emptyset \Rightarrow y = z$.

A join-space for which $(A_1) - (A_4)$ and (J) hold, as well as
(G) $\forall x, y \in J, \ x \neq y : \{x, y\} \subseteq x \circ y$ is called a *(g) - join space.*

Proposition 2. *Let $(J, \circ)$ be a (g)-join-space. Then for each $x \in J$ there exists $y \in J$ such that $x \circ y = \{x, y\}$.*

Proof. Let $y \in J$ be an element of $J \backslash \{x\}$ for which $card(x \circ y)$ is minimal. By hypothesis $(J, \circ)$ is a (g)-join- space; so that $card(x \circ y) \geq 2$. If $card(x \circ y) = 2$ than $(x \circ y) = \{x, y\}$. If $card(x \circ y) \geq 3$ there exists $z \in x \circ y \backslash \{x, y\}$. Now, $z \in x \circ y$ gives $x \circ z \subseteq x \circ (x \circ y) = (x \circ x) \circ y = x \circ y$. The case $x \circ z = x \circ y$ yields $y \in x \circ z \subset x \circ y$ (where $\subset$ stands for the strict inclusion) implies $card(x \circ z) < card(x \circ y)$, contradiction. This completes the proof.

Let $(J, \circ)$ be a (g)-join-space. We define a relation $\sim$ over J by:

$$x \sim y \leftrightarrow \{x \neq y \text{ and } x \circ y = \{x, y\}\}.$$

0	a	b	c	d	e
a	a	a,b	a,b,c	a, d ,e	a,e
b	a,b	b	b,c	b,c,d	a,b,e
c	a,b,c	b,c	c	c,d	c,d,e
d	a,d,e	b,c,d	c,d	d	d,e
e	a,e	a,b,e	c, d, e	d,e	e

One can easily see that $\sim$ is symmetrical whenever $\circ$ is commutative. Any (g)-join-space $(J, \circ)$ which satisfies the condition:

(P) $\forall k \in N$, $(\forall)\{x_1, x_2, \ldots, x_k\} \subseteq J$ with $x_1 \sim x_2, \ldots x_{k-1} \sim x_k$ then

$$\{x_1, \ldots x_k\} \subseteq x_1 \circ x_k$$

is called a *path join-space.*

Remark 4. i) From $\{x_1, ..., x_k\}$ being a set it follows $x_i \neq x_j$ for all $i, j \in \{1, \ldots k\}$, $i \neq j$.

ii) The above definition is not trivial. This is shown from the following example: Let $(J, \circ), J = \{a, b, c, d, e\}$ be a (g)- join-space as below:

Note that $x \circ y$ is the minimal path that joins x and y in the graph from fig.6. This (g)-join-space is not a path join-space.

Theorem 2. *Let $(J, \circ)$ be a path join-space. Then there is a tree G=(V,E) such that $J = H_G$.*

Proof. Let $V = J$, $E = \{(x, y) \in J^2 \mid x \sim y\}$, $G = (V, E)$. We shall show that G is a tree and $J = H_G$. To prove there is no cycle in G, suppose by contradiction that there exists $\{x_1, \ldots, x_k\} \subseteq J$ such that $x_1 \sim x_2$, $x_2 \sim x_3, \ldots, x_{k-1} \sim x_k$, $x_k \sim x_1 (k \leq 3)$. Then:

$$x_2 \in x_1/x_k (\text{because } x_k \sim x_1, \ x_1 \sim x_2 \Rightarrow x_1 \in x_2 \circ x_k)$$

$$x_2 \in x_k/x_1 (\text{because } x_1 \sim x_k, \ x_k \sim x_{k-1}, \ldots, x_3 \sim x_2 \Rightarrow x_k \in x_1 \circ x_2).$$

Hence $x_1/x_k \cap x_k/x_1 \neq \emptyset$ where from $x_1 \circ x_1 \cap x_k \circ x_k \neq \emptyset$, which is impossible.

Now suppose that G is not connected, i.e. there exist at least two connected components in G. Let x_i and x_j be such that $card(x_i \circ x_j)$ is minimal with the property that x_i and x_j are not in the same component. It follows that $x_i \not\sim x_j$; so $(\exists) z \in x_i \circ x_j \backslash \{x_i, x_j\}$. We have seen that $x_i \circ z \subset x_i \circ x_j$, $z \circ x_j \subset x_i \circ x_j$ implies $card(x_i \circ z) < card(x_i \circ x_j)$, $card(z \circ x_j) < card(x_i \circ x_j)$. Since x_i, x_j are not in the same component at least one of the pairs $(x_i, z), (x_j, z)$ has the same property, in contradiction with the minimality of $card(x_i \circ x_j)$. So far G

is a tree. Let $\square$ be the hyperoperation associated to the tree-structure of G. By condition (P), $x\square y \subseteq x \circ y$ for each $x, y \in J$. Suppose that:

$$(*) \quad c\square y \subset x \circ y \ (obviously\ x \neq y) \ for\ some\ x, y \in J,$$

and take (x, y) with the property that $card(x \circ y)$ has the minimum value among all the pairs which satisfy (*). Let $z \in x \circ y \backslash x\square y$ (of course $z \notin \{x, y\}$). Since $z \in x \circ y$ and $z \notin \{x, y\}$, $card(x \circ z) < card(x \circ y)$, $card(z \circ y) < card(x \circ y)$ and, by minimality, $x\square z = x \circ z$ and $z\square y = z \circ y$.

In G, $x\square z$ intersects $z\square y$ at least in one point (for example z). If $x\square z \cap z\square y = \{z\}$ we obtain $z \in x\square y$, contradiction. It follows that $\exists t \in (x\square z \cap z\square y)\backslash\{z\}$. Let us choose t such that $x\square t \cap y\square t = \{t\}$. Then $t \in x\square z \subseteq x \circ z$, and this yields $x \in t/z$. Since $z \in x \circ y \Rightarrow x \in z/y$ it follows that:

$$z/y \cap t/z \neq \emptyset \Rightarrow z \circ z \cap t \circ y \neq \emptyset \Rightarrow z \in y \circ t.$$

On the other hand we have in $(J, \square)$ $y\square x = y\square t \cup t\square x$. This yields $y\square t \subseteq y\square x$ where from $z \in y\square x$, contradiction. Hence $x\square y = x \circ y$, $\forall x, y \in J$ and consequently, $H = H_G$.

Remark 5. The result above shows that - supposing we keep the set V and we make no difference among paths joining the same end-points - the construction in Section **2** is unique and is possible only for trees.

5. Let G=(V,E) be a 2-connected graph. We have seen in Section 3 that in this case a new construction is needed. This is founded on an equivalent definition of a 2-connected graph according to which for any two different elements $x, y \in V$ there is a cycle passing through x,y. Define a hyperoperation $o : V \times V \to \mathcal{P}^*(V)$ by:

a. $x \circ x = \{x\} (\forall)\ x \in V$;

b. $x \circ y = \{z \in V |$ there is a cycle through $x, y, z\} \forall x, y \in V\ x \neq y$.

G being 2-connected o is well defined. It is easy to see that o is commutative and

(**) if $\{x, y, z\} \subseteq V$ then $x \in y \circ z \Leftrightarrow y \in x \circ z \Leftrightarrow z \in x \circ y$.

In general, o is not associative. This is shown by the following

Example 2. Let $H = \{a_1, a_2, b_1, b_2, c_1, c_2, m_1, m_2, n_1, n_2, p_1, p_2\}$ and the graph (fig. 8).

We notice that $(a_1 \circ a_1) \circ b_1 = a_1 \circ b_1 = H\backslash\{a_2, b_2\}$ and $a_1 \circ (a_1 \circ b_1) = a_1 \circ H\backslash\{a_2, b_2\} = H$, so o is not associative.

Theorem 3. *Let G= (V, E) be a 2-connected graph and o be the hyperoperation just defined. Then o is associative if and only if $x \circ y = V$ for each $x, y \in V$, $x \neq y$.*

Proof. One can easily see that o is reproducible and so the implication $\Rightarrow$ is obvious. Conversely, assume that o is associative and let $a, b, c \in V$ with $b \neq c$. We will shall prove $a \in b \circ c$. Suppose on the contrary $a \notin b \circ c$. If there exists $f \in b \circ c\backslash\{b, c\}$ with $f \in a \circ b$ then $a \neq f$ (because $a \notin b \circ c$). But $f \in a \circ b \Rightarrow a \in b \circ f$; hence $a \in b \circ f \subseteq b \circ (b \circ c) = (b \circ b) \circ c = b \circ c$, contradiction. It follows $a \circ b \cap b \circ c = \{b\}$ ($c \in a \circ b$ would imply $a \in b \circ c$). Similarly, we may show that $a \circ c \cap b \circ c = \{c\}$. Suppose now that there exists $x \in a \circ c \cap a \circ b$, $x \neq a$. Of course, if $x = b$ then $b \in a \circ c$, impossible. Hence $x \neq b$ and in the same way $x \neq c$. Now $x \in a \circ c$, $x \neq a, c \Rightarrow a \in c \circ x$ *and* $x \in a \circ b$, $x \neq a, b \Rightarrow b \in a \circ x$. So, $b \in (c \circ x) \circ x$ where from $b \in c \circ x$ which in turn gives $x \in b \circ c$, in contradiction with $x \in a \circ c$. We therefore proved that:

$$a \circ b \cap b \circ c = \{b\};\ a \circ b \cap a \circ c = \{a\};\ b \circ c \cap a \circ c = \{c\}. \qquad (2)$$

In $a \circ b$ there is a path P_{ab} joining a and b; let also P_{ac} and P_{cb} be paths in $a \circ c$ and $b \circ c$ respectively. We obtain from (2) that $P_{ab} \cup P_{ac} \cup P_{cb}$ is a cycle containing a,b,c; hence $a \in b \circ c$, contradiction. Summarizing up, $a \notin b \circ c$ leads to a contradiction; so $a \in b \circ c$ for any $b, c \in V$, $b \neq c$, and the proof is complete.

Remark 6. i) The results above can be restated as: o is associative if and only if for any three points in V there is a cycle containing them.

ii) An interesting class of 2-connected graphs is the one consisting of all such objects that contain a hamiltonian cycle. In this case o is associative but this property is not sufficient for a 2-connected graph to have a hamiltonian cycle as one can see from the example in fig. 9.

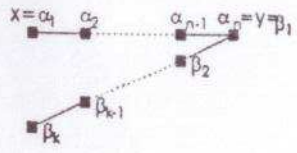

Fig. 1

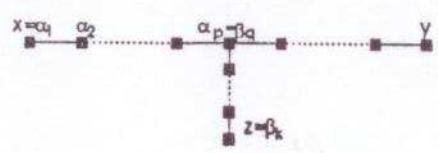

Fig. 2

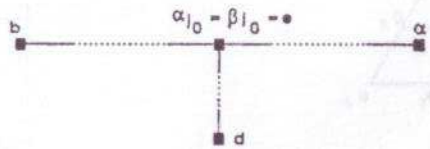

Fig. 3

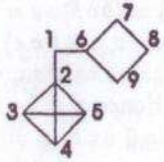

Fig. 4

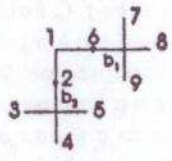

Fig. 5

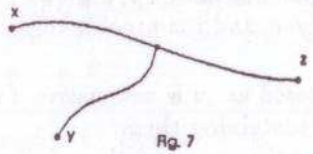

Fig. 7

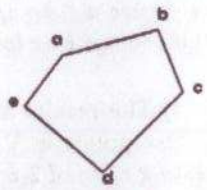

Fig. 6

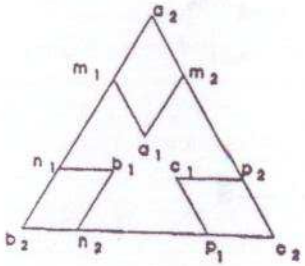

Fig. 8

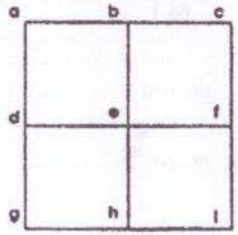

Fig. 9

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