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# Bedload transport in shallow water models: why splitting (may) fail, how hyperbolicity (can) help

S. Cordier<sup>a,\*</sup>, M.H. Le<sup>a,b</sup>, T. Morales de Luna<sup>c</sup>

<sup>a</sup>*Laboratoire de Mathématiques et Applications,  
Physique Mathématique d'Orléans (MAPMO)*

<sup>b</sup>*Bureau de Recherches Géologiques et Minières (BRGM)*

<sup>c</sup>*Departamento de Matemáticas, Universidad de Córdoba*

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## Abstract

In this paper, we are concerned with models for sedimentation transport consisting of a shallow water system coupled with a so called Exner equation that described the evolution of the topography. We show that, for some model of the bedload transport rate including the well-known Meyer-Peter and Müller model, the system is hyperbolic and, thus, linearly stable, only under some constraint on the velocity. In practical situations, this condition is hopefully fulfilled. The numerical approximations of such system are often based on a splitting method, solving first shallow water equation on a time step and, after updating the topography. It is proved that this strategy can create spurious/unphysical oscillations which are related to the study of hyperbolicity e.g. the sign of some eigenvalue of the coupled system differs from the splitting one. Some numerical results are given to illustrate these problems and the way to overcome them in some cases using an stronger C.F.L. condition.

*Keywords:* Shallow Water system, Exner equation, erosion modeling, sediment transport, hyperbolicity, stability, splitting method.

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## 1. Introduction

Soil erosion is caused by the movement of sediment due to mechanical actions of flow. In the context of bedload transport, a mass conservation also called Exner equation [1] is used to updated the bed elevation. This equation is often coupled with the shallow water equations describing the overland flows (see [2] and references therein). The complete system of PDE is written as

$$(1) \quad \begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) = -gh\partial_x z_b + gh\partial_x H \\ \partial_t z_b + \xi\partial_x q_b = 0 \end{cases}$$

where  $h$  is the water depth,  $u$  is the flow velocity and  $z_b$  is the thickness of sediment layer which can be modified by the fluid. The sediment layer is assumed to be located on a fixed bedrock layer at depth  $H$  which is not modified by the fluid (see figure 1).  $\xi = (1 - \gamma)^{-1}$  with  $\gamma$  the porosity of the sediment layer.  $q_b$  is the volumetric bedload sediment transport rate per unit time and width and  $g$  is the acceleration due to gravity. The conservative variable  $hu$  is also called debit water and noted by  $q$ . In what follows and for the sake of simplicity, we shall assume that  $H$  is constant and the layer  $z_b$  is large enough so that the term  $\partial_x H$  can be neglected. We shall also assume that  $\xi = 1$ .

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\*Corresponding author

*Email addresses:* [cordier@math.cnrs.fr](mailto:cordier@math.cnrs.fr) (S. Cordier), [mh.le@brgm.fr](mailto:mh.le@brgm.fr) (M.H. Le), [Tomas.Morales@uco.es](mailto:Tomas.Morales@uco.es) (T. Morales de Luna)

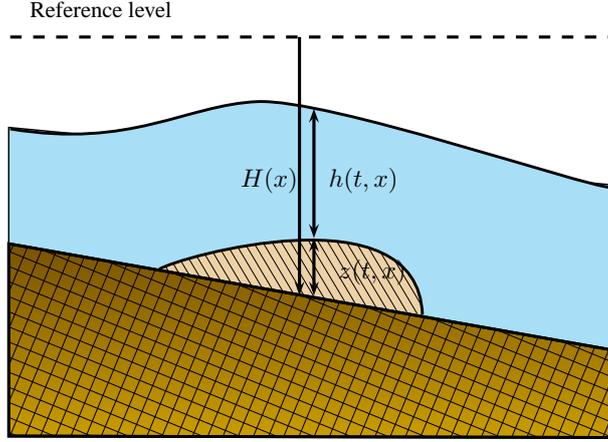


Figure 1: Sketch of shallow water over an erodible bed.

Let us firstly quote some of the expressions proposed for the bedload transport rate in the literature. Many researches are devoted to developer the methods of predicting and estimating bedload transport rate. Its expressions were proposed for granular non-cohesive sediments and were quantified by empirical laws.

A simple expression proposed by Grass [3] considers  $q_b$  as a function of the flow velocity and a coefficient which depends on soil properties

$$(2) \quad \text{Grass (1981): } q_b = A_g u |u|^{m_g - 1}, \quad 1 \leq m_g \leq 4,$$

where  $A_g$  is a constant.

Practically, the estimations of bedload transport rate are mainly based on the bottom shear stress  $\tau_b$ , i.e the force of water acting on the bed during its routing. In context of laminar flows, the bottom shear stress is given as

$$(3) \quad \tau_b = \rho g h S_f,$$

where  $\rho$  is the density of water,  $S_f$  is the friction term quantified by different empirical laws, as the Darcy-Weisbach or Manning formulas (see [4])

- Darcy-Weisbach:

$$(4) \quad S_f = \frac{f u |u|}{8 g h},$$

where  $f$  is the Darcy-Weisbach's coefficient.

- Manning:

$$(5) \quad S_f = \frac{n^2 u |u|}{h^{4/3}},$$

where  $n$  is the Manning's coefficient.

The bottom shear stress is usually used in dimensionless form, noted  $\tau_b^*$ , which is also called Shields parameter. It is defined in terms of the ratio between drag forces and the summerged weight as

$$(6) \quad \tau_b^* = \frac{\tau_b}{(\rho_s - \rho) g d_s},$$

where  $\rho_s$  is the sediment density,  $d_s$  is the diameter of sediments. The principle is that  $\tau_b^*$  must be exceed a threshold value  $\tau_{cr}^*$  to initiate motion. The threshold value  $\tau_{cr}^*$  depend on the characteristic of sediment

and its determination have been performed experimentally. The earliest one is the graphical presentation given by Shields (1935) in which  $\tau_{cr}^*$  is determined in relation with boundary Reynolds number at beginning of motion.

The bedload transport rate is represented as a function of  $\tau_b^*$  via a non-dimensional parameter  $\Phi$  as

$$(7) \quad q_b = \Phi \sqrt{\left(\frac{\rho_s}{\rho} - 1\right)} g d_s^3.$$

The followings expressions, illustrated by figure 2, have been often applied [5–8]:

$$(8) \quad \text{Meyer-Peter \& Müller (1948): } \Phi = 8(\tau_b^* - \tau_{cr}^*)^{3/2}$$

$$(9) \quad \text{Fernández Luque \& Van Beek (1976): } \Phi = 5.7(\tau_b^* - \tau_{cr}^*)^{3/2}$$

$$(10) \quad \text{Nielsen (1992): } \Phi = 12\sqrt{\tau_b^*}(\tau_b^* - \tau_{cr}^*)$$

$$(11) \quad \text{Ribberink (1998): } \Phi = 11(\tau_b^* - \tau_{cr}^*)^{1.65}$$

$$(12) \quad \text{Camenen and Larson (2005): } \Phi = 12(\tau_b^*)^{1.5} \exp(-4.5\tau_{cr}^*/\tau_b^*)$$

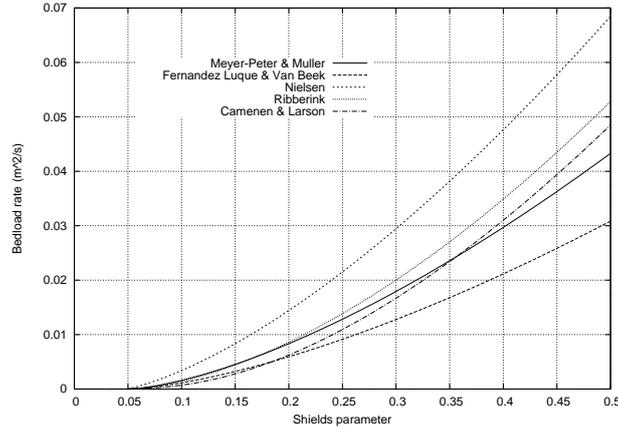


Figure 2: Bedload rate in function of Shields parameter with  $d_s = 32mm$ ,  $\rho = 1000kg/m^3$ ,  $\rho_s = 2600kg/m^3$  and  $\tau_{cr}^* = 0.05$ .

**Remark.** Classical formulae for bedload transport do not take into account gravity effects. As a consequence, particles located at the advancing front of a dune do not fall due to gravity as the motion of particles only depends on the hydrodynamical variables. This means that vertical profiles may be observed in numerical simulations which are not found in physical situations, for example at the front of an advancing dune. In order to obtain more realistic profiles, a diffusion term may be added to the third equation of (1) or one could consider some a modification of the classical formulae by including gravity effects as it was done in [9].

Returning to the PDE system (1). It can be written in vectorial form as

$$(13) \quad \partial_t W + \partial_x F(W) = S(W),$$

where  $W = (h, q, z_b)^T$  is the state vector in conservative form. The system can be written in non conservative form as

$$(14) \quad \partial_t W + A(W)\partial_x W = S(W),$$

where  $A(W) = D_W F$  is the matrix of transport coefficients. The main properties of system of PDE of this form called transport equations is the hyperbolicity [10] which required that the matrix is  $\mathbf{R}$  diagonalisable (and strictly hyperbolic when eigenvalues are distincts).

Let us quickly give an interpretation of this property as a stability condition for the linearized system. Assume  $W$  is a small wave perturbation of a constant state, i.e of the form

$$(15) \quad W(x, t) = W_0 + \epsilon W_1 e^{i(kx - \omega t)}; \quad \epsilon \ll 1.$$

This is a solution of the linearized problem

$$(16) \quad -i\omega W_1 + A(W_0)(ik)W_1 = 0,$$

if and only if  $W_1$  is an eigenvector associated with a velocity ( $\frac{\omega}{k}$ ). In other words, if one can write any arbitrary perturbation  $W_1$  as a linear combination of such eigenvector associated with real velocity, the solution will propagate without amplification of the perturbation. On the contrary, if some of the eigenvalue are complex, this will lead to instability.

Let us also remark that these eigenvalues are also important when using explicit upwind scheme to insure stability. Indeed, the time step, for a constant mesh size  $\Delta x$ , has to satisfy the so called Courant-Friedrichs-Lewy condition (CFL condition) in what follows, (see [10], [11] and references therein)

$$(17) \quad |\lambda_{\max}| \Delta t < \Delta x$$

where  $|\lambda_{\max}|$  is the maximum of the modulus of the eigenvalues or in other words,  $\lambda_{\max}$  is the maximum velocity for propagation of information. We recall that when using a finite volume scheme technique for solving (13), this condition can be seen as the definition of a time step sufficiently small so that the different Riemann problems at each intercell do not interact between each other, that is, the information of each Riemann problem does not cross more than one cell.

The paper is organized as follow: In the next sections, we study the hyperbolicity of system (1) and establish the main result. Next, we consider the numerical solving of (1) by time-splitting strategy or a coupled scheme. The last section presents some numerical tests to justify the arguments mentioned.

## 2. Domain of hyperbolicity

We are concerned here by the hyperbolicity of system (1) for the different models proposed (2-11) for the bedload transport rate  $q_b$ . It is known that, for Grass model, the system is always hyperbolic [2]. Nevertheless, the study the hyperbolicity of system (1) for more general bedload transport fluxes has not been done up to the authors knowledge. Moreover, Castro *et al.* [2] have stated numerically that hyperbolicity may be lost in some cases for the Meyer-Peter&Müller bedload transport flux. Proposition 2.1 will give an answer to this fact and states the domain of hyperbolicity for the different bedload transport fluxes proposed before.

Note that  $q_b = q_b(h, q)$  and thus the matrix of transport coefficients is of the form

$$(18) \quad A(W) = \begin{pmatrix} 0 & 1 & 0 \\ gh - u^2 & 2u & gh \\ a & b & 0 \end{pmatrix},$$

with  $a = \frac{\partial q_b}{\partial h}$  and  $b = \frac{\partial q_b}{\partial q}$ . The characteristic polynomial of  $A(W)$  can be written as

$$(19) \quad \begin{aligned} p_A(\lambda) &= -\lambda \begin{vmatrix} -\lambda & 1 \\ -u^2 + gh & 2u - \lambda \end{vmatrix} - gh \begin{vmatrix} -\lambda & 1 \\ a & b \end{vmatrix} \\ &= -\lambda[(u - \lambda)^2 - gh] + gh(b\lambda + a). \end{aligned}$$

The system (1) is thus strictly hyperbolic if and only if  $p_A(\lambda)$  has three solutions distincts noted  $\lambda_1 < \lambda_2 < \lambda_3$ . In other words if the curve  $f(\lambda) = \lambda[(u - \lambda)^2 - gh]$  and the line  $d(\lambda) = gh(b\lambda + a)$  have three points of intersection. This is illustrated by the figure 3 for the case of a subcritical flow.

Let us find relation between  $a$  and  $b$  in the different models: (2) and (8-12).

- Grass:  $q_b = A_g u |u|^{m-1}$

$$\left. \begin{aligned} u = \frac{q}{h} &\Rightarrow \frac{\partial u}{\partial q} = \frac{1}{h}; & \frac{\partial u}{\partial h} &= -\frac{q}{h^2}. \\ \frac{\partial q_b}{\partial q} &= (q_b)'_u \cdot \frac{\partial u}{\partial q}; & \frac{\partial q_b}{\partial h} &= (q_b)'_u \cdot \frac{\partial u}{\partial h} \end{aligned} \right\} \Rightarrow \frac{\partial q_b}{\partial h} = -\frac{q}{h} \frac{\partial q_b}{\partial q}$$

- Meyer-Peter & Müller, Fernández Luque & Van Beek, Nielsen, Ribberink, Camenen & Larson:  $q_b = q_b(\tau_b)$

**Darcy-Weisbach** :  $\tau_b = \rho g h \frac{f|u|}{8gh} = \alpha u |u|$  ( $\alpha = \text{cst}$ )

$$\left. \begin{aligned} \frac{\partial q_b}{\partial q} &= (q_b)'_{\tau_b} \cdot \frac{\partial \tau_b}{\partial q} = (q_b)'_{\tau_b} \cdot (\tau_b)'_u \frac{\partial u}{\partial q} \\ \frac{\partial q_b}{\partial h} &= (q_b)'_{\tau_b} \cdot \frac{\partial \tau_b}{\partial h} = (q_b)'_{\tau_b} \cdot (\tau_b)'_u \frac{\partial u}{\partial h} \end{aligned} \right\} \Rightarrow \frac{\partial q_b}{\partial h} = -\frac{q}{h} \frac{\partial q_b}{\partial q}$$

**Manning** :  $\tau_b = \rho g h \frac{n^2 u |u|}{h^{4/3}} = \alpha \frac{u |u|}{h^{1/3}} = \alpha \frac{q |q|}{h^{7/3}}$

$$\left. \begin{aligned} \frac{\partial \tau_b}{\partial q} &= \frac{2\alpha |q|}{h^{7/3}} \\ \frac{\partial \tau_b}{\partial h} &= -\frac{7}{6} \frac{q}{h} \frac{2\alpha |q|}{h^{7/3}} = -\frac{7}{6} \frac{q}{h} \frac{\partial \tau_b}{\partial q} \end{aligned} \right\} \Rightarrow \frac{\partial q_b}{\partial h} = -\frac{7}{6} \frac{q}{h} \frac{\partial q_b}{\partial q}$$

Finally, we have two types of relation:  $a = -ub$  or  $a = -\frac{7}{6}ub$ . We shall assume that  $b > 0$  since the sediment rate increases with increasing of the flow rate. The line  $d(\lambda)$  is thus can be rewritten as  $d(\lambda) = ghb(\lambda - ku)$  with  $k = 1$  ou  $k = 7/6$  and the slope  $ghb > 0$ .

**Proposition 2.1.** *Consider system (1) with  $q_b = q_b(h, q)$  such that*

$$(20) \quad \frac{\partial q_b}{\partial h} = -k \frac{q}{h} \frac{\partial q_b}{\partial q}.$$

*For a given state  $(h, q)$ , the system is strictly hyperbolic if and only if*

$$(21) \quad \alpha_- < ku < \alpha_+,$$

*with  $\alpha_{\pm}$  will be defined by expression (25) in the proof. Especially,*

- *In the case  $k = 1$  (or  $a = -ub$ ), the system is always strictly hyperbolic.*
- *In the case  $k = 7/6$  (or  $a = -\frac{7}{6}ub$ ), a sufficient condition for the system (1) to be strictly hyperbolic is*

$$(22) \quad |u| < 6\sqrt{gh}.$$

*Proof.* For simplicity, let us consider the case  $u > 0$  so  $a = -kub < 0$ . The case  $u < 0$  can be treated with the same arguments. We define the two tangents of the curve  $f(\lambda)$  which are parallel to  $d(\lambda)$ . Their intersections with  $f(\lambda)$  are characterized by  $f'(\lambda) = ghb$  which yields to two values of  $\lambda$  of the form

$$(23) \quad \lambda_{\pm} \stackrel{\text{def}}{=} \frac{2u \pm \sqrt{u^2 + gh + ghb}}{3}.$$

The two tangents are such that  $d_{\pm}(\lambda_{\pm}) = f(\lambda_{\pm})$ . This implies the equations for the tangents given by

$$(24) \quad d_{\pm}(\lambda) = ghb(\lambda - \alpha_{\pm})$$

with

$$(25) \quad \alpha_{\pm} \stackrel{\text{def}}{=} \lambda_{\pm} - \frac{f(\lambda_{\pm})}{ghb}.$$

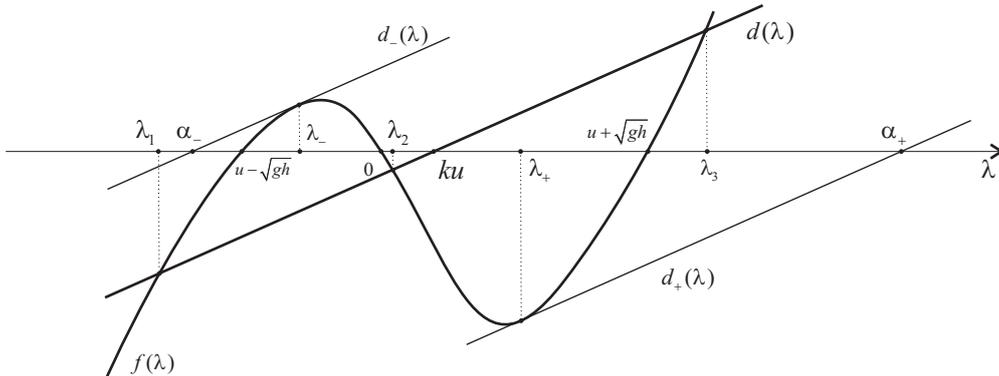


Figure 3: Eigenvalues of the transport coefficients matrix.

The roots of  $p_A(\lambda)$  which are the eigenvalues of  $A(W)$  are given as the intersection of  $f(\lambda)$  and  $d(\lambda)$  (see figure 3 for the illustration). Recall that  $f(\lambda)$  is a third order polynomial with roots  $\{0, u \pm \sqrt{gh}\}$ . The equation  $p_A(\lambda) = d(\lambda) - f(\lambda)$  will have 3 distinct solutions if and only if the line  $d(\lambda)$  lies in between  $d_-(\lambda)$  and  $d_+(\lambda)$ . This can be equivalently written as  $\alpha_- < ku < \alpha_+$ .

It can be checked that we always have  $\alpha_- < u - \sqrt{gh} < u < u + \sqrt{gh} < \alpha_+$  so the system is always hyperbolic in the case  $k = 1$ . In the case  $k = 7/6$ , if  $|u| < 6\sqrt{gh}$  so we have  $u - \sqrt{gh} < \frac{7}{6}u < u + \sqrt{gh}$  and thus the hyperbolicity condition is verified. This concludes the proof.

Note that similar arguments have been used by one of the author in [12–14] to characterize the hyperbolicity domains for systems of moments equations arising in plasma physics.  $\square$

**Remark.** Condition (22) is interesting because the subcritical flows verify this hyperbolic condition. Moreover, the condition (22) is easily satisfied with a general flow in reality.

### 3. Time-splitting method

The strategy of time-splitting consists in solving separately the shallow water system during a time step  $\Delta t$  for a fixed topography  $z_b$ , and after updating the topography using the third equation for the same time step. See, for example, the references [15, 16] or chapt. 3 of [17]. It appears to be natural considering the topography slowly varying for hydrodynamical characteristic time. This suggests the possibility of using a time-splitting method to solve the system (1).

Using the time-splitting strategy allows to minimize computational costs and to develop the computation code in modular way. The time-splitting solving can inherit easily the robust and efficient solution of the shallow water equations developed in the few last years, for example, those presented in the publication [18]. We are interested specially in the code FULLSWOF\_2D (licence DL 03434-01) developed in the framework of the multi-disciplinary project METHODE ANR-07-BLAN-0232 (see [19] and <http://www.univ-orleans.fr/mapmo/methode/>). However, the use of time-splitting method requires to pay attention as the following remarks:

1. As explained in the introduction, the stability condition for shallow water equations only requires the CFL limitation associated with

$$(26) \quad (|u| + \sqrt{gh}) \Delta t < \Delta x.$$

Remark that the CFL condition (26) is not enough to insure the stability condition (17) of the coupled system (1) because the maximum eigenvalue  $\lambda_3$  of the matrix of transport coefficients  $A(W)$  is always larger than  $|u| + \sqrt{gh}$  (see figure 9). This will result in instabilities in some numerical simulations when a splitting technique is used. These instabilities are less frequent or do not exist if the numerical scheme

used is sufficiently diffusive. For instance, when Lax-Friedrichs scheme is used it is very difficult to find instabilities while for Roe type schemes they are easily found. We recall that something similar has been studied in [20] and [21] where the splitting technique for the two-layer shallow water system was studied. Instead of computing the exact eigenvalues of  $A(W)$ , an upper bound first order  $\lambda_3^0$  of  $\lambda_3$  may be used, defined as the intersection of  $d(\lambda)$  and the tangent  $d_T(\lambda)$  of  $f(\lambda)$  at  $u + \sqrt{gh}$  (see the figure 4)

$$(27) \quad d_T(\lambda) = f'(u + \sqrt{gh})(\lambda - u - \sqrt{gh}).$$

$\lambda_3^0$  is thus of the form

$$(28) \quad \lambda_3^0 \stackrel{\text{def}}{=} \frac{(u + \sqrt{gh})f'(u + \sqrt{gh}) + gha}{f'(u + \sqrt{gh}) - ghb}.$$

This value can be used to impose an CFL condition associated to the full coupled system (1). With

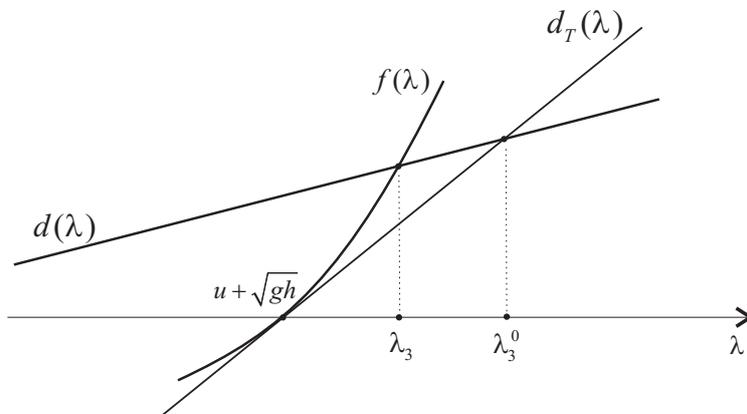


Figure 4: Upper bound first order of the maximum eigenvalue.

this upper-bound the true CFL condition is now granted. Nevertheless we remark that now the true maximum propagation speed is overestimated.

2. Another problem is related to the sign of eigenvalues of the matrix  $A(W)$ . Rewriting the characteristic polynomial  $p_A(\lambda)$  in form

$$p_A(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3),$$

the product of three eigenvalues, in the case  $u > 0$ , satisfies

$$\lambda_1 \lambda_2 \lambda_3 = p_A(0) = gha < 0.$$

In a supercritical flow, i.e.  $|u| > \sqrt{gh}$ , this means that  $\lambda_2$  and  $\lambda_3$  are positive and consequently  $\lambda_1 < 0$ . This can be interpreted as there is a wave propagating to the left. The using of splitting strategy in such situation cannot account this information as we would have  $0 < u - \sqrt{gh} < u + \sqrt{gh}$ . This will lead to instability when using splitting method with upwind schemes.

#### 4. Numerical results

In the following tests, we shall use two different schemes for (1):

- Splitting scheme: We use a Roe scheme for the shallow water system plus an upwinding technique based on the sign of velocity for the bedload transport equation, that is,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} \left[ A_{i-1/2}^+(U_i^n - U_{i-1}^n) + A_{i+1/2}^-(U_{i+1}^n - U_i^n) \right],$$

where  $U = (h, q)^T$  and  $A_{i+1/2}$  is a Roe matrix for the shallow water system with non-flat topography,

$$(29) \quad (z_b)_{i+1/2}^{n+1} = (z_b)_i^n - \frac{\Delta t}{\Delta x} \left[ (q_b)_{i+1/2}^{n+1} - (q_b)_{i-1/2}^{n+1} \right],$$

where

$$(30) \quad (q_b)_{i+1/2}^{n+1} = \begin{cases} q_b(h_i^{n+1}, q_i^{n+1}), & \text{if } u_{i+1/2} \geq 0, \\ q_b(h_{i+1}^{n+1}, q_{i+1}^{n+1}), & \text{if } u_{i+1/2} < 0, \end{cases}$$

with  $u_{i+1/2}$  the velocity at the interface of the cells  $i$  and  $i+1$ , for instance the value used in the Roe's matrix.

- Coupled Scheme: We use a Roe scheme for the full system which can be written in the form

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} (\mathcal{A}_{i-1/2}^+(W_i^n - W_{i-1}^n) + \mathcal{A}_{i+1/2}^-(W_{i+1}^n - W_i^n)),$$

where  $\mathcal{A}_{i+1/2}$  is a Roe matrix for (1).

We refer to [20, 22–24] for the details of the numerical schemes.

First, we shall show a case where the splitting technique gives the same result as the fully coupled scheme. Let us consider as initial condition a subcritical steady state for the classical shallow water system given by

$$(31) \quad \begin{cases} hu(x, t = 0) = 0.5, \\ z_b(x, t = 0) = 0.1 + 0.1e^{-(x-5)^2}, \\ \frac{u^2}{2} + g(h + z) = 6.386. \end{cases}$$

The water surface and topography corresponding to this initial condition are shown in figure 5. We shall consider Grass model (2) with  $A_g = 0.005$ . As we see in figure 6, no major differences are observed between

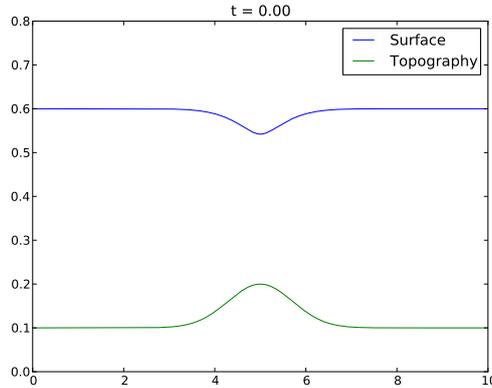
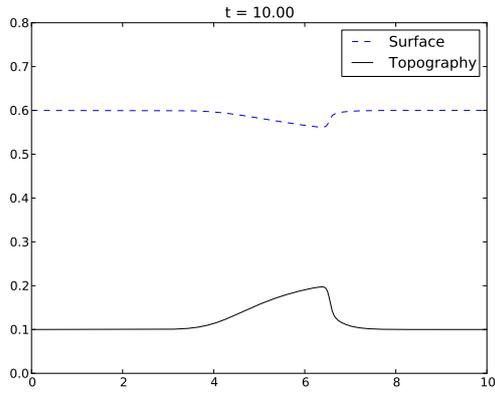


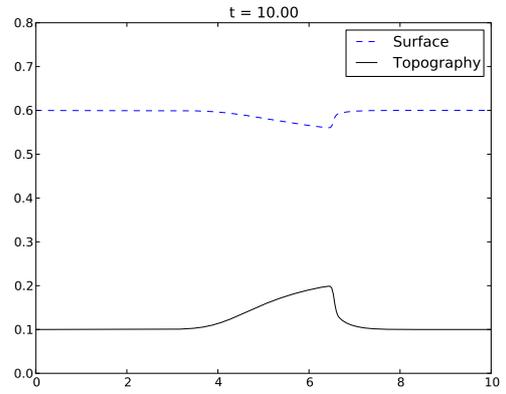
Figure 5: Initial condition for (31).

the two schemes. Both simulations have been computed with  $CFL = 0.95$ .

Now, consider the same initial condition 5 but we set  $A_g = 0.07$ . We remark in figure 7 that some instabilities arise which disappear if the  $CFL$  is reduced. This is not the case when the coupled scheme is used as we see in figure 8.

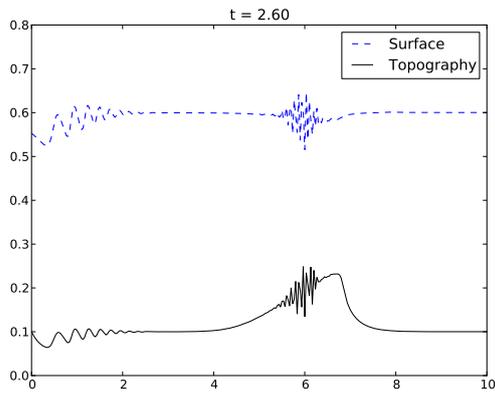


(a) Coupled scheme

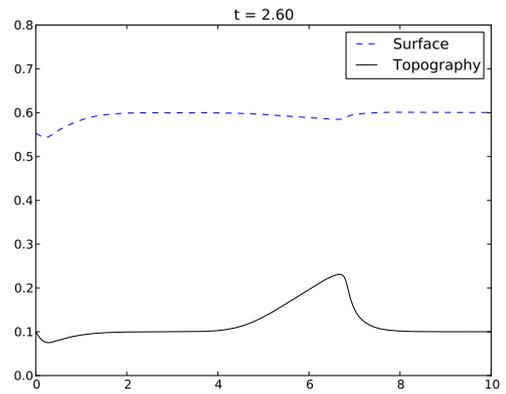


(b) Splitting scheme

Figure 6: Solution for (31) with  $A_g = 0.005$



(a) Splitting scheme  $CFL = 0.95$



(b) Splitting scheme  $CFL = 0.5$

Figure 7: Solution for (31) with  $A_g = 0.07$

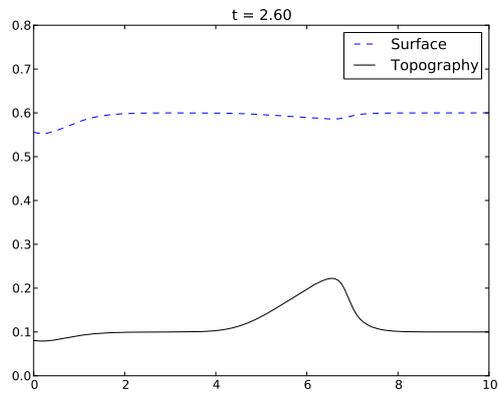


Figure 8: Solution for (31),  $A_g = 0.07$  with a coupled scheme.

The main difference between the experiment with  $A_g = 0.005$  and  $A_g = 0.07$  is that, in the second case, the true eigenvalues of the full system are far from the approximations  $u + \sqrt{gh}$  and  $u - \sqrt{gh}$  given by the splitting scheme. In particular, the maximum of the modulus of the eigenvalue is larger than  $|u| + \sqrt{gh}$ . This implies that a  $CFL$  condition close to 1 should not be used with the splitting scheme. (See figure 9)

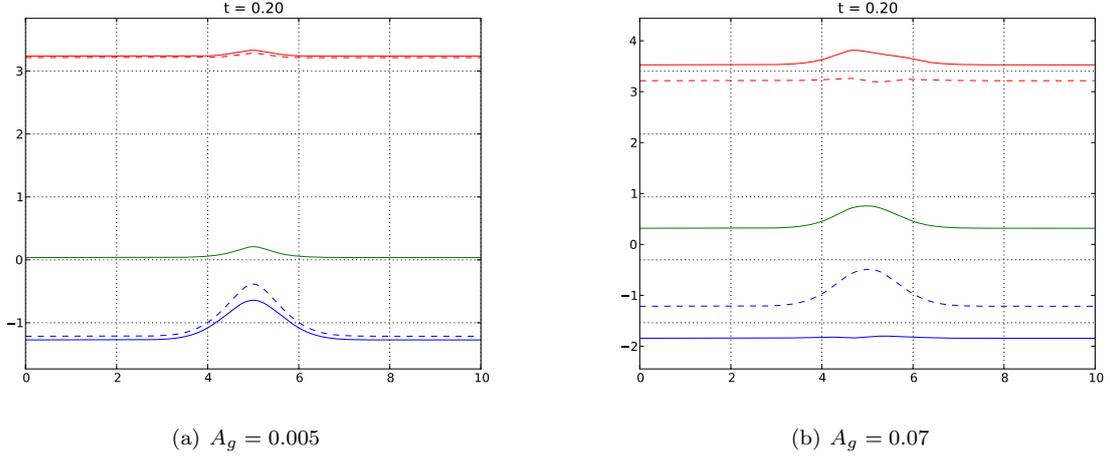


Figure 9: Eigenvalues for (31). True eigenvalues of the system (Continuous line) and  $u \pm \sqrt{gh}$  approximation (Dashed line)

Finally, let us consider an example with subcritical and supercritical regions. In order to do so, we begin with the initial condition

$$(32) \quad \begin{cases} hu(x, t = 0) = 0.6, \\ z_b(x, t = 0) = 0.1 + 0.1e^{-(x-5)^2}, \\ h(x, t = 0) + z_b(x, t = 0) = 0.4. \end{cases}$$

We solve the shallow water system ( $A_g = 0.0$ ) with this initial data until a steady state solution is reached (see Fig. 10) Once the steady state is reached, we let the sediment to evolve by using a Grass bedload flux

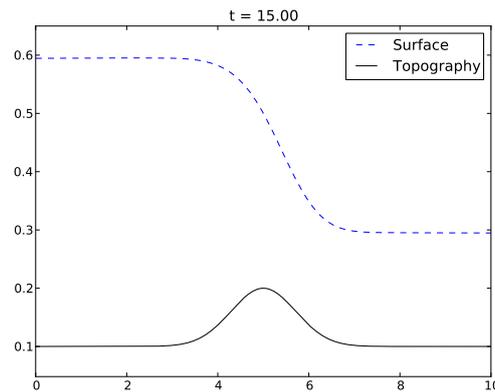


Figure 10: Transcritical steady-state for shallow water system.

with  $A_g = 0.0005$ . Figure 11 shows that again we find instabilities by using the splitting technique. These instabilities remain even for  $CFL = 0.05$ . Note that these instabilities do not appear when using the full coupled system even for  $CFL = 0.95$  as it is shown in figure 12.

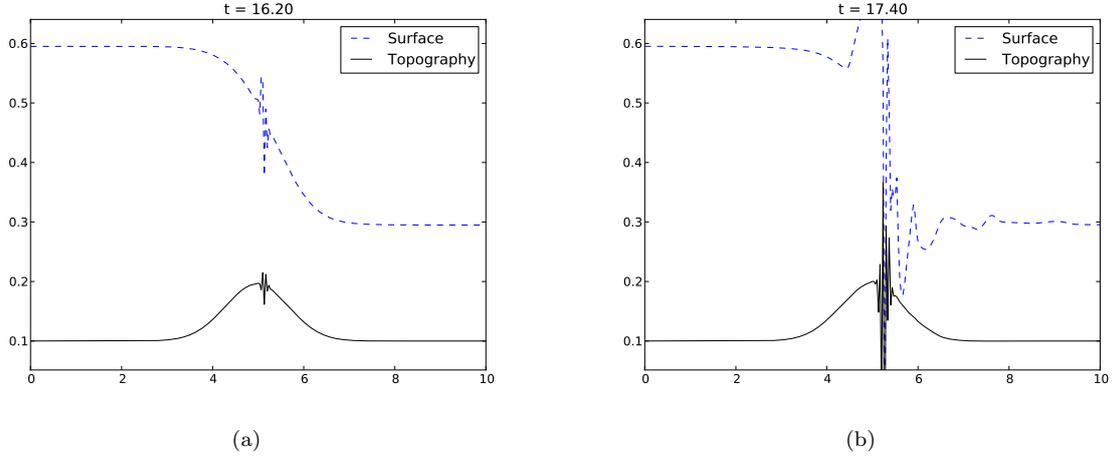


Figure 11: Solution for (32) with  $A_g = 0.0005$  using splitting scheme

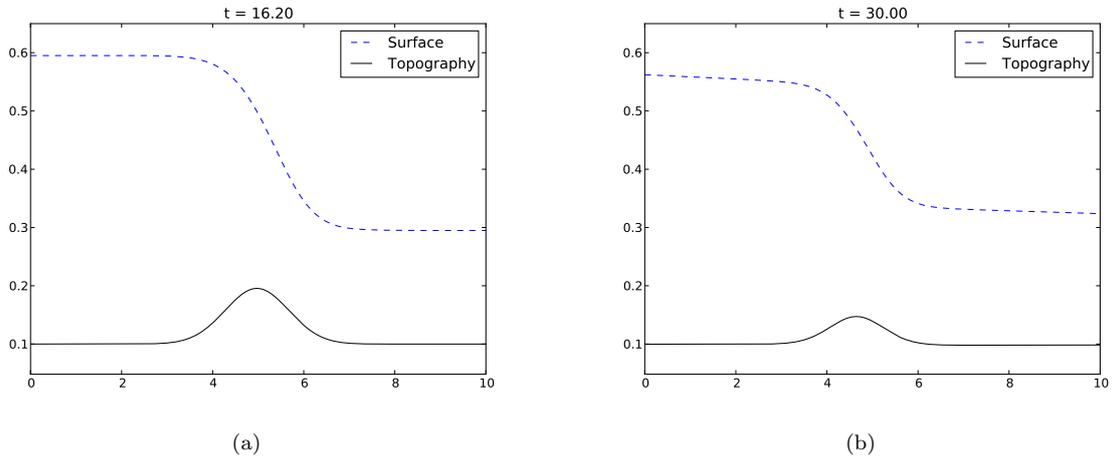


Figure 12: Solution for (32) with  $A_g = 0.0005$  using full system

If we study the eigenvalues of the system in Fig 13, we remark that  $0$  and  $u \pm \sqrt{gh}$  are good approximations of the true eigenvalues of the system. Nevertheless, in the supercritical region we see that, while  $u \pm \sqrt{gh}$  is always positive, we find a negative eigenvalue of the system. As a consequence, the time-splitting method with upwind scheme does not take into account the information travelling backwards in general which explains the instabilities in the simulation shown before.

## 5. Conclusions

Exner equation coupled with shallow water equations result as a system that may lose hyperbolicity in some cases at least theoretically. A practical criterium has been introduced for the study of the hyperbolicity region. Nevertheless, the system remains hyperbolic in most of physical situations for classical definitions of the bedload transport flux.

While it seems natural to do a splitting approach by solving first the shallow water system and then updating the topography, it is shown that it could result as instability. The resulting instabilities may be

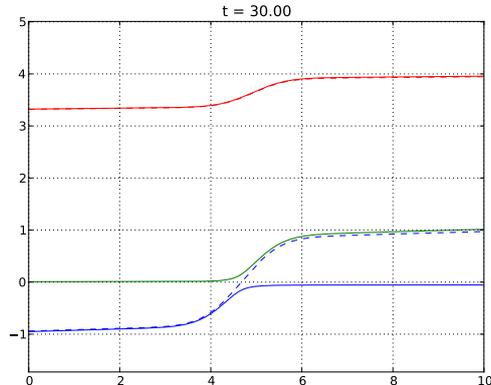


Figure 13: Eigenvalues for (32). True eigenvalues of the system (Continuous line) and  $u \pm \sqrt{gh}$  approximation (Dashed line).

avoided by reducing the CFL condition in certain cases but in some simulations a numerical scheme for the full coupled system may be needed.

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