



# Bedload transport in shallow water models: why splitting (may) fail, how hyperbolicity (can) help

Stéphane Cordier, Minh H. Le, Tomas Morales de Luna

## ► To cite this version:

Stéphane Cordier, Minh H. Le, Tomas Morales de Luna. Bedload transport in shallow water models: why splitting (may) fail, how hyperbolicity (can) help. 2010. hal-00536267v1

**HAL Id: hal-00536267**

**<https://hal.science/hal-00536267v1>**

Preprint submitted on 15 Nov 2010 (v1), last revised 23 Jan 2011 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Bedload transport in shallow water models: why splitting (may) fail, how hyperbolicity (can) help

S. Cordier<sup>a,\*</sup>, M.H. Le<sup>a,b</sup>, T. Morales de Luna<sup>c</sup>

<sup>a</sup>*Laboratoire de Mathématiques et Applications,  
Physique Mathématique d'Orléans (MAPMO)*

<sup>b</sup>*Bureau de Recherches Géologiques et Minières (BRGM)*

<sup>c</sup>*Departamento de Matemáticas Universidad de Córdoba*

---

## Abstract

In this paper, we are concerned with models for sedimentation transport consisting of a shallow water system coupled with a so called Exner equation that described the evolution of the topography. We show that, for some model of the bedload transport rate including the well-known Meyer-Peter and Müller model, the system is hyperbolic and, thus, linearly stable, only under some constraint on the velocity. In practical situations, this condition is hopefully fulfilled. The numerical approximations of such system are often based on a splitting method, solving first shallow water equation on a time step and, after updating the topography. It is proved that this strategy can create spurious/unphysical oscillations which are related to the study of hyperbolicity e.g. the sign of some eigenvalue of the coupled system differs from the splitting one. Some numerical results are given to illustrate these problems and the way to overcome them in some cases using an stronger C.F.L. condition.

*Keywords:* Shallow water equations, sediment transport, hyperbolicity, splitting method.

---

## 1. Introduction

Soil erosion is caused by the movement of sediment due to mechanical actions of flow. In the context of bedload transport, a mass conservation also called Exner equation [1] is used to updated the bed elevation. This equation is often coupled with the shallow water equations describing the overland flows (see [2] and references therein). The complete system of PDE is written as

$$(1) \quad \begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) = -gh\partial_x z_b + gh\partial_x H \\ \partial_t z_b + \xi\partial_x q_b = 0 \end{cases}$$

where  $h$  is the water depth,  $u$  is the flow velocity and  $z_b$  is the thickness of sediment layer which can be modified by the fluid. The sediment layer is assumed to be located on a fixed bedrock layer at depth  $H$  which is not modified by the fluid (see figure 1).  $\xi = (1 - \gamma)^{-1}$  with  $\gamma$  the porosity of the sediment layer.  $q_b$  is the volumetric bedload sediment transport rate per unit time and width and  $g$  is the acceleration due to gravity. The conservative variable  $hu$  is also called debit water and noted by  $q$ . In what follows and for the sake of simplicity, we shall assume that  $H$  is constant and the layer  $z_b$  is large enough so that the term  $\partial_x H$  can be neglected. We shall also assume that  $\xi = 1$ .

Let us firstly quote some of the expressions proposed for the bedload transport rate in the literature. Many researches are devoted to developer the methods of predicting and estimating bedload discharge. The

---

\*Corresponding author

*Email addresses:* [cordier@math.cnrs.fr](mailto:cordier@math.cnrs.fr) (S. Cordier), [mh.le@brgm.fr](mailto:mh.le@brgm.fr) (M.H. Le), [Tomas.Morales@uco.es](mailto:Tomas.Morales@uco.es) (T. Morales de Luna)

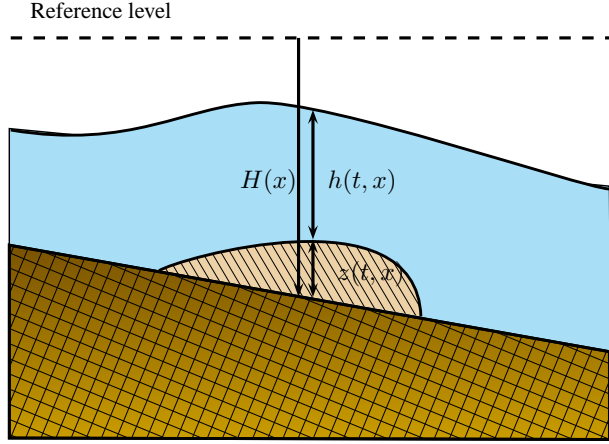


Figure 1: Sketch of shallow water over an erodible bed.

development of bedload transport models has a strong empirical character and relies heavily on physical insights and quantitative data obtained in laboratory studies. Bedload transport rate expressions were proposed for granular non-cohesive sediments.

A simple expression proposed by Grass [3] considers  $q_b$  as a function of the flow velocity and a coefficient which depends on soil properties

$$(2) \quad \text{Grass (1981): } q_b = A_g u |u|^{m_g - 1}, \quad 1 \leq m_g \leq 4,$$

where  $A_g$  is a constant.

Practically, the estimations of bedload transport rate are mainly based on the bottom shear stress  $\tau_b$ , i.e. the force of water acting on the bed during its routing. In context of laminar flows, the bottom shear stress is given as

$$(3) \quad \tau_b = \rho g h S_f,$$

where  $\rho$  is the density of water,  $S_f$  is the friction term quantified by different laws, as the Darcy-Weisbach or Manning formulas (see [4])

- Darcy-Weisbach:

$$(4) \quad S_f = \frac{F u |u|}{8h},$$

where  $F$  is the Darcy-Weisbach's friction coefficient.

- Manning:

$$(5) \quad S_f = \frac{u |u|}{K^2 h^{4/3}},$$

where  $K$  is the Manning's coefficient.

The bottom shear stress is usually used in dimensionless form, also called Shields parameter, noted  $\tau_b^*$  and given as

$$(6) \quad \tau_b^* = \frac{\tau_b}{(\rho_s - \rho) g d_s},$$

where  $\rho_s$  is the sediment density,  $d_s$  is the diameter of sediments. The principle is that  $\tau_b^*$  must be exceed a critical value  $\tau_{cr}^*$  called critical Shields parameter which is required to initiate motion of the grains of non-cohesive particles. So, the bedload transport rate  $q_b$  is represented as a function of  $\tau_b^*$  via a non-dimensional parameter  $\Phi$  as

$$(7) \quad q_b = \Phi \sqrt{\left(\frac{\rho_s}{\rho} - 1\right) g d_s^3}.$$

The followings expressions, illustrated by figure 2, have been often applied [5–7]:

$$(8) \quad \text{Meyer-Peter and Müller (1948): } \Phi = 8(\tau_b^* - \tau_{cr}^*)^{3/2}$$

$$(9) \quad \text{Nielsen (1992): } \Phi = 12\sqrt{\tau_b^*}(\tau_b^* - \tau_{cr}^*)$$

$$(10) \quad \text{Ribberink (1998): } \Phi = 11(\tau_b^* - \tau_{cr}^*)^{1.65}$$

$$(11) \quad \text{Camenen and Larson (2005): } \Phi = 12(\tau_b^*)^{1.5} \exp\left(\frac{-4.5\tau_{cr}^*}{\tau_b^*}\right)$$

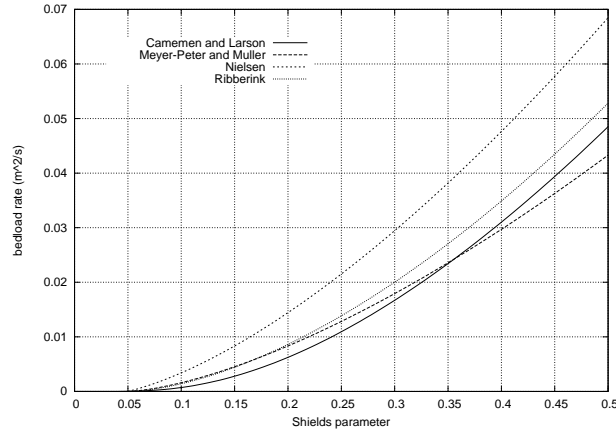


Figure 2: Volumetric bedload rate  $q_b$  in function of Shields parameter for the very coarse sediments ( $d_s = 32mm$ ).

Returning to the PDE system (1). It can be written in vectorial form as

$$(12) \quad \partial_t W + \partial_x F(W) = S(W)$$

where  $W = (h, q, z_b)^T$  is the state vector in conservative form. The system can be written in non conservative form as

$$(13) \quad \partial_t W + A(W)\partial_x W = S(W)$$

where  $A(W) = D_W F$  is the matrix of transport coefficients. The main properties of system of PDE of this form called transport equations is the hyperbolicity [8] which required that the matrix is  $\mathbf{R}$  diagonalisable (and strictly hyperbolic when eigenvalues are distincts).

Let us quickly give an interpretation of this property as a stability condition for the linearized system. Assume  $W$  is a small wave perturbation of a constant state, i.e of the form

$$(14) \quad W(x, t) = W_0 + \epsilon W_1 e^{i(kx - \omega t)}; \quad \epsilon \ll 1.$$

This is a solution of the linearized problem

$$(15) \quad -i\omega W_1 + A(W_0)(ik)W_1 = 0$$

if and only if  $W_1$  is an eigenvector associated with a velocity ( $\frac{u}{k}$ ). In other words, if one can write any arbitrary perturbation  $W_1$  as a linear combination of such eigenvector associated with real velocity, the solution will propagate without amplification of the perturbation. On the contrary, if some of the eigenvalue are complex, this will lead to instability.

Let us also remark that these eigenvalues are also important when using explicit upwind scheme to insure stability. Indeed, the time step, for a constant mesh size  $h$ , has to satisfy the so called Courant-Friedrichs-Lewy condition (CFL condition) in what follows, (see [8], [9] and references therein)

$$(16) \quad |\lambda_{\max}| \Delta t < \Delta x$$

where  $|\lambda_{\max}|$  is the maximum of the modulus of the eigenvalues or in other words,  $\lambda_{\max}$  is the maximum velocity for propagation of information. We recall that when using a finite volume scheme technique for solving 12, this condition can be seen as the definition of a time step sufficiently small so that the different Riemann problems at each intercell do not interact between each other, that is, the information of each Riemann problem does not cross more than one cell.

The paper is organized as follow: In the next sections, we study the hyperbolicity of system (1) and establish the main result. Next, we consider the numerical solving of (1) by time-splitting strategy or a coupled scheme. The last section presents some numerical tests to justify the arguments mentioned.

## 2. Domain of hyperbolicity

We are concerned here by the hyperbolicity of system (1) for the different models proposed (2-10) for the bedload transport rate  $q_b$ . It is known that, for Grass model, the system is always hyperbolic [2]. Nevertheless, the study the hyperbolicity of system (1) for more general bedload transport fluxes has not been done to the authors knowledge. Moreover, Castro *et al.* [2] have stated numerically that hyperbolicity may be lost in some cases for the Meyer-Peter&Müller bedload transport flux. Proposition 2.1 will give an answer to this fact and states the domain of hyperbolicity for the different bedload transport fluxes proposed before.

Note that  $q_b = q_b(h, q)$  and thus the matrix of transport coefficients is of the form

$$(17) \quad A(W) = \begin{pmatrix} 0 & 1 & 0 \\ gh - u^2 & 2u & gh \\ a & b & 0 \end{pmatrix}$$

with  $a = \frac{\partial q_b}{\partial h}$  and  $b = \frac{\partial q_b}{\partial q}$ . The characteristic polynomial of  $A(W)$  can be written as

$$(18) \quad \begin{aligned} p_A(\lambda) &= -\lambda \begin{vmatrix} -\lambda & 1 \\ -u^2 + gh & 2u - \lambda \end{vmatrix} - gh \begin{vmatrix} -\lambda & 1 \\ a & b \end{vmatrix} \\ &= -\lambda[(u - \lambda)^2 - gh] + gh(b\lambda + a). \end{aligned}$$

The system (1) is thus strictly hyperbolic if and only if  $p_A(\lambda)$  has three solutions distincts noted  $\lambda_1 < \lambda_2 < \lambda_3$ . In other words if the curve  $f(\lambda) = \lambda[(u - \lambda)^2 - gh]$  and the line  $d(\lambda) = gh(b\lambda + a)$  have three points of intersection. This is illustrated by the figure 3 for the case of a subcritical flow.

Let us find relation between  $a$  and  $b$  in the different models (2-10).

- Grass:  $q_b = A_g u |u|^{m-1}$

$$\left. \begin{aligned} u = \frac{q}{h} &\Rightarrow \frac{\partial u}{\partial q} = \frac{1}{h}; & \frac{\partial u}{\partial h} &= -\frac{q}{h^2}. \\ \frac{\partial q_b}{\partial q} &= (q_b)'_u \cdot \frac{\partial u}{\partial q}; & \frac{\partial q_b}{\partial h} &= (q_b)'_u \cdot \frac{\partial u}{\partial h} \end{aligned} \right\} \Rightarrow \frac{\partial q_b}{\partial h} = -\frac{q}{h} \frac{\partial q_b}{\partial q}$$

- Meyer-Peter and Müller, Nielsen, Ribberink, Camenen and Larson:  $q_b = q_b(\tau_b)$

$$\text{Darcy-Weisback} : \tau_b = \rho g h \frac{F u |u|}{8 g h} = \alpha u |u| \quad (\alpha = \text{cst})$$

$$\left. \begin{aligned} \frac{\partial q_b}{\partial q} &= (q_b)'_{\tau_b} \cdot \frac{\partial \tau_b}{\partial q} = (q_b)'_{\tau_b} \cdot (\tau_b)'_u \frac{\partial u}{\partial q} \\ \frac{\partial q_b}{\partial h} &= (q_b)'_{\tau_b} \cdot \frac{\partial \tau_b}{\partial h} = (q_b)'_{\tau_b} \cdot (\tau_b)'_u \frac{\partial u}{\partial h} \end{aligned} \right\} \Rightarrow \frac{\partial q_b}{\partial h} = -\frac{q}{h} \frac{\partial q_b}{\partial q}$$

$$\text{Manning} : \tau_b = \rho g h \frac{u |u|}{K^2 h^{4/3}} = \alpha \frac{u |u|}{h^{1/3}} = \alpha \frac{q |q|}{h^{7/3}}$$

$$\left. \begin{aligned} \frac{\partial \tau_b}{\partial q} &= \frac{2 \alpha |q|}{h^{7/3}} \\ \frac{\partial \tau_b}{\partial h} &= -\frac{7}{6} \frac{q}{h} \frac{2 \alpha |q|}{h^{7/3}} = -\frac{7}{6} \frac{q}{h} \frac{\partial \tau_b}{\partial q} \end{aligned} \right\} \Rightarrow \frac{\partial q_b}{\partial h} = -\frac{7}{6} \frac{q}{h} \frac{\partial q_b}{\partial q}$$

Finally, we have two types of relation:  $a = -ub$  or  $a = -\frac{7}{6}ub$ . We shall assume that  $b > 0$  since the sediment rate increases with increasing of the flow rate. The line  $d(\lambda)$  is thus can be rewritten as  $d(\lambda) = ghb(\lambda - ku)$  with  $k = 1$  ou  $k = 7/6$  and the slope  $ghb > 0$ .

**Proposition 2.1.** *Consider system (1) with  $q_b = q_b(h, q)$  such that*

$$(19) \quad \frac{\partial q_b}{\partial h} = -k \frac{q}{h} \frac{\partial q_b}{\partial q}.$$

*For a given state  $(h, q)$ , the system is strictly hyperbolic if and only if*

$$(20) \quad \alpha_- < ku < \alpha_+,$$

*with  $\alpha_{\pm}$  will be defined by expression (23) in the proof. Moreover, in the case  $k = 1$  (or  $a = -ub$ ), the system is always strictly hyperbolic.*

*Proof.* For simplicity, let us consider the case  $u > 0$  so  $a = -kub < 0$ . The case  $u < 0$  can be treated with the same arguments. We define the two tangents of the curve  $f(\lambda)$  which are parallel to  $d(\lambda)$ . Their intersections with  $f(\lambda)$  are characterized by  $f'(\lambda) = ghb$  which yields to two values of  $\lambda$  of the form

$$(21) \quad \lambda_{\pm} \stackrel{\text{def}}{=} \frac{2u \pm \sqrt{u^2 + gh + ghb}}{3}.$$

The two tangents are such that  $d_{\pm}(\lambda_{\pm}) = f(\lambda_{\pm})$ . This implies the equations for the tangents given by

$$(22) \quad d_{\pm}(\lambda) = ghb(\lambda - \alpha_{\pm})$$

with

$$(23) \quad \alpha_{\pm} \stackrel{\text{def}}{=} \lambda_{\pm} - \frac{f(\lambda_{\pm})}{ghb}.$$

The roots of  $p_A(\lambda)$  which are the eigenvalues of  $A(W)$  are given as the intersection of  $f(\lambda)$  and  $d(\lambda)$  (see figure 3 for the illustration).

Recall that  $f(\lambda)$  is a third order polynomial with roots  $\{0, u \pm \sqrt{gh}\}$ . The equation  $p_A(\lambda) = d(\lambda) - f(\lambda)$  will have 3 distinct solutions if and only if the line  $d(\lambda)$  lies in between  $d_-(\lambda)$  and  $d_+(\lambda)$ . This can be equivalently written as  $\alpha_- < ku < \alpha_+$ .

It can be checked that we always have  $\alpha_- < u - \sqrt{gh} < u < u + \sqrt{gh} < \alpha_+$  so the system is always hyperbolic in the case  $k = 1$ . This concludes the proof.

Note that similar arguments have been used by one of the author in [10–12] to characterized the hyperbolicity domains for systems of moments equations arising in plasma physics.  $\square$

**Remark.** In the case  $k = 7/6$ , a sufficient condition for the system (1) to be strictly hyperbolic is

$$(24) \quad |u| < 6\sqrt{gh}.$$

This result is interesting because the subcritical flow always verify hyperbolic condition. Moreover, the condition (24) is easily satisfied with a general flow in reality.

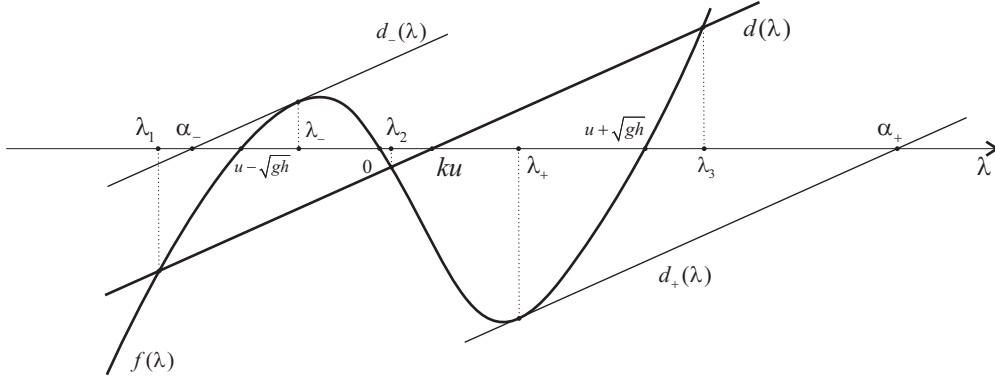


Figure 3: Eigenvalues of the transport coefficients matrix.

### 3. Time-splitting method

The strategy of time-splitting consists in solving separately the shallow water system during a time step  $\Delta t$  for a fixed topography  $z_b$ , and after updating the topography using the third equation for the same time step. See, for example, the references [13, 14] or chapt. 3 of [15]. It appears to be natural considering the topography slowly varying for hydrodynamical characteristic time. This suggests the possibility of using a time-splitting method to solve the system (1).

Using the time-splitting strategy permits to minimize computational costs and to develop the computation code in modular way. The time-splitting solving can inherit easily the robust and efficient solution of the shallow water equations developed in the few last years, for example, those presented in the publication [16]. We interest specially to the code FullSWOF\_2D (licence DL 03434-01) developed by projet METHODE ANR-07-BLAN-0232 (see [17] and <http://www.univ-orleans.fr/mapmo/methode/>). However, the use of time-splitting method requires to pay attention as the following remarks:

1. As explained in the introduction, the stability condition for shallow water equations only requires the CFL limitation associated with

$$(25) \quad (|u| + \sqrt{gh}) \Delta t < \Delta x.$$

Remark that the CFL condition (25) is not enough to insure the stability condition (16) of the coupled system (1) because the maximum eigenvalue  $\lambda_3$  of the matrix of transport coefficients  $A(W)$  is always larger than  $|u| + \sqrt{gh}$  (see figure 9). This will result as instabilities in some numerical simulations when a splitting technique is used. These instabilities are less frequent or do not exist if the numerical scheme used is sufficiently diffusive. For instance, when Lax-Friedrichs scheme is used it is very difficult to find instabilities while for Roe type schemes they are easily found. We recall that something similar has been studied in [18] and [19] where the splitting technique for the two-layer shallow water system was studied. Instead of computing the exact eigenvalues of  $A(W)$ , an upper bound first order  $\lambda_3^0$  of  $\lambda_3$  may be used, defined as the intersection of  $d(\lambda)$  and the tangent  $d_T(\lambda)$  of  $f(\lambda)$  at  $u + \sqrt{gh}$  (see the figure 4)

$$(26) \quad d_T(\lambda) = f'(u + \sqrt{gh})(\lambda - u - \sqrt{gh}).$$

$\lambda_3^0$  is thus of the form

$$(27) \quad \lambda_3^0 \stackrel{\text{def}}{=} \frac{(u + \sqrt{gh})f'(u + \sqrt{gh}) + gha}{f'(u + \sqrt{gh}) - ghb}.$$

This value can be used to impose an CFL condition associated to the full coupled system (1). With this upper-bound the true CFL condition is now granted. Nevertheless we remark that now the true maximum propagation speed is overestimated.

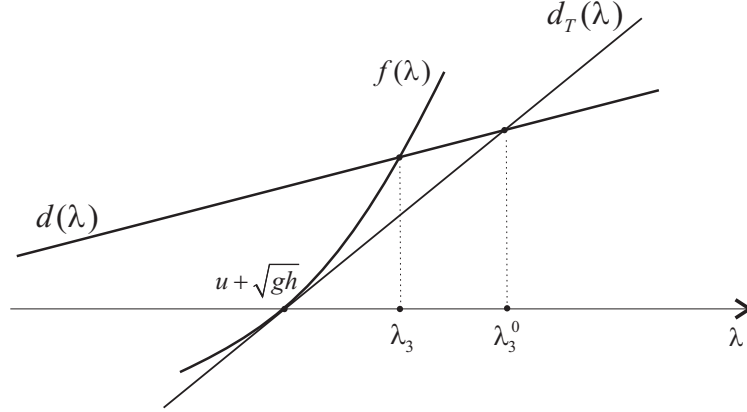


Figure 4: Upper bound first order of the maximum eigenvalue.

2. Another problem is related to the sign of eigenvalues of the matrix  $A(W)$ . Rewriting the characteristic polynomial  $p_A(\lambda)$  in form

$$p_A(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3),$$

the product of three eigenvalues, in the case  $u > 0$ , satisfies

$$\lambda_1 \lambda_2 \lambda_3 = p_A(0) = gha < 0.$$

In a supercritical flow, i.e.  $|u| > \sqrt{gh}$ , this means that  $\lambda_2$  and  $\lambda_3$  are positive and consequently  $\lambda_1 < 0$ . This can be interpreted as there is a wave propagating to the left. The using of splitting strategy in such situation cannot account this information as we would have  $0 < u - \sqrt{gh} < u + \sqrt{gh}$ . This will lead to instability when using splitting method with upwind schemes.

#### 4. Numerical results

In the following tests, we shall use two different schemes for (1):

- Splitting scheme: We use a Roe scheme for the shallow water system plus an upwinding technique based on the sign of velocity for the bedload transport equation, that is,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} (A_{i-1/2}^+ (U_i^n - U_{i-1}^n) + A_{i+1/2}^- (U_{i+1}^n - U_i^n)),$$

where  $U = (h, q)^T$  and  $A_{i+1/2}$  is a Roe matrix for the shallow water system with non-flat topography,

$$(28) \quad (z_b)_i^{n+1} = (z_b)_i^n - \frac{\Delta t}{\Delta x} ((q_b)_{i+1/2}^n - (q_b)_{i-1/2}^n),$$

where

$$(29) \quad (q_b)_{i+1/2}^n = \begin{cases} q_b(h_i^n, q_i^n), & \text{if } u_{i+1/2} \geq 0, \\ q_b(h_{i+1/2}^n, q_{i+1/2}^n), & \text{if } u_{i+1/2} < 0, \end{cases}$$

- Coupled Scheme: We use a Roe scheme for the full system which can be written in the form

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} (\mathcal{A}_{i-1/2}^+ (W_i^n - W_{i-1}^n) + \mathcal{A}_{i+1/2}^- (W_{i+1}^n - W_i^n)),$$

where  $\mathcal{A}_{i+1/2}$  is a Roe matrix for (1).



We refer to [20],[21], [18], [22] for the details of the numerical schemes.

First, we shall show a case where the splitting technique gives the same result as the fully coupled scheme. Let us consider as initial condition a subcritical steady state for the classical shallow water system given by

$$(30) \quad \begin{cases} hu(x, t = 0) = 0.5, \\ z_b(x, t = 0) = 0.1 + 0.1e^{-(x-5)^2}, \\ \frac{u^2}{2} + g(h + z) = 6.386. \end{cases}$$

The water surface and topography corresponding to this initial condition are shown in figure 5. We shall consider Grass model (2) with  $A_g = 0.005$ . As we see in figure 6, no major differences are observed between

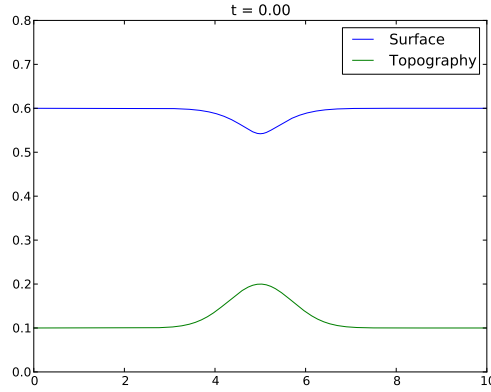
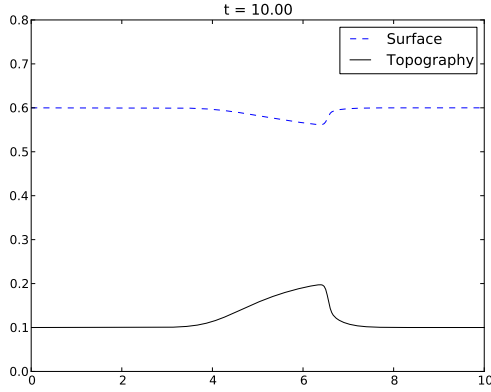
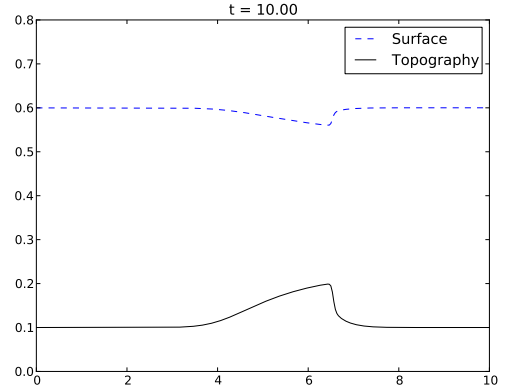


Figure 5: Initial condition for (30).



(a) Coupled scheme



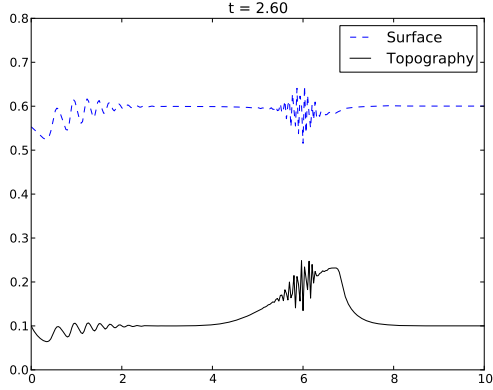
(b) Splitting scheme

Figure 6: Solution for (30) with  $A_g = 0.005$

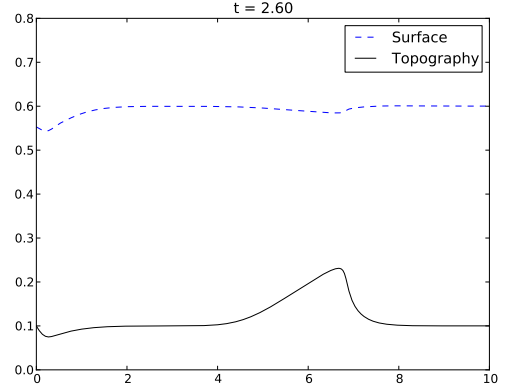
the two schemes. Both simulations have been computed with  $CFL = 0.95$ .

Now, consider the same initial condition 5 but we set  $A_g = 0.07$ . We remark in figure 7 that some instabilities arise which disappear if the  $CFL$  is reduced. This is not the case when the coupled scheme is used as we see in figure 8.

The main difference between the experiment with  $A_g = 0.005$  and  $A_g = 0.07$  is that, in the second case, the true eigenvalues of the full system are far from the approximations  $u + \sqrt{gh}$  and  $u - \sqrt{gh}$  given by the



(a) Splitting scheme  $CFL = 0.95$



(b) Splitting scheme  $CFL = 0.5$

Figure 7: Solution for (30) with  $A_g = 0.07$

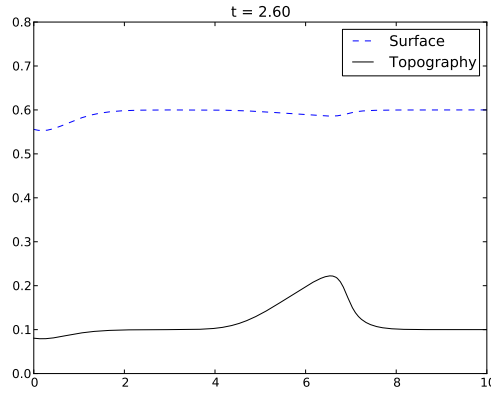


Figure 8: Solution for (30),  $A_g = 0.07$  with a coupled scheme.

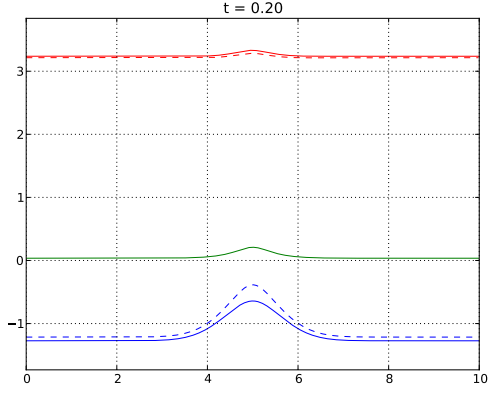
splitting scheme. In particular, the maximum of the modulus of the eigenvalue is larger than  $|u| + \sqrt{gh}$ . This implies that a  $CFL$  condition close to 1 should not be used with the splitting scheme. (See figure 9)

Finally, let us consider an example with subcritical and supercritical regions. In order to do so, we begin with the initial condition

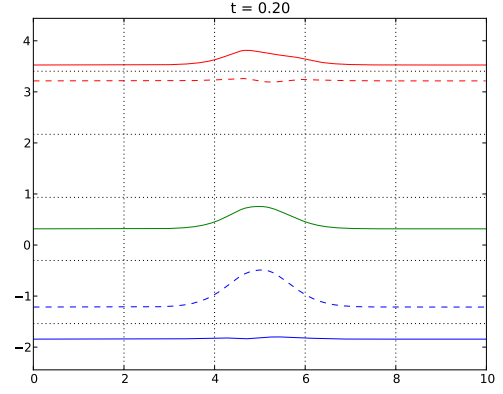
$$(31) \quad \begin{cases} hu(x, t = 0) = 0.6, \\ z_b(x, t = 0) = 0.1 + 0.1e^{-(x-5)^2}, \\ h(x, t = 0) + z_b(x, t = 0) = 0.4. \end{cases}$$

We solve the shallow water system ( $A_g = 0.0$ ) with this initial data until a steady state solution is reached (see Fig. 10) Once the steady state is reached, we let the sediment to evolve by using a Grass bedload flux with  $A_g = 0.0005$ . Figure 11 shows that again we find instabilities by using the splitting technique. These instabilities remain even for  $CFL = 0.05$ . Note that theses instabilities do not appear when using the full coupled system even for  $CFL = 0.95$  as it is shown in figure 12.

If we study the eigenvalues of the system in Fig 13, we remark that  $0$  and  $u \pm \sqrt{gh}$  are good approximations of the true eigenvalues of the system. Nevertheless, in the supercritical region we see that, while  $u \pm \sqrt{gh}$  is always positive, we find a negative eigenvalue of the system. As a consequence, the upwinding in the



(a)  $A_g = 0.005$



(b)  $A_g = 0.07$

Figure 9: Eigenvalues for (30). True eigenvalues of the system (Continuous line) and  $u \pm \sqrt{gh}$  approximation (Dashed line)

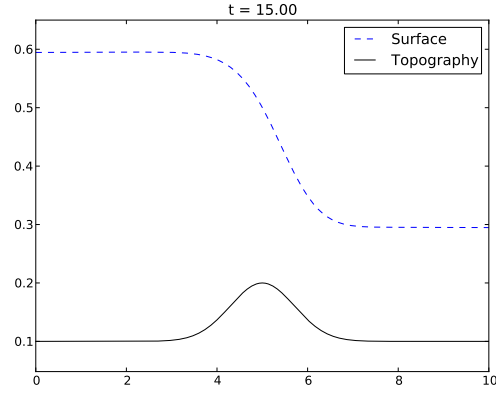


Figure 10: Transcritical steady-state for shallow water system.

splitting technique does not take into account the information travelling backwards in general which explains the instabilities in the simulation shown before.

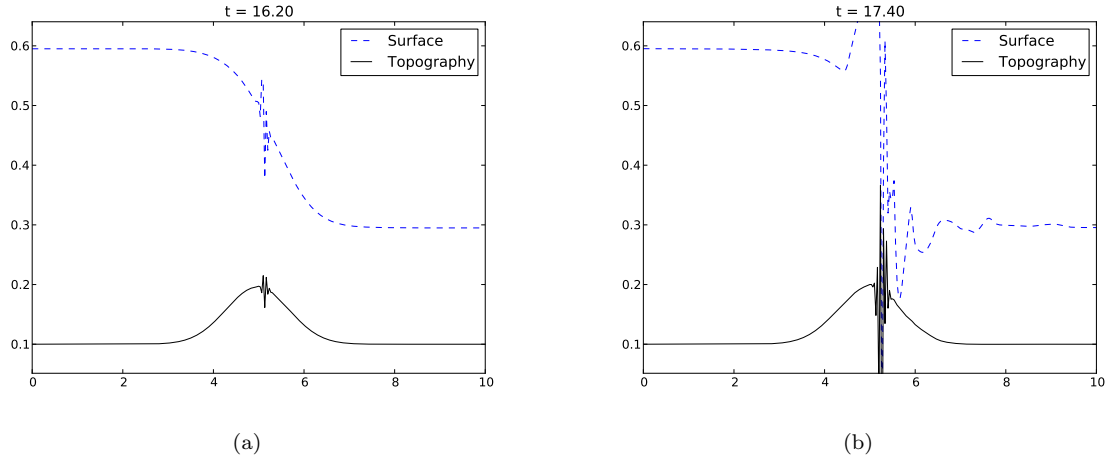


Figure 11: Solution for (31) with  $A_g = 0.0005$  using splitting scheme

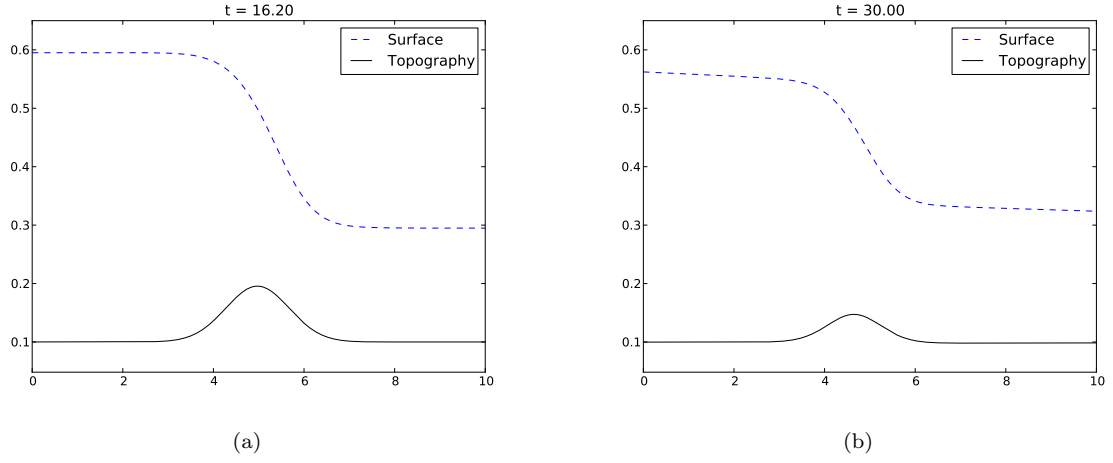


Figure 12: Solution for (31) with  $A_g = 0.0005$  using full system

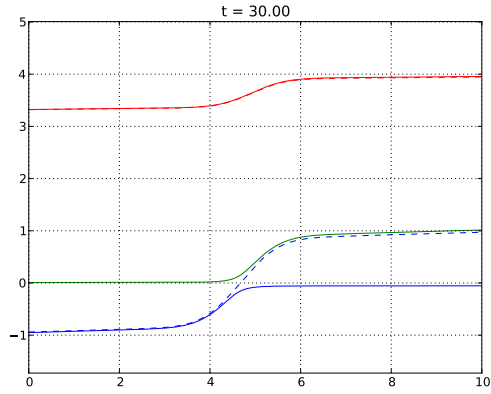


Figure 13: Eigenvalues for (31). True eigenvalues of the system (Continuous line) and  $u \pm \sqrt{gh}$  approximation (Dashed line).

## 5. Conclusions

Exner equation coupled with shallow water equations result as a system that may lose hyperbolicity in some cases at least theoretically. A practical criterium has been introduced for the study of the hyperbolicity region. Nevertheless, the system remains hyperbolic in most of physical situations for classical definitions of the bedload transport flux.

While it seems natural to do a splitting approach by solving first the shallow water system and then updating the topography, it is shown that it could result as instability. The resulting instabilities may be avoided by reducing the CFL condition in certain cases but in some simulations a numerical scheme for the full coupled system may be needed.

## Acknowledgements

This work has been partially supported by projet METHODE ANR-07-BLAN-0232. One of the authors has been partially supported by the Spanish Government Research project MTM2009-11293. The authors also would like to acknowledge constructive comments of C. Lucas, P. Sochala, O. Cerdan, O. Deleste and C. Laguerre.

## References

- [1] F. Exner, Über die wechselwirkung zwischen wasser und geschiebe in flüssen, Sitzungsber., Akad. Wissenschaften pt. IIa (1925) Bd. 134.
- [2] M. C. Díaz, E. Fernández-Nieto, A.M.Ferreiro, Sediment transport models in shallow water equations and numerical approach by high order finite volume methods, Computers & Fluids 37 (3) (2008) 299–316. doi:10.1016/j.compfluid.2007.07.017.  
URL <http://www.sciencedirect.com/science/article/B6V26-4PSK8R5-1/2/9043410cc90abc43b410c70660a4efd6>
- [3] A. Grass, Sediment transport by waves and currents, SERC London Cent. Mar. Technol Report No. FL29.
- [4] J. Hervouet, Les équations de Navier-Stokes à surface libre et leurs formes simplifiées en eau peu profonde, Support de cours: Ecoulements peu profonds, INRIA, 2002.
- [5] E. Meyer-Peter, R. Müller, Formulas for bed-load transport, in: 2nd meeting IAHSR, Stockholm, Sweden, 1948, pp. 1–26.
- [6] P. Nielsen, Coastal Bottom Boundary Layers and Sediment Transport, World Scientific Pub Co Inc, 1992.
- [7] B. Camenen, M. Larson, A general formula for non-cohesive bed load sediment transport, Estuarine Coastal and Shelf Science 63 (2005) 249–260.
- [8] E. Godlewski, P. Raviart, Numerical Approximation of Hyperbolic Systems of Conservation Laws, Applied Mathematical Sciences, Vol. 118, 1996.
- [9] F. Bouchut, Nonlinear stability of finite volume methods for hyperbolic conservation laws, and well-balanced schemes for sources, Frontiers in Mathematics, Birkhauser, 2004.
- [10] S. Cordier, Hyperbolicity of grad's extension of hydrodynamic models of ionospheric plasma. part one: The single species case., Math. Models and Methods in Applied Science (M3AS) 4 (5) (1994) 625–645.
- [11] S. Cordier, Hyperbolicity of grad's extension of hydrodynamic models of ionospheric plasma. part two: The two species case, Math. Models and Methods in Applied Science (M3AS) 4 (5) (1994) 647–667.
- [12] S. Cordier, Hyperbolicity of the hydrodynamical model of plasmas under the quasineutrality hypothesis, Math. Models in Appl. Sci. (M2AS) 18 (1995) 627–647.
- [13] E. Kubatko, J. Westerink, C. Dawson, An unstructured grid morphodynamic model with a discontinuous galerkin method for bed evolution, Ocean modelling 15 (2006) 71–89.
- [14] G. Nord, M. Esteves, Psem.2d: A physically based model of erosion processes at the plot scale, Water resources research 41 (2005) 14PP. doi:200510.1029/2004WR003690.  
URL <http://www.agu.org/journals/ABS/2005/2004WR003690.shtml>
- [15] J. Yang, Assimilation de données variationnelle pour les problèmes de transport des sédiments en rivière, PhD. Thesis, Joseph-Fourier - Grenoble I university, 2004.
- [16] E. F. Toro, Shock-Capturing Methods for Free-Surface Shallow Flows, Wiley and Sons Ltd., 2001.
- [17] O. Deleste, Rain water overland flow on agricultural fields simulation, PhD. Thesis, Orleans university, 2010.  
URL <http://tel.archives-ouvertes.fr/tel-00531377/fr/>
- [18] M. Castro, J. Macías, C. Parés, A  $Q$ -scheme for a class of systems of coupled conservation laws with source term. Application to a two-layer 1-D shallow water system, M2AN Math. Model. Numer. Anal. 35 (1) (2001) 107–127.
- [19] F. Bouchut, T. Morales de Luna, An entropy satisfying scheme for two-layer shallow water equations with uncoupled treatment, ESAIM: Mathematical Modelling and Numerical Analysis (ESAIM: M2AN) 42 (2008) 683–698. doi:10.1051/m2an:2008019.  
URL <http://www.esaim-m2an.org/index.php?option=article&access=standard&Itemid=129&url=/articles/m2an/abs/first/m2an0716/m2an0716.html>

- [20] M. E. Vázquez-Cendón, Improved treatment of source terms in upwind schemes for the shallow water equations in channels with irregular geometry, *J. Comput. Phys.* 148 (2) (1999) 497–526.
- [21] I. Tóth, A weak formulation of Roe’s approximate Riemann solver, *J. Comput. Phys.* 102 (2) (1992) 360–373.
- [22] C. Parés, M. Castro, On the well-balance property of Roe’s method for nonconservative hyperbolic systems. Applications to shallow-water systems, *M2AN Math. Model. Numer. Anal.* 38 (5) (2004) 821–852.