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# Free convolution with a semicircular distribution and eigenvalues of spiked deformations of Wigner matrices <sup>\*</sup>

M. Capitaine<sup>†</sup>, C. Donati-Martin<sup>‡</sup>, D. Féral<sup>§</sup> and M. Février<sup>¶</sup>

## Abstract

We investigate the asymptotic behavior of the eigenvalues of spiked perturbations of Wigner matrices defined by  $M_N = \frac{1}{\sqrt{N}}W_N + A_N$ , where  $W_N$  is a  $N \times N$  Wigner Hermitian matrix whose entries have a distribution  $\mu$  which is symmetric and satisfies a Poincaré inequality and  $A_N$  is a deterministic Hermitian matrix whose spectral measure converges to some probability measure  $\nu$  with compact support. We assume that  $A_N$  has a fixed number of fixed eigenvalues (spikes) outside the support of  $\nu$  whereas the distance between the other eigenvalues and the support of  $\nu$  uniformly goes to zero as  $N$  goes to infinity. We establish that only a particular subset of the spikes will generate some eigenvalues of  $M_N$  which will converge to some limiting points outside the support of the limiting spectral measure. This phenomenon can be fully described in terms of free probability involving the subordination function related to the free additive convolution of  $\nu$  by a semicircular distribution. Note that only finite rank perturbations had been considered up to now (even in the deformed GUE case).

## 1 Introduction

In the fifties, in order to describe the energy levels of a complex nuclei system by the eigenvalues of large Hermitian matrices, E. Wigner introduced the so-called Wigner  $N \times N$  matrix  $W_N$ . According to Wigner's work [37], [38] and further results of different authors (see [4] for a review), provided the common

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distribution  $\mu$  of the entries is centered with variance  $\sigma^2$ , the large  $N$ -limiting spectral distribution of the rescaled complex Wigner matrix  $X_N = \frac{1}{\sqrt{N}}W_N$  is the semicircle distribution  $\mu_\sigma$  whose density is given by

$$\frac{d\mu_\sigma}{dx}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} \mathbf{1}_{[-2\sigma, 2\sigma]}(x). \quad (1.1)$$

Moreover, if the fourth moment of the measure  $\mu$  is finite, the largest (resp. smallest) eigenvalue of  $X_N$  converges almost surely towards the right (resp. left) endpoint  $2\sigma$  (resp.  $-2\sigma$ ) of the semicircular support (cf. [7] or Theorem 2.12 in [4]).

Now, how does the spectrum behave under a deterministic Hermitian perturbation  $A_N$ ? The set of possible spectra for  $M_N = X_N + A_N$  depends in a complicated way on the spectra of  $X_N$  and  $A_N$  (see [22]). Nevertheless, when  $N$  becomes large, free probability provides us a good understanding of the global behaviour of the spectrum of  $M_N$ . Indeed, if the spectral measure of  $A_N$  weakly converges to some probability measure  $\nu$  and  $\|A_N\|$  is uniformly bounded in  $N$ , the spectral distribution of  $M_N$  weakly converges to the free convolution  $\mu_\sigma \boxplus \nu$  almost surely and in expectation (cf [1], [28] and [34], [20] for pioneering works). We refer the reader to [36] for an introduction to free probability theory. Note that when  $A_N$  is of finite rank, the spectral distribution of  $M_N$  still converges to the semicircular distribution ( $\nu \equiv \delta_0$  and  $\mu_\sigma \boxplus \nu = \mu_\sigma$ ).

In [30], S. Péché investigated the deformed G.U.E. model  $M_N^G = W_N^G/\sqrt{N} + A_N$ , where  $W_N^G$  is a G.U.E. matrix, that is a Wigner matrix associated to a centered Gaussian measure with variance  $\sigma^2$  and  $A_N$  is a deterministic perturbation of finite rank with fixed eigenvalues. This model is the additive analogue of the Wishart matrices with spiked covariance matrix previously considered by J. Baik, G. Ben Arous and S. Péché [8] who exhibited a striking phase transition phenomenon for the fluctuations of the largest eigenvalue according to the values of the spikes. S. Péché pointed out an analogue phase transition phenomenon for the fluctuations of the largest eigenvalue of  $M_N^G$  with respect to the largest eigenvalue  $\theta$  of  $A_N$  [30]. These investigations imply that, if  $\theta$  is far enough from zero ( $\theta > \sigma$ ), then the largest eigenvalue of  $M_N^G$  jumps above the support  $[-2\sigma, 2\sigma]$  of the limiting spectral measure and converges (in probability) towards  $\rho_\theta = \theta + \frac{\sigma^2}{\theta}$ . Note that Z. Füredi and J. Komlós already exhibited such a phenomenon in [23] dealing with non-centered symmetric matrices.

In [21], D. Féral and S. Péché proved that the results of [30] still hold for a non-necessarily Gaussian Wigner Hermitian matrix  $W_N$  with sub-Gaussian moments and in the particular case of a rank one perturbation matrix  $A_N$  whose entries are all  $\frac{\theta}{N}$  for some real number  $\theta$ . In [19], we considered a deterministic Hermitian matrix  $A_N$  of arbitrary fixed finite rank  $r$  and built from a family of  $J$  fixed non-null real numbers  $\theta_1 > \dots > \theta_J$  independent of  $N$  and such that each  $\theta_j$  is an eigenvalue of  $A_N$  of fixed multiplicity  $k_j$  (with  $\sum_{j=1}^J k_j = r$ ). In the following, the  $\theta_j$ 's are referred as the spikes of  $A_N$ . We dealt with general Wigner matrices associated to some symmetric measure satisfying a Poincaré inequality. We proved that eigenvalues of  $A_N$  with absolute value strictly greater than  $\sigma$  generate some eigenvalues of  $M_N$  which converge to some limiting points

outside the support of  $\mu_\sigma$ . To be more precise, we need to introduce further notations. Given an arbitrary Hermitian matrix  $B$  of size  $N$ , we denote by  $\lambda_1(B) \geq \dots \geq \lambda_N(B)$  its  $N$  ordered eigenvalues. For each spike  $\theta_j$ , we denote by  $n_{j-1} + 1, \dots, n_{j-1} + k_j$  the descending ranks of  $\theta_j$  among the eigenvalues of  $A_N$  (multiplicities of eigenvalues are counted) with the convention that  $k_1 + \dots + k_{j-1} = 0$  for  $j = 1$ . One has that

$$n_{j-1} = k_1 + \dots + k_{j-1} \quad \text{if } \theta_j > 0 \quad \text{and} \quad n_{j-1} = N - r + k_1 + \dots + k_{j-1} \quad \text{if } \theta_j < 0.$$

Letting  $J_{+\sigma}$  (resp.  $J_{-\sigma}$ ) be the number of  $j$ 's such that  $\theta_j > \sigma$  (resp.  $\theta_j < -\sigma$ ), we established in [19] that, when  $N$  goes to infinity,

- a) for all  $j$  such that  $1 \leq j \leq J_{+\sigma}$  (resp.  $j \geq J - J_{-\sigma} + 1$ ), the  $k_j$  eigenvalues  $(\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j)$  converge almost surely to  $\rho_{\theta_j} = \theta_j + \frac{\sigma^2}{\theta_j}$  which is  $> 2\sigma$  (resp.  $< -2\sigma$ ).

- b)  $\lambda_{k_1+\dots+k_{J_{+\sigma}+1}}(M_N) \xrightarrow{a.s.} 2\sigma$  and  $\lambda_{N-(k_J+\dots+k_{J-J_{-\sigma}+1})}(M_N) \xrightarrow{a.s.} -2\sigma$ .

Actually, this phenomenon may be described in terms of free probability involving the subordination function related to the free convolution of  $\nu = \delta_0$  by a semicircular distribution. Let us present it briefly. For a probability measure  $\tau$  on  $\mathbb{R}$ , let us denote by  $g_\tau$  its Stieltjes transform, defined for  $z \in \mathbb{C} \setminus \mathbb{R}$  by

$$g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z - x}.$$

Let  $\nu$  and  $\tau$  be two probability measures on  $\mathbb{R}$ . It is proved in [14] Theorem 3.1 that there exists an analytic map  $F : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ , called subordination function, such that

$$\forall z \in \mathbb{C}^+, g_{\tau \boxplus \nu}(z) = g_\nu(F(z)),$$

where  $\mathbb{C}^+$  denotes the set of complex numbers  $z$  such that  $\Im z > 0$ . When  $\tau = \mu_\sigma$ , let us denote by  $F_{\sigma, \nu}$  the corresponding subordination function. When  $\nu = \delta_0$  and  $\tau = \mu_\sigma$ , the subordination function is given by  $F_{\sigma, \delta_0} = 1/g_{\mu_\sigma}$ . According to Lemma 4.4 in [19], one may notice that the complement of the support of  $\mu_\sigma \boxplus \delta_0 (= \mu_\sigma)$  can be described as:

$$\mathbb{R} \setminus [-2\sigma, 2\sigma] = \{x, \exists u \in \mathbb{R}^*, |u| > \sigma \text{ such that } x = H_{\sigma, \delta_0}(u)\},$$

where  $H_{\sigma, \delta_0}(z) = z + \frac{\sigma^2}{z}$  is the inverse function of the subordination function  $F_{\sigma, \delta_0}$  on  $\mathbb{R} \setminus [-2\sigma, 2\sigma]$ . Now, the characterization of the spikes of  $A_N$  that generate jumps of eigenvalues of  $M_N$  i.e.  $|\theta_j| > \sigma$  is obviously equivalent to the following

$$\theta_j \in \mathbb{R} \setminus \text{supp}(\delta_0) (= \mathbb{R}^*) \quad \text{and} \quad H'_{\sigma, \delta_0}(\theta_j) > 0.$$

Moreover the relationship between a spike  $\theta_j$  of  $A_N$  such that  $|\theta_j| > \sigma$  and the limiting point  $\rho_{\theta_j}$  of the corresponding eigenvalues of  $M_N$  (which is then outside  $[-2\sigma; 2\sigma]$ ) is actually described by the inverse function of the subordination function as:

$$\rho_{\theta_j} = H_{\sigma, \delta_0}(\theta_j).$$

Actually this very interpretation in terms of subordination function of the characterization of the spikes of  $A_N$  that generate jumps of eigenvalues of  $M_N$  as well as the values of the jumps provides the intuition to imagine the generalization of the phenomenon dealing with non-finite rank perturbations just by replacing  $\delta_0$  by the limiting spectral distribution  $\nu$  of  $A_N$  in the previous lines. Up to now, no result has been established for non-finite rank additive spiked perturbation. Moreover, this paper shows up that free probability can also shed light on the asymptotic behaviour of the eigenvalues of the deformed Wigner model and strengthens the fact that free probability theory and random matrix theory are closely related.

More precisely, in this paper, we consider the following general deformed Wigner models  $M_N = \frac{1}{\sqrt{N}}W_N + A_N$  such that:

- $W_N$  is a  $N \times N$  Wigner Hermitian matrix associated to a distribution  $\mu$  of variance  $\sigma^2$  and mean zero:  
 $(W_N)_{ii}$ ,  $\sqrt{2}\Re((W_N)_{ij})_{i < j}$ ,  $\sqrt{2}\Im((W_N)_{ij})_{i < j}$  are i.i.d., with distribution  $\mu$  which is symmetric and satisfies a Poincaré inequality (the definition of such an inequality is recalled in the Appendix).
- $A_N$  is a deterministic Hermitian matrix whose eigenvalues  $\gamma_i^{(N)}$ , denoted for simplicity by  $\gamma_i$ , are such that the spectral measure  $\mu_{A_N} := \frac{1}{N} \sum_{i=1}^N \delta_{\gamma_i}$  converges to some probability measure  $\nu$  with compact support. We assume that there exists a fixed integer  $r \geq 0$  (independent from  $N$ ) such that  $A_N$  has  $N - r$  eigenvalues  $\beta_j(N)$  satisfying

$$\max_{1 \leq j \leq N-r} \text{dist}(\beta_j(N), \text{supp}(\nu)) \xrightarrow{N \rightarrow \infty} 0,$$

where  $\text{supp}(\nu)$  denotes the support of  $\nu$ . We also assume that there are  $J$  fixed real numbers  $\theta_1 > \dots > \theta_J$  independent of  $N$  which are outside the support of  $\nu$  and such that each  $\theta_j$  is an eigenvalue of  $A_N$  with a fixed multiplicity  $k_j$  (with  $\sum_{j=1}^J k_j = r$ ). The  $\theta_j$ 's will be called the spikes or the spiked eigenvalues of  $A_N$ .

According to [1], the spectral distribution of  $M_N$  weakly converges to the free convolution  $\mu_\sigma \boxplus \nu$  almost surely (cf. Remark 4.1 below). It turns out that the spikes of  $A_N$  that will generate jumps of eigenvalues of  $M_N$  will be the  $\theta_j$ 's such that  $H'_{\sigma,\nu}(\theta_j) > 0$  where  $H_{\sigma,\nu}(z) = z + \sigma^2 g_\nu(z)$  and the corresponding limiting points outside the support of  $\mu_\sigma \boxplus \nu$  will be given by

$$\rho_{\theta_j} = H_{\sigma,\nu}(\theta_j).$$

It is worth noticing that the set  $\{u \in \mathbb{R} \setminus \text{supp}(\nu), H'_{\sigma,\nu}(u) > 0\}$  is actually the complement of the closure of the open set

$$U_{\sigma,\nu} := \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}$$

introduced by P. Biane in [13] to describe the support of the free additive convolution of a probability measure  $\nu$  on  $\mathbb{R}$  by a semicircular distribution. Note that the deep study of P. Biane of the free convolution by a semicircular distribution will be of fundamental use in our approach. In Theorem 8.1, which is the main result of the paper, we present a complete description of the convergence of the eigenvalues of  $M_N$  depending on the location of the  $\theta_j$ 's with respect to  $\overline{U_{\sigma,\nu}}$  and to the connected components of the support of  $\nu$ .

Our approach also allows us to study the “non-spiked” deformed Wigner models i.e. such that  $r = 0$ . Up to now, the results which can be found in the literature for such a situation concern the so-called Gaussian matrix models with external source where the underlying Wigner matrix is from the G.U.E.. Many works on these models deal with the local behaviour of the eigenvalues of  $M_N$  (see for instance [15], [2] and [16] for details). Moreover, the recent results of [27] (which investigate several matrices in a free probability context) imply that the operator norm (i.e. the largest singular value) of some non-spiked deformed G.U.E.  $M_N^G = W_N^G/N + A_N$  converges almost surely to the  $L^\infty$ -norm of a  $(\mu_\sigma \boxplus \nu)$ -distributed random variable, under some conditions on the convergence of the Stieltjes transform of  $A_N$  to that of  $\nu$ . Here, we readily deduce (cf. Proposition 8.1 below) from our results the almost sure convergence of the extremal eigenvalues of general non-spiked deformed Wigner models to the corresponding endpoints of the compact support of the free convolution  $\mu_\sigma \boxplus \nu$ .

The asymptotic behaviour of the eigenvalues of the deformed Wigner model  $M_N$  actually comes from two phenomena involving free convolution:

1. the inclusion of the spectrum of  $M_N$  in an  $\epsilon$ -neighborhood of the support of  $\mu_\sigma \boxplus \mu_{A_N}$ , for all large  $N$  almost surely;
2. an exact separation phenomenon between the spectrum of  $M_N$  and the spectrum of  $A_N$ , involving the subordination function  $F_{\sigma,\nu}$  of  $\mu_\sigma \boxplus \nu$  (i.e. to a gap in the spectrum of  $M_N$ , it corresponds through  $F_{\sigma,\nu}$  a gap in the spectrum of  $A_N$  which splits the spectrum of  $A_N$  exactly as that of  $M_N$ ).

The key idea to prove the first point is to obtain a precise estimate of order  $\frac{1}{N}$  of the difference between the respective Stieltjes transforms of the mean spectral measure of the deformed model and of  $\mu_\sigma \boxplus \mu_{A_N}$ . To get such an estimate, we prove an “approximative subordination equation” satisfied by the Stieltjes transform of the deformed model. Note that, even if the ideas and tools are very close to those developed in [19], the proof in [19] does not use the above analysis from free probability whereas this very analysis allows us to extend the results of [19] to non-finite rank deformations. In particular, we didn't consider in [19]  $\mu_\sigma \boxplus \mu_{A_N}$  whose support actually makes the asymptotic values of the eigenvalues that will be outside the limiting support of the spectral measure of  $M_N$  appear.

Note that phenomena 1. and 2. are actually the additive analogues of those described in [5], [6] in the framework of spiked population models, even if the authors do not refer to free probability. In [9], the authors use the results of

[5], [6] to establish the almost sure convergence of the eigenvalues generated by the spikes in a spiked population model where all but finitely many eigenvalues of the covariance matrix are equal to one. Thus, they generalize the pioneering result of [8] in the Gaussian setting. Recently, [31], [3] extended this theory to a generalized spiked population model where the base population covariance matrix is arbitrary. Our results are exactly the additive analogues of theirs. It is worth noticing that one may check that these results on spiked population models could also be fully described in terms of free probability involving the subordination function related to the free multiplicative convolution of  $\nu$  by a Marchenko-Pastur distribution. Moreover, the results of F. Benaych-Georges and R. N. Rao in [12] about the convergence of the extremal eigenvalues of a matrix  $X_N + A_N$ ,  $A_N$  being a finite rank perturbation whereas  $X_N$  is a unitarily invariant matrix with some compactly supported limiting spectral distribution  $\mu$ , could be rewritten in terms of the subordination function related to the free additive convolution of  $\delta_0$  by  $\mu$ . Hence, we think that subordination property in free probability definitely sheds light on spiked deformed models.

Finally, one can expect that our results hold true in a more general setting than the one considered here, namely only requires the existence of a finite fourth moment on the measure  $\mu$  of the Wigner entries. Nevertheless, the assumption that  $\mu$  satisfies a Poincaré inequality is fundamental in our approach since we need several variance estimates.

The paper is organized as follows. In Section 2, we first recall some results on free additive convolution and subordination property as well as the description by P. Biane of the support of the free convolution of some probability measure  $\nu$  by a semicircular distribution. We then deduce a characterization of this support via the subordination function when  $\nu$  is compactly supported and we exhibit relationships between the steps of the distribution functions of  $\nu$  and  $\mu_\sigma \boxplus \nu$ . In Section 3, we establish an approximative subordination equation for the Stieltjes transform  $g_N$  of the mean spectral distribution of the deformed model  $M_N$  and explain in Section 4 how to deduce an estimation up to the order  $\frac{1}{N^2}$  of the difference between  $g_N$  and the Stieltjes transform of  $\mu_\sigma \boxplus \mu_{A_N}$  when  $N$  goes to infinity. In Section 5, we show how to deduce the almost sure inclusion of the spectrum of  $M_N$  in a neighborhood of the support of  $\mu_\sigma \boxplus \mu_{A_N}$  for all large  $N$ ; we use the ideas (based on inverse Stieltjes transform) of [24] and [32] in the non-deformed Gaussian complex, real or symplectic Wigner setting; nevertheless, since  $\mu_\sigma \boxplus \mu_{A_N}$  depends on  $N$ , we need here to apply the inverse Stieltjes transform to functions depending on  $N$  and we therefore give the details of the proof to convince the reader that the approach developed by [24] and [32] still holds. In Section 6, we show how the support of  $\mu_\sigma \boxplus \mu_{A_N}$  makes the asymptotic values of the eigenvalues that will be outside the support of the limiting spectral measure appear since we prove that, for any  $\epsilon > 0$ ,  $\text{supp}(\mu_\sigma \boxplus \mu_{A_N})$  is included in an  $\epsilon$ -neighborhood of  $\text{supp}(\mu_\sigma \boxplus \nu) \cup \{\rho_{\theta_j}, \theta_j \text{ such that } H'_{\sigma, \nu}(\theta_j) > 0\}$ , when  $N$  is large enough. Section 7 is devoted to the proof of the exact separation phenomenon between the spectrum of  $M_N$  and the spectrum of  $A_N$ , involving the subordination function  $F_{\sigma, \nu}$ . In the last section, we show how to deduce our main result (Theorem 8.1) about the convergence of the eigenvalues of the

deformed model  $M_N$ . Finally we present in an Appendix the proofs of some technical estimates on variances used throughout the paper.

Throughout this paper, we will use the following notations.

- For a probability measure  $\tau$  on  $\mathbb{R}$ , we denote by  $g_\tau$  its Stieltjes transform defined for  $z \in \mathbb{C} \setminus \mathbb{R}$  by

$$g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z - x}.$$

- $G_N$  denotes the resolvent of  $M_N$  and  $g_N$  the mean of the Stieltjes transform of the spectral measure of  $M_N$ , that is

$$g_N(z) = \mathbb{E}(\text{tr}_N G_N(z)), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where  $\text{tr}_N$  is the normalized trace:  $\text{tr}_N = \frac{1}{N} \text{Tr}$ .

We recall some useful properties of the resolvent (see [26], [18]).

**Lemma 1.1.** *For a  $N \times N$  Hermitian or symmetric matrix  $M$ , for any  $z \in \mathbb{C} \setminus \text{Spect}(M)$ , we denote by  $G(z) := (zI_N - M)^{-1}$  the resolvent of  $M$ . Let  $z \in \mathbb{C} \setminus \mathbb{R}$ ,*

- (i)  $\|G(z)\| \leq |\Im z|^{-1}$  where  $\|\cdot\|$  denotes the operator norm.
- (ii)  $|G(z)_{ij}| \leq |\Im z|^{-1}$  for all  $i, j = 1, \dots, N$ .
- (iii) For  $p \geq 2$ ,

$$\frac{1}{N} \sum_{i,j=1}^N |G(z)_{ij}|^p \leq (|\Im z|^{-1})^p. \quad (1.2)$$

- (iv) The derivative with respect to  $M$  of the resolvent  $G(z)$  satisfies:

$$G'_M(z).B = G(z)BG(z) \quad \text{for any matrix } B.$$

- (v) Let  $z \in \mathbb{C}$  such that  $|z| > \|M\|$ ; we have

$$\|G(z)\| \leq \frac{1}{|z| - \|M\|}.$$

- $\tilde{g}_N$  denotes the Stieltjes transform of the probability measure  $\mu_\sigma \boxplus \mu_{A_N}$ .
- When we state that some quantity  $\Delta_N(z)$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ , is  $O(\frac{1}{N^p})$ , this means precisely that:

$$|\Delta_N(z)| \leq \frac{P(|\Im z|^{-1})}{N^p},$$

for some polynomial  $P$  with nonnegative coefficients.

- For any set  $S$  in  $\mathbb{R}$ , we denote the set  $\{x \in \mathbb{R}, \text{dist}(x, S) \leq \epsilon\}$  (resp.  $\{x \in \mathbb{R}, \text{dist}(x, S) < \epsilon\}$ ) by  $S + [-\epsilon, +\epsilon]$  (resp.  $S + (-\epsilon, +\epsilon)$ ).



## 2 Free convolution

### 2.1 Definition and subordination property

Let  $\tau$  be a probability measure on  $\mathbb{R}$ . Its Stieltjes transform  $g_\tau$  is analytic on the complex upper half-plane  $\mathbb{C}^+$ . There exists a domain

$$D_{\alpha,\beta} = \{u + iv \in \mathbb{C}, |u| < \alpha v, v > \beta\}$$

on which  $g_\tau$  is univalent. Let  $K_\tau$  be its inverse function, defined on  $g_\tau(D_{\alpha,\beta})$ , and

$$R_\tau(z) = K_\tau(z) - \frac{1}{z}.$$

Given two probability measures  $\tau$  and  $\nu$ , there exists a unique probability measure  $\lambda$  such that

$$R_\lambda = R_\tau + R_\nu$$

on a domain where these functions are defined. The probability measure  $\lambda$  is called the free convolution of  $\tau$  and  $\nu$  and denoted by  $\tau \boxplus \nu$ .

The free convolution of probability measures has an important property, called subordination, which can be stated as follows: let  $\tau$  and  $\nu$  be two probability measures on  $\mathbb{R}$ ; there exists an analytic map  $F : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  such that

$$\forall z \in \mathbb{C}^+, \quad g_{\tau \boxplus \nu}(z) = g_\nu(F(z)).$$

This phenomenon was first observed by D. Voiculescu under a genericity assumption in [35], and then proved in generality in [14] Theorem 3.1. Later, a new proof of this result was given in [11], using a fixed point theorem for analytic self-maps of the upper half-plane.

### 2.2 Free convolution by a semicircular distribution

In [13], P. Biane provides a deep study of the free convolution by a semicircular distribution. We first recall here some of his results that will be useful in our approach.

Let  $\nu$  be a probability measure on  $\mathbb{R}$ . P. Biane [13] introduces the set

$$\Omega_{\sigma,\nu} := \{u + iv \in \mathbb{C}^+, v > v_{\sigma,\nu}(u)\},$$

where the function  $v_{\sigma,\nu} : \mathbb{R} \rightarrow \mathbb{R}^+$  is defined by

$$v_{\sigma,\nu}(u) = \inf \left\{ v \geq 0, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} \leq \frac{1}{\sigma^2} \right\}$$

and proves the following

**Proposition 2.1.** [13] *The map*

$$H_{\sigma,\nu} : z \mapsto z + \sigma^2 g_\nu(z)$$

is a homeomorphism from  $\overline{\Omega_{\sigma,\nu}}$  to  $\mathbb{C}^+ \cup \mathbb{R}$  which is conformal from  $\Omega_{\sigma,\nu}$  onto  $\mathbb{C}^+$ . Let  $F_{\sigma,\nu} : \mathbb{C}^+ \cup \mathbb{R} \rightarrow \overline{\Omega_{\sigma,\nu}}$  be the inverse function of  $H_{\sigma,\nu}$ . One has,

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_\sigma \boxplus \nu}(z) = g_\nu(F_{\sigma,\nu}(z))$$

and then

$$F_{\sigma,\nu}(z) = z - \sigma^2 g_{\mu_\sigma \boxplus \nu}(z). \quad (2.1)$$

Note that in particular the Stieltjes transform  $\tilde{g}_N$  of  $\mu_\sigma \boxplus \mu_{A_N}$  satisfies

$$\forall z \in \mathbb{C}^+, \quad \tilde{g}_N(z) = g_{\mu_{A_N}}(z - \sigma^2 \tilde{g}_N(z)). \quad (2.2)$$

Considering  $H_{\sigma,\nu}$  as an analytic map defined in the whole upper half-plane  $\mathbb{C}^+$ , P. Biane identifies  $\Omega_{\sigma,\nu}$  as the connected component of the set  $H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$  which contains  $iy$  for large  $y$ . In fact, it is proved in [10] that  $\Omega_{\sigma,\nu} = H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$ , or equivalently, that  $H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$  is connected. We give here another proof of this result in the particular case of the free convolution by a semicircular distribution:

**Proposition 2.2.**

$$\Omega_{\sigma,\nu} = H_{\sigma,\nu}^{-1}(\mathbb{C}^+).$$

**Proof of Proposition 2.2:** It is clear from the discussion above that  $\Omega_{\sigma,\nu}$  is included in  $H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$ .

Fix  $u + iv \in H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$ ; we have to prove that  $v > v_{\sigma,\nu}(u)$ .

Since  $u + iv \in H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$ , we have:

$$\Im H_{\sigma,\nu}(u + iv) = v(1 - \sigma^2 \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2}) > 0.$$

This, together with  $v > 0$ , implies

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} < \frac{1}{\sigma^2}$$

and consequently  $v \geq v_{\sigma,\nu}(u)$ . If we assume that  $v = v_{\sigma,\nu}(u)$ , then  $v_{\sigma,\nu}(u) > 0$  and finally

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} = \frac{1}{\sigma^2}$$

by Lemma 2 in [13]. This is a contradiction : necessarily  $v > v_{\sigma,\nu}(u)$  or, in other words,  $u + iv \in \Omega_{\sigma,\nu}$  and we are done.  $\square$

The previous results of P. Biane allow him to conclude that  $\mu_\sigma \boxplus \nu$  is absolutely continuous with respect to the Lebesgue measure and to obtain the following description of the support.

**Theorem 2.1.** [13] Define  $\Psi_{\sigma,\nu} : \mathbb{R} \rightarrow \mathbb{R}$  by:

$$\Psi_{\sigma,\nu}(u) = H_{\sigma,\nu}(u + iv_{\sigma,\nu}(u)) = u + \sigma^2 \int_{\mathbb{R}} \frac{(u-x)d\nu(x)}{(u-x)^2 + v_{\sigma,\nu}(u)^2}.$$

$\Psi_{\sigma,\nu}$  is a homeomorphism and, at the point  $\Psi_{\sigma,\nu}(u)$ , the measure  $\mu_\sigma \boxplus \nu$  has a density given by

$$p_{\sigma,\nu}(\Psi_{\sigma,\nu}(u)) = \frac{v_{\sigma,\nu}(u)}{\pi\sigma^2}.$$

Define the set

$$U_{\sigma,\nu} := \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\} = \{u \in \mathbb{R}, v_{\sigma,\nu}(u) > 0\}.$$

The support of the measure  $\mu_\sigma \boxplus \nu$  is the image of the closure of the open set  $U_{\sigma,\nu}$  by the homeomorphism  $\Psi_{\sigma,\nu}$ .  $\Psi_{\sigma,\nu}$  is strictly increasing on  $U_{\sigma,\nu}$ .

Hence,

$$\mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \nu) = \Psi_{\sigma,\nu}(\mathbb{R} \setminus \overline{U_{\sigma,\nu}}).$$

One has  $\Psi_{\sigma,\nu} = H_{\sigma,\nu}$  on  $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$  and  $\Psi_{\sigma,\nu}^{-1} = F_{\sigma,\nu}$  on  $\mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \nu)$ . In particular, we have the following description of the complement of the support:

$$\mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \nu) = H_{\sigma,\nu}(\mathbb{R} \setminus \overline{U_{\sigma,\nu}}). \quad (2.3)$$

Let  $\nu$  be a compactly supported probability measure. We are going to establish a characterization of the complement of the support of  $\mu_\sigma \boxplus \nu$  involving the support of  $\nu$  and  $H_{\sigma,\nu}$ . We will need the following preliminary lemma.

**Lemma 2.1.** *The support of  $\nu$  is included in  $\overline{U_{\sigma,\nu}}$ .*

**Proof of Lemma 2.1:** Let  $x_0$  be in  $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ . Then, there is some  $\epsilon > 0$  such that  $[x_0 - \epsilon, x_0 + \epsilon] \subset \mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ . For any integer  $n \geq 1$ , we define  $\alpha_k = x_0 - \epsilon + 2k\epsilon/n$  for all  $0 \leq k \leq n$ . Then, as the sets  $[\alpha_k, \alpha_{k+1}]$  are trivially contained in  $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ , one has that:

$$\forall u \in [\alpha_k, \alpha_{k+1}], \quad \frac{1}{\sigma^2} \geq \int_{\alpha_k}^{\alpha_{k+1}} \frac{d\nu(x)}{(u-x)^2} \geq \frac{\nu([\alpha_k, \alpha_{k+1}])}{(\alpha_{k+1} - \alpha_k)^2}.$$

This readily implies that

$$\nu([x_0 - \epsilon, x_0 + \epsilon]) \leq \sum_{k=0}^{n-1} \nu([\alpha_k, \alpha_{k+1}]) \leq \frac{(2\epsilon)^2}{\sigma^2 n}.$$

Letting  $n \rightarrow \infty$ , we get that  $\nu([x_0 - \epsilon, x_0 + \epsilon]) = 0$ , which implies that  $x_0 \in \mathbb{R} \setminus \text{supp}(\nu)$ .  $\square$

From the continuity and strict convexity of the function  $u \rightarrow \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2}$  on  $\mathbb{R} \setminus \text{supp}(\nu)$ , it follows that

$$\overline{U_{\sigma,\nu}} = \text{supp}(\nu) \cup \left\{ u \in \mathbb{R} \setminus \text{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} \geq \frac{1}{\sigma^2} \right\} \quad (2.4)$$

and

$$\mathbb{R} \setminus \overline{U_{\sigma,\nu}} = \left\{ u \in \mathbb{R} \setminus \text{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2} \right\}.$$

Now, as  $H_{\sigma,\nu}$  is analytic on  $\mathbb{R} \setminus \text{supp}(\nu)$ , the following characterization readily follows:

$$\mathbb{R} \setminus \overline{U_{\sigma,\nu}} = \{u \in \mathbb{R} \setminus \text{supp}(\nu), H'_{\sigma,\nu}(u) > 0\}.$$

and thus, according to (2.3), we get

**Proposition 2.3.**

$x \in \mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \nu) \Leftrightarrow \exists u \in \mathbb{R} \setminus \text{supp}(\nu)$  such that  $x = H_{\sigma,\nu}(u)$ ,  $H'_{\sigma,\nu}(u) > 0$ .

**Remark 2.1.** Note that  $H_{\sigma,\nu}$  is strictly increasing on  $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$  since, if  $a < b$  are in  $\mathbb{R} \setminus \text{supp}(\nu)$ , one has, by Cauchy-Schwarz inequality, that

$$\begin{aligned} H_{\sigma,\nu}(b) - H_{\sigma,\nu}(a) &= (b-a) \left[ 1 - \sigma^2 \int_{\mathbb{R}} \frac{d\nu(x)}{(a-x)(b-x)} \right] \\ &\geq (b-a) \left[ 1 - \sigma^2 \sqrt{(-g'_\nu(a))(-g'_\nu(b))} \right]. \end{aligned}$$

which is nonnegative if  $a$  and  $b$  belong to  $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ .

**Remark 2.2.** Each connected component of  $\overline{U_{\sigma,\nu}}$  contains at least one connected component of  $\text{supp}(\nu)$ .

Indeed, let  $[s_l, t_l]$  be a connected component of  $\overline{U_{\sigma,\nu}}$ . If  $s_l$  or  $t_l$  is in  $\text{supp}(\nu)$ ,  $[s_l, t_l]$  contains at least a connected component of  $\text{supp}(\nu)$  since  $\text{supp}(\nu)$  is included in  $\overline{U_{\sigma,\nu}}$ . Now, if neither  $s_l$  nor  $t_l$  is in  $\text{supp}(\nu)$ , according to (2.4), we have

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(s_l-x)^2} = \int_{\mathbb{R}} \frac{d\nu(x)}{(t_l-x)^2} = \frac{1}{\sigma^2}.$$

Assume that  $[s_l, t_l] \subset \mathbb{R} \setminus \text{supp}(\nu)$ , then, by strict convexity of the function  $u \mapsto \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2}$  on  $\mathbb{R} \setminus \text{supp}(\nu)$ , one obtains that, for any  $u \in ]s_l, t_l[$ ,

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2},$$

which leads to a contradiction.  $\square$

**Remark 2.3.** One can readily see that

$$\overline{U_{\sigma,\nu}} \subset \{u, \text{dist}(u, \text{supp}(\nu)) \leq \sigma\}$$

and deduce, since  $\text{supp}(\nu)$  is compact, that  $U_{\sigma,\nu}$  is a relatively compact open set. Hence,  $\overline{U_{\sigma,\nu}}$  has a finite number of connected components and may be written as the following finite disjoint union

$$\overline{U_{\sigma,\nu}} = \bigcup_{l=m}^1 [s_l, t_l] \quad \text{with } s_m < t_m < \dots < s_1 < t_1. \quad (2.5)$$

We close this section with a proposition pointing out a relationship between the distribution functions of  $\nu$  and  $\mu_\sigma \boxplus \nu$ .

**Proposition 2.4.** *Let  $[s_l, t_l]$  be a connected component of  $\overline{U_{\sigma, \nu}}$ , then*

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma, \nu}(s_l), \Psi_{\sigma, \nu}(t_l)]) = \nu([s_l, t_l]).$$

**Proof of Proposition 2.4:** Let  $]a, b[$  be a connected component of  $U_{\sigma, \nu}$ . Since  $a$  and  $b$  are not atoms of  $\nu$  and  $\mu_\sigma \boxplus \nu$  is absolutely continuous, it is enough to show

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma, \nu}(a), \Psi_{\sigma, \nu}(b)]) = \nu([a, b]).$$

From Cauchy's inversion formula,  $\mu_\sigma \boxplus \nu$  has a density given by  $p_\sigma(x) = -\frac{1}{\pi} \Im(g_\nu(F_{\nu, \sigma}(x)))$  and

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma, \nu}(a), \Psi_{\sigma, \nu}(b)]) = -\frac{1}{\pi} \Im \left( \int_{\Psi_{\sigma, \nu}(a)}^{\Psi_{\sigma, \nu}(b)} g_\nu(F_{\nu, \sigma}(x)) dx \right).$$

We set  $z = F_{\sigma, \nu}(x)$ , then  $x = H_{\sigma, \nu}(z)$  and  $z = u + iv_{\sigma, \nu}(u)$ . Note that  $v_{\sigma, \nu}(u) > 0$  for  $u \in ]a, b[$  and  $v_{\sigma, \nu}(a) = v_{\sigma, \nu}(b) = 0$  (see [13]). Then,

$$\begin{aligned} & (\mu_\sigma \boxplus \nu)([\Psi_{\sigma, \nu}(a), \Psi_{\sigma, \nu}(b)]) \\ &= -\frac{1}{\pi} \Im \left( \int_a^b g_\nu(u + iv_{\sigma, \nu}(u)) H'_{\sigma, \nu}(u + iv_{\sigma, \nu}(u)) (1 + iv'_{\sigma, \nu}(u)) du \right) \\ &= -\frac{1}{\pi} \Im \left( \int_a^b g_\nu(u + iv_{\sigma, \nu}(u)) (1 + \sigma^2 g'_\nu(u + iv_{\sigma, \nu}(u))) (1 + iv'_{\sigma, \nu}(u)) du \right) \\ &= -\frac{1}{\pi} \left( \Im \int_a^b g_\nu(u + iv_{\sigma, \nu}(u)) (1 + iv'_{\sigma, \nu}(u)) du + \frac{\sigma^2}{2} \Im [g_\nu^2(u + iv_{\sigma, \nu}(u))]_a^b \right) \\ &= -\frac{1}{\pi} \Im \int_a^b g_\nu(u + iv_{\sigma, \nu}(u)) (1 + iv'_{\sigma, \nu}(u)) du = -\frac{1}{\pi} \Im \int_\gamma g_\nu(z) dz, \end{aligned}$$

where

$$\gamma = \{z = u + iv_{\sigma, \nu}(u), u \in [a, b]\}.$$

Now, we recall that, since  $a$  and  $b$  are points of continuity of the distribution function of  $\nu$ ,

$$\nu([a, b]) = \lim_{\epsilon \rightarrow 0} -\frac{1}{\pi} \Im \left( \int_a^b g_\nu(u + i\epsilon) du \right) = \lim_{\epsilon \rightarrow 0} -\frac{1}{\pi} \Im \left( \int_{\gamma_\epsilon} g_\nu(z) dz \right),$$

where  $\gamma_\epsilon = \{z = u + i\epsilon, u \in [a, b]\}$ . Thus, it remains to prove that:

$$\lim_{\epsilon \rightarrow 0} \left( \Im \left( \int_\gamma g_\nu(z) dz \right) - \Im \left( \int_{\gamma_\epsilon} g_\nu(z) dz \right) \right) = 0. \quad (2.6)$$

Let  $\epsilon > 0$  such that  $\epsilon < \sup_{[a,b]} v_{\sigma,\nu}(u)$ . We introduce the contour

$$\hat{\gamma}_\epsilon = \{z = u + i(v_{\sigma,\nu}(u) \wedge \epsilon), u \in [a, b]\}.$$

From the analyticity of  $g_\nu$  on  $\mathbb{C}^+$ , we have

$$\int_\gamma g_\nu(z) dz = \int_{\hat{\gamma}_\epsilon} g_\nu(z) dz.$$

Let  $I_\epsilon = \{u \in [a, b], v_{\sigma,\nu}(u) < \epsilon\} = \cup C_i(\epsilon)$ , where  $C_i(\epsilon)$  are the connected components of  $I_\epsilon$ . Then,  $I_\epsilon \downarrow_{\epsilon \rightarrow 0} \{a, b\}$ . For  $u \in I_\epsilon$ ,

$$|\Im g_\nu(u + i\epsilon)| = \epsilon \int \frac{d\nu(x)}{(u-x)^2 + \epsilon^2} \leq \epsilon \int \frac{d\nu(x)}{(u-x)^2 + v_{\sigma,\nu}^2(u)} \leq \frac{\epsilon}{\sigma^2}$$

and

$$\int_{I_\epsilon} |\Im g_\nu(u + i\epsilon)| du \leq \frac{\epsilon}{\sigma^2} (b-a).$$

On the other hand, for  $u \in I_\epsilon$ ,

$$|\Re g_\nu(u + iv_{\sigma,\nu}(u))| = v_{\sigma,\nu}(u) \int \frac{d\nu(x)}{(u-x)^2 + v_{\sigma,\nu}(u)^2} \leq \frac{\epsilon}{\sigma^2}.$$

Moreover,

$$\Re g_\nu(u + iv_{\sigma,\nu}(u)) v'_{\sigma,\nu}(u) = \frac{\Psi_{\sigma,\nu}(u) - u}{\sigma^2} v'_{\sigma,\nu}(u)$$

and

$$\begin{aligned} \int_{I_\epsilon} \Re g_\nu(u + iv_{\sigma,\nu}(u)) v'_{\sigma,\nu}(u) du &= \int_{I_\epsilon} \frac{\Psi_{\sigma,\nu}(u) - u}{\sigma^2} v'_{\sigma,\nu}(u) du \\ &= \frac{1}{\sigma^2} \sum_i [(\Psi_{\sigma,\nu}(u) - u) v_{\sigma,\nu}(u)]_{C_i(\epsilon)} \\ &\quad - \frac{1}{\sigma^2} \int_{I_\epsilon} (\Psi'_{\sigma,\nu}(u) - 1) v_{\sigma,\nu}(u) du, \end{aligned}$$

by integration by parts. Now (see [13] or Theorem 2.1),

$$\int_{I_\epsilon} \Psi'_{\sigma,\nu}(u) v_{\sigma,\nu}(u) du = \pi \sigma^2 (\mu_\sigma \boxplus \nu)(\Psi_{\sigma,\nu}(I_\epsilon)) \xrightarrow{\epsilon \rightarrow 0} 0.$$

$$\int_{I_\epsilon} v_{\sigma,\nu}(u) du \leq \epsilon(b-a).$$

Since  $\Psi_{\sigma,\nu}$  is increasing on  $[a, b]$ ,

$$\sum_i [\Psi_{\sigma,\nu}(u) v_{\sigma,\nu}(u)]_{C_i(\epsilon)} \leq \epsilon(\Psi_{\sigma,\nu}(b) - \Psi_{\sigma,\nu}(a))$$

and

$$\sum_i [uv_{\sigma,\nu}(u)]_{C_i(\epsilon)} \leq \epsilon(b-a).$$

The above inequalities imply (2.6).  $\square$

### 3 Approximate subordination equation for $g_N(z)$

We look for an approximative equation for  $g_N(z)$  of the form (2.2). To estimate  $g_N(z)$ , we first handle the simplest case where  $W_N$  is a GUE matrix and then see how the equation is modified in the general Wigner case. We shall rely on an integration by parts formula. The first integration by parts formula concerns the Gaussian case; the distribution  $\mu$  associated to  $W_N$  is a centered Gaussian distribution with variance  $\sigma^2$  and the resulting distribution of  $X_N = W_N/\sqrt{N}$  is denoted by  $\text{GUE}(N, \sigma^2/N)$ . Then, the integration by parts formula can be expressed in a matricial form.

**Lemma 3.1.** *Let  $\Phi$  be a complex-valued  $\mathcal{C}^1$  function on  $(M_N(\mathbb{C}))_{sa}$  and  $X_N \sim \text{GUE}(N, \frac{\sigma^2}{N})$ . Then,*

$$\mathbb{E}[\Phi'(X_N).H] = \frac{N}{\sigma^2} \mathbb{E}[\Phi(X_N)\text{Tr}(X_N H)], \quad (3.1)$$

for any Hermitian matrix  $H$ , or by linearity for  $H = E_{jk}$ ,  $1 \leq j, k \leq N$ , where  $(E_{jk})_{1 \leq j, k \leq N}$  is the canonical basis of the complex space of  $N \times N$  matrices.

For a general distribution  $\mu$ , we shall use an ‘‘approximative’’ integration by parts formula, applied to the variable  $\xi = \sqrt{2}\Re((X_N)_{kl})$  or  $\sqrt{2}\Im((X_N)_{kl})$ ,  $k < l$ , or  $(X_N)_{kk}$ . Note that for  $k < l$  the derivative of  $\Phi(X_N)$  with respect to  $\sqrt{2}\Re((X_N)_{kl})$  (resp.  $\sqrt{2}\Im((X_N)_{kl})$ ) is  $\Phi'(X_N).e_{kl}$  (resp.  $\Phi'(X_N).f_{kl}$ ), where  $e_{kl} = \frac{1}{\sqrt{2}}(E_{kl} + E_{lk})$  (resp.  $f_{kl} = \frac{i}{\sqrt{2}}(E_{kl} - E_{lk})$ ) and for any  $k$ , the derivative of  $\Phi(X_N)$  with respect to  $(X_N)_{kk}$  is  $\Phi'(X_N).E_{kk}$ .

**Lemma 3.2.** *Let  $\xi$  be a real-valued random variable such that  $\mathbb{E}(|\xi|^{p+2}) < \infty$ . Let  $\phi$  be a function from  $\mathbb{R}$  to  $\mathbb{C}$  such that the first  $p+1$  derivatives are continuous and bounded. Then,*

$$\mathbb{E}(\xi\phi(\xi)) = \sum_{a=0}^p \frac{\kappa_{a+1}}{a!} \mathbb{E}(\phi^{(a)}(\xi)) + \epsilon, \quad (3.2)$$

where  $\kappa_a$  are the cumulants of  $\xi$ ,  $|\epsilon| \leq C \sup_t |\phi^{(p+1)}(t)| \mathbb{E}(|\xi|^{p+2})$ ,  $C$  only depends on  $p$ .

Let  $U$  be a unitary matrix such that

$$A_N = U^* \text{diag}(\gamma_1, \dots, \gamma_N) U$$

and let  $G$  stand for  $G_N(z)$ . Consider  $\tilde{G} = UGU^*$ . We describe the approach in the Gaussian case and present the corresponding results in the general Wigner case but detail some technical proofs in the Appendix.

**a) Gaussian case:** We apply (3.1) to  $\Phi(X_N) = G_{jl}$ ,  $H = E_{il}$ ,  $1 \leq i, j, l \leq N$ , and then take  $\frac{1}{N} \sum_l$  to obtain, using the resolvent equation  $GX_N = -I + zG - GA_N$  (see [19]),

$$Z_{ji} := \sigma^2 \mathbb{E}[G_{ji} \text{tr}_N(G)] + \delta_{ij} - z \mathbb{E}(G_{ji}) + \mathbb{E}[(GA_N)_{ji}] = 0.$$

Now, let  $1 \leq k, p \leq N$  and consider the sum  $\sum_{i,j} U_{ik}^* U_{pj} Z_{ji}$ . We obtain from the previous equation

$$\sigma^2 \mathbb{E}[\tilde{G}_{pk} \text{tr}_N(G)] + \delta_{pk} - z \mathbb{E}(\tilde{G}_{pk}) + \gamma_k \mathbb{E}[\tilde{G}_{pk}] = 0. \quad (3.3)$$

Hence, using Lemma 9.2 in the Appendix stating that

$$|\mathbb{E}[\tilde{G}_{pk} \text{tr}_N(G)] - \mathbb{E}[\tilde{G}_{pk}] \mathbb{E}[\text{tr}_N(G)]| = O\left(\frac{1}{N^2}\right),$$

we finally get the following estimation

$$\mathbb{E}(\tilde{G}_{pk}) = \frac{\delta_{pk}}{(z - \sigma^2 g_N(z) - \gamma_k)} + O\left(\frac{1}{N^2}\right), \quad (3.4)$$

where we use that  $|\frac{1}{z - \sigma^2 g_N(z) - \gamma_i}| \leq |\Im z|^{-1}$ , and then

$$\begin{aligned} g_N(z) &= \frac{1}{N} \sum_{k=1}^N \mathbb{E}[\tilde{G}_{kk}] = \frac{1}{N} \sum_{i=1}^N \frac{1}{z - \sigma^2 g_N(z) - \gamma_k} + O\left(\frac{1}{N^2}\right) \\ &= \int_{\mathbb{R}} \frac{1}{z - \sigma^2 g_N(z) - x} d\mu_{A_N}(x) + O\left(\frac{1}{N^2}\right) \\ &= g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + O\left(\frac{1}{N^2}\right). \end{aligned}$$

In the Gaussian case, we have thus proved:

**Proposition 3.1.** *For  $z \in \mathbb{C}^+$ ,  $g_N(z)$  satisfies:*

$$g_N(z) = g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + \frac{P(|\Im z|^{-1})}{N^2}. \quad (3.5)$$

**b) Non-Gaussian case:** In this case, the integration by parts formula gives the following generalization of (3.4):

**Lemma 3.3.**

$$\mathbb{E}(\tilde{G}_{pk}) = \frac{\delta_{pk}}{(z - \sigma^2 g_N(z) - \gamma_k)} + \frac{\kappa_4}{2N^2} \frac{\mathbb{E}[\tilde{A}(p, k)]}{(z - \sigma^2 g_N(z) - \gamma_k)} + O\left(\frac{1}{N^2}\right), \quad (3.6)$$

where

$$\begin{aligned} \tilde{A}(p, k) &= \sum_{i,j} U_{ik}^* U_{pj} \left\{ \sum_l G_{jl} G_{il}^3 + \sum_l G_{ji} G_{il} G_{li} G_l \right. \\ &\quad \left. + \sum_l G_{jl} G_{ii} G_{li} G_l + \sum_l G_{ji} G_{ii} G_{li}^2 \right\} \end{aligned} \quad (3.7)$$

and  $\frac{1}{N^2} \tilde{A}(p, k) \leq C \frac{|\Im z|^{-4}}{N}$ .



**Proof** Lemma 3.3 readily follows from (9.4), Lemma 9.2 and (9.3) established in the Appendix.  $\square$

Thus,

$$\begin{aligned} g_N(z) &= \frac{1}{N} \sum_{k=1}^N \mathbb{E}[\tilde{G}_{kk}] = \frac{1}{N} \sum_{k=1}^N \frac{1}{z - \sigma^2 g_N(z) - \gamma_k} \\ &+ \frac{\kappa_4}{2N^3} \sum_{k=1}^N \frac{\mathbb{E}[\tilde{A}(k, k)]}{z - \sigma^2 g_N(z) - \gamma_k} + O\left(\frac{1}{N^2}\right). \end{aligned}$$

Let us show that the first three terms in  $\frac{1}{N} \sum_k \mathbb{E}[\tilde{A}(k, k)] / (z - \sigma^2 g_N(z) - \gamma_k)$  coming from the decomposition (3.7) are bounded and thus give a  $O(\frac{1}{N^2})$  contribution in  $g_N(z)$ . We denote by  $G_D$  the diagonal matrix with  $k$ -th diagonal entry equal to  $\frac{1}{z - \sigma^2 g_N(z) - \gamma_k}$ .

$$\begin{aligned} \left| \sum_{i,j,k} U_{ik}^* U_{kj} \frac{1}{z - \sigma^2 g_N(z) - \gamma_k} \mathbb{E} \left[ \sum_l G_{jl} G_{il}^3 \right] \right| &= \left| \mathbb{E} \left[ \sum_{i,l} (U^* G_D U G)_{il} G_{il}^3 \right] \right| \\ &\leq |\Im z|^{-2} \mathbb{E} \left[ \sum_{i,l} |G_{il}^3| \right] \\ &\leq |\Im z|^{-5} N, \end{aligned}$$

using Lemma 1.1. The second term is of the same kind. For the third term, we obtain

$$\left| \sum_i (U^* G_D U G^2 G^{(d)})_{ii} G_{ii} \right| \leq |\Im z|^{-5} N$$

where  $G^{(d)}$  is the diagonal matrix with  $l$ -th diagonal entry equal to  $G_{ll}$ . It follows that

$$g_N(z) = g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + \frac{1}{N} \hat{L}_N(z) + O\left(\frac{1}{N^2}\right),$$

where

$$\hat{L}_N(z) = \frac{\kappa_4}{2N^2} \sum_{i,j,k,l} U_{ik}^* U_{kj} \frac{1}{z - \sigma^2 g_N(z) - \gamma_k} \mathbb{E}[G_{ji} G_{ii} G_{ll}^2]. \quad (3.8)$$

It is easy to see that  $\hat{L}_N(z)$  is bounded by  $C|\Im z|^{-5}$ .

**Proposition 3.2.**  $\hat{L}_N$  defined by (3.8) can be written as

$$\begin{aligned} \hat{L}_N(z) &= L_N(z) + O\left(\frac{1}{N}\right), \text{ where } L_N(z) = \\ &\frac{\kappa_4}{2N^2} \sum_{i,l} [(G_{A_N}(z - \sigma^2 g_N(z)))^2]_{ii} [G_{A_N}(z - \sigma^2 g_N(z))]_{ii} ([G_{A_N}(z - \sigma^2 g_N(z))]_{ll})^2. \end{aligned} \quad (3.9)$$

**Proof of Proposition 3.2:**

*Step 1:* We first show that for  $1 \leq a, b \leq N$ ,

$$\mathbb{E}[G_{ab}] = [G_{A_N}(z - \sigma^2 g_N(z))]_{ab} + O\left(\frac{1}{N}\right). \quad (3.10)$$

From Lemma 3.3, for any  $1 \leq p, k \leq N$ ,

$$\mathbb{E}[\tilde{G}_{pk}] = \frac{\delta_{pk}}{(z - \sigma^2 g_N(z) - \gamma_k)} + \frac{\kappa_4}{2N^2} \frac{\mathbb{E}[\tilde{A}(p, k)]}{(z - \sigma^2 g_N(z) - \gamma_k)} + O\left(\frac{1}{N^2}\right).$$

Let  $1 \leq a, b \leq N$ ,

$$\begin{aligned} \mathbb{E}[G_{ab}] &= \sum_{p,k} U_{ap}^* \mathbb{E}[\tilde{G}_{pk}] U_{kb} \\ &= \sum_k U_{ak}^* \frac{1}{(z - \sigma^2 g_N(z) - \gamma_k)} U_{kb} \\ &\quad + \frac{\kappa_4}{2N^2} \sum_{p,k} U_{ap}^* \frac{\mathbb{E}[\tilde{A}(p, k)]}{(z - \sigma^2 g_N(z) - \gamma_k)} U_{kb} \\ &\quad + O\left(\frac{1}{N}\right), \end{aligned}$$

since  $\sum_{p,k} |U_{ap}^* U_{kb}| \leq N$ . The first term in the right-hand side of the above equation is equal to  $[G_{A_N}(z - \sigma^2 g_N(z))]_{ab}$ . It remains to show that the term involving  $\mathbb{E}[\tilde{A}(p, k)]$  is of order  $\frac{1}{N}$ . Let us consider the ‘‘worst term’’ in the decomposition (3.7) of  $\tilde{A}(p, k)$ , namely the last one.

$$\begin{aligned} &\frac{1}{2N^2} \sum_{p,k,i,j,l} U_{ap}^* \frac{1}{(z - \sigma^2 g_N(z) - \gamma_k)} U_{kb} U_{ik}^* U_{pj} \mathbb{E}[G_{ji} G_{ii} G_{ll}^2] \\ &= \frac{1}{2N^2} \mathbb{E}\left[\sum_{k,i,l} \frac{1}{(z - \sigma^2 g_N(z) - \gamma_k)} U_{kb} U_{ik}^* G_{ai} G_{ii} G_{ll}^2\right] \\ &= \frac{1}{2N^2} \mathbb{E}\left[\sum_{i,l} (U^* G_D U)_{ib} G_{ai} G_{ii} G_{ll}^2\right] \\ &= \frac{1}{2N^2} \mathbb{E}\left[\sum_l (G G^{(d)} U^* G_D U)_{ab} G_{ll}^2\right] \leq \frac{1}{2N} |\Im z|^{-5}. \end{aligned}$$

*Step 2:*  $\hat{L}_N$  defined by (3.8) can be written as

$$\frac{\kappa_4}{2N^2} \sum_{i,l} \mathbb{E}[(U^* G_D U G)_{ii} G_{ii} G_{ll}^2].$$

First notice the following bound (see Appendix)

$$\mathbb{E}[(U^* G_D U G)_{ii} G_{ii} G_{ll}^2] - \mathbb{E}[(U^* G_D U G)_{ii}] \mathbb{E}[G_{ii}] \mathbb{E}[G_{ll}]^2 = O\left(\frac{1}{N}\right). \quad (3.11)$$

Thus,

$$\hat{L}_N(z) = \frac{\kappa_4}{2N^2} \sum_{i,l} \mathbb{E}[(U^* G_D U G)_{ii}] \mathbb{E}[G_{ii}] \mathbb{E}[G_{ll}]^2 + O\left(\frac{1}{N}\right).$$

Now, note that  $\mathbb{E}[(U^* G_D U G)_{ii}] = \mathbb{E}[(U^* G_D \tilde{G} U)_{ii}]$  and, according to Lemma 3.3,

$$\begin{aligned} \mathbb{E}[(U^* G_D \tilde{G} U)_{ii}] &= \sum_{p,k} (U^* G_D)_{ip} \mathbb{E}[\tilde{G}_{pk}] U_{ki} \\ &= (U^* G_D^2 U)_{ii} + \frac{\kappa_4}{2N^2} \sum_{p,k} (U^* G_D)_{ip} \mathbb{E}[\tilde{A}(p,k)] (G_D U)_{ki} \\ &\quad + \sum_{p,k} (U^* G_D)_{ip} O_{pk} \left(\frac{1}{N^2}\right) U_{ki}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\kappa_4}{2N^2} \sum_{i,l} \mathbb{E}[(U^* G_D U G)_{ii}] \mathbb{E}[G_{ii}] \mathbb{E}[G_{ll}]^2 \\ = \frac{\kappa_4}{2N^2} \sum_{i,l} [(G_{A_N}(z - \sigma^2 g_N(z)))_{ii}]^2 \mathbb{E}[G_{ii}] \mathbb{E}[G_{ll}]^2 \end{aligned} \quad (3.12)$$

$$+ \frac{\kappa_4^2}{4N^4} \sum_{i,l,p,k} (U^* G_D)_{ip} \mathbb{E}[\tilde{A}(p,k)] (G_D U)_{ki} \mathbb{E}[G_{ii}] \mathbb{E}[G_{ll}]^2 \quad (3.13)$$

$$+ \frac{1}{N^2} \sum_{i,l,p,k} (U^* G_D)_{ip} O_{pk} \left(\frac{1}{N^2}\right) U_{ki} \mathbb{E}[G_{ii}] \mathbb{E}[G_{ll}]^2. \quad (3.14)$$

The last term (3.14) can be rewritten as

$$\frac{1}{N^2} \sum_{l,p,k} (U \mathbb{E}[G^{(d)}] U^* G_D)_{kp} O_{pk} \left(\frac{1}{N^2}\right) \mathbb{E}[G_{ll}]^2,$$

so that one can easily see that it is a  $O\left(\frac{1}{N}\right)$ .

The second term (3.13) can be rewritten as

$$\begin{aligned} \frac{\kappa_4^2}{4N^4} \sum_{t,l,s} \mathbb{E}[G_{ll}]^2 \\ \times \left\{ [U^* G_D U \mathbb{E}[G^{(d)}] U^* G_D U G]_{ts} [G_{ts}^3 + G_{tt} G_{st} G_{ss}] \right. \\ \left. + [U^* G_D U \mathbb{E}[G^{(d)}] U^* G_D U G]_{tt} [G_{ts} G_{st} G_{ss} + G_{tt} G_{ss}^2] \right\}, \end{aligned}$$

which is obviously a  $O\left(\frac{1}{N}\right)$ .

Hence, Proposition 3.2 follows by rewriting the first term (3.12) using (3.10).  $\square$

From the above computations, we can state the following :

**Proposition 3.3.** *For  $z \in \mathbb{C}^+$ ,  $g_N(z)$  satisfies:*

$$g_N(z) = g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + \frac{1}{N} L_N(z) + \frac{P(|\Im z|^{-1})}{N^2} \quad (3.15)$$

where  $L_N(z)$  is given by (3.9).

## 4 Estimation of $g_N - \tilde{g}_N$

**Proposition 4.1.** For  $z \in \mathbb{C}^+$ ,

$$g_N(z) - \tilde{g}_N(z) + \frac{\tilde{E}_N(z)}{N} = O\left(\frac{1}{N^2}\right), \quad (4.1)$$

where  $\tilde{E}_N(z)$  is given by

$$\tilde{E}_N(z) = \{\sigma^2 \tilde{g}'_N(z) - 1\} \tilde{L}_N(z) \quad (4.2)$$

with  $\tilde{L}_N(z) =$

$$\frac{\kappa_4}{2N^2} \sum_{i,l} [(G_{A_N}(z - \sigma^2 \tilde{g}_N(z)))^2]_{ii} [G_{A_N}(z - \sigma^2 \tilde{g}_N(z))]_{ii} ([G_{A_N}(z - \sigma^2 \tilde{g}_N(z))]_{ll})^2. \quad (4.3)$$

**Proof of proposition 4.1:** First, we are going to prove that for  $z \in \mathbb{C}^+$ ,

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O\left(\frac{1}{N^2}\right), \quad (4.4)$$

where  $E_N(z)$  is given by

$$E_N(z) = \{\sigma^2 g'_N(z) - 1\} L_N(z). \quad (4.5)$$

For a fixed  $z \in \mathbb{C}^+$ , one may write the subordination equation (2.2) :

$$\tilde{g}_N(z) = g_{\mu_{A_N}}(F_{\sigma, \mu_{A_N}}(z)) = g_{\mu_{A_N}}(z - \sigma^2 \tilde{g}_N(z)),$$

and the approximative matricial subordination equation (3.15) :

$$g_N(z) = g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + \frac{1}{N} L_N(z) + \frac{P(|\Im z|^{-1})}{N^2}.$$

The main idea is to simplify the difference  $g_N(z) - \tilde{g}_N(z)$  by introducing a complex number  $z'$  likely to satisfy

$$F_{\sigma, \mu_{A_N}}(z') = z - \sigma^2 g_N(z). \quad (4.6)$$

We know by Proposition 2.1 that  $F_{\sigma, \mu_{A_N}}$  is a homeomorphism from  $\mathbb{C}^+$  to  $\Omega_{\sigma, \mu_{A_N}}$  whose inverse  $H_{\sigma, \mu_{A_N}}$  has an analytic continuation to the whole upper half-plane  $\mathbb{C}^+$ . Since  $z - \sigma^2 g_N(z) \in \mathbb{C}^+$ ,  $z' \in \mathbb{C}$  is well-defined by the formula :

$$z' := H_{\sigma, \mu_{A_N}}(z - \sigma^2 g_N(z)).$$

One has

$$\begin{aligned} z' - z &= -\sigma^2 (g_N(z) - g_{\mu_{A_N}}(z - \sigma^2 g_N(z))) \\ &= -\sigma^2 \frac{L_N(z)}{N} + O\left(\frac{1}{N^2}\right) \\ &= O\left(\frac{1}{N}\right) \end{aligned}$$

There exists thus a polynomial  $P$  with nonnegative coefficients such that

$$|z' - z| \leq \frac{P(|\Im z|^{-1})}{N}.$$

On the one hand, if

$$\frac{P(|\Im z|^{-1})}{N} \geq \frac{|\Im z|}{2},$$

or equivalently

$$1 \leq \frac{2|\Im z|^{-1}P(|\Im z|^{-1})}{N}, \quad (4.7)$$

it is enough to prove that

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O(1). \quad (4.8)$$

Indeed, if we assume that (4.7) and (4.8) hold, then there exists a polynomial  $Q$  with nonnegative coefficients such that

$$\begin{aligned} |g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N}| &\leq Q(|\Im z|^{-1}) \\ &\leq Q(|\Im z|^{-1}) \frac{2|\Im z|^{-1}P(|\Im z|^{-1})}{N} \\ &\leq Q(|\Im z|^{-1}) \left( \frac{2|\Im z|^{-1}P(|\Im z|^{-1})}{N} \right)^2. \end{aligned}$$

Hence,

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O\left(\frac{1}{N^2}\right).$$

To prove (4.8), one can notice that both  $g_N(z)$  and  $\tilde{g}_N(z)$  are bounded by  $\frac{1}{|\Im z|}$ , and that

$$|E_N(z)| \leq \left\{ \frac{\sigma^2}{|\Im z|^2} + 1 \right\} |L_N(z)|,$$

where  $L_N(z) = O(1)$ .

On the other hand, if

$$\frac{P(|\Im z|^{-1})}{N} \leq \frac{|\Im z|}{2},$$

one has :

$$|\Im z' - \Im z| \leq |z' - z| \leq \frac{|\Im z|}{2}$$

which implies  $\Im z' \geq \frac{\Im z}{2}$  and therefore  $z' \in \mathbb{C}^+$ . As a consequence of Proposition 2.2,  $z - \sigma^2 g_N(z) \in \Omega_{\sigma, \mu_{A_N}}$  and (4.6) is satisfied. Thus,

$$|g_N(z) - \tilde{g}_N(z') - \frac{L_N(z)}{N}| \leq \frac{P(|\Im z|^{-1})}{N^2},$$

or, in other words,

$$g_N(z) - \tilde{g}_N(z') - \frac{L_N(z)}{N} = O\left(\frac{1}{N^2}\right). \quad (4.9)$$

On the other hand,

$$\begin{aligned} \tilde{g}_N(z') - \tilde{g}_N(z) &= (z - z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z' - x)(z - x)} \\ &= (z - z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z - x)^2} \\ &\quad + (z - z')^2 \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z' - x)(z - x)^2}. \end{aligned}$$

Taking into account the estimation of  $z' - z$  above, one has :

$$(z - z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z - x)^2} = -\sigma^2 \tilde{g}'_N(z) \frac{L_N(z)}{N} + O\left(\frac{1}{N^2}\right)$$

and

$$(z - z')^2 \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z' - x)(z - x)^2} = O\left(\frac{1}{N^2}\right).$$

Hence

$$\tilde{g}_N(z') - \tilde{g}_N(z) + \sigma^2 \tilde{g}'_N(z) \frac{L_N(z)}{N} = O\left(\frac{1}{N^2}\right). \quad (4.10)$$

(4.4) follows from (4.9) and (4.10) since

$$\begin{aligned} \left|g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N}\right| &\leq \left|g_N(z) - \tilde{g}_N(z') - \frac{L_N(z)}{N}\right| \\ &\quad + \left|\tilde{g}_N(z') - \tilde{g}_N(z) + \sigma^2 \tilde{g}'_N(z) \frac{L_N(z)}{N}\right|. \end{aligned}$$

Now, since  $E_N(z) = O(1)$ , we can deduce from (4.4) that  $g_N(z) - \tilde{g}_N(z) = O\left(\frac{1}{N}\right)$  and then that  $E_N(z) - \tilde{E}_N(z) = O\left(\frac{1}{N}\right)$ . (4.1) readily follows.  $\square$

**Remark 4.1.** *By combining the estimation proved above for the difference between  $g_N$  and the Stieltjes transform of  $\mu_\sigma \boxplus \mu_{A_N}$  with some classical arguments developed in [29], one can recover the almost sure convergence of the spectral distribution of  $M_N$  to the free convolution  $\mu_\sigma \boxplus \nu$ .*

## 5 Inclusion of the spectrum of $M_N$ in a neighborhood of the support of $\mu_\sigma \boxplus \mu_{A_N}$

The purpose of this section is to prove the following Theorem 5.1.

**Theorem 5.1.**  $\forall \epsilon > 0$ ,

$$\mathbb{P}(\text{For all large } N, \text{Spect}(M_N) \subset \{x, \text{dist}(x, \text{supp}(\mu_\sigma \boxplus \mu_{A_N})) \leq \epsilon\}) = 1.$$

The proof still uses the ideas of [24] and [32] but, since  $\mu_\sigma \boxplus \mu_{A_N}$  depends on  $N$ , we need here to apply the inverse Stieltjes transform to functions depending on  $N$ . Therefore we give the details of the proof to convince the reader that the approach still holds.

**Lemma 5.1.** *For any fixed large  $N$ ,  $\tilde{E}_N$  defined in Proposition 4.1 is the Stieltjes transform of a compactly supported distribution  $\Lambda_N$  on  $\mathbb{R}$  whose support is included in the support of  $\mu_\sigma \boxplus \mu_{A_N}$ .*

The proof relies on the following characterization already used in [32].

**Theorem 5.2.** [33]

- Let  $\Lambda$  be a distribution on  $\mathbb{R}$  with compact support. Define the Stieltjes transform of  $\Lambda$ ,  $l : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$  by

$$l(z) = \Lambda \left( \frac{1}{z - x} \right).$$

Then  $l$  is analytic on  $\mathbb{C} \setminus \mathbb{R}$  and has an analytic continuation to  $\mathbb{C} \setminus \text{supp}(\Lambda)$ . Moreover

(c<sub>1</sub>)  $l(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ ,

- (c<sub>2</sub>) there exists a constant  $C > 0$ , an integer  $n \in \mathbb{N}$  and a compact set  $K \subset \mathbb{R}$  containing  $\text{supp}(\Lambda)$ , such that for any  $z \in \mathbb{C} \setminus \mathbb{R}$ ,

$$|l(z)| \leq C \max\{\text{dist}(z, K)^{-n}, 1\},$$

- (c<sub>3</sub>) for any  $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$  with compact support

$$\Lambda(\phi) = -\frac{1}{\pi} \lim_{y \rightarrow 0^+} \Im \int_{\mathbb{R}} \phi(x) l(x + iy) dx.$$

- Conversely, if  $K$  is a compact subset of  $\mathbb{R}$  and if  $l : \mathbb{C} \setminus K \rightarrow \mathbb{C}$  is an analytic function satisfying (c<sub>1</sub>) and (c<sub>2</sub>) above, then  $l$  is the Stieltjes transform of a compactly supported distribution  $\Lambda$  on  $\mathbb{R}$ . Moreover,  $\text{supp}(\Lambda)$  is exactly the set of singular points of  $l$  in  $K$ .

We use here the notations and results of Section 2. If  $u \in \mathbb{R}$  is not in the support of  $\mu_\sigma \boxplus \mu_{A_N}$ , according to (2.3),  $u - \sigma^2 \tilde{g}_N(u) = F_{\sigma, \mu_{A_N}}(u)$  belongs to  $\mathbb{R} \setminus \overline{U_{\sigma, \mu_{A_N}}}$  and then cannot belong to  $\text{Spect}(A_N)$  since  $\text{Spect}(A_N) \subset U_{\sigma, \mu_{A_N}}$ . Hence the singular points of  $\tilde{E}_N$  are included in the support of  $\mu_\sigma \boxplus \mu_{A_N}$ . Now, we are going to show that for any fixed large  $N$ ,  $\tilde{E}_N$  satisfies (c<sub>1</sub>) and (c<sub>2</sub>) of Theorem 5.2. Let  $C > 0$  be such that, for all large  $N$ ,  $\text{supp}(\mu_\sigma \boxplus \mu_{A_N}) \subset [-C; C]$  and  $\text{supp}(\mu_{A_N}) \subset [-C; C]$ .

Let  $\alpha > C + \sigma$ . For any  $z \in \mathbb{C}$  such that  $|z| > \alpha$ ,

$$|\sigma^2 \tilde{g}_N(z)| \leq \frac{\sigma^2}{|z| - C} \leq \frac{\sigma^2}{\alpha - C} < \frac{(\alpha - C)^2}{\alpha - C} = \alpha - C$$

and

$$|z - \sigma^2 \tilde{g}_N(z)| \geq \left| |z| - |\sigma^2 \tilde{g}_N(z)| \right| > |z| - (\alpha - C) > C.$$

Thus we get that for any  $z \in \mathbb{C}$  such that  $|z| > \alpha$ ,

$$\begin{aligned} \|G_{A_N}(z - \sigma^2 \tilde{g}_N(z))\| &\leq \frac{1}{|z - \sigma^2 \tilde{g}_N(z)| - C} \\ &< \frac{1}{|z| - (\alpha - C) - C} \\ &< \frac{1}{|z| - \alpha}. \end{aligned}$$

We get readily that, for  $|z| > \alpha$ ,

$$|\tilde{E}_N(z)| \leq \frac{\kappa_4}{2} \frac{1}{(|z| - \alpha)^5} \left( \frac{\sigma^2}{(|z| - C)^2} + 1 \right).$$

Then, it is clear that  $|\tilde{E}_N(z)| \rightarrow 0$  when  $|z| \rightarrow +\infty$  and  $(c_1)$  is satisfied.

Now we are going to prove  $(c_2)$  using the approach of [32](Lemma 5.5). Denote by  $\mathcal{E}_N$  the convex envelope of the support of  $\mu_\sigma \boxplus \mu_{A_N}$  and define

$$K_N := \{x \in \mathbb{R}; \text{dist}(x, \mathcal{E}_N) \leq 1\}$$

and

$$D_N = \{z \in \mathbb{C}; 0 < \text{dist}(z, K_N) \leq 1\}.$$

- Let  $z \in D_N \cap (\mathbb{C} \setminus \mathbb{R})$  with  $\Re(z) \in K_N$ . We have  $\text{dist}(z, K_N) = |\Im z| \leq 1$ . We have

$$|\tilde{E}_N(z)| \leq \frac{\kappa_4}{2} \left( \sigma^2 \frac{1}{|\Im z|^2} + 1 \right) \frac{1}{|\Im z|^5}.$$

Noticing that  $1 \leq \frac{1}{|\Im z|^2}$ , we easily deduce that there exists some constant  $C_0$  such that for any  $z \in D_N \cap \mathbb{C} \setminus \mathbb{R}$  with  $\Re(z) \in K_N$ ,

$$\begin{aligned} |\tilde{E}_N(z)| &\leq C_0 |\Im z|^{-7} \\ &\leq C_0 \text{dist}(z, K_N)^{-7} \\ &\leq C_0 \max(\text{dist}(z, K_N)^{-7}; 1). \end{aligned}$$

- Let  $z \in D_N \cap (\mathbb{C} \setminus \mathbb{R})$  with  $\Re(z) \notin K_N$ . Then  $\text{dist}(z, \text{supp}(\mu_\sigma \boxplus \mu_{A_N})) \geq 1$ . Since  $\tilde{E}_N$  is bounded on compact subsets of  $\mathbb{C} \setminus \text{supp}(\mu_\sigma \boxplus \mu_{A_N})$ , we easily deduce that there exists some constant  $C_1(N)$  such that for any  $z \in D_N$  with  $\Re(z) \notin K_N$ ,

$$|\tilde{E}_N(z)| \leq C_1(N) \leq C_1(N) \max(\text{dist}(z, K_N)^{-7}; 1).$$

- Since  $|\tilde{E}_N(z)| \rightarrow 0$  when  $|z| \rightarrow +\infty$ ,  $\tilde{E}_N$  is bounded on  $\mathbb{C} \setminus \overline{D_N}$ . Thus, there exists some constant  $C_2(N)$  such that for any  $z \in \mathbb{C} \setminus \overline{D_N}$ ,

$$|\tilde{E}_N(z)| \leq C_2(N) = C_2(N) \max(\text{dist}(z, K_N)^{-7}; 1).$$



Hence  $(c_2)$  is satisfied with  $C(N) = \max(C_0, C_1(N), C_2(N))$  and  $n = 7$  and Lemma 5.1 follows from Theorem 5.2.  $\square$

**Proof of Theorem 5.1:** Using the inverse Stieltjes transform, we get respectively that, for any  $\varphi_N$  in  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$  with compact support,

$$\begin{aligned} \mathbb{E}[\mathrm{tr}_N(\varphi_N(M_N))] &- \int_{\mathbb{R}} \varphi_N d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\varphi_N)}{N} \\ &= \frac{1}{\pi} \lim_{y \rightarrow 0^+} \Im \int_{\mathbb{R}} \varphi_N(x) r_N(x + iy) dx, \end{aligned}$$

where  $r_N(z) = \tilde{g}_N(z) - g_N(z) + \frac{1}{N} \tilde{E}_N(z)$  satisfies, according to Proposition 4.1, for any  $z \in \mathbb{C} \setminus \mathbb{R}$ ,

$$|r_N(z)| \leq \frac{1}{N^2} P(|\Im z|^{-1})$$

for some integer  $k$ . We refer the reader to the Appendix of [18] where it is proved using the ideas of [24] that

$$\begin{aligned} &\limsup_{y \rightarrow 0^+} \left| \int_{\mathbb{R}} \varphi_N(x) h(x + iy) dx \right| \\ &\leq \int_{\mathbb{R}} \int_0^{+\infty} |(1 + D)^{k+1} \varphi_N(x)| (|x| + \sqrt{2}t + K)^\alpha Q(t) \exp(-t) dt dx, \end{aligned}$$

when  $h$  is an analytic function on  $\mathbb{C} \setminus \mathbb{R}$  which satisfies

$$|h(z)| \leq (|z| + K)^\alpha P(|\Im z|^{-1}).$$

Hence, if there exists  $K > 0$  such that, for all large  $N$ , the support of  $\varphi_N$  is included in  $[-K, K]$  and  $\sup_N \sup_{x \in [-K, K]} |D^p \varphi_N(x)| = C_p < \infty$  for any  $p \leq k + 1$ , dealing with  $h(z) = N^2 r_N(z)$ , we deduce that for all large  $N$ ,

$$\limsup_{y \rightarrow 0^+} \left| \int_{\mathbb{R}} \varphi_N(x) r_N(x + iy) dx \right| \leq \frac{C}{N^2}$$

and then

$$\mathbb{E}[\mathrm{tr}_N(\varphi_N(M_N))] - \int_{\mathbb{R}} \varphi_N d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\varphi_N)}{N} = O\left(\frac{1}{N^2}\right). \quad (5.1)$$

Let  $\rho \geq 0$  be in  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$  such that its support is included in  $\{|x| \leq 1\}$  and  $\int \rho(x) dx = 1$ . Let  $0 < \epsilon < 1$ . Define

$$\rho_{\frac{\epsilon}{2}}(x) = \frac{2}{\epsilon} \rho\left(\frac{2x}{\epsilon}\right),$$

$$K_N(\epsilon) = \{x, \mathrm{dist}(x, \mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N})) \leq \epsilon\}$$

and

$$f_N(\epsilon)(x) = \int_{\mathbb{R}} \mathbf{1}_{K_N(\epsilon)}(y) \rho_{\frac{\epsilon}{2}}(x-y) dy.$$

the function  $f_N(\epsilon)$  is in  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ ,  $f_N(\epsilon) \equiv 1$  on  $K_N(\frac{\epsilon}{2})$ ; its support is included in  $K_N(2\epsilon)$ . Since there exists  $K$  such that, for all large  $N$ , the support of  $\mu_\sigma \boxplus \mu_{A_N}$  is included in  $[-K; K]$ , for all large  $N$  the support of  $f_N(\epsilon)$  is included in  $[-K-2; K+2]$  and for any  $p > 0$ ,

$$\sup_{x \in [-K-2; K+2]} |D^p f_N(\epsilon)(x)| \leq \sup_{x \in [-K-2; K+2]} \int_{-K-1}^{K+1} |D^p \rho_{\frac{\epsilon}{2}}(x-y)| dy \leq C_p(\epsilon).$$

Thus, according to (5.1),

$$\mathbb{E}[\text{tr}_N(f_N(\epsilon)(M_N))] - \int_{\mathbb{R}} f_N(\epsilon) d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(f_N(\epsilon))}{N} = O_\epsilon\left(\frac{1}{N^2}\right) \quad (5.2)$$

and

$$\mathbb{E}[\text{tr}_N((f'_N(\epsilon))^2(M_N))] - \int_{\mathbb{R}} (f'_N(\epsilon))^2 d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N((f'_N(\epsilon))^2)}{N} = O_\epsilon\left(\frac{1}{N^2}\right). \quad (5.3)$$

Moreover, following the proof of Lemma 5.6 in [32], one can show that  $\Lambda_N(1) = 0$ . Then, the function  $\psi_N(\epsilon) \equiv 1 - f_N(\epsilon)$  also satisfies

$$\mathbb{E}[\text{tr}_N(\psi_N(\epsilon)(M_N))] - \int_{\mathbb{R}} \psi_N(\epsilon) d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\psi_N(\epsilon))}{N} = O_\epsilon\left(\frac{1}{N^2}\right). \quad (5.4)$$

Moreover, since  $\psi'_N(\epsilon) = -f'_N(\epsilon)$ , it comes readily from (5.3) that

$$\mathbb{E}[\text{tr}_N((\psi'_N(\epsilon))^2(M_N))] - \int_{\mathbb{R}} (\psi'_N(\epsilon))^2 d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N((\psi'_N(\epsilon))^2)}{N} = O_\epsilon\left(\frac{1}{N^2}\right).$$

Now, since  $\psi_N(\epsilon) \equiv 0$  on the support of  $\mu_\sigma \boxplus \mu_{A_N}$ , we deduce that

$$\mathbb{E}[\text{tr}_N(\psi_N(\epsilon)(M_N))] = O_\epsilon\left(\frac{1}{N^2}\right) \quad (5.5)$$

and

$$\mathbb{E}[\text{tr}_N((\psi'_N(\epsilon))^2(M_N))] = O_\epsilon\left(\frac{1}{N^2}\right). \quad (5.6)$$

By Lemma 9.1 (sticking to the proof of Proposition 4.7 in [24]), we have

$$\mathbf{V}[\text{tr}_N(\psi_N(\epsilon)(M_N))] \leq \frac{C_\epsilon}{N^2} \mathbb{E}[\text{tr}_N\{(\psi'_N(\epsilon)(M_N))^2\}].$$

Hence, using (5.6), one can deduce that

$$\mathbf{V}[\text{tr}_N(\psi_N(\epsilon)(M_N))] = O_\epsilon\left(\frac{1}{N^4}\right) \quad (5.7)$$

Set

$$Z_{N,\epsilon} := \operatorname{tr}_N(\psi_N(\epsilon)(M_N))$$

and

$$\Omega_{N,\epsilon} = \{Z_{N,\epsilon} > N^{-\frac{4}{3}}\}$$

From (5.5) and (5.7), we deduce that

$$\mathbb{E}\{|Z_{N,\epsilon}|^2\} = O_\epsilon\left(\frac{1}{N^4}\right).$$

Hence

$$P(\Omega_{N,\epsilon}) \leq N^{\frac{8}{3}} \mathbb{E}\{|Z_{N,\epsilon}|^2\} = O_\epsilon\left(\frac{1}{N^{\frac{4}{3}}}\right).$$

By Borel-Cantelli lemma, we deduce that, almost surely for all large  $N$ ,  $Z_{N,\epsilon} \leq N^{-\frac{4}{3}}$ . Since  $Z_{N,\epsilon} \geq \mathbf{1}_{\mathbb{R} \setminus K_N(2\epsilon)}$ , it follows that, almost surely for all large  $N$ , the number of eigenvalues of  $M_N$  which are in  $\mathbb{R} \setminus K_N(2\epsilon)$  is lower than  $N^{-\frac{1}{3}}$  and thus obviously has to be equal to zero. The proof of Theorem 5.1 is complete.  $\square$

## 6 Study of $\mu_\sigma \boxplus \mu_{A_N}$

The aim of this section is to show the following inclusion of the support of  $\mu_\sigma \boxplus \mu_{A_N}$  (see Theorem 6.1 below). To this aim, we will use the notations and results of Section 2. We define

$$\Theta = \{\theta_j, 1 \leq j \leq J\} \quad \text{and} \quad \Theta_{\sigma,\nu} = \Theta \cap (\mathbb{R} \setminus \overline{U_{\sigma,\nu}}). \quad (6.1)$$

Furthermore, for all  $\theta_j \in \Theta_{\sigma,\nu}$ , we set

$$\rho_{\theta_j} := H_{\sigma,\nu}(\theta_j) = \theta_j + \sigma^2 g_\nu(\theta_j) \quad (6.2)$$

which is outside the support of  $\mu_\sigma \boxplus \nu$  according to (2.3), and we define

$$K_{\sigma,\nu}(\theta_1, \dots, \theta_J) := \operatorname{supp}(\mu_\sigma \boxplus \nu) \cup \{\rho_{\theta_j}, \theta_j \in \Theta_{\sigma,\nu}\}. \quad (6.3)$$

**Theorem 6.1.** *For any  $\epsilon > 0$ ,*

$$\operatorname{supp}(\mu_\sigma \boxplus \mu_{A_N}) \subset K_{\sigma,\nu}(\theta_1, \dots, \theta_J) + (-\epsilon, \epsilon),$$

*when  $N$  is large enough.*

Let us decompose  $\mu_{A_N}$  as

$$\mu_{A_N} = \hat{\mu}_{\beta,N} + \hat{\mu}_{\Theta,N},$$

$$\text{where } \hat{\mu}_{\beta,N} = \frac{1}{N} \sum_{j=1}^{N-r} \delta_{\beta_j(N)} \quad \text{and} \quad \hat{\mu}_{\Theta,N} = \frac{1}{N} \sum_{j=1}^J k_j \delta_{\theta_j}.$$

In the following, we will denote by  $D(x, \delta)$  the open disk centered on  $x$  and with radius  $\delta$ . We begin with a trivial technical lemma we will need in the following.

**Lemma 6.1.** *Let  $\mathcal{K}$  be a compact set included in  $\mathbb{R} \setminus \text{supp}(\nu)$ . Then  $g'_{\hat{\mu}_{\beta,N}}$  (which is well defined on  $\mathcal{K}$  for large  $N$ ) converges to  $g'_\nu$  uniformly on  $\mathcal{K}$ .*

**Proof of Lemma 6.1:** We first prove that for all  $u \in \mathcal{K}$ ,

$$-g'_{\hat{\mu}_{\beta,N}}(u) = \frac{1}{N} \sum_{j=1}^{N-r} \frac{1}{(u - \beta_j)^2} \xrightarrow{N \rightarrow +\infty} \int \frac{d\nu(x)}{(u - x)^2} = -g'_\nu(u). \quad (6.4)$$

Let  $\epsilon > 0$  be such that  $\text{dist}(\mathcal{K}, \text{supp}(\nu)) \geq \epsilon$ . For all  $u \in \mathcal{K}$ , let  $h_u$  be a bounded continuous function defined on  $\mathbb{R}$  which coincides with  $f_u(x) = 1/(u - x)^2$  on  $\text{supp}(\nu) + [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ . As  $\max_{1 \leq j \leq N-r} \text{dist}(\beta_j(N), \text{supp}(\nu))$  tends to zero as  $N \rightarrow \infty$ , one can find  $N_0$  such that, for all  $N \geq N_0$ ,  $\beta_j(N) \in \text{supp}(\nu) + [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$  for all  $1 \leq j \leq N - r$ . Since the sequence of measures  $\hat{\mu}_{\beta,N}$  weakly converges to  $\nu$ , (6.4) follows, observing that  $-g'_{\hat{\mu}_{\beta,N}}(u) = \int h_u(x) d\hat{\mu}_{\beta,N}(x)$  and  $-g'_\nu(u) = \int h_u(x) d\nu(x)$ .

The uniform convergence follows from Montel's theorem, since  $g'_{\hat{\mu}_{\beta,N}}$  and  $g'_\nu$  are analytic on  $D = \{z \in \mathbb{C}, \text{dist}(z, \text{supp}(\nu)) > \frac{\epsilon}{2}\}$  and uniformly bounded on  $D$  by  $\frac{4}{\epsilon^2}$  for  $N \geq N_0$ .  $\square$

We are now in position to give the proof of Theorem 6.1. We recall that, from (2.3),

$$\mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \mu_{A_N}) = H_{\sigma, \mu_{A_N}}(\mathbb{R} \setminus \overline{U_{\sigma, \mu_{A_N}}}). \quad (6.5)$$

In the proofs, we will write for simplicity  $U_N$ ,  $H_N$  and  $F_N$  instead of  $U_{\sigma, \mu_{A_N}}$ ,  $H_{\sigma, \mu_{A_N}}$  and  $F_{\sigma, \mu_{A_N}}$  respectively.

The main step of the proof consists in observing the following inclusion of the open set  $U_{\sigma, \mu_{A_N}}$ .

**Lemma 6.2.** *For any  $\epsilon' > 0$ ,*

$$U_{\sigma, \mu_{A_N}} \subset \{u, \text{dist}(u, \overline{U_{\sigma, \nu}}) < \epsilon'\} \cup \{u, \text{dist}(u, \Theta_{\sigma, \nu}) < \epsilon'\}, \quad (6.6)$$

*for all large  $N$  (since the compact sets  $\overline{U_{\sigma, \nu}}$  and  $\Theta_{\sigma, \nu}$  are disjoint, the previous union is disjoint once  $\epsilon'$  is small enough).*

**Proof of Lemma 6.2:** Define

$$\mathcal{F}_{\epsilon'} = \{u, \text{dist}(u, \overline{U_{\sigma, \nu}}) \geq \epsilon'\} \cap \{u, \text{dist}(u, \Theta_{\sigma, \nu}) \geq \epsilon'\}.$$

We shall show that for all large  $N$ ,  $\mathcal{F}_{\epsilon'} \subset \mathbb{R} \setminus \overline{U_N}$ .

Since  $\max_{1 \leq j \leq N-r} \text{dist}(\beta_j(N), \text{supp}(\nu)) \rightarrow 0$  when  $N$  goes to infinity, there exists  $N_0$  such that for all  $N \geq N_0$ , the  $\beta_j(N)$ 's are in  $\text{supp}(\nu) + (-\epsilon', \epsilon')$ . Since  $\text{supp}(\nu) \subset \overline{U_{\sigma, \nu}}$ , it is clear that for all  $N \geq N_0$ ,  $\mathcal{F}_{\epsilon'}$  is included in  $\mathbb{R} \setminus \text{Spect}A_N$ . Moreover, one can readily observe that if  $u$  satisfies  $\text{dist}(u, \text{supp}(\nu) + (-\epsilon', \epsilon')) \geq \sigma$  and  $\text{dist}(u, \Theta) \geq \sigma$  then, for all  $N \geq N_0$ ,  $-g'_{\mu_{A_N}}(u) \leq \frac{1}{\sigma^2}$ . This implies that, for all  $N \geq N_0$ , the open set  $U_N$  is included in the compact set

$$\mathcal{F}'_{\epsilon'} = \{u, \text{dist}(u, \text{supp}(\nu) + (-\epsilon', \epsilon')) \leq \sigma\} \cup \{u, \text{dist}(u, \Theta) \leq \sigma\}.$$

Hence, it is sufficient to show that for  $N$  large enough, the compact set  $\mathcal{K}_{\epsilon'} := \mathcal{F}_{\epsilon'} \cap \mathcal{F}'_{\epsilon'}$  is contained in  $\mathbb{R} \setminus \overline{U}_N$ .

As  $\nu$  is compactly supported, the function  $u \mapsto -g'_\nu(u) = \int_{\mathbb{R}} d\nu(x)/(u-x)^2$  is continuous on  $\mathbb{R} \setminus \text{supp}(\nu)$ . Hence it reaches its bounds on the compact set  $\mathcal{K}_{\epsilon'}$  (which is obviously included in  $\mathbb{R} \setminus \overline{U}_{\sigma,\nu}$ ) so that there exists  $\alpha > 0$  such that  $-g'_\nu(u) \leq \frac{1}{\sigma^2} - 2\alpha$  for any  $u$  in  $\mathcal{K}_{\epsilon'}$ .

According to Lemma 6.1, there exists  $N_0$  such that for all  $N \geq N_0$  and for all  $u$  in  $\mathcal{K}_{\epsilon'}$ ,

$$|g'_{\hat{\mu}_{\beta,N}}(u) - g'_\nu(u)| \leq \frac{3\alpha}{4}. \quad (6.7)$$

At last, one can notice that  $N_0$  may be chosen large enough so that

$$\forall N \geq N_0, \quad -g'_{\hat{\mu}_{\Theta,N}}(u) = \frac{1}{N} \sum_{j=1}^J \frac{k_j}{(u - \theta_j)^2} \leq \frac{\alpha}{4}. \quad (6.8)$$

This is just because for all  $u \in \mathcal{F}_{\epsilon'}$ , one has that:  $-g'_{\hat{\mu}_{\Theta,N}}(u) \leq \frac{r}{N\epsilon'^2}$  which converges uniformly on  $\mathcal{K}'_{\epsilon'}$  to 0 as  $N$  goes to infinity.

Combining all the preceding gives that, on  $\mathcal{K}_{\epsilon'}$ , the function  $-g'_{\hat{\mu}_{A_N}}$  is bounded from above by  $\frac{1}{\sigma^2} - \alpha$ . This implies that  $\mathcal{K}_{\epsilon'}$  is included in  $\mathbb{R} \setminus \overline{U}_{\sigma,\mu_{A_N}}$  which is what we wanted to show.  $\square$

Now we shall establish the following inclusion.

**Lemma 6.3.** *For all  $\epsilon > 0$ , for all  $\epsilon' > 0$  small enough,*

$$\mathbb{R} \setminus (K_\sigma(\theta_1, \dots, \theta_J) + [-\epsilon, \epsilon]) \subset H_N(\{u, \text{dist}(u, \Theta_{\sigma,\nu} \cup \overline{U}_{\sigma,\nu}) > \epsilon'\}), \quad (6.9)$$

when  $N$  is large enough.

Combined with Lemma 6.2, this result leads to Theorem 6.1.

**Proof of Lemma 6.3:** According to (2.3), (2.5) and Remark 2.1, we have that

$$\begin{aligned} & \mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \nu) = \\ & ]-\infty, H_{\sigma,\nu}(s_m) [ \bigcup \left( \bigcup_{l=m}^2 ]H_{\sigma,\nu}(t_l), H_{\sigma,\nu}(s_{l-1}) [ \right) \bigcup ]H_{\sigma,\nu}(t_1), +\infty [ \end{aligned}$$

i.e.

$$\text{supp}(\mu_\sigma \boxplus \nu) = \bigcup_{l=m}^1 \left[ H_{\sigma,\nu}(s_l), H_{\sigma,\nu}(t_l) \right]. \quad (6.10)$$

Note that there exists some finite integer  $q$  such that, for  $\epsilon$  small enough,  $\mathbb{R} \setminus (K_\sigma(\theta_1, \dots, \theta_J) + [-\epsilon, \epsilon])$  is the following disjoint union of intervals

$$]-\infty, h_0 [ \bigcup_{i=1, \dots, q} ]k_i, h_i [ \cup ]k_{q+1}, +\infty [,$$

where  $h_i = H_{\sigma,\nu}(s_{p_i}) - \epsilon$  and  $k_{i+1} = H_{\sigma,\nu}(t_{p_i}) + \epsilon$  for some  $p_i$  or  $h_i = H_{\sigma,\nu}(\theta_{j_i}) - \epsilon$  and  $k_{i+1} = H_{\sigma,\nu}(\theta_{j_i}) + \epsilon$  for some  $\theta_{j_i}$  in  $\Theta_{\sigma,\nu}$ .

For such an  $\epsilon > 0$ , since  $H_{\sigma,\nu}$  coincides on  $\mathbb{R} \setminus U_{\sigma,\nu}$  with the homeomorphism  $\Psi_{\sigma,\nu}$  defined in Theorem 2.1, we can deduce in particular that  $H_{\sigma,\nu}$  is right-continuous (resp. left-continuous) at each  $t_l$  (resp.  $s_l$ ) for  $1 \leq l \leq m$ , and  $H_{\sigma,\nu}$  is continuous at each  $\theta_i$  in  $\Theta_{\sigma,\nu}$ . Thus, there exists  $\epsilon' > 0$  such that: for all  $1 \leq l \leq m$ ,

$$H_{\sigma,\nu}(s_l - \epsilon') \geq H_{\sigma,\nu}(s_l) - \frac{\epsilon}{2} \quad \text{and} \quad H_{\sigma,\nu}(t_l + \epsilon') \leq H_{\sigma,\nu}(t_l) + \frac{\epsilon}{2} \quad (6.11)$$

and for all  $\theta_j$  in  $\Theta_{\sigma,\nu}$ ,

$$H_{\sigma,\nu}(\theta_j - \epsilon') \geq H_{\sigma,\nu}(\theta_j) - \frac{\epsilon}{2} \quad \text{and} \quad H_{\sigma,\nu}(\theta_j + \epsilon') \leq H_{\sigma,\nu}(\theta_j) + \frac{\epsilon}{2}. \quad (6.12)$$

Now  $H_N$  being increasing on  $\mathbb{R} \setminus \overline{U_N}$ , for  $N$  large enough, the image by  $H_N$  of

$$\{u, d(u, \Theta_{\sigma,\nu}) > \epsilon'\} \cap \{u, d(u, \overline{U_{\sigma,\nu}}) > \epsilon'\} \subseteq \mathbb{R} \setminus \overline{U_N}$$

is the following disjoint union of intervals

$$]-\infty, h_0(N)[ \bigcup_{i=1, \dots, q} ]k_i(N), h_i(N)[ \cup ]k_{q+1}(N), +\infty[,$$

where  $h_i(N) = H_N(s_{p_i} - \epsilon')$  and  $k_{i+1}(N) = H_N(t_{p_i} + \epsilon')$  or  $h_i(N) = H_N(\theta_{j_i} - \epsilon')$  and  $k_{i+1}(N) = H_N(\theta_{j_i} + \epsilon')$ .

One can see that it only remains to state that for all large  $N$ :  $\forall 1 \leq l \leq m$ ,

$$H_N(s_l - \epsilon') \geq H_{\sigma,\nu}(s_l) - \epsilon \quad \text{and} \quad H_N(t_l + \epsilon') \leq H_{\sigma,\nu}(t_l) + \epsilon. \quad (6.13)$$

$$H_N(\theta_i - \epsilon') \geq H_{\sigma,\nu}(\theta_i) - \epsilon \quad \text{and} \quad H_N(\theta_i + \epsilon') \leq H_{\sigma,\nu}(\theta_i) + \epsilon. \quad (6.14)$$

Moreover, as  $\mu_{A_N}$  weakly converges to  $\nu$ , it is not hard to see that for all  $1 \leq l \leq m$ , and all  $\theta_i$  in  $\Theta_{\sigma,\nu}$ ,  $H_N(s_l - \epsilon')$ ,  $H_N(t_l + \epsilon')$ ,  $H_N(\theta_i - \epsilon')$  and  $H_N(\theta_i + \epsilon')$  converge as  $N \rightarrow \infty$  to  $H_{\sigma,\nu}(s_l - \epsilon')$ ,  $H_{\sigma,\nu}(t_l + \epsilon')$ ,  $H_{\sigma,\nu}(\theta_i - \epsilon')$  and  $H_{\sigma,\nu}(\theta_i + \epsilon')$  respectively. So, there exists  $N_0$  such that for all  $N \geq N_0$ :  $H_N(s_l - \epsilon') \geq H_{\sigma,\nu}(s_l - \epsilon') - \frac{\epsilon}{2}$  and  $H_N(t_l + \epsilon') \leq H_{\sigma,\nu}(t_l + \epsilon') + \frac{\epsilon}{2}$  as well as  $H_N(\theta_i - \epsilon') \geq H_{\sigma,\nu}(\theta_i - \epsilon') - \frac{\epsilon}{2}$  and  $H_N(\theta_i + \epsilon') \leq H_{\sigma,\nu}(\theta_i + \epsilon') + \frac{\epsilon}{2}$ . We can then deduce (6.13) and (6.14) from (6.11) and (6.12).  $\square$

## 7 Exact separation of eigenvalues

Before stating the fundamental exact separation phenomenon between the spectrum of  $M_N$  and the spectrum of  $A_N$ , we need a preliminary lemma (see Lemma 7.1 below).

From Section 2, we readily deduce the following

**Proposition 7.1.**

$$\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \dots, \theta_J) = \{x \in \mathbb{R}, F_{\sigma,\nu}(x) \in \mathbb{R} \setminus \{\overline{U_{\sigma,\nu}} \cup \Theta\}\}$$

and  $F_{\sigma,\nu}$  is a homeomorphism from  $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \dots, \theta_J)$  onto  $\mathbb{R} \setminus \{\overline{U_{\sigma,\nu}} \cup \Theta\}$  with inverse  $H_{\sigma,\nu}$ .

**Remark 7.1.** : For all  $\hat{\sigma} < \sigma$ ,  $\mathbb{R} \setminus \overline{U_{\sigma,\nu}} \subset \mathbb{R} \setminus \overline{U_{\hat{\sigma},\nu}}$  so that it makes sense to consider the following composition of homeomorphism

$$H_{\hat{\sigma},\nu} \circ F_{\sigma,\nu} : \mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \dots, \theta_J) \rightarrow H_{\hat{\sigma},\nu}(\mathbb{R} \setminus \{\overline{U_{\sigma,\nu}} \cup \Theta\}) \subset \mathbb{R} \setminus K_{\hat{\sigma},\nu}(\theta_1, \dots, \theta_J),$$

which is strictly increasing on each connected component of  $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \dots, \theta_J)$ .

**Lemma 7.1.** Let  $[a, b]$  be a compact set contained in  $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \dots, \theta_J)$ . Then,

- (i) For all large  $N$ ,  $[F_{\sigma,\nu}(a), F_{\sigma,\nu}(b)] \subset \mathbb{R} \setminus \text{Spect}(A_N)$ .
- (ii) For all  $0 < \hat{\sigma} < \sigma$ , the interval  $[H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a)), H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b))]$  is contained in  $\mathbb{R} \setminus K_{\hat{\sigma},\nu}(\theta_1, \dots, \theta_J)$  and  $H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b)) - H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a)) \geq b - a$ .

**Proof of Lemma 7.1:** For simplicity, we define  $K_{\sigma,J}^\epsilon = K_\sigma(\theta_1, \dots, \theta_J) + [-\epsilon, \epsilon]$ . As  $[a, b]$  is a compact set, there exist  $\epsilon > 0$  and  $\alpha > 0$  such that

$$[a - \alpha, b + \alpha] \subset \mathbb{R} \setminus K_{\sigma,J}^\epsilon \quad \text{and} \quad \text{dist}([a - \alpha, b + \alpha]; K_{\sigma,J}^\epsilon) \geq \alpha.$$

As before, we let  $\tilde{\mu}_N = \mu_\sigma \boxplus \mu_{A_N}$ . According to Theorem 6.1, there exists some  $N_0$  such that for all  $N \geq N_0$ ,  $\text{supp}(\tilde{\mu}_N)$  is contained in  $K_{\sigma,J}^\epsilon$ . Thus, using (2.3) and since  $F_N$  is continuous strictly increasing on  $[a - \alpha, b + \alpha]$ , we have

$$\forall N \geq N_0, \quad [F_N(a - \alpha), F_N(b + \alpha)] \subset \mathbb{R} \setminus \overline{U}_N \subset \mathbb{R} \setminus \text{Spect}(A_N). \quad (7.1)$$

As  $F_{\sigma,\nu}$  is strictly increasing on the compact set  $[a - \alpha, b + \alpha]$  ( $\text{supp}(\mu_\sigma \boxplus \nu) \subset K_{\sigma,J}^\epsilon$ ), one can consider  $\delta > 0$  such that

$$F_{\sigma,\nu}(a - \alpha) \leq F_{\sigma,\nu}(a) - \delta \quad \text{and} \quad F_{\sigma,\nu}(b + \alpha) \geq F_{\sigma,\nu}(b) + \delta. \quad (7.2)$$

Now, the weak convergence of the probability measures  $\tilde{\mu}_N$  to  $\mu_\sigma \boxplus \nu$  will lead to the result, recalling from the definition of the subordination functions that for all  $x \in [a - \alpha, b + \alpha]$ :  $F_{\sigma,\nu}(x) = x - \sigma^2 g_{\mu_\sigma \boxplus \nu}(x)$  and  $F_N(x) = x - \sigma^2 g_{\tilde{\mu}_N}(x)$  (at least for all  $N \geq N_0$ ). Indeed, observing that for any  $x$  in  $[a - \alpha, b + \alpha]$ , the map  $h : t \mapsto \frac{1}{x-t}$  is bounded on  $K_{\sigma,J}^\epsilon$ , one readily gets the simple convergence of  $g_{\tilde{\mu}_N}$  to  $g_{\mu_\sigma \boxplus \nu}$  as well as the one of the corresponding subordination functions, by considering a bounded continuous function which coincides with  $h$  on  $K_{\sigma,J}^\epsilon$ . We then deduce that there exists  $N'_0 \geq N_0$  such that, for all  $N \geq N'_0$ ,

$$F_N(a - \alpha) \leq F_{\sigma,\nu}(a - \alpha) + \delta \quad \text{and} \quad F_N(b + \alpha) \geq F_{\sigma,\nu}(b + \alpha) - \delta. \quad (7.3)$$

Combining (7.1), (7.2) and (7.3) proves that the inclusion of point (i) holds true for all  $N \geq N'_0$ .

The first part of (ii) is obvious from Remark 7.1. The second part mainly follows from the fact that  $F_{\sigma,\nu}$  is strictly increasing on  $\mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \nu)$ . More precisely, if we set  $a' = H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a))$  and  $b' = H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b))$ , then

$$\begin{aligned} b' - a' &= F_{\sigma,\nu}(b) - F_{\sigma,\nu}(a) + \hat{\sigma}^2(g_\nu(F_{\sigma,\nu}(b)) - g_\nu(F_{\sigma,\nu}(a))) \\ &\geq F_{\sigma,\nu}(b) - F_{\sigma,\nu}(a) + \sigma^2(g_\nu(F_{\sigma,\nu}(b)) - g_\nu(F_{\sigma,\nu}(a))) \\ &\geq H_{\sigma,\nu}(F_{\sigma,\nu}(b)) - H_{\sigma,\nu}(F_{\sigma,\nu}(a)) = b - a \end{aligned}$$

since  $F_{\sigma,\nu}(a) < F_{\sigma,\nu}(b)$  and then  $g_\nu(F_{\sigma,\nu}(b)) - g_\nu(F_{\sigma,\nu}(a)) < 0$ .  $\square$

The exact separation result involving the subordination function related to the free convolution of  $\mu_\sigma$  and  $\nu$  can now be stated. Let  $[a, b]$  be a compact interval contained in  $\mathbb{R} \setminus K_\sigma(\theta_1, \dots, \theta_J)$ . By Theorems 5.1 and 6.1, almost surely for all large  $N$ ,  $[a, b]$  is outside the spectrum of  $M_N$ . Moreover, from Lemma 7.1 (i), it corresponds an interval  $I = [a', b']$  outside the spectrum of  $A_N$  for all large  $N$  i.e., with the convention that  $\lambda_0(M_N) = \lambda_0(A_N) = +\infty$  and  $\lambda_{N+1}(M_N) = \lambda_{N+1}(A_N) = -\infty$ , there is  $i_N \in \{0, \dots, N\}$  such that

$$\lambda_{i_N+1}(M_N) < F_{\sigma,\nu}(a) := a' \quad \text{and} \quad \lambda_{i_N}(M_N) > F_{\sigma,\nu}(b) := b'. \quad (7.4)$$

The numbers  $a$  and  $a'$  (resp.  $b$  and  $b'$ ) are linked as follows:

$$\begin{aligned} a &= \rho_{a'} := H_{\sigma,\nu}(a') = a' + \sigma^2 g_\nu(a'), \\ b &= \rho_{b'} := H_{\sigma,\nu}(b') = b' + \sigma^2 g_\nu(b'). \end{aligned}$$

We claim that  $[a, b]$  splits the spectrum of  $M_N$  exactly as  $I$  splits the spectrum of  $A_N$ . In other words,

**Theorem 7.1.** *With  $i_N$  satisfying (7.4), one has*

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < a \quad \text{and} \quad \lambda_{i_N}(M_N) > b, \text{ for all large } N] = 1. \quad (7.5)$$

The proof closely follows the proof of Theorem 4.5 in [19] by introducing in a fit way the subordination functions or their inverses. For the reader's convenience, we rewrite the whole proof. The key idea is to introduce a continuum of matrices  $M_N^{(k)}$  interpolating from  $M_N$  to  $A_N$ :

$$M_N^{(k)} := \frac{\sigma_k}{\sigma} \frac{W_N}{\sqrt{N}} + A_N,$$

where

$$\sigma_k^2 = \sigma^2 \left( \frac{1}{1 + kC_{a,b}} \right),$$

and  $C_{a,b}$  being a positive constant which has to be chosen small enough to ensure that the matrices  $M_N^{(k)}$  and  $M_N^{(k+1)}$  are close enough to each other. More precisely,  $C_{a,b}$  is chosen such that

$$\max \left( \sigma^2 C_{a,b} |g_{\mu_\sigma \boxplus \nu}(a)|; \sigma^2 C_{a,b} |g_{\mu_\sigma \boxplus \nu}(b)|; 3\sigma C_{a,b} \right) < \frac{b-a}{4}. \quad (7.6)$$



In particular,  $\sigma_0 = \sigma$  and  $\sigma_k \rightarrow 0$  when  $k$  goes to infinity.

We first prove that the intervals  $[H_{\sigma_k, \nu}(F_{\sigma, \nu}(a)), H_{\sigma_k, \nu}(F_{\sigma, \nu}(b))]$  split respectively the spectrum of  $M_N^{(k)}$  in exactly the same way. Moreover, we also prove that for  $k$  large enough, the interval  $[H_{\sigma_k, \nu}(F_{\sigma, \nu}(a)), H_{\sigma_k, \nu}(F_{\sigma, \nu}(b))]$  splits the spectrum of  $M_N^{(k)}$  as  $[F_{\sigma, \nu}(a), F_{\sigma, \nu}(b)]$  splits the spectrum of  $A_N$ , this means roughly that we extend the first statement to  $k = \infty$  and the result follows.

As in [19], this proof is inspired by the work [6] and mainly relies on results on eigenvalues of the rescaled Wigner matrix  $X_N$  combined with the following classical result (due to Weyl).

**Lemma 7.2.** (cf. Theorem 4.3.7 of [25]) *Let  $B$  and  $C$  be two  $N \times N$  Hermitian matrices. For any pair of integers  $j, k$  such that  $1 \leq j, k \leq N$  and  $j + k \leq N + 1$ , we have*

$$\lambda_{j+k-1}(B + C) \leq \lambda_j(B) + \lambda_k(C).$$

For any pair of integers  $j, k$  such that  $1 \leq j, k \leq N$  and  $j + k \geq N + 1$ , we have

$$\lambda_j(B) + \lambda_k(C) \leq \lambda_{j+k-N}(B + C).$$

**Proof of Theorem 7.1:** Given  $k \geq 0$ , define

$$a_k = H_{\sigma_k}(F_{\sigma, \nu}(a)) \text{ and } b_k = H_{\sigma_k}(F_{\sigma, \nu}(b)).$$

**Remark 7.2.** *Note that in [19] where  $\nu = \delta_0$ , we considered  $a_k = z_{\sigma_k}(g_\sigma(a))$  where  $g_\sigma$  denoted the Stieltjes transform of  $\mu_\sigma$  and  $z_{\sigma_k}$  the inverse of  $g_{\sigma_k}$ . Actually, when  $\nu = \delta_0$ , then  $H_{\sigma_k, \nu}(z) = z + \sigma_k^2/z = z_{\sigma_k}(1/z)$  and  $F_{\sigma, \nu} = 1/g_\sigma$  so that  $z_{\sigma_k}(g_\sigma) = H_{\sigma_k, \nu}(F_{\sigma, \nu})$ . This very interpretation of the composition  $z_{\sigma_k} \circ g_\sigma$  in terms of subordination function allows us to extend the result of exact separation to non-finite rank perturbations.*

The last point of (ii) in Lemma 7.1 yields  $b_k - a_k \geq b - a$ . Moreover

$$\begin{aligned} a_{k+1} - a_k &= (\sigma_{k+1}^2 - \sigma_k^2) g_{\mu_\sigma \boxplus \nu}(a) \\ &= -C_{a,b} \frac{\sigma^2}{(1 + kC_{a,b})(1 + (k+1)C_{a,b})} g_{\mu_\sigma \boxplus \nu}(a), \end{aligned}$$

so that  $|a_{k+1} - a_k| \leq \sigma^2 C_{a,b} |g_{\mu_\sigma \boxplus \nu}(a)|$ . Similarly  $|b_{k+1} - b_k| \leq \sigma^2 C_{a,b} |g_{\mu_\sigma \boxplus \nu}(b)|$ . Hence, we deduce from (7.6) that

$$|a_{k+1} - a_k| < \frac{b-a}{4} \quad \text{and} \quad |b_{k+1} - b_k| < \frac{b-a}{4}. \quad (7.7)$$

Now, we shall show by induction on  $k$  that, with probability 1, for large  $N$ , the  $M_N^{(k)}$  have respectively the same amount of eigenvalues to the left sides of the interval  $[a_k, b_k]$ . For all  $k \geq 0$ , set

$$E_k = \{\text{no eigenvalues of } M_N^{(k)} \text{ in } [a_k, b_k], \text{ for all large } N\}.$$

By Lemma 7.1 (ii) and Theorems 5.1 and 6.1, we know that  $\mathbb{P}(E_k) = 1$  for all  $k$ . In particular, one has for all  $\omega \in E_0$  and for all large  $N$ ,

$$\exists j_N(\omega) \in \{0, \dots, N\} \text{ such that } \lambda_{j_N(\omega)+1}(M_N) < a \text{ and } \lambda_{j_N(\omega)}(M_N) > b. \quad (7.8)$$

Extending the random variable  $j_N$ , by setting for instance  $j_N := -1$  on the complementary of  $E_0$ , we want to show that for all  $k$ ,

$$\mathbb{P}[\lambda_{j_N+1}(M_N^{(k)}) < a_k \text{ and } \lambda_{j_N}(M_N^{(k)}) > b_k, \text{ for all large } N] = 1. \quad (7.9)$$

We proceed by induction. By (7.8), this is true for  $k = 0$ . Now, let us assume that (7.9) holds true. Since

$$M_N^{(k+1)} = M_N^{(k)} + \left( \frac{1}{\sqrt{1 + (k+1)C_{a,b}}} - \frac{1}{\sqrt{1 + kC_{a,b}}} \right) X_N,$$

we can deduce from Lemma 7.2 that

$$\lambda_{j_N+1}(M_N^{(k+1)}) \leq \lambda_{j_N+1}(M_N^{(k)}) + (-\lambda_N(X_N))C_{a,b}.$$

Since, for  $N$  large enough,  $0 < -\lambda_N(X_N) \leq 3\sigma$  almost surely, it follows using (7.6) that

$$\lambda_{j_N+1}(M_N^{(k+1)}) < a_k + \frac{b-a}{4} := \hat{a}_k \quad \text{a.s..}$$

Similarly, one can show that

$$\lambda_{j_N}(M_N^{(k+1)}) > b_k - \frac{b-a}{4} := \hat{b}_k \quad \text{a.s..}$$

Inequalities (7.7) ensure that

$$[\hat{a}_k, \hat{b}_k] \subset [a_{k+1}, b_{k+1}].$$

As  $\mathbb{P}(E_{k+1}) = 1$ , we deduce that, with probability 1,

$$\lambda_{j_N+1}(M_N^{(k+1)}) < a_{k+1} \text{ and } \lambda_{j_N}(M_N^{(k+1)}) > b_{k+1}, \quad \text{for all large } N.$$

This completes the proof by induction of (7.9).

Now, we are going to show that there exists  $K$  large enough so that, for all  $k \geq K$ , there is exact separation of the eigenvalues of the matrices  $A_N$  and  $M_N^{(k)}$  i.e.

$$\mathbb{P}[\lambda_{i_N+1}(M_N^{(k)}) < a_k \text{ and } \lambda_{i_N}(M_N^{(k)}) > b_k, \text{ for all large } N] = 1. \quad (7.10)$$

There exists  $\alpha > 0$  such that  $[a-\alpha; b+\alpha] \subset \mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \dots, \theta_J)$ . Thus according to Lemma 7.1 (i) for all large  $N$ ,

$$[F_{\sigma,\nu}(a-\alpha); F_{\sigma,\nu}(b+\alpha)] \subset \mathbb{R} \setminus \text{Spect}(A_N).$$

Now, there exists  $\epsilon' > 0$  such that  $F_{\sigma,\nu}(a - \alpha) < F_{\sigma,\nu}(a) - \epsilon'$  and  $F_{\sigma,\nu}(b + \alpha) > F_{\sigma,\nu}(b) + \epsilon'$ . It follows that, for all large  $N$ ,

$$\lambda_{i_N+1}(A_N) < F_{\sigma,\nu}(a) - \epsilon' \quad \text{and} \quad \lambda_{i_N}(A_N) > F_{\sigma,\nu}(b) + \epsilon'. \quad (7.11)$$

Using Lemma 7.2, (7.11) and the fact that, almost surely, for all large  $N$ ,

$$0 < \max(-\lambda_N(X_N), \lambda_1(X_N)) < 3\sigma,$$

we get the following inequalities.

If  $i_N < N$ , for all large  $N$ ,

$$\begin{aligned} \lambda_{i_N+1}(M_N^{(k)}) &\leq \lambda_{i_N+1}(A_N) + \frac{\sigma_k}{\sigma} \lambda_1(X_N) \\ &< F_{\sigma,\nu}(a) - \epsilon' + \frac{\sigma_k}{\sigma} \lambda_1(X_N) \\ &= a_k - \sigma_k^2 g_{\mu_\sigma \boxplus \nu}(a) + \frac{\sigma_k}{\sigma} \lambda_1(X_N) - \epsilon' \\ &< a_k - \sigma_k^2 g_{\mu_\sigma \boxplus \nu}(a) + 3\sigma_k - \epsilon'. \end{aligned}$$

If  $i_N > 0$ , for all large  $N$ ,

$$\begin{aligned} \lambda_{i_N}(M_N^{(k)}) &\geq \lambda_{i_N}(A_N) + \frac{\sigma_k}{\sigma} \lambda_N(X_N) \\ &> F_{\sigma,\nu}(b) + \epsilon' + \frac{\sigma_k}{\sigma} \lambda_N(X_N) \\ &= b_k - \sigma_k^2 g_{\mu_\sigma \boxplus \nu}(b) + \frac{\sigma_k}{\sigma} \lambda_N(X_N) + \epsilon' \\ &> b_k - \sigma_k^2 g_{\mu_\sigma \boxplus \nu}(b) - 3\sigma_k + \epsilon'. \end{aligned}$$

As  $\sigma_k \rightarrow 0$  when  $k \rightarrow +\infty$ , there is  $K$  large enough such that for all  $k \geq K$ ,

$$\max(|-\sigma_k^2 g_{\mu_\sigma \boxplus \nu}(a) + 3\sigma_k|, |-\sigma_k^2 g_{\mu_\sigma \boxplus \nu}(b) - 3\sigma_k|) < \epsilon'$$

and then, almost surely, for all  $N$  large enough

$$\lambda_{i_N+1}(M_N^{(k)}) < a_k \quad \text{if } i_N < N, \quad (7.12)$$

$$\text{and } \lambda_{i_N}(M_N^{(k)}) > b_k \quad \text{if } i_N > 0. \quad (7.13)$$

Since  $\lambda_{N+1}(M_N^{(k)}) = -\lambda_0(M_N^{(k)}) = -\infty$ , (7.12) (resp. (7.13)) is obviously satisfied if  $i_N = N$  (resp.  $i_N = 0$ ). Thus, we have established that for any  $i_N \in \{0, \dots, N\}$  satisfying (7.4), (7.10) holds for all  $k \geq K$  when  $K$  is large enough. Comparing this with (7.9), we deduce that  $j_N = i_N$  almost surely and

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < a \text{ and } \lambda_{i_N}(M_N) > b, \quad \text{for all large } N] = 1.$$

This ends the proof of Theorem 7.1.  $\square$

We readily deduce the following

**Corollary 7.1.** *Let  $\epsilon > 0$ . Let us fix  $u$  in  $\Theta_{\sigma,\nu} \cup \{t_l, l = 1, \dots, m\}$  (resp. in  $\Theta_{\sigma,\nu} \cup \{s_l, l = 1, \dots, m\}$ ). Let us choose  $\delta > 0$  small enough so that for large  $N$ ,  $[u + \delta; u + 2\delta]$  (resp.  $[u - 2\delta; u - \delta]$ ) is included in  $(\mathbb{R} \setminus \overline{U_{\sigma,\nu}}) \cap (\mathbb{R} \setminus \text{Spect}(A_N))$  and for any  $0 \leq \delta' \leq 2\delta$ ,  $H_{\sigma,\nu}(u + \delta') - H_{\sigma,\nu}(u) < \epsilon$  (resp.  $H_{\sigma,\nu}(u) - H_{\sigma,\nu}(u - \delta') < \epsilon$ ). Let  $i_N = i_N(u)$  be such that*

$$\lambda_{i_N+1}(A_N) < u + \delta \text{ and } \lambda_{i_N}(A_N) > u + 2\delta$$

(resp.  $\lambda_{i_N+1}(A_N) < u - 2\delta$  and  $\lambda_{i_N}(A_N) > u - \delta$ ). Then

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < H_{\sigma,\nu}(u) + \epsilon \text{ and } \lambda_{i_N}(M_N) > H_{\sigma,\nu}(u), \text{ for all large } N] = 1.$$

(resp.  $\mathbb{P}[\lambda_{i_N+1}(M_N) < H_{\sigma,\nu}(u)$  and  $\lambda_{i_N}(M_N) > H_{\sigma,\nu}(u) - \epsilon$  for all large  $N$ ] = 1.)

## 8 Convergence of eigenvalues

In the non-spiked case  $\Theta = \emptyset$  i.e.  $r = 0$ , the results of Theorems 6.1 and 5.1 read as:  $\forall \epsilon > 0$ ,

$$\mathbb{P}[\text{Spect}(M_N) \subset \text{supp}(\mu_\sigma \boxplus \nu) + (-\epsilon, \epsilon), \text{ for all } N \text{ large}] = 1. \quad (8.1)$$

This readily leads to the following asymptotic result for the extremal eigenvalues.

**Proposition 8.1.** *Assume that the deformed model  $M_N$  is without spike i.e.  $r = 0$ . Let  $k \geq 0$  be a fixed integer.*

*The first largest (resp. last smallest) eigenvalues  $\lambda_{1+k}(M_N)$  (resp.  $\lambda_{N-k}(M_N)$ ) converge almost surely to the right (resp. left) endpoint of the support of  $\mu_\sigma \boxplus \nu$ .*

**Proof of Proposition 8.1:** We here only focus on the convergence of the first largest eigenvalues since the other case is similar. Recalling that  $\text{supp}(\mu_\sigma \boxplus \nu) = \cup_{l=m}^1 [H_{\sigma,\nu}(s_l), H_{\sigma,\nu}(t_l)]$ , from (8.1), one has that, for all  $\epsilon > 0$ ,

$$\mathbb{P}[\limsup_N \lambda_1(M_N) \leq H_{\sigma,\nu}(t_1) + \epsilon] = 1.$$

But as  $H_{\sigma,\nu}(t_1)$  is a boundary point of  $\text{supp}(\mu_\sigma \boxplus \nu)$ , the number of eigenvalues of  $M_N$  falling into  $[H_{\sigma,\nu}(t_1) - \epsilon, H_{\sigma,\nu}(t_1) + \epsilon]$  tends almost surely to infinity as  $N \rightarrow \infty$ . Thus, almost surely,

$$\liminf_N \lambda_{1+k}(M_N) \geq H_{\sigma,\nu}(t_1) - \epsilon.$$

The result then follows by letting  $\epsilon \rightarrow 0$ .  $\square$

In the spiked case where  $r \geq 1$  ( $\Theta \neq \emptyset$ ), the spectral measure  $\mu_{M_N}$  still converges almost surely to  $\mu_\sigma \boxplus \nu$ . We shall study the impact of the spiked eigenvalues  $\theta_i$ 's on the local behavior of some eigenvalues of  $M_N$ .

In particular, we shall prove that once the largest spike  $\theta_1$  is sufficiently big, the

largest eigenvalue of  $M_N$  jumps almost surely above the right endpoint  $H_{\sigma,\nu}(t_1)$ . Once  $m \geq 2$ , that is when  $\text{supp}(\mu_\sigma \boxplus \nu)$  has at least two connected components, we prove that there may also exist some jumps into the gap(s) of this support. This phenomenon holds for any  $\theta_j \in \Theta_{\sigma,\nu}$ .

For  $\theta_j \notin \Theta_{\sigma,\nu}$ , that is if  $\theta_j \in \overline{U_{\sigma,\nu}}$ , two situations may occur. To explain this, let us consider the connected component  $[s_{l_j}, t_{l_j}]$  of  $\overline{U_{\sigma,\nu}}$  which contains  $\theta_j$ . If  $\text{supp}(\nu) \cap [\theta_j, t_{l_j}] = \emptyset$  (resp.  $\text{supp}(\nu) \cap [s_{l_j}, \theta_j] = \emptyset$ ) then the  $k_j$  corresponding eigenvalues of  $M_N$  converge almost surely to the corresponding boundary point  $H_{\sigma,\nu}(t_{l_j})$  (resp.  $H_{\sigma,\nu}(s_{l_j})$ ) of the support of  $\mu_\sigma \boxplus \nu$ . Otherwise, namely when  $\theta_j$  is between two connected components of  $\text{supp}(\nu)$  included in  $[s_{l_j}, t_{l_j}]$ , the convergence occurs towards a point inside the (interior) of  $\text{supp}(\mu_\sigma \boxplus \nu)$ .

Here is the precise formulation of our result. This is the additive analogue of the main result of [3] on the almost sure convergence of the eigenvalues generated by the spikes in a generalized spiked population model.

**Theorem 8.1.** *For each spiked eigenvalue  $\theta_j$ , we denote by  $n_{j-1}+1, \dots, n_{j-1}+k_j$  the descending ranks of  $\theta_j$  among the eigenvalues of  $A_N$ .*

- 1) *If  $\theta_j \in \mathbb{R} \setminus \overline{U_{\sigma,\nu}}$  (i.e.  $\in \Theta_{\sigma,\nu}$ ), the  $k_j$  eigenvalues  $(\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j)$  converge almost surely outside the support of  $\mu_\sigma \boxplus \nu$  towards  $\rho_{\theta_j} = H_{\sigma,\nu}(\theta_j)$ .*
- 2) *If  $\theta_j \in \overline{U_{\sigma,\nu}}$  then we let  $[s_{l_j}, t_{l_j}]$  (with  $1 \leq l_j \leq m$ ) be the connected component of  $\overline{U_{\sigma,\nu}}$  which contains  $\theta_j$ .*
  - a) *If  $\theta_j$  is on the right (resp. on the left) of any connected component of  $\text{supp}(\nu)$  which is included in  $[s_{l_j}, t_{l_j}]$  then the  $k_j$  eigenvalues  $(\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j)$  converge almost surely to  $H_{\sigma,\nu}(t_{l_j})$  (resp.  $H_{\sigma,\nu}(s_{l_j})$ ) which is a boundary point of the support of  $\mu_\sigma \boxplus \nu$ .*
  - b) *If  $\theta_j$  is between two connected components of  $\text{supp}(\nu)$  which are included in  $[s_{l_j}, t_{l_j}]$  then the  $k_j$  eigenvalues  $(\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j)$  converge almost surely to the  $\alpha_j$ -th quantile of  $\mu_\sigma \boxplus \nu$  (that is to  $q_{\alpha_j}$  defined by  $\alpha_j = (\mu_\sigma \boxplus \nu)(]-\infty, q_{\alpha_j}])$ ) where  $\alpha_j$  is such that  $\alpha_j = 1 - \lim_N \frac{n_{j-1}}{N} = \nu(]-\infty, \theta_j])$ .*

**Proof of Theorem 8.1:** 1) Choosing  $u = \theta_j$  in Corollary 7.1 gives, for any  $\epsilon > 0$ ,

$$\rho_{\theta_j} - \epsilon \leq \lambda_{n_{j-1}+k_j}(M_N) \leq \dots \leq \lambda_{n_{j-1}+1}(M_N) \leq \rho_{\theta_j} + \epsilon, \text{ for large } N \quad (8.2)$$

holds almost surely. Hence

$$\forall 1 \leq i \leq k_j, \quad \lambda_{n_{j-1}+i}(M_N) \xrightarrow{a.s.} \rho_{\theta_j}.$$

2) a) We only focus on the case where  $\theta_j$  is on the right of any connected component of  $\text{supp}(\nu)$  which is included in  $[s_{l_j}, t_{l_j}]$  since the other case may be considered with similar arguments. Let us consider the set  $\{\theta_{j_0} > \dots > \theta_{j_p}\}$  of all the  $\theta_i$ 's being in  $[s_{l_j}, t_{l_j}]$  and on the right of any connected component of

$\text{supp}(\nu)$  which is included in  $[s_{l_j}, t_{l_j}]$ . Note that we have for all large  $N$ , for any  $0 \leq h \leq p$ ,

$$n_{j_{h-1}} + k_{j_h} = n_{j_h}$$

and  $\theta_{j_0}$  is the largest eigenvalue of  $A_N$  which is lower than  $t_{l_j}$ . Let  $\epsilon > 0$ . Applying Corollary 7.1 with  $u = t_{l_j}$ , we get that, almost surely,

$$\lambda_{n_{j_0-1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \epsilon \text{ and } \lambda_{n_{j_0-1}}(M_N) > H_{\sigma,\nu}(t_{l_j}) \text{ for all large } N.$$

Now, almost surely, the number of eigenvalues of  $M_N$  being in  $]H_{\sigma,\nu}(t_{l_j}) - \epsilon, H_{\sigma,\nu}(t_{l_j})]$  should tend to infinity when  $N$  goes to infinity. Since almost surely for all large  $N$ ,  $\lambda_{n_{j_0-1}}(M_N) > H_{\sigma,\nu}(t_{l_j})$  and  $\lambda_{n_{j_0-1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \epsilon$ , we should have

$$H_{\sigma,\nu}(t_{l_j}) - \epsilon \leq \lambda_{n_{j_{p-1}}+k_{j_p}}(M_N) \leq \dots \leq \lambda_{n_{j_0-1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \epsilon.$$

Hence, we deduce that:  $\forall 0 \leq l \leq p$  and  $\forall 1 \leq i \leq k_{j_p}$ ,  $\lambda_{n_{j_{p-1}}+i}(M_N) \xrightarrow{a.s.} H_{\sigma,\nu}(t_{l_j})$ . The result then follows since  $j \in \{j_0, \dots, j_p\}$ .

b) Let  $\alpha_j = 1 - \lim_N \frac{n_{j-1}}{N} = \nu(]-\infty, \theta_j])$ . Denote by  $Q$  (resp.  $Q_N$ ) the distribution function of  $\mu_\sigma \boxplus \nu$  (resp. of the spectral measure of  $M_N$ ). Since  $\mu_\sigma \boxplus \nu$  is absolutely continuous,  $Q$  is continuous on  $\mathbb{R}$  and strictly increasing on each interval  $[\Psi_{\sigma,\nu}(s_l), \Psi_{\sigma,\nu}(t_l)]$ ,  $1 \leq l \leq m$ .

From Proposition 2.4 and the hypothesis on  $\theta_j$ ,  $\alpha_j \in ]Q(\Psi_{\sigma,\nu}(s_{l_j})), Q(\Psi_{\sigma,\nu}(t_{l_j}))]$  and there exists a unique  $q_j \in ]\Psi_{\sigma,\nu}(s_{l_j}), \Psi_{\sigma,\nu}(t_{l_j})[$  such that  $Q(q_j) = \alpha_j$ . Moreover,  $Q$  is strictly increasing in a neighborhood of  $q_i$ .

Let  $\epsilon > 0$ . From the almost sure convergence of  $\mu_{M_N}$  to  $\mu_\sigma \boxplus \nu$ , we deduce

$$Q_N(q_j + \epsilon) \xrightarrow{N \rightarrow \infty} Q(q_j + \epsilon) > \alpha_j, \quad \text{a.s.}$$

From the definition of  $\alpha_j$ , it follows that for large  $N$ ,  $N, N-1, \dots, n_{j-1} + k_j, \dots, n_{j-1} + 1$  belong to the set  $\{k, \lambda_k(M_N) \leq q_j + \epsilon\}$  and thus,

$$\limsup_{N \rightarrow \infty} \lambda_{n_{j-1}+1}(M_N) \leq q_j + \epsilon.$$

In the same way, since  $Q_N(q_j - \epsilon) \xrightarrow{N \rightarrow \infty} Q(q_j - \epsilon) < \alpha_j$ ,

$$\liminf_{N \rightarrow \infty} \lambda_{n_{j-1}+k_j}(M_N) \geq q_j - \epsilon.$$

Thus, the  $k_j$  eigenvalues  $(\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j)$  converge almost surely to  $q_j$ .  $\square$

## 9 Appendix

We present in this appendix the different estimates on the variance used throughout the paper. They rely on the Poincaré hypothesis on the distribution  $\mu$  of

the entries of the Wigner matrix  $W_N$ . We assume that  $\mu$  satisfies a Poincaré inequality, that is there exists a positive constant  $C$  such that for any  $C^\infty$  function  $f : \mathbb{R} \rightarrow \mathbb{C}$  such that  $f$  and  $f'$  are in  $L^2(\mu)$ ,

$$\mathbf{V}(f) \leq C \int |f'|^2 d\mu,$$

with  $\mathbf{V}(f) = \mathbb{E}(|f - \mathbb{E}(f)|^2)$ .

We refer the reader to [17] for a characterization of such measures on  $\mathbb{R}$ . This inequality translates in the matricial case as follows:

For any matrix  $M$ , define  $\|M\|_2 = (\text{Tr}(M^*M))^{\frac{1}{2}}$  the Hilbert-Schmidt norm. Let  $\Psi : (M_N(\mathbb{C})_{sa}) \rightarrow \mathbb{R}^{N^2}$  (resp.  $\Psi : (M_N(\mathbb{R})_s) \rightarrow \mathbb{R}^{N(N+1)/2}$ ) be the canonical isomorphism which maps a Hermitian (resp. symmetric) matrix  $M$  to the real parts and the imaginary parts of its entries (resp. to the entries)  $M_{ij}, i \leq j$ .

**Lemma 9.1.** *Let  $M_N$  be the complex (resp. real) Wigner Deformed matrix introduced in Section 1. For any  $C^\infty$  function  $f : \mathbb{R}^{N^2}$  (resp.  $\mathbb{R}^{N(N+1)/2}$ )  $\rightarrow \mathbb{C}$  such that  $f$  and its gradient  $\nabla(f)$  are both polynomially bounded,*

$$\mathbf{V}[f \circ \Psi(M_N)] \leq \frac{C}{N} \mathbb{E}\{\|\nabla[f \circ \Psi(M_N)]\|_2^2\}. \quad (9.1)$$

From this Lemma and the properties of the resolvent  $G$  (see Lemma 1.1), we obtain:

- $\mathbf{V}((G_N(z))_{ij}) \leq \frac{C}{N} P(|\Im z|^{-1})$
- $\mathbf{V}((G_N(z))_{ii}^2) \leq \frac{C}{N} P(|\Im z|^{-1})$
- Let  $H$  be a deterministic Hermitian matrix with norm  $\|H\|$ , then,

$$\mathbf{V}((HG_N(z))_{ii}) \leq \frac{C}{N} \|H\|^2 P(|\Im z|^{-1})$$

- $\mathbf{V}(\text{tr}_N(G_N(z))) \leq \frac{C}{N^2} P(|\Im z|^{-1})$

where  $P$  is a polynomial. It follows that:

$$\mathbb{E}[(U^* G_D U G)_{ii} G_{ii} G_{ii}^2] = \mathbb{E}[(U^* G_D U G)_{ii}] \mathbb{E}[G_{ii}] \mathbb{E}[G_{ii}]^2 + \frac{1}{N} P(|\Im z|^{-1}),$$

proving (3.11).

We now prove

**Lemma 9.2.** *Let  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then,*

$$|\mathbb{E}[\tilde{G}_{pk} \text{tr}_N(G)] - \mathbb{E}[\tilde{G}_{pk}] \mathbb{E}[\text{tr}_N(G)]| \leq \frac{P(|\Im z|^{-1})}{N^2}.$$

**Proof:** The cumulant expansion gives

$$z\mathbb{E}(G_{ji}) = \sigma^2\mathbb{E}(\text{tr}_N(G)G_{ji}) + \delta_{ij} + \mathbb{E}[(GA_N)_{ji}] + \frac{\kappa_4}{2N^2}\mathbb{E}[T(i, j)] + O_{ji}\left(\frac{1}{N^2}\right),$$

where

$$\begin{aligned} T(i, j) &= \frac{1}{3} \left\{ \frac{1}{\sqrt{2}} \sum_{l < i} \left( G_{jl}^{(3)} \cdot (e_{li}, e_{li}, e_{li}) + \sqrt{-1} G_{jl}^{(3)} \cdot (f_{li}, f_{li}, f_{li}) \right) \right. \\ &\quad + \frac{1}{\sqrt{2}} \sum_{l > i} \left( G_{jl}^{(3)} \cdot (e_{il}, e_{il}, e_{il}) - \sqrt{-1} G_{jl}^{(3)} \cdot (f_{il}, f_{il}, f_{il}) \right) \\ &\quad \left. + G_{jl}^{(3)} \cdot (E_{ii}, E_{ii}, E_{ii}) \right\}. \end{aligned}$$

Straightforward computations give that

$$\begin{aligned} T(i, j) &= \sum_l G_{jl} G_{li}^3 + \sum_l G_{ji} G_{il} G_{li} G_{ll} \\ &\quad + \sum_l G_{jl} G_{ii} G_{li} G_{ll} + \sum_l G_{ji} G_{ii} G_{ll}^2 - 2G_{ii}^3 G_{ji}. \end{aligned}$$

We now compute the sum  $\sum U_{ik}^* U_{pj} \dots$  to obtain:

$$\begin{aligned} (z - \gamma_k)\mathbb{E}[\tilde{G}_{pk}] &= \sigma^2\mathbb{E}[\text{tr}_N(G)\tilde{G}_{pk}] + \delta_{pk} + \frac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p, k)] \\ &\quad - \frac{\kappa_4}{N^2} \sum_{i,j} U_{ik}^* U_{pj} \mathbb{E}[G_{ii}^3 G_{ji}] + \sum_{i,j} U_{ik}^* U_{pj} O_{ji}\left(\frac{1}{N^2}\right), \end{aligned} \quad (9.2)$$

where

$$\tilde{A}(p, k) = \sum_{i,j} U_{ik}^* U_{pj} A(i, j)$$

and

$$\begin{aligned} A(i, j) &= \sum_l G_{jl} G_{li}^3 + \sum_l G_{ji} G_{il} G_{li} G_{ll} \\ &\quad + \sum_l G_{jl} G_{ii} G_{li} G_{ll} + \sum_l G_{ji} G_{ii} G_{ll}^2. \end{aligned}$$

Since  $\frac{\kappa_4}{N^2} \sum_{i,j} U_{ik}^* U_{pj} G_{ii}^3 G_{ji} = \frac{\kappa_4}{N^2} (UG(G^{(d)})^3 U^*)_{pk}$ , this term is obviously a  $O\left(\frac{1}{N^2}\right)$ .

Let us verify the following bound for  $\tilde{A}$ :

$$\left| \frac{1}{N^2} \tilde{A}(p, k) \right| \leq C \frac{|\Im z|^{-4}}{N}. \quad (9.3)$$

Such a bound for the first term in the decomposition of  $A$  can be readily deduced from (1.2). We write the computation for the fourth term in the decomposition of  $A$ , the other two terms are similar:

$$\begin{aligned} &\frac{1}{N^2} \sum_{i,j,l} U_{ik}^* U_{pj} G_{ji} G_{ii} G_{ll}^2 \\ &= \frac{1}{N^2} \sum_l (UGG^{(d)}U^*)_{pk} G_{ll}^2 = O\left(\frac{1}{N}\right). \end{aligned}$$



We prove now that the last term in (9.2) is of order  $O(\frac{1}{N^2})$ . This term is a linear combination of terms of the form:

$$\frac{\kappa_6}{N^3} \sum_{i,j,l} U_{ik}^* U_{pj} \mathbb{E}[G_{jl}^{(5)} \cdot (v_1, \dots, v_5)],$$

where  $v_u = E_{mn}$  with  $(m, n) = (i, l)$  or  $(m, n) = (l, i)$ . The fifth derivative is a product of six  $G$ . If there are  $G_{il}^2$  or  $G_{il}G_{li}$  in the product, we can conclude thanks to Lemma 1.1. The only term without any  $G_{il}$  is

$$G_{ji}G_{il}G_{ii}G_{il}G_{ii}G_{il}$$

which gives the contribution

$$\frac{1}{N^3} \sum_l (UG(G^{(d)})^2U^*)_{pk} G_{il}^3 = O(\frac{1}{N^2}).$$

The term with one  $G_{il}$  (or  $G_{li}$ ) will also give a contribution in  $\frac{1}{N^2}$ . Hence

$$(z - \gamma_k) \mathbb{E}[\tilde{G}_{pk}] = \sigma^2 \mathbb{E}[\text{tr}_N(G) \tilde{G}_{pk}] + \delta_{pk} + \frac{\kappa_4}{2N^2} \mathbb{E}[\tilde{A}(p, k)] + O(\frac{1}{N^2}). \quad (9.4)$$

We now apply (3.1) (or its extension (3.2)) to  $\Phi(X_N) = G_{jl}G_{qq}$  and  $H = E_{il}$  and take the sum in  $l$ . We obtain

$$\begin{aligned} z \mathbb{E}(G_{ji}G_{qq}) &= \sigma^2 \mathbb{E}(\text{tr}_N(G)G_{ji}G_{qq}) + \frac{\sigma^2}{N} \mathbb{E}[G_{qi}(G^2)_{jq}] + \mathbb{E}[G_{qq}\delta_{ij}] \\ &\quad + \mathbb{E}[(GA_N)_{ji}G_{qq}] + \frac{\kappa_4}{2N^2} \mathbb{E}[T(i, j)G_{qq}] \\ &\quad + \frac{\kappa_4}{2N^2} \mathbb{E}[B(i, j, q)] + O_{j,i}(\frac{1}{N^2}), \end{aligned}$$

where  $B(i, j, q)$  stands for all the terms coming from the third derivative of the product  $(G_{jl}G_{qq})$  except  $G_{qq}G_{jl}^{(3)}$ . Now, we consider  $\frac{1}{N} \sum_q$  of the above equalities to obtain:

$$\begin{aligned} z \mathbb{E}(G_{ji} \text{tr}_N(G)) &= \sigma^2 \mathbb{E}(\text{tr}_N(G)^2 G_{ji}) + \frac{\sigma^2}{N^2} \mathbb{E}[(G^3)_{ji}] + \mathbb{E}[\text{tr}_N(G)\delta_{ij}] \\ &\quad + \mathbb{E}[(GA_N)_{ji} \text{tr}_N(G)] + \frac{\kappa_4}{2N^2} \mathbb{E}[T(i, j) \text{tr}_N(G)] \\ &\quad + \frac{\kappa_4}{2N^2} \frac{1}{N} \sum_q \mathbb{E}[B(i, j, q)] + O_{j,i}(\frac{1}{N^2}). \end{aligned}$$

We now compute the sum  $\sum U_{ik}^* U_{pj} \dots$  and obtain

$$\begin{aligned} (z - \gamma_k) \mathbb{E}(\tilde{G}_{pk} \text{tr}_N(G)) &= \sigma^2 \mathbb{E}(\text{tr}_N(G)^2 \tilde{G}_{pk}) + \frac{\sigma^2}{N^2} \mathbb{E}[(UG^3U^*)_{pk}] \\ &\quad + \mathbb{E}[\text{tr}_N(G)\delta_{pk}] + \frac{\kappa_4}{2N^2} \mathbb{E}[\tilde{A}(p, k) \text{tr}_N(G)] \\ &\quad + \frac{\kappa_4}{2N^2} \frac{1}{N} \sum_q \mathbb{E}[\tilde{B}(p, k, q)] + O(\frac{1}{N^2}), \end{aligned}$$

where

$$\tilde{B}(p, k, q) = \sum U_{ik}^* U_{pj} B(i, j, q)$$

and the terms  $\frac{\kappa_4}{2N^2} \sum U_{ik}^* U_{pj} \mathbb{E}[(T(i, j) - A(i, j)) \text{tr}_N(G)]$  and  $\sum U_{ik}^* U_{pj} O_{j,i}(\frac{1}{N^2})$  remain a  $O(\frac{1}{N^2})$  by the same arguments used to handle the analogue terms in (9.2).

Now, consider the difference between the above equation and  $g_N(z) \times (9.2)$ :

$$\begin{aligned} & (z - \gamma_k) \mathbb{E}[(\tilde{G}_{pk}(\text{tr}_N(G) - \mathbb{E}[\text{tr}_N(G)]))] = \\ & \frac{\sigma^2}{N^2} \mathbb{E}[(UG^3U^*)_{pk}] + \sigma^2 \mathbb{E}[\text{tr}_N(G)(\text{tr}_N(G) - \mathbb{E}[\text{tr}_N(G)]) \tilde{G}_{pk}] \\ & + \frac{\kappa_4}{2N^2} \mathbb{E}[\tilde{A}(p, k)(\text{tr}_N(G) - \mathbb{E}[\text{tr}_N(G)])] \\ & + \frac{\kappa_4}{2N^2} \frac{1}{N} \sum_q \mathbb{E}[\tilde{B}(p, k, q)] + O(\frac{1}{N^2}) \end{aligned}$$

and

$$\begin{aligned} & (z - \gamma_k - \sigma^2 g_N(z)) \mathbb{E}[\tilde{G}_{pk}(\text{tr}_N(G) - \mathbb{E}[\text{tr}_N(G)])] = \\ & \sigma^2 \mathbb{E}[(\text{tr}_N(G) - \mathbb{E}[\text{tr}_N(G)])^2 \tilde{G}_{pk}] + \frac{\sigma^2}{N^2} \mathbb{E}[(UG^3U^*)_{pk}] \\ & + \frac{\kappa_4}{2N^2} \mathbb{E}[\tilde{A}(p, k)(\text{tr}_N(G) - \mathbb{E}[\text{tr}_N(G)])] \\ & + \frac{\kappa_4}{2N^2} \frac{1}{N} \sum_q \mathbb{E}[\tilde{B}(p, k, q)] + O(\frac{1}{N^2}). \end{aligned}$$

We now prove that the right-hand side of the above equation is of order  $\frac{1}{N^2}$ . This is obvious for the second and first term (since  $\mathbf{V}(\text{tr}_N(G_N(z))) = O(\frac{1}{N^2})$ ). Now, we have seen that

$$\frac{1}{N^2} \tilde{A}(p, k) \leq \frac{C|\Im z|^{-4}}{N}.$$

By Cauchy-Schwarz inequality,

$$\frac{1}{N^2} \mathbb{E}[\tilde{A}(p, k)(\text{tr}_N(G) - \mathbb{E}[\text{tr}_N(G)])] = O(\frac{1}{N^2}).$$

It remains to study the last term

$$\frac{1}{N^3} \sum_q \mathbb{E}[\tilde{B}(p, k, q)] = \frac{1}{N^3} \sum_{i,j,q} U_{ik}^* U_{pj} \mathbb{E}[B(i, j, q)].$$

This term contains derivatives of  $G_{qq}$  of order  $a$  with  $a$  strictly positive ( $a = 1, 2, 3$ ) applied to a 3-tuple  $(v_1, v_2, v_3)$  where  $v_u = E_{il}$  or  $E_{li}$  (with a product of the derivative of order  $3 - a$  of  $G_{jl}$ ). Thus, the index  $q$  appears in  $\tilde{B}(p, k, q)$  under the form of a product  $G_{qm} G_{nq}$  with  $m, n \in \{i, l\}$ . Thus, the sum in  $q$  will give  $G_{nm}^2$ . Moreover, the term in  $j$  in the derivative appears as  $G_{jm}$  with

$m \in \{i, l\}$  and we can do the sum in  $j$  to obtain  $(UG)_{pm}$ . Thus,  $\frac{1}{N^3} \sum_q \tilde{B}(p, k, q)$  can be written as  $\frac{1}{N^3} \sum_{i,l}$  of terms of the form

$$U_{ik}^*(G^2)_{i_1 j_1} (UG)_{p j_2} G_{i_3 j_3} G_{i_4 j_4},$$

where  $i_r, j_r \in \{i, l\}$  and  $j_2 = l$  for  $a = 3$  (no derivative in  $G_{jl}$ ),  $j_4 = l$  for  $a < 3$ . As in the previous computations, either the product  $G_{il}^2$  (or  $G_{il}G_{li}$ ) appears and we can apply Lemma 1.1 (the others terms are bounded). In the other cases, we can always perform one sum in  $i$  (or  $l$ ) and obtain  $\frac{1}{N^3} \sum_{l(\text{ or } i)}$  of bounded terms. Let us just give an example of terms which can be obtained (for  $a = 1$ ):

$$U_{ik}^*(G^2)_{li} (UG)_{pl} G_{ii} G_{ll}.$$

Then,

$$\frac{1}{N^3} \sum_{i,l} U_{ik}^*(G^2)_{li} (UG)_{pl} G_{ii} G_{ll} = \frac{1}{N^3} \sum_i U_{ik}^*(UGG^{(d)}G^2)_{pi} G_{ii}.$$

Therefore,  $\frac{1}{N^3} \sum_q \mathbb{E}[\tilde{B}(p, k, q)]$  is of order  $\frac{1}{N^2}$ . This proves Lemma 9.2 since  $|\frac{1}{z - \gamma_k - \sigma^2 g_N(z)}| \leq |\Im z|^{-1}$ .  $\square$

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