Lattice solitons for linear finite difference schemes

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Spurious lattice solitons for linear finite difference schemes

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Abstract

The goal of this work is to show that lattice traveling solitary wave are solution of the general linear finite-differenced version of the linear advection equation. The occurance of such a spurious solitary waves, which exhibits a very long life time, results in a non-vanishing numerical error for arbitrary time in unbounded numerical domain. Such a behavior is referred here to has a structural instability of the scheme, since the space of solutions spanned by the numerical scheme encompasses types of solutions (lattice solitary waves in the present case) that are not solution of the original continuous equations.

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1 Introduction

The analysis and the control of numerical error in discretized propagation-type equations is of major importance for both theoretical analysis and practical applications. A huge amount of works has been devoted to the analysis of the numerical errors, its dynamics and its influence on the computed solution (the reader is referred to classical books, among which [1, 5]). It appears that existing works are mostly devoted to linear, one-dimensional numerical models, such as the linear advection equation

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \tag{1}$$

where c is a constant uniform advection velocity. A striking observation is that, despite the tremendous efforts devoted to the analysis of numerical schemes in this simple case, the full exact non-homogeneous error equation has been derived only very recently [8].
The two sources of numerical error are the dispersive and dissipative properties of the numerical scheme, which are very often investigated in unbounded or periodic domains thanks to a spectral analysis. In previous work [9], we analyzed of a linear dispersive mechanism which results in local error focusing, i.e. to a sudden local error burst in the L_∞ norm for polychromatic solutions, referred to as the spurious caustic phenomenon. We showed that, for some specific values of the Courant number, spurious caustics can exist for some popular finite-difference schemes.

In another work [10], [11] we have determined classes of traveling solitary wave solutions for a differential approximation of a finite difference scheme by means of a hyperbolic ansatz. We showed that spurious solitary waves can occur in finite-difference solutions of nonlinear wave equation. The occurance of such a spurious solitary wave, which exhibits a very long life time, results in a non-vanishing numerical error for arbitrary time in unbounded numerical domain. Such a behavior is referred here to has a structural instability of the scheme, since the space of solutions spanned by the numerical scheme encompasses types of solutions (solitary waves in the present case) that are not solution of the original continuous equations.

In physics, mathematics or engineering, a spurious solution refers to a non-physical one, which bears no resemblance to real one (see, for, instance, [12].) Sometimes, a spurious solution may just mean an unwelcome or unexpected artifact. To avoid an unwelcome spurious solution, one has to find out its true origin, which in general can be due to either physical and mathematical modelings, or numerical methods. Spurious solutions induced by mathematical modeling are usually due to the

missing of certain analytical constraints in mathematical formulations. Yet, the most interesting spurious solutions are those that are directly related to numerical methods, i.e., spurious solutions induced by algorithm.

Analysis of spurious waves in the numerical approximation of a onedimensional advection equation has been extensively investigated over the past few years by means of Fourier Analysis, through the concept of group velocity (see [13], [14], [15]). It was also the kind of technique we used in [9].

Yet, in the present study, we investigate a new kind of spurious solution, which-might appear in the process of the numerical scheme, when a solution which is not the searched one appears to satisfy the recursive difference relation and emerges through the numerical computation. Following [16], and the fact that a wide class of numerical instabilities can be interpreted as spurious triad wave interactions generated by discretization, we show that such a spurious solution could then be related to a kind of "locking" of the scheme, since it takes the place of the searched one and gives birth to a wrong solution.

We thus presently extend our previous works, in so far as we exhibit lattice solitary waves solution of the general linear finite-differenced version of the linear advection equation, rejoining the fact that there exists travelling solitary wave solutions for a differential approximation of a finite difference scheme.

So far, we would like to lay the emphasis on the fact that, contrary to most beliefs, solitary waves and solitons can not uniquely be obtained as solutions of nonlinear differential equations and as solutions of linear differential equations, as it is very well shown in the very interesting paper of C. Radhakrishnan [17], where, taking the example of the Korteweg-de Vries equation, it is shown that soliton solutions need not always be the consequence of the trade-off between the nonlinear terms and the dispersive term in the nonlinear differential equation, and that even the ordinary one dimensional linear partial differential equation can produce a soliton. The author explains that solutions of both linear and nonlinear differential equations are functions which depend nonlinearly on the independent variable, and that one can construct linear as well as nonlinear differential equations from the same function, as it is the case for the linear advection equation. Thus, as it is explained, the claim that a particular physical phenomenon can be described only by a nonlinear differential equation, and not by any linear differential equation is not tenable, provided a linear differential equation with the same solution as that of the nonlinear differential equation exists, and that, incidentally, linearization is the oldest and most popular method of solving nonlinear differential equations. In

the same way, in [18], Liu et al. proved the existence of solitary waves in in Linear ODE with Variable Coefficients.
In the present paper, we consider the linear advection equation (1), which happens to be obtained by linearizing the nonlinear Burgers equation.
Our analysis is restricted to interior stencil, and the influence of boundary conditions will not be considered.

2 Test numerical schemes

A linear finite difference scheme for the Burgers equation can be written under the following general form:

$$F({u_l}^m, h, \tau) = 0, \tag{2}$$

where the discrete solution is denoted

$${u_l}^m = u\,(l\,dx, m\,dt) \tag{3}$$

$l \in \{j-1,\, j,\, j+1\}$, $m \in \{n-1,\, n,\, n+1\}$, $j = 1,\, ...,\, n_x$, $n = 1,\, ...,\, n_t$, h, τ denoting respectively the mesh size and time step, and σ the Courant-Friedrichs-Lewy number (cfl) coefficient, defined as $\sigma = c\,\tau/h$.

A numerical scheme is specified by selecting appropriate expression of the linear function F in equation (2).

For the sake of simplicity, the analysis will be restricted to schemes which involves at most two time levels and three grid points. The extension of the present analysis to other schemes is straightforward. For this class of schemes, the general finite-differenced version of the linear advection equation (1) can therefore be written as follows

$$\frac{{u_j}^{n+1} - {u_j}^n}{\tau} + \sum_{k=j-1}^{j+1} m_{jk}\, u_k^n = 0 \tag{4}$$

where the m_{jk} are coefficients related to the numerical scheme. Those coefficients are constants, and can take only three different values, that we will denote m^-, m, m^+, which implies that the matrix

$$[m_{jk}]_{1 \le j \le n,\, 1 \le k \le n}$$

is a block cyclic one, of the form:

$$M = \begin{bmatrix} m & m^+0 & \dots & \dots & \dots & 0 & \\ m^- & m & m^+ & 0 & \dots & 0 & \\ 0 & m^- & m & m^+ & 0 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \dots & \dots & \end{bmatrix} \tag{5}$$

If one writes it under the form:

$$\frac{du_i}{dt} + \sum_{k=j-1}^{j+1} m_{jk}\, u_k^n = 0 \tag{6}$$

where the m_{jk} denote the linear scheme coefficients, it appears as a special case of a more general equation, the *Discrete Self-Trapping* (*DST*) equation (see, among numerous references, [19], [20], [21]):

$$\frac{dA_j}{dt} + \gamma\, A_j^2\, A_j + \varepsilon \sum m_{jk}\, A_k = 0 \tag{7}$$

with, here:

$$\gamma = 0 \tag{8}$$

Nonlinear localized modes in discrete systems exist due to the interplay between lattice coupling and nonlinearity effects. We presently aim at proving that finite difference schemes admit spatially localized modes or discrete solitons. It is a fundamental property of discrete systems, or, more generally, spatially periodic systems with a band-gap structure for the dispersion relation of linear waves, that bright and dark localized modes may appear for the same physical system.
It is well known that depending upon of the parameters and the chosen initial condition the *DST* equation can lead either to self-trapping (i.e. local modes or solitons), or to chaos, or to a mixture of the above two behaviors (see [19], [22], [23]).

To begin with, we search exact solutions of the form

$$\phi_j(x,t) = \psi_j(x)\, e^{i\,\Lambda\, t} \tag{9}$$

By substituting this form in (6), one obtains:

$$i\,\Lambda\,\psi_j(x)\, e^{i\,\Lambda\, t} + \sum_{k=j-1}^{j+1} m_{jk}\,\psi_k(x)\, e^{i\,\Lambda\, t} = 0 \tag{10}$$

wich leads to:

$$i\,\Lambda\,\psi_j(x) + \sum_{k=j-1}^{j+1} m_{jk}\,\psi_k(x) = 0 \tag{11}$$

We search exact stationary solutions of the form

$$\phi_j = \psi_j\,e^{i\,\Lambda\,t} \tag{12}$$

where Λ represents a phase shift among successive discrete sites, and, in essence, translates the spatial carrier wave number within the Brillouin zone, which yields:

$$\Lambda\,\psi_j\,e^{i\,\Lambda\,t} + \sum_{k=j-1}^{j+1} m_{jk}\,\psi_k\,e^{i\,\Lambda\,t} = 0 \tag{13}$$

and, thus:

$$\Lambda\,\psi_j + \sum_{k=j-1}^{j+1} m_{jk}\,\psi_k = 0 \tag{14}$$

or, due to the fact that the characteristic scheme coefficients are constants and cyclically take the values m^-, m, m^+:

$$\Lambda\,\psi_j + m^-\,\psi_{j-1} + m\,\psi_j + m^+\,\psi_{j+1} = 0 \tag{15}$$

or:

$$(m + \Lambda)\,\psi_j + m^-\,\psi_{j-1} + m^+\,\psi_{j+1} = 0 \tag{16}$$

which appears to be a second-order recurrence solution with regards to ψ_j, whose characteristic equation is given by:

$$m^+\,r^2 + (\Lambda + m)\,r + m^- = 0 \tag{17}$$

whose solutions are:

$$r = \pm\frac{\Lambda + m - \sqrt{(\Lambda + m)^2 - 4\,m^+\,m^-}}{2} \tag{18}$$

$$r = \frac{-\Lambda - m \pm \sqrt{(\Lambda + m)^2 - 4\,m^+\,m^-}}{2} \tag{19}$$

If m^- and m^+ are of opposite signs, real solutions will thus exist; if they are of the same sign, real solutions will exist if:

$$\Lambda > 2|\, m^+\, m^-| - m \tag{20}$$

which corresponds to modes starting from the entire part of $2|\, m^+\, m^-| - m$.

From now on, we will set:

$$\left\{ \begin{array}{rcl} r_1 & = & \frac{-\Lambda - m - \sqrt{(\Lambda+m)^2 - 4\, m^+\, m^-}}{2} \\ r_2 & = & \frac{-\Lambda - m + \sqrt{(\Lambda+m)^2 - 4\, m^+\, m^-}}{2} \end{array} \right. \tag{21}$$

which ensures, for any ψ_n:

$$\psi_n = a\, r_1^n + b\, r_2^n \tag{22}$$

where a, b are real constants that will be determined using the boundary conditions.

Set:

$$\psi_\infty = \lim_{n \to +\infty} \psi_n = \psi(n_x) \tag{23}$$

Since we deal with the Burgers equation, we assume that the boundary conditions of Dirichlet type, periodic:

$$u(x = 0, t) = u(x = n_x\, h, t) \tag{24}$$

with an initial condition:

$$u(x, 0) = u_0(x) \tag{25}$$

which leads to:

$$\psi_0 = \psi_\infty \tag{26}$$

It ensures:

$$\left\{ \begin{array}{rcl} a + b & = & \psi_\infty \\ a\, r_{1\,x}^n + b\, r_2^{-n_x} & = & \psi_\infty \end{array} \right. \tag{27}$$

which yields:

$$\left\{ \begin{array}{rcl} a & = & \frac{\psi_\infty\, (1 - r_2)}{r_1 - r_2} \\ b & = & \frac{\psi_\infty\, (r_1 - 1)}{r_1 - r_2} \end{array} \right. \tag{28}$$

To study the type of the solution, we examine the first mode, which corresponds to $\Lambda = 1$:

3 Numerical example: the case of the Lax scheme

For the classical Lax scheme, the finite difference relation writes:

$$\frac{1}{\tau}\,u_j^{n+1} + \left(\frac{-1}{2\tau} + \frac{c}{2h}\right)\,u_{j+1}^n - \left(\frac{1}{2\tau} + \frac{c}{2h}\right)\,u_{j-1}^n = 0 \tag{29}$$

which is equivalent to

$$\frac{1}{\tau}\,u_j^{n+1} + \left(\frac{1}{\tau} - \frac{1}{\tau}\right)\,u_j^n + \left(\frac{-1}{2\tau} + \frac{c}{2h}\right)\,u_{j+1}^n - \left(\frac{1}{2\tau} + \frac{c}{2h}\right)\,u_{j-1}^n = 0 \tag{30}$$

so that one obtains:

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{\tau}\,u_j^n + \left(\frac{-1}{2\tau} + \frac{c}{2h}\right)\,u_{j+1}^n - \left(\frac{1}{2\tau} + \frac{c}{2h}\right)\,u_{j-1}^n = 0 \tag{31}$$

that we can write under the form:

$$\frac{du_j^n}{dt} + \frac{1}{\tau}\,u_j^n + \left(\frac{-1}{2\tau} + \frac{c}{2h}\right)\,u_{j+1}^n - \left(\frac{1}{2\tau} + \frac{c}{2h}\right)\,u_{j-1}^n = 0 \tag{32}$$

Identification leads then to the following values for the characteristic scheme coefficients m^-, m, m^+:

$$\left\{\begin{array}{rcl} m^- & = & \frac{1}{2\tau} - \frac{c}{2h} \\ m & = & \frac{1}{\tau} \\ m^+ & = & -\frac{1}{2\tau} - \frac{c}{2h} \end{array}\right. \tag{33}$$

Hence, the solutions of (17) are:

$$\left\{\begin{array}{rcl} r_1 & = & \frac{-\Lambda - \frac{1}{\tau} - \sqrt{\Lambda^2 + \frac{2\,\Lambda}{\tau} - \frac{c^2}{h^2} + \frac{2}{\tau^2}}}{2} \\ r_2 & = & \frac{-\Lambda - \frac{1}{\tau} + \sqrt{\Lambda^2 + \frac{2\,\Lambda}{\tau} - \frac{c^2}{h^2} + \frac{2}{\tau^2}}}{2} \end{array}\right. \tag{34}$$

Thus, high modes always exist.

One has then:

$$\left\{\begin{array}{rcl} a & = & \frac{\psi_\infty\,(1 - r_2)}{r_1 - r_2} \\ b & = & \frac{\psi_\infty\,(r_1 - 1)}{r_1 - r_2} \end{array}\right. \tag{35}$$

and:

$$\psi_n = a\,r_1^n + b\,r_2^n \tag{36}$$

4 Concluding remarks

The existence of spurious numerical lattice solitary waves in linear advection schemes has been proved. Such lattice solitary waves, which are not solutions of the exact continuous original equation, nevertheless satisfy the numerical scheme, appearing as parasitic solutions of the correct one. Such schemes will be referred to as structurally instable ones. Such spurious solitary waves have constant energy, and therefore the numerical error norm does not vanish at arbitrary long integration times on unbounded numerical domains.

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