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Wavelet estimation of a density in a GARCH-type model

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Abstract We consider the GARCH-type model: $S = \sigma^2 Z$, where σ^2 and Z are independent random variables. The density of σ^2 is unknown whereas the one of Z is known. We want to estimate the density of σ^2 from n observations of S under some dependence assumption (the *exponentially strongly mixing dependence*). Adopting the wavelet methodology, we construct a nonadaptive estimator based on projections and an adaptive estimator based on the hard thresholding rule. Taking the mean integrated squared error over Besov balls, we prove that the adaptive one attains a sharp rate of convergence.

Keywords GARCH models · Adaptive density estimation · Rate of convergence · Wavelets · Hard thresholding.

2000 Mathematics Subject Classification 62G07, 62G20.

1 Motivations

We consider the random sequence $(S_i)_{i \in \mathbb{Z}}$ where, for any $i \in \mathbb{Z}$,

$$S_i = \sigma_i^2 Z_i, \quad (1)$$

$(Z_i)_{i \in \mathbb{Z}}$ is a sequence of identically distributed random variables with common known density $f_Z : [0, 1] \rightarrow (0, \infty)$ and $(\sigma_i^2)_{i \in \mathbb{Z}}$ is a sequence of identically distributed random variables with common unknown density $f_{\sigma^2} : [0, 1] \rightarrow (0, \infty)$. For any $i \in \mathbb{Z}$, Z_i and σ_i^2 are independent. We suppose that $(S_i)_{i \in \mathbb{Z}}$ is strictly stationary and exponentially strongly mixing (to be defined in Section 2). We aim to estimate f_{σ^2} when only n random variables S_1, \dots, S_n are observed. The model (1) belongs to the family of the GARCH-type time series models classically encountered in financial models when the volatility process $(\sigma_i^2)_{i \in \mathbb{Z}}$ is unobserved (see for instance Carrasco and Chen (2002) for an overview).

In the literature, the most common approach is to rewrite (1) as a convolution model via the logarithmic transformation: $\ln S_i = \ln \sigma_i^2 + \ln Z_i$, $i \in \mathbb{Z}$. Then the density of $\ln \sigma_1^2$ is deconvolved and estimated by using Fourier transform and nonparametric methods (kernel, splines, wavelets, ...). See e.g. Masry (1991), Comte et al. (2008) and Van Zanten and Zareba (2008). In the independent case, methods and results on the classical density convolution model can be found in Carroll and Hall (1988), Devroye (1989), Fan (1991), Pensky and Vidakovic (1999), Fan and Koo (2002), Butucea and Matias (2005), Comte et al. (2006), Delaigle and Gijbels (2006) and Lacour (2006). However, note that the density of $\ln \sigma_1^2$ is obviously not f_{σ^2} and the estimation of f_{σ^2} from $\ln S_1, \dots, \ln S_n$ seems not really natural (see for instance (Comte et al. 2008, 3.5)).

In this study, the “direct” estimation of f_{σ^2} from S_1, \dots, S_n is investigated. Adopting a methodology similar to the one of Chaubey et al. (2010a) (for the density estimation problem under multiplicative censoring), we construct two new estimators based on wavelet basis. The first one, linear and nonadaptive, uses projections, and the second one, non-linear and adaptive, uses the hard thresholding rule introduced by Donoho et al. (1996). We evaluate their performances by determining upper bounds of the mean integrated squared error (MISE) over a wide range of smoothness function classes: the Besov balls. We prove that our adaptive estimator attains a sharp rate of convergence. “Sharp” in the sense that it is close to the one attained by the linear wavelet estimator constructed to minimize the MISE.

The paper is organized as follows. Assumptions on the model and some notations are introduced in Section 2. Section 3 briefly describes the wavelet basis and the Besov balls. The estimators are presented in Section 4. The results are set in Section 5. Technical proofs are collected in Section 6.

2 Assumptions and notations

Let us now clarify the assumptions made on (1).

Assumption on $(S_i)_{i \in \mathbb{Z}}$. For any $m \in \mathbb{Z}$, we define the m -th strongly mixing coefficient of $(S_i)_{i \in \mathbb{Z}}$ by

$$a_m = \sup_{(A,B) \in \mathcal{F}_{-\infty,0}^S \times \mathcal{F}_{m,\infty}^S} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

where, for any $u \in \mathbb{Z}$, $\mathcal{F}_{-\infty,u}^S$ is the σ -algebra generated by the random variables \dots, S_{u-1}, S_u and $\mathcal{F}_{u,\infty}^Y$ is the σ -algebra generated by the random variables S_u, S_{u+1}, \dots .

We consider the exponentially strongly mixing case i.e. there exist three known constants, $\gamma > 0$, $c > 0$ and $\theta > 0$, such that, for any $m \in \mathbb{Z}$,

$$a_m \leq \gamma \exp(-c|m|^\theta). \quad (2)$$

This assumption is satisfied by a large class of GARCH processes. See e.g. Withers (1981), Doukhan (1994), Modha and Masry (1996) and Carrasco and Chen (2002).

Note that, when $\theta \rightarrow \infty$, we are in the standard i.i.d. case. So (1) becomes the classical density convolution model.

Assumption of f_Z . We suppose that there exists a positive integer ν such that

$$f_Z(x) = \frac{1}{(\nu-1)!} (-\ln x)^{\nu-1}, \quad x \in [0, 1]. \quad (3)$$

Note that f_Z is the density of $\prod_{i=1}^{\nu} U_i$, where U_1, \dots, U_n are n i.i.d. random variables having the common uniform distribution on $[0, 1]$.

Assumption on f_{σ^2} . We assume w.o.l.g. that the support of f_{σ^2} is $[0, 1]$ and $f_{\sigma^2} \in \mathbb{L}^2([0, 1])$, where

$$\mathbb{L}^2([0, 1]) = \left\{ h : [0, 1] \rightarrow \mathbb{R}; \left(\int_0^1 h^2(x) dx \right)^{1/2} < \infty \right\}.$$

Assumption on the density of S_1 . Note that, thanks to the independence of σ_1^2 and Z_1 , the density of S_1 is

$$f_S(x) = \int_x^1 f_Z\left(\frac{x}{y}\right) f_{\sigma^2}(y) \frac{1}{y} dy, \quad x \in [0, 1].$$

We suppose that there exists a known constant $C_* > 0$ such that

$$\sup_{x \in [0, 1]} f_S(x) \leq C_*. \quad (4)$$

3 Wavelets and Besov balls

Wavelet basis. Let N be an integer such that $N > \nu$ (where ν is the one in (3)), ϕ and ψ be the initial wavelets of dbN . In particular, we have

- $\text{supp}(\phi) = \text{supp}(\psi) = [1 - N, N]$,
- ϕ and ψ are of class \mathcal{C}^{ν} .

Set

$$\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k), \quad \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k).$$

With an appropriate treatments at the boundaries, there exists an integer τ satisfying $2^\tau \geq 2N$ such that the collection $\mathcal{B} = \{\phi_{\tau,k}(\cdot), k \in \{0, \dots, 2^\tau - 1\}; \psi_{j,k}(\cdot); j \in \mathbb{N} - \{0, \dots, \tau - 1\}, k \in \{0, \dots, 2^j - 1\}\}$, is an orthonormal basis of $\mathbb{L}^2([0, 1])$. We refer to Cohen et al. (1993).

For any integer $\ell \geq \tau$, any $h \in \mathbb{L}^2([0, 1])$ can be expanded on \mathcal{B} as

$$h(x) = \sum_{k=0}^{2^\ell - 1} \alpha_{\ell,k} \phi_{\ell,k}(x) + \sum_{j=\ell}^{\infty} \sum_{k=0}^{2^j - 1} \beta_{j,k} \psi_{j,k}(x), \quad x \in [0, 1],$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ are the wavelet coefficients of h defined by

$$\alpha_{j,k} = \int_0^1 h(x)\phi_{j,k}(x)dx, \quad \beta_{j,k} = \int_0^1 h(x)\psi_{j,k}(x)dx. \quad (5)$$

Besov balls. Let $M > 0$, $s > 0$, $p \geq 1$ and $r \geq 1$. A function h belongs to $B_{p,r}^s(M)$ if and only if there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients (5) satisfy

$$2^{\tau(1/2-1/p)} \left(\sum_{k=0}^{2^\tau-1} |\alpha_{\tau,k}|^p \right)^{1/p} + \left(\sum_{j=\tau}^{\infty} \left(2^{j(s+1/2-1/p)} \left(\sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \right)^{1/p} \right)^r \right)^{1/r} \leq M^*.$$

In this expression, s is a smoothness parameter and p and r are norm parameters. For a particular choice of s , p and r , $B_{p,r}^s(M)$ contain the Hölder and Sobolev balls. See Meyer (1992).

4 Estimators

Estimators of the wavelet coefficients. The first step to estimate f_{σ^2} consists in expanding f_{σ^2} on \mathcal{B} and estimating its unknown wavelet coefficients. For any positive integer ℓ and any $h \in \mathcal{C}^\ell([0, 1])$, set

$$T(h)(x) = (xh(x))', \quad T_\ell(h)(x) = T(T_{\ell-1}(h))(x), \quad x \in [0, 1]. \quad (6)$$

For any integer $j \geq \tau$ and any $k \in \{0, \dots, 2^j - 1\}$,

– we estimate $\alpha_{j,k} = \int_0^1 f_{\sigma^2}(x)\phi_{j,k}(x)dx$ by

$$\hat{\alpha}_{j,k} = \frac{1}{n} \sum_{i=1}^n T_\nu(\phi_{j,k})(S_i), \quad (7)$$

(ν is the one in (3))

– we estimate $\beta_{j,k} = \int_0^1 f_{\sigma^2}(x)\psi_{j,k}(x)dx$ by

$$\hat{\beta}_{j,k} = \frac{1}{n} \sum_{i=1}^n T_\nu(\psi_{j,k})(S_i). \quad (8)$$

Some statistical properties of $\hat{\alpha}_{j,k}$ and $\hat{\beta}_{j,k}$ are studied in Propositions 2 and 3.

We consider two wavelets estimators for f_{σ^2} : a linear estimator and a hard thresholding estimator.

Linear estimator. Assuming that $f_{\sigma^2} \in B_{p,r}^s(M)$ with $p \geq 2$, we define the linear estimator \widehat{f}^L by

$$\widehat{f}^L(x) = \sum_{k=0}^{2^{j_0}-1} \widehat{\alpha}_{j_0,k} \phi_{j_0,k}(x), \quad x \in [0, 1], \quad (9)$$

where $\widehat{\alpha}_{j,k}$ is defined by (7) and j_0 is an integer which will be chosen later (see Theorem 1 below).

For a survey on wavelet linear estimators for various density models, we refer to Chaubey et al. (2010a).

Hard thresholding estimator. We define the hard thresholding estimator \widehat{f}^H by

$$\widehat{f}^H(x) = \sum_{k=0}^{2^\tau-1} \widehat{\alpha}_{\tau,k} \phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \widehat{\beta}_{j,k} \mathbf{1}_{\{|\widehat{\beta}_{j,k}| \geq \kappa \lambda_j\}} \psi_{j,k}(x), \quad (10)$$

$x \in [0, 1]$, where $\widehat{\alpha}_{\tau,k}$ is defined by (7), $\widehat{\beta}_{j,k}$ by (8), j_1 is the integer satisfying

$$\frac{1}{2} \left(\frac{n_*}{\ln n_*} \right)^{1/(2\nu+1)} < 2^{j_1} \leq \left(\frac{n_*}{\ln n_*} \right)^{1/(2\nu+1)},$$

$$n_* = n^{\theta/(\theta+1)},$$

θ is the one in (2), κ is a large enough constant (the one in Proposition 3) and

$$\lambda_j = 2^{\nu j} \sqrt{\frac{\ln n_*}{n_*}}. \quad (11)$$

The feature of the hard thresholding estimator is to only estimate the ‘‘large’’ unknown wavelet coefficients of f_{σ^2} which contain the main characteristics of f_{σ^2} .

For the construction of hard thresholding wavelet estimators in the standard density model, see e.g. Donoho et al. (1996) and Delyon and Juditsky (1996), in the (standard) convolution density model, see e.g. Pensky and Vidakovic (1999) and Fan and Koo (2002), and in the density model under multiplicative censoring, see Chaubey et al. (2010b).

5 Results

Upper bounds for \widehat{f}^L and \widehat{f}^H are given in Theorems 1 and 2 below.

Theorem 1 *Consider (1) under the assumptions of Section 2. Suppose that $f_{\sigma^2} \in B_{p,r}^s(M)$ with $s > 0$, $p \geq 2$ and $r \geq 1$. For any $q \in (0, 1)$, let \widehat{f}^L be (9) with j_0 such that*

$$\frac{1}{2} n^{1/(2s+2\nu+1+q)} < 2^{j_0} \leq n^{1/(2s+2\nu+1+q)}.$$

Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_0^1 \left(\widehat{f}^L(x) - f_{\sigma^2}(x) \right)^2 dx \right) \leq C n^{-2s/(2s+2\nu+1+q)}.$$

Theorem 2 Consider (1) under the assumptions of Section 2. Let \widehat{f}^H be (10). Suppose that $f_{\sigma^2} \in B_{p,r}^s(M)$ with $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > (2\nu + 1)/p\}$. Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_0^1 \left(\widehat{f}^H(x) - f_{\sigma^2}(x) \right)^2 dx \right) \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)},$$

where $n_* = n^{\theta/(\theta+1)}$ and θ is the one in (2).

The proof of Theorem 2 is based on several probability results (moment inequalities, concentration inequality, ...) and a suitable decomposition of the MISE.

Theorem 2 shows that, besides being adaptive, \widehat{f}^H attains a rate of convergence close to the one of \widehat{f}^L .

Note that, if we restrict our study to the independent case i.e. $\theta \rightarrow \infty$ and $\nu = 1$, the rate of convergence attained by \widehat{f}^H becomes the one for the classical density model under multiplicative censoring i.e. $(\ln n/n)^{2s/(2s+3)}$. See (Chaubey et al. 2010b, Theorem 5.2) with $\rho_n = 1$.

Conclusion and perspectives. We construct a new adaptive estimator \widehat{f}^H for f_{σ^2} from (1). It is based on wavelets and thresholding. It attains a sharp rate of convergence over Besov balls.

A possible perspective of this work is to consider other realistic dependence conditions (polynomial strongly mixing dependence, associated sequences, ...).

Moreover, perhaps we can improve the estimation of f_{σ^2} by considering other kinds of thresholding rules as the block thresholding one (BlockJS, ...). See e.g. Cai (1999, 2002) and Chesneau et al. (2010). However, to bound the MISE of such block thresholding estimators (as mentioned in (Van Zanten and Zareba 2008, 3.3)), it is not immediately clear how to extend some technical results (Talagrand's inequality, ...) to the dependent case.

All these aspects need further investigations that we leave for a future work.

6 Proofs

In this section, we consider (1) under the assumptions of Section 2. Moreover, C denotes any constant that does not depend on j , k and n . Its value may change from one term to another and may depend on ϕ or ψ .

6.1 Auxiliary results

Lemma 1 For any positive integer ℓ and any $h \in \mathcal{C}^\ell([0, 1])$, set

$$G(h)(x) = -xh'(x), \quad G_\ell(h)(x) = G(G_{\ell-1}(h))(x).$$

1. We have

$$f_{\sigma^2}(x) = G_\nu(f_S)(x), \quad x \in [0, 1],$$

2. Let T_ν be (6). We have, for any $h \in \mathcal{C}^\nu([0, 1])$,

$$\int_0^1 f_{\sigma^2}(x)h(x)dx = \int_0^1 f_S(x)T_\nu(h)(x)dx.$$

Proof of Lemma 1.

1. By definition of f_Z and the Binomial theorem, we obtain

$$\begin{aligned} f_S(x) &= \int_x^1 f_Z\left(\frac{x}{y}\right) f_{\sigma^2}(y) \frac{1}{y} dy \\ &= \frac{1}{(\nu-1)!} \int_x^1 (\ln y - \ln x)^{\nu-1} f_{\sigma^2}(y) \frac{1}{y} dy \\ &= \frac{1}{(\nu-1)!} \sum_{u=0}^{\nu-1} \binom{\nu-1}{u} (-\ln x)^u \int_x^1 (\ln y)^{\nu-1-u} f_{\sigma^2}(y) \frac{1}{y} dy. \end{aligned}$$

Therefore, using $\sum_{u=0}^{\nu-1} \binom{\nu-1}{u} (-1)^u = 0$,

$$\begin{aligned} f'_S(x) &= -\frac{1}{(\nu-1)!} \sum_{u=1}^{\nu-1} \binom{\nu-1}{u} \frac{u}{x} (-\ln x)^{u-1} \int_x^1 (\ln y)^{\nu-1-u} f_{\sigma^2}(y) \frac{1}{y} dy \\ &\quad - \frac{1}{(\nu-1)!} \sum_{u=0}^{\nu-1} \binom{\nu-1}{u} (-\ln x)^u (\ln x)^{\nu-1-u} f_{\sigma^2}(x) \frac{1}{x} \\ &= -\frac{1}{x} \left(\frac{1}{(\nu-1)!} \sum_{u=1}^{\nu-1} \binom{\nu-1}{u} u (-\ln x)^{u-1} \int_x^1 (\ln y)^{\nu-1-u} f_{\sigma^2}(y) \frac{1}{y} dy \right) \\ &\quad - (\ln x)^{\nu-1} f_{\sigma^2}(x) \frac{1}{x} \frac{1}{(\nu-1)!} \sum_{u=0}^{\nu-1} \binom{\nu-1}{u} (-1)^u \\ &= -\frac{1}{x} \left(\frac{1}{(\nu-1)!} \sum_{u=1}^{\nu-1} \binom{\nu-1}{u} u (-\ln x)^{u-1} \int_x^1 (\ln y)^{\nu-1-u} f_{\sigma^2}(y) \frac{1}{y} dy \right). \end{aligned}$$

So

$$\begin{aligned} G(f_S)(x) &= -x f'_S(x) \\ &= \frac{1}{(\nu-1)!} \sum_{u=1}^{\nu-1} \binom{\nu-1}{u} u (-\ln x)^{u-1} \int_x^1 (\ln y)^{\nu-1-u} f_{\sigma^2}(y) \frac{1}{y} dy. \end{aligned}$$

For any $\ell \in \{1, \dots, \nu - 1\}$, proceeding ℓ times in a similar fashion to the above i.e. using derivations and the equality $\sum_{u=m}^{\nu-1} \binom{\nu-1}{u} \frac{u!}{(u-m)!} (-1)^u = \frac{(\nu-1)!}{(\nu-m-1)!} (-1)^m \sum_{u=0}^{\nu-m-1} \binom{\nu-m-1}{u} (-1)^u = 0$, $m \in \{0, \dots, \ell\}$, we obtain

$$G_\ell(f_S)(x) = \frac{1}{(\nu-1)!} \sum_{u=\ell}^{\nu-1} \binom{\nu-1}{u} \frac{u!}{(u-\ell)!} (-\ln x)^{u-\ell} \int_x^1 (\ln y)^{\nu-1-u} f_{\sigma^2}(y) \frac{1}{y} dy.$$

Hence

$$G_{\nu-1}(f_S)(x) = \int_x^1 f_{\sigma^2}(y) \frac{1}{y} dy$$

and, by a derivation,

$$f_{\sigma^2}(x) = -x(G_{\nu-1}(f_S)(x))' = G_\nu(f_S)(x). \quad (12)$$

2. Using (12), an integration by parts and the fact that $G_{\nu-1}(f_S)(1) = 0$, we obtain

$$\begin{aligned} \int_0^1 f_{\sigma^2}(x)h(x)dx &= \int_0^1 G_\nu(f_S)(x)h(x)dx \\ &= \int_0^1 -x(G_{\nu-1}(f_S)(x))'h(x)dx \\ &= [-xG_{\nu-1}(f_S)(x)h(x)]_0^1 + \int_0^1 G_{\nu-1}(f_S)(x)(xh(x))'dx \\ &= \int_0^1 G_{\nu-1}(f_S)(x)T(h)(x)dx. \end{aligned}$$

Proceeding $\nu - 1$ times in a similar fashion to the above i.e. using $\nu - 1$ integrations by parts, we have

$$\begin{aligned} \int_0^1 G_{\nu-1}(f_S)(x)T(h)(x)dx &= \int_0^1 G_{\nu-2}(f_S)(x)T_2(h)(x)dx = \dots \\ &= \int_0^1 f_S(x)T_\nu(h)(x)dx. \end{aligned}$$

Therefore

$$\int_0^1 f_{\sigma^2}(x)h(x)dx = \int_0^1 f_S(x)T_\nu(h)(x)dx.$$

This ends the proof of Lemma 1. □

Proposition 1 *Let $q \geq 2$, for any $j \geq \tau$ and any $k \in \{0, \dots, 2^j - 1\}$, $\alpha_{j,k} = \int_0^1 f_{\sigma^2}(x)\phi_{j,k}(x)dx$ and T_ν be (6). Then*

1.

$$\mathbb{E}(T_\nu(\phi_{j,k})(S_1)) = \alpha_{j,k}.$$

2.

$$\sup_{x \in [0,1]} |T_\nu(\phi_{j,k})(x)| \leq \theta_1 2^{(2\nu+1)j/2},$$

where $\theta_1 = \nu! \sum_{u=0}^{\nu} \sup_{x \in [1-N, N]} |\phi^{(u)}(x)|$.

3.

$$\mathbb{E}((T_\nu(\phi_{j,k})(S_1))^2) \leq \theta_2 2^{2\nu j},$$

where $\theta_2 = C_* \nu (\nu!)^2 \sum_{u=0}^{\nu} \int_{1-N}^N (\phi^{(u)}(y))^2 dy$.

4. for any $q \in (0, 1)$, there exists a constant $C > 0$ such that

$$\mathbb{V} \left(\sum_{i=1}^n T_\nu(\phi_{j,k})(S_i) \right) \leq C n 2^{(2\nu+q)j}.$$

These results hold for ψ instead of ϕ (and, a fortiori, $\beta_{j,k} = \int_0^1 f_{\sigma^2}(x) \psi_{j,k}(x) dx$ instead of $\alpha_{j,k}$).

Proof of Proposition 1.

1. Since $\phi \in \mathcal{C}^\nu([0, 1])$, it follows from Lemma 1 that

$$\mathbb{E}(T_\nu(\phi_{j,k})(S_1)) = \int_0^1 T_\nu(\phi_{j,k})(x) f_S(x) dx = \int_0^1 f_{\sigma^2}(x) \phi_{j,k}(x) dx = \alpha_{j,k}. \quad (13)$$

2. For any $u \in \{0, \dots, \nu\}$, we have $(\phi_{j,k})^{(u)}(x) = 2^{(2u+1)j/2} \phi^{(u)}(2^j x - k)$.
Hence

$$\begin{aligned} \sup_{x \in [0,1]} |T_\nu(\phi_{j,k})(x)| &\leq \nu! \sum_{u=0}^{\nu} \sup_{x \in [0,1]} |x^u (\phi_{j,k})^{(u)}(x)| \\ &\leq \nu! \sum_{u=0}^{\nu} 2^{(2u+1)j/2} \sup_{x \in [1-N, N]} |\phi^{(u)}(x)| \\ &\leq \theta_1 2^{(2\nu+1)j/2}. \end{aligned} \quad (14)$$

3. Since $S_1(\Omega) = [0, 1]$, we have

$$\begin{aligned} \mathbb{E}((T_\nu(\phi_{j,k})(S_1))^2) &\leq \nu (\nu!)^2 \sum_{u=0}^{\nu} \mathbb{E} \left(S_1^{2u} ((\phi_{j,k})^{(u)}(S_1))^2 \right) \\ &\leq \nu (\nu!)^2 \sum_{u=0}^{\nu} \mathbb{E} \left(((\phi_{j,k})^{(u)}(S_1))^2 \right). \end{aligned} \quad (15)$$

Using (4), $(\phi_{j,k})^{(u)}(x) = 2^{(2u+1)j/2}\phi^{(u)}(2^j x - k)$ and doing the change of variables $y = 2^j x - k$, we obtain

$$\begin{aligned} \mathbb{E}\left(\left((\phi_{j,k})^{(u)}(S_1)\right)^2\right) &= \int_0^1 \left((\phi_{j,k})^{(u)}(x)\right)^2 f_S(x) dx \leq C_* \int_0^1 \left((\phi_{j,k})^{(u)}(x)\right)^2 dx \\ &\leq C_* 2^{(2u+1)j} \int_0^1 (\phi^{(u)}(2^j x - k))^2 dx \\ &= C_* 2^{2uj} \int_{-k}^{2^j - k} (\phi^{(u)}(y))^2 dy \\ &\leq C_* 2^{2uj} \int_{1-N}^N (\phi^{(u)}(y))^2 dy. \end{aligned} \quad (16)$$

Putting (15) and (16) together, we obtain

$$\begin{aligned} \mathbb{E}\left(\left(T_\nu(\phi_{j,k})(S_1)\right)^2\right) &\leq C_* \nu (\nu!)^2 \sum_{u=0}^{\nu} 2^{2uj} \int_{1-N}^N (\phi^{(u)}(y))^2 dy \\ &\leq \theta_2 2^{2\nu j}. \end{aligned} \quad (17)$$

4. We have

$$\begin{aligned} &\mathbb{V}\left(\sum_{i=1}^n T_\nu(\phi_{j,k})(S_i)\right) \\ &= n\mathbb{V}(T_\nu(\phi_{j,k})(S_1)) + 2 \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbb{C}(T_\nu(\phi_{j,k})(S_v), T_\nu(\phi_{j,k})(S_\ell)) \\ &\leq n\mathbb{V}(T_\nu(\phi_{j,k})(S_1)) + 2 \left| \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbb{C}(T_\nu(\phi_{j,k})(S_v), T_\nu(\phi_{j,k})(S_\ell)) \right|. \end{aligned} \quad (18)$$

Using (17), we have

$$\mathbb{V}(T_\nu(\phi_{j,k})(S_1)) \leq \mathbb{E}\left(\left(T_\nu(\phi_{j,k})(S_1)\right)^2\right) \leq C 2^{2\nu j}. \quad (19)$$

The stationarity of $(S_i)_{i \in \mathbb{Z}}$ implies that

$$\begin{aligned} &\left| \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbb{C}(T_\nu(\phi_{j,k})(S_v), T_\nu(\phi_{j,k})(S_\ell)) \right| \\ &= \left| \sum_{m=1}^n (n-m) \mathbb{C}(T_\nu(\phi_{j,k})(S_0), T_\nu(\phi_{j,k})(S_m)) \right| \\ &\leq n \sum_{m=1}^n |\mathbb{C}(T_\nu(\phi_{j,k})(S_0), T_\nu(\phi_{j,k})(S_m))|. \end{aligned}$$

By the Davydov inequality (see Davydov (1970)), it holds that

$$\begin{aligned} & |\mathbb{C}(T_\nu(\phi_{j,k})(S_0), T_\nu(\phi_{j,k})(S_m))| \\ & \leq 10a_m^q \left(\mathbb{E} \left((T_\nu(\phi_{j,k})(S_0))^{2/(1-q)} \right) \right)^{1-q} \\ & \leq 10a_m^q \left(\sup_{x \in [0,1]} |T_\nu(\phi_{j,k})(x)| \right)^{2q} \left(\mathbb{E} \left((T_\nu(\phi_{j,k})(S_0))^2 \right) \right)^{1-q}. \end{aligned}$$

Using (14) and (17) we obtain

$$\begin{aligned} |\mathbb{C}(T_\nu(\phi_{j,k})(S_0), T_\nu(\phi_{j,k})(S_m))| & \leq Ca_m^q \left(2^{(2\nu+1)j/2} \right)^{2q} (2^{2\nu j})^{1-q} \\ & \leq C2^{(2\nu+q)j} a_m^q. \end{aligned}$$

Since $\sum_{m=1}^n a_m^q \leq \sum_{m=1}^\infty a_m^q = C$, we have

$$\begin{aligned} \left| \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbb{C}(T_\nu(\phi_{j,k})(S_v), T_\nu(\phi_{j,k})(S_\ell)) \right| & \leq Cn2^{(2\nu+q)j} \sum_{m=1}^n a_m^q \\ & \leq Cn2^{(2\nu+q)j}. \end{aligned} \quad (20)$$

It follows from (18), (19) and (20) that

$$\mathbb{V} \left(\sum_{i=1}^n T_\nu(\phi_{j,k})(S_i) \right) \leq Cn2^{(2\nu+q)j}.$$

The proof of Proposition 1 is complete. □

Proposition 2 For any integer $j \geq \tau$ and any $k \in \{0, \dots, 2^j - 1\}$, let $\alpha_{j,k} = \int_0^1 f_{\sigma^2}(x)\phi_{j,k}(x)dx$ and $\hat{\alpha}_{j,k}$ be (7). Then, for any $q \in (0, 1)$,

1. there exists a constant $C > 0$ such that

$$\mathbb{E} \left((\hat{\alpha}_{j,k} - \alpha_{j,k})^2 \right) \leq C2^{(2\nu+q)j} \frac{1}{n}.$$

2. there exists a constant $C > 0$ such that

$$\mathbb{E} \left((\hat{\alpha}_{j,k} - \alpha_{j,k})^4 \right) \leq C2^{(4\nu+1+q)j} \frac{1}{n}.$$

These inequalities hold for $\hat{\beta}_{j,k}$ defined by (8) instead of $\hat{\alpha}_{j,k}$, and $\beta_{j,k} = \int_0^1 f_{\sigma^2}(x)\psi_{j,k}(x)dx$ instead of $\alpha_{j,k}$.

Proof of Proposition 2.

1. Using Proposition 1, we have

$$\begin{aligned} \mathbb{E} \left((\widehat{\alpha}_{j,k} - \alpha_{j,k})^2 \right) &= \mathbb{E} \left((\widehat{\alpha}_{j,k} - \mathbb{E}(\widehat{\alpha}_{j,k}))^2 \right) = \mathbb{V}(\widehat{\alpha}_{j,k}) \\ &= \frac{1}{n^2} \mathbb{V} \left(\sum_{i=1}^n T_\nu(\phi_{j,k})(S_i) \right) \\ &\leq C \frac{1}{n^2} n 2^{(2\nu+q)j} = C 2^{(2\nu+q)j} \frac{1}{n}. \end{aligned} \quad (21)$$

2. The triangular inequality yields

$$|\widehat{\alpha}_{j,k} - \alpha_{j,k}| \leq |\widehat{\alpha}_{j,k}| + |\alpha_{j,k}|.$$

Using again Proposition 1, we obtain

$$|\widehat{\alpha}_{j,k}| \leq \frac{1}{n} \sum_{i=1}^n |T_\nu(\phi_{j,k})(S_i)| \leq \sup_{x \in [0,1]} |T_\nu(\phi_{j,k})(x)| \leq C 2^{(2\nu+1)j/2}.$$

We have $\sup_{x \in [0,1]} |\phi_{j,k}(x)| \leq 2^{j/2} \sup_{x \in [1-N, N]} |\phi(x)| = C 2^{j/2}$. So

$$\begin{aligned} |\alpha_{j,k}| &\leq \int_0^1 f_{\sigma^2}(x) |\phi_{j,k}(x)| dx \leq C 2^{j/2} \int_0^1 f_{\sigma^2}(x) dx = C 2^{j/2} \\ &\leq C 2^{(2\nu+1)j/2}. \end{aligned} \quad (22)$$

Therefore

$$|\widehat{\alpha}_{j,k} - \alpha_{j,k}| \leq C 2^{(2\nu+1)j/2}. \quad (23)$$

It follows from (23) and (21) that

$$\mathbb{E} \left((\widehat{\alpha}_{j,k} - \alpha_{j,k})^4 \right) \leq C 2^{(2\nu+1)j} \mathbb{E} \left((\widehat{\alpha}_{j,k} - \alpha_{j,k})^2 \right) \leq C 2^{(4\nu+1+q)j} \frac{1}{n}.$$

The proof of Proposition 2 is complete. \square

Proposition 3 For any $j \in \{\tau, \dots, j_1\}$ and any $k \in \{0, \dots, 2^j - 1\}$, let $\beta_{j,k} = \int_0^1 f_{\sigma^2}(x) \psi_{j,k}(x) dx$, $\widehat{\beta}_{j,k}$ be (8) and λ_j be (11). Then there exist two constants, $\kappa > 0$ and $C > 0$, such that

$$\mathbb{P} \left(|\widehat{\beta}_{j,k} - \beta_{j,k}| \geq \kappa \lambda_j / 2 \right) \leq C \frac{1}{n_*^4},$$

where $n_* = n^{\theta/(\theta+1)}$.

Proof of Proposition 3. Lemma 2 below presents a Bernstein inequality for exponentially strongly mixing process. This is a slightly modified version of (Modha and Masry 1996, Theorem 4.2).

Lemma 2 (Modha and Masry (1996)) *Let $\gamma > 0$, $c > 0$, $\theta > 1$ and $(Z_i)_{i \in \mathbb{Z}}$ be a stationary process such that, for any $m \in \mathbb{Z}$, the associated m -th strongly mixing coefficient satisfies*

$$a_m \leq \gamma \exp(-c|m|^\theta).$$

Let $n \in \mathbb{N}^$, $h : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function and, for any $i \in \mathbb{Z}$, $U_i = h(Z_i)$. We assume that there exists a constant $M > 0$ satisfying $|U_1| \leq M < \infty$. Then, for any $\lambda > 0$, we have*

$$\mathbb{P} \left(\left| \frac{1}{n} \sum_{i=1}^n U_i \right| \geq \lambda \right) \leq 2(1 + 4e^{-2}\gamma) \exp \left(-\frac{u\lambda^2 n^{\theta/(\theta+1)}}{2(\mathbb{E}(U_1^2) + \lambda M/3)} \right),$$

where $u = (1/2)(c/8)^{1/(\theta+1)}$.

For any $i \in \{1, \dots, n\}$, set

$$U_i = T_\nu(\psi_{j,k})(S_i) - \beta_{j,k}.$$

Proposition 1 applied with ψ instead of ϕ implies that

$$\mathbb{E}(U_1) = 0, \quad \mathbb{E}(U_1^2) \leq \mathbb{E}((T_\nu(\psi_{j,k})(S_1))^2) \leq \theta_2 2^{2\nu j}$$

and, using (22),

$$|U_1| \leq \sup_{x \in [0,1]} |T_\nu(\psi_{j,k})(x)| + |\beta_{j,k}| \leq (\theta_1 + \sup_{x \in [1-N,N]} |\psi(x)|) 2^{(2\nu+1)j/2}.$$

It follows from Lemma 2 applied with U_1, \dots, U_n , $\lambda = \kappa \lambda_j / 2$ and $M = (\theta_1 + \sup_{x \in [1-N,N]} |\psi(x)|) 2^{(2\nu+1)j/2}$, and the inequality $2^{j_1} \leq n_* / \ln n_*$ that

$$\begin{aligned} \mathbb{P} \left(|\widehat{\beta}_{j,k} - \beta_{j,k}| \geq \kappa \lambda_j / 2 \right) &= \mathbb{P} \left(\left| \frac{1}{n} \sum_{i=1}^n U_i \right| \geq \kappa \lambda_j / 2 \right) \\ &\leq C \exp \left(-\frac{u\kappa^2 \lambda_j^2 n_*}{8 \left(\theta_2 2^{2\nu j} + \kappa(\theta_1 + \sup_{x \in [1-N,N]} |\psi(x)|) \lambda_j 2^{(2\nu+1)j/2} / 6 \right)} \right) \\ &\leq C \exp \left(-\frac{u\kappa^2 \ln n_*}{8 \left(\theta_2 + \kappa(\theta_1 + \sup_{x \in [1-N,N]} |\psi(x)|) 2^{j/2} \sqrt{\ln n_* / n_*} / 6 \right)} \right) \\ &\leq C \frac{1}{n_*^{\rho(\kappa)}}, \end{aligned} \tag{24}$$

where

$$\rho(\kappa) = \frac{u\kappa^2}{8 \left(\theta_2 + \kappa(\theta_1 + \sup_{x \in [1-N,N]} |\psi(x)|) / 6 \right)}.$$

Since $\lim_{x \rightarrow \infty} \rho(x) = \infty$, there exists a $\kappa > 0$ such that $\rho(\kappa) = 4$. Therefore, for such a κ , we have

$$\mathbb{P} \left(|\widehat{\beta}_{j,k} - \beta_{j,k}| \geq \kappa \lambda_j / 2 \right) \leq C \frac{1}{n_*^4}.$$

This completes the proof of Proposition 3.

□

6.2 Proofs of the main results

Proof of Theorem 1. We expand the function f_{σ^2} on \mathcal{B} as

$$f_{\sigma^2}(x) = \sum_{k=0}^{2^{j_0}-1} \alpha_{j_0,k} \phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x), \quad x \in [0, 1],$$

where

$$\alpha_{j_0,k} = \int_0^1 f_{\sigma^2}(x) \phi_{j_0,k}(x) dx, \quad \beta_{j,k} = \int_0^1 f_{\sigma^2}(x) \psi_{j,k}(x) dx.$$

We have, for any $x \in [0, 1]$,

$$\widehat{f}^L(x) - f_{\sigma^2}(x) = \sum_{k=0}^{2^{j_0}-1} (\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}) \phi_{j_0,k}(x) - \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x).$$

Since \mathcal{B} is an orthonormal basis of $\mathbb{L}^2([0, 1])$, we have

$$\mathbb{E} \left(\int_0^1 \left(\widehat{f}^L(x) - f_{\sigma^2}(x) \right)^2 dx \right) = \sum_{k=0}^{2^{j_0}-1} \mathbb{E} \left((\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k})^2 \right) + \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2.$$

Using Proposition 2, we obtain

$$\sum_{k=0}^{2^{j_0}-1} \mathbb{E} \left((\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k})^2 \right) \leq C 2^{j_0(2\nu+1+q)} \frac{1}{n} \leq C n^{-2s/(2s+2\nu+1+q)}.$$

Since $p \geq 2$, we have $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$. Hence

$$\sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C 2^{-2j_0s} \leq C n^{-2s/(2s+2\nu+1+q)}.$$

Therefore

$$\mathbb{E} \left(\int_0^1 \left(\widehat{f}^L(x) - f_{\sigma^2}(x) \right)^2 dx \right) \leq C n^{-2s/(2s+2\nu+1+q)}.$$

The proof of Theorem 1 is complete.

□

Proof of Theorem 2. We expand the function f_{σ^2} on \mathcal{B} as

$$f_{\sigma^2}(x) = \sum_{k=0}^{2^\tau-1} \alpha_{\tau,k} \phi_{\tau,k}(x) + \sum_{j=\tau}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x), \quad x \in [0, 1],$$

where

$$\alpha_{\tau,k} = \int_0^1 f_{\sigma^2}(x) \phi_{\tau,k}(x) dx, \quad \beta_{j,k} = \int_0^1 f_{\sigma^2}(x) \psi_{j,k}(x) dx.$$

We have, for any $x \in [0, 1]$,

$$\begin{aligned} & \widehat{f}^H(x) - f_{\sigma^2}(x) \\ &= \sum_{k=0}^{2^\tau-1} (\widehat{\alpha}_{\tau,k} - \alpha_{\tau,k}) \phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \left(\widehat{\beta}_{j,k} \mathbf{1}_{\{|\widehat{\beta}_{j,k}| \geq \kappa \lambda_j\}} - \beta_{j,k} \right) \psi_{j,k}(x) \\ & - \sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x). \end{aligned}$$

Since \mathcal{B} is an orthonormal basis of $L^2([0, 1])$, we have

$$\mathbb{E} \left(\int_0^1 \left(\widehat{f}^H(x) - f_{\sigma^2}(x) \right)^2 dx \right) = R + S + T, \quad (25)$$

where

$$R = \sum_{k=0}^{2^\tau-1} \mathbb{E} \left((\widehat{\alpha}_{\tau,k} - \alpha_{\tau,k})^2 \right), \quad S = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left(\left(\widehat{\beta}_{j,k} \mathbf{1}_{\{|\widehat{\beta}_{j,k}| \geq \kappa \lambda_j\}} - \beta_{j,k} \right)^2 \right)$$

and

$$T = \sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2.$$

Let us bound R , T and S , in turn.

Using Proposition 2, $n_* < n$ and $2s/(2s + 2\nu + 1) < 1$, we obtain

$$R \leq C 2^{\tau(2\nu+1+q)} \frac{1}{n} \leq C \frac{1}{n} \leq C \frac{1}{n_*} \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \quad (26)$$

For $r \geq 1$ and $p \geq 2$, we have $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$. Since $2s/(2s + 2\nu + 1) < 2s/(2\nu + 1)$, we have

$$T \leq C \sum_{j=j_1+1}^{\infty} 2^{-2js} \leq C 2^{-2j_1s} \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2\nu+1)} \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}.$$

For $r \geq 1$ and $p \in [1, 2)$, we have $B_{p,r}^s(M) \subseteq B_{2,\infty}^{s+1/2-1/p}(M)$. Since $s > (2\nu + 1)/p$, we have $(s + 1/2 - 1/p)/(2\nu + 1) > s/(2s + 2\nu + 1)$. So

$$\begin{aligned} T &\leq C \sum_{j=j_1+1}^{\infty} 2^{-2j(s+1/2-1/p)} \leq C 2^{-2j_1(s+1/2-1/p)} \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{2(s+1/2-1/p)/(2\nu+1)} \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \end{aligned}$$

Hence, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1, 2)$ and $s > (2\nu + 1)/p\}$, we have

$$T \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \quad (27)$$

We can write the term S as

$$S = S_1 + S_2 + S_3 + S_4, \quad (28)$$

where

$$\begin{aligned} S_1 &= \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left(\left(\widehat{\beta}_{j,k} - \beta_{j,k} \right)^2 \mathbf{1}_{\{|\widehat{\beta}_{j,k}| \geq \kappa \lambda_j\}} \mathbf{1}_{\{|\beta_{j,k}| < \kappa \lambda_j / 2\}} \right), \\ S_2 &= \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left(\left(\widehat{\beta}_{j,k} - \beta_{j,k} \right)^2 \mathbf{1}_{\{|\widehat{\beta}_{j,k}| \geq \kappa \lambda_j\}} \mathbf{1}_{\{|\beta_{j,k}| \geq \kappa \lambda_j / 2\}} \right), \\ S_3 &= \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left(\beta_{j,k}^2 \mathbf{1}_{\{|\widehat{\beta}_{j,k}| < \kappa \lambda_j\}} \mathbf{1}_{\{|\beta_{j,k}| \geq 2\kappa \lambda_j\}} \right) \end{aligned}$$

and

$$S_4 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left(\beta_{j,k}^2 \mathbf{1}_{\{|\widehat{\beta}_{j,k}| < \kappa \lambda_j\}} \mathbf{1}_{\{|\beta_{j,k}| < 2\kappa \lambda_j\}} \right).$$

Let us investigate the bounds of S_1 , S_2 , S_3 and S_4 in turn.

Upper bounds for S_1 and S_3 . We have

$$\left\{ |\widehat{\beta}_{j,k}| < \kappa \lambda_j, |\beta_{j,k}| \geq 2\kappa \lambda_j \right\} \subseteq \left\{ |\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa \lambda_j / 2 \right\},$$

$$\left\{ |\widehat{\beta}_{j,k}| \geq \kappa \lambda_j, |\beta_{j,k}| < \kappa \lambda_j / 2 \right\} \subseteq \left\{ |\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa \lambda_j / 2 \right\}$$

and

$$\left\{ |\widehat{\beta}_{j,k}| < \kappa \lambda_j, |\beta_{j,k}| \geq 2\kappa \lambda_j \right\} \subseteq \left\{ |\beta_{j,k}| \leq 2|\widehat{\beta}_{j,k} - \beta_{j,k}| \right\}.$$

So

$$\max(S_1, S_3) \leq C \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left(\left(\widehat{\beta}_{j,k} - \beta_{j,k} \right)^2 \mathbf{1}_{\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa \lambda_j / 2\}} \right).$$

It follows from the Cauchy-Schwarz inequality, Proposition 2 with $q \in (0, \max(1, (2\nu + 1)/\theta))$ which implies $2^{qj} \leq 2^{qj_1} \leq n_*^{q/(2\nu+1)} \leq n/n_*$, $2^j \leq 2^{j_1} \leq n_*$ and Proposition 3 that

$$\begin{aligned} & \mathbb{E} \left(\left(\widehat{\beta}_{j,k} - \beta_{j,k} \right)^2 \mathbf{1}_{\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_j/2\}} \right) \\ & \leq \left(\mathbb{E} \left(\left(\widehat{\beta}_{j,k} - \beta_{j,k} \right)^4 \right) \right)^{1/2} \left(\mathbb{P} \left(|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_j/2 \right) \right)^{1/2} \\ & \leq C \left(2^{(4\nu+1+q)j} \frac{1}{n} \right)^{1/2} \left(\frac{1}{n_*^4} \right)^{1/2} = C 2^{2\nu j} \frac{1}{n_*^2} \left(2^{(1+q)j} \frac{1}{n} \right)^{1/2} \leq C 2^{2\nu j} \frac{1}{n_*^2}. \end{aligned}$$

Since $2s/(2s + 2\nu + 1) < 1$, we have

$$\begin{aligned} \max(S_1, S_3) & \leq C \frac{1}{n_*^2} \sum_{j=\tau}^{j_1} 2^{j(1+2\nu)} \leq C \frac{1}{n_*^2} 2^{j_1(1+2\nu)} \leq C \frac{1}{n_*} \\ & \leq C \frac{\ln n_*}{n_*} \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \end{aligned} \quad (29)$$

Upper bound for S_2 . Using again Proposition 2 with $q \in (0, \max(1, (2\nu + 1)/\theta))$ which implies $2^{qj} \leq 2^{qj_1} \leq n_*^{q/(2\nu+1)} \leq n/n_*$, we obtain

$$\begin{aligned} \mathbb{E} \left(\left(\widehat{\beta}_{j,k} - \beta_{j,k} \right)^2 \right) & \leq C 2^{(2\nu+q)j} \frac{1}{n} \leq C 2^{2\nu j} \left(2^{qj} \frac{1}{n} \right) \leq C 2^{2\nu j} \frac{1}{n_*} \\ & \leq C 2^{2\nu j} \frac{\ln n_*}{n_*}. \end{aligned}$$

Hence

$$S_2 \leq C \frac{\ln n_*}{n_*} \sum_{j=\tau}^{j_1} 2^{2\nu j} \sum_{k=0}^{2^j-1} \mathbf{1}_{\{|\beta_{j,k}| > \kappa\lambda_j/2\}}.$$

Let j_2 be the integer defined by

$$\frac{1}{2} \left(\frac{n_*}{\ln n_*} \right)^{1/(2s+2\nu+1)} < 2^{j_2} \leq \left(\frac{n_*}{\ln n_*} \right)^{1/(2s+2\nu+1)}. \quad (30)$$

We have

$$S_2 \leq S_{2,1} + S_{2,2},$$

where

$$S_{2,1} = C \frac{\ln n_*}{n_*} \sum_{j=\tau}^{j_2} 2^{2\nu j} \sum_{k=0}^{2^j-1} \mathbf{1}_{\{|\beta_{j,k}| > \kappa\lambda_j/2\}}$$

and

$$S_{2,2} = C \frac{\ln n_*}{n_*} \sum_{j=j_2+1}^{j_1} 2^{2\nu j} \sum_{k=0}^{2^j-1} \mathbf{1}_{\{|\beta_{j,k}| > \kappa\lambda_j/2\}}.$$

We have

$$S_{2,1} \leq C \frac{\ln n_*}{n_*} \sum_{j=\tau}^{j_2} 2^{j(1+2\nu)} \leq C \frac{\ln n_*}{n_*} 2^{j_2(1+2\nu)} \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}.$$

For $r \geq 1$ and $p \geq 2$, since $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$,

$$\begin{aligned} S_{2,2} &\leq C \frac{\ln n_*}{n_*} \sum_{j=j_2+1}^{j_1} 2^{2\nu j} \frac{1}{\lambda_j^2} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C \sum_{j=j_2+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C 2^{-2j_2 s} \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \end{aligned}$$

For $r \geq 1$, $p \in [1, 2)$ and $s > (2\nu + 1)/p$, using $\mathbf{1}_{\{|\beta_{j,k}| > \kappa \lambda_j / 2\}} \leq C |\beta_{j,k}|^p / \lambda_j^p$, $B_{p,r}^s(M) \subseteq B_{2,\infty}^{s+1/2-1/p}(M)$ and $(2s + 2\nu + 1)(2 - p)/2 + (s + 1/2 - 1/p + \nu - 2\nu/p)p = 2s$, we have

$$\begin{aligned} S_{2,2} &\leq C \frac{\ln n_*}{n_*} \sum_{j=j_2+1}^{j_1} 2^{2\nu j} \frac{1}{\lambda_j^p} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{j\nu(2-p)} 2^{-j(s+1/2-1/p)p} \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{(2-p)/2} 2^{-j_2(s+1/2-1/p+\nu-2\nu/p)p} \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \end{aligned}$$

So, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > (2\nu + 1)/p\}$, we have

$$S_2 \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \quad (31)$$

Upper bound for S_4 . We have

$$S_4 \leq \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \mathbf{1}_{\{|\beta_{j,k}| < 2\kappa \lambda_j\}}.$$

Let j_2 be the integer (30). Then

$$S_4 \leq S_{4,1} + S_{4,2},$$

where

$$S_{4,1} = \sum_{j=\tau}^{j_2} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \mathbf{1}_{\{|\beta_{j,k}| < 2\kappa \lambda_j\}}, \quad S_{4,2} = \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \mathbf{1}_{\{|\beta_{j,k}| < 2\kappa \lambda_j\}}.$$

We have

$$\begin{aligned} S_{4,1} &\leq C \sum_{j=\tau}^{j_2} 2^j \lambda_j^2 = C \frac{\ln n_*}{n_*} \sum_{j=\tau}^{j_2} 2^{j(1+2\nu)} \leq C \frac{\ln n_*}{n_*} 2^{j_2(1+2\nu)} \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \end{aligned}$$

For $r \geq 1$ and $p \geq 2$, since $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$, we have

$$S_{4,2} \leq \sum_{j=j_2+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C 2^{-2j_2s} \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}.$$

For $r \geq 1$, $p \in [1, 2)$ and $s > (2\nu+1)/p$, using $\beta_{j,k}^2 \mathbf{1}_{\{|\beta_{j,k}| < 2\kappa \lambda_j\}} \leq C \lambda_j^{2-p} |\beta_{j,k}|^p$, $B_{p,r}^s(M) \subseteq B_{2,\infty}^{s+1/2-1/p}(M)$ and $(2s+2\nu+1)(2-p)/2 + (s+1/2-1/p+\nu-2\nu/p)p = 2s$, we have

$$\begin{aligned} S_{4,2} &\leq C \sum_{j=j_2+1}^{j_1} \lambda_j^{2-p} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \\ &= C \left(\frac{\ln n_*}{n_*} \right)^{(2-p)/2} \sum_{j=j_2+1}^{j_1} 2^{j\nu(2-p)} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{j\nu(2-p)} 2^{-j(s+1/2-1/p)p} \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{(2-p)/2} 2^{-j_2(s+1/2-1/p+\nu-2\nu/p)p} \\ &\leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \end{aligned}$$

So, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > (2\nu+1)/p\}$, we have

$$S_4 \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \quad (32)$$

It follows from (28), (29), (31) and (32) that

$$S \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}. \quad (33)$$

Combining (25), (26), (27) and (33), we have, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > (2\nu+1)/p\}$,

$$\mathbb{E} \left(\int_0^1 \left(\widehat{f}^H(x) - f_{\sigma^2}(x) \right)^2 dx \right) \leq C \left(\frac{\ln n_*}{n_*} \right)^{2s/(2s+2\nu+1)}.$$

The proof of Theorem 2 is complete. \square

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