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# BEYOND PRESSURELESS GAS DYNAMICS : QUADRATURE-BASED VELOCITY MOMENT MODELS

C. CHALONS <sup>\*</sup>, DAMIEN KAH <sup>†</sup>, AND MARC MASSOT <sup>‡</sup>

**Abstract.** Following the seminal work of F. Bouchut on zero pressure gas dynamics which has been extensively used for gas particle-flows, the present contribution investigates quadrature-based velocity moments models for kinetic equations in the framework of the infinite Knudsen number limit, that is, for dilute clouds of small particles where the collision or coalescence probability asymptotically approaches zero. Such models define a hierarchy based on the number of moments and associated quadrature nodes, the first level of which leads to pressureless gas dynamics. We focus in particular on the four moment model where the flux closure is provided by a two-node quadrature in the velocity phase space and provide the right framework for studying both smooth and singular solutions. The link with both the kinetic underlying equation as well as with zero pressure gas dynamics is provided and we define the notion of measure solutions as well as the mathematical structure of the resulting system of four PDEs. We exhibit a family of entropies and entropy fluxes and define the notion of entropic solution. We study the Riemann problem and provide a series of entropic solutions in particular cases. This leads to a rigorous link with the possibility of the system of macroscopic PDEs to allow particle trajectory crossing (PTC) in the framework of smooth solutions. Generalized  $\delta$ -choc solutions resulting from Riemann problem are also investigated. Finally, using a kinetic scheme proposed in the literature without mathematical background in several areas, we validate such a numerical approach in the framework of both smooth and singular solutions.

**Key words.** Quadrature-based moment methods, gas-particle flows, kinetic theory, Particle trajectory crossing, entropic measure solution

**subject classifications** 76N15 (35L65 65M99 76M25 82C40).

## 1. Introduction

The physics of particles and droplets in a carrier gaseous flow field are described in many applications (fluidized beds, spray dynamics, alumina particles in rocket boosters, ...) by a number density function (NDF) satisfying a kinetic equation introduced by [Williams (1958)]. Solving such a kinetic equation relies either on a sample of discrete numerical parcels of particles through a Lagrangian–Monte-Carlo approach or on a moment approach resulting in a Eulerian system of conservation laws on velocity moments eventually conditioned on size. In the latter case investigated in the present contribution, the main difficulty for particle flows with high Knudsen numbers (i.e. weakly collisional flows), where the velocity distribution can be very far from equilibrium, is the closure of the convective transport at the macroscopic level. One way to proceed is to use quadrature-based moment methods where the higher-order moments required for closure are evaluated from the lower-order transported moments using quadratures in the form of a sum of Dirac delta functions in velocity phase space (see [Yuan & Fox (2010), Kah (2010)] and the references therein for a series of advances within this framework).

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Such an approach also allows for a well-behaved kinetic numerical scheme in the spirit of Bouchut [Bouchut & al (2003)] (See references from [Fréret & al (2008), de Chaisemartin (2009), Massot & al (2009), de Chaisemartin & al (2008)] to [Kah *et al.* (2010), Fréret *et al.* (2010), Yuan & Fox (2010)]) where the fluxes in a cell-centered finite-volume formulation are directly evaluated from the knowledge of the quadrature abscissas and weights with guaranteed realizability conditions and singularity treatment. Such a quadrature approach and the related numerical methods have been shown to be able to capture PTC in a Direct Numerical Simulation (DNS) context, where the distribution in the exact kinetic equation remains at all times in the form of a sum of Dirac delta functions, and can be extended to partially high-order numerical schemes ([Vikas *et al.* (2010)]).

In another component of the literature devoted to multiphase semiclassical limits of the Schrödinger equation [Gosse *et al.* (2003), Gosse & Runborg (2005)], the series of Wigner measures obtained from the Wigner transform for studying the semiclassical limits can be shown to converge towards a measure solution of the Liouville equation. Such an equation naturally unfolds the caustics and can generate the proper multiphase solutions globally in time. Two approaches have been used to solve this equation with a moment approach, either the *Heaviside closure* [Brenier & Corrias (1998)] as it is called in [Gosse *et al.* (2003)], or, the one which is related to the present work, the *delta closure* (see [Gosse *et al.* (2003)] and references therein). It leads to a weakly hyperbolic systems of conservation laws by taking moments of a Liouville equation exactly identical to the Williams-Boltzmann equation studies in gas-particle flows previously mentioned. Such approaches naturally degenerate towards the pressureless gas system of equation in the context of monokinetic velocity distributions [Massot & al (2009), Kah *et al.* (2010), Kah (2010), Runborg (2000)].

Numerical algorithms in order to simulate such systems of conservations laws with the related *delta closure* or *quadrature-based closure* have been proposed in [Gosse *et al.* (2003)] and [Desjardins *et al.* (2008)] independently, from the work for [Bouchut & al (2003)] using naturally kinetic scheme with finite volume methods. However, many issues are still to be tackled in order to reach high order numerical schemes that preserve the whole vector of moments. In fact, such models are meant to capture a given level of complexity in the phase space which is fixed in advance by the number of moments and quadrature nodes. In some rare or particular situations, when one controls perfectly such dynamics, it can be guaranteed that the solutions will remain smooth. However, in most cases the numerical schemes have to tackle the possibility of singular solutions when the dynamics complexity goes beyond the one allowed by the model. In such cases the solution of the resulting system of PDEs is the viscosity solution and does not reproduce the exact dynamics in phase space and measure solutions are expected, for which we need a precise framework. More specifically, even if for the pressureless gas system, the work of [Bouchut (2004)] had set the correct mathematical background in order to define general entropic solutions, such a work had not yet been performed for higher order moment methods in the several cited publications. This is the purpose of the present contribution.

The paper is organized as follows. First we introduce the *quadrature-based* or *delta closure* velocity moment models for kinetic equations and focus on the four moment model in order to provide a complete picture of what can be done more generically to higher orders but which would be difficult to expose due to algebra complications. We then define entropy conditions and provide, for smooth solution, the one-to-one kinetic-macroscopic relation. We then tackle the Riemann problem and define entropy

measure solutions Two examples of smooth and singular solution are then provided for which we rigorously identify the entropic character of the solution and which are then reproduced numerically.

## 2. Quadrature-based velocity moment models for kinetic equations

Consider the solution  $f = f(t, x, v)$  of the free transport kinetic equation

$$\partial_t f + v \partial_x f = 0, \quad t > 0, x \in \mathbb{R}, v \in \mathbb{R} \quad (2.1)$$

with initial condition

$$f(0, x, v) = f_0(x, v).$$

The exact solution is given by

$$f(t, x, v) = f(0, x - vt, v) = f_0(x - vt, v).$$

Defining the  $i$ -order moment

$$M_i = \int_v f(t, x, v) v^i dv, \quad i = 1, \dots, N, \quad N \in \mathbb{N},$$

the associated governing equations are easily obtained from (2.1) after multiplication by  $v^i$  and integration over  $v$ , and write

$$\partial_t M_i + \partial_x M_{i+1} = 0, \quad i \geq 0.$$

For the sake of simplicity, but without any restriction, we will focus our attention hereafter on the four-moment model

$$\begin{cases} \partial_t M_0 + \partial_x M_1 = 0, \\ \partial_t M_1 + \partial_x M_2 = 0, \\ \partial_t M_2 + \partial_x M_3 = 0, \\ \partial_t M_3 + \partial_x \overline{M}_4 = 0. \end{cases} \quad (2.2)$$

With clear notations, it will be convenient to write (2.2) under the following abstract form

$$\partial_t \mathbf{M} + \partial_x \mathbf{F}(\mathbf{M}) = 0, \quad (2.3)$$

with  $\mathbf{M} = (M_0, M_1, M_2, M_3)^t$  and  $\mathbf{F}(\mathbf{M}) = (M_1, M_2, M_3, \overline{M}_4)^t$ .

This model is closed provided that  $\overline{M}_4$  is defined as a function of  $\mathbf{M}$ . In quadrature-based moment methods, the starting point to define this closure relation consists in representing the velocity distribution of  $f(t, x, v)$  by a set of two Dirac delta functions :

$$f(t, x, v) = \rho_1(t, x) \delta(v - v_1(x, t)) + \rho_2(t, x) \delta(v - v_2(x, t)), \quad (2.4)$$

where the weights  $\rho_1(t, x) > 0$ ,  $\rho_2(t, x) > 0$  and the velocity abscissas  $v_1(t, x)$ ,  $v_2(t, x)$  are expected to be uniquely determined from the knowledge of  $\mathbf{M}(x, t)$ . Dropping the  $(x, t)$ -dependance to avoid cumbersome notations, such a function  $f$  has exact moments of order  $i = 0, \dots, 4$  given by  $\rho_1 v_1^i + \rho_2 v_2^i$ . The next step then naturally consists in setting

$$\overline{M}_4 = \rho_1 v_1^4 + \rho_2 v_2^4 \quad (2.5)$$

where  $\rho_1, \rho_2$  and  $v_1, v_2$  are defined from  $\mathbf{M}$  by the following nonlinear system :

$$\begin{cases} M_0 = \rho_1 + \rho_2, \\ M_1 = \rho_1 v_1 + \rho_2 v_2, \\ M_2 = \rho_1 v_1^2 + \rho_2 v_2^2, \\ M_3 = \rho_1 v_1^3 + \rho_2 v_2^3. \end{cases} \quad (2.6)$$

At last, it remains to prove that this system is well-posed, which is the matter of the next proposition. We refer to [Gosse *et al.* (2003), Desjardins *et al.* (2008)] for the proof.

**PROPOSITION 2.1.** *System (2.3)-(2.5)-(2.6) is well-defined on the phase space  $\Omega$  given by*

$$\Omega = \{\mathbf{M} = (M_0, M_1, M_2, M_3)^t, M_0 > 0, M_0 M_2 - M_1^2 > 0\}.$$

*Moreover, setting  $\mathbf{U} = (\rho_1, \rho_2, \rho_1 v_1, \rho_2 v_2)^t$ , the function  $\mathbf{U} = \mathbf{U}(\mathbf{M})$  is one-to-one and onto as soon as  $v_1 \neq v_2$ , and whatever  $v_1$  and  $v_2$  are provided that we set  $\rho_1 = \rho_2$  in the case  $v_1 = v_2$ .*

Proposition 2.1 can be extended to the more general case of a  $2k$ -moment models. The velocity distribution is represented in this situation by a set of  $k$  Dirac delta functions, leading to  $M_i = \sum_{j=1}^k \rho_j v_j^i$ ,  $i = 0, \dots, 2k-1$ , and  $\overline{M}_{2k} = \sum_{j=1}^k \rho_j v_j^{2k}$ . In particular, the two-moment model writes

$$\begin{cases} \partial_t \rho + \partial_x \rho v = 0, \\ \partial_t \rho v + \partial_x \rho v^2 = 0, \end{cases}$$

which is nothing but the well-known pressureless gas dynamics system. Recall that this model is only weakly hyperbolic (the jacobian matrix is not diagonalizable) with  $v$  as unique eigenvalue, the characteristic field being linearly degenerate. Actually, we will observe in the course of the next section that the four-moment model (2.3) is equivalent for smooth solutions (only) to two decoupled pressureless gas dynamics systems associated with  $(\rho_1, \rho_1 v_1)$  and  $(\rho_2, \rho_2 v_2)$  respectively. Then (2.3) is expected to admit two eigenvalues  $v_1$  and  $v_2$  and to be weakly hyperbolic with linearly degenerate characteristic fields, as stated in the following proposition.

**PROPOSITION 2.2.** *([Gosse et al. (2003)]) System (2.3)-(2.5)-(2.6) is weakly hyperbolic on  $\Omega$  and admits the two eigenvalues  $v_1$  and  $v_2$ . The associated characteristic fields are linearly degenerate.*

**Proof.** For the sake of completeness, we propose here a direct proof of the eigenvalues  $v_1$  and  $v_2$  of (2.3)-(2.5)-(2.6). By (2.6), we first easily get

$$\begin{cases} M_0 = \rho_1 + \rho_2, \\ M_1 - v_1 M_0 = \rho_2 (v_2 - v_1), \\ M_2 - v_1 M_1 = \rho_2 v_2 (v_2 - v_1), \\ M_3 - v_1 M_2 = \rho_2 v_2^2 (v_2 - v_1), \end{cases}$$

and then, setting  $\sigma_0 = v_1 v_2$  and  $\sigma_1 = -(v_1 + v_2)$ ,

$$\begin{pmatrix} M_0 & M_1 \\ M_1 & M_2 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \end{pmatrix} = - \begin{pmatrix} M_2 \\ M_3 \end{pmatrix}.$$

This system is invertible in the phase space  $\Omega$  ( $M_0M_2 - M_1^2 \neq 0$ ) and uniquely defines  $\sigma_0$  and  $\sigma_1$  with respect to  $\mathbf{M}$  :

$$\begin{pmatrix} \sigma_0 \\ \sigma_1 \end{pmatrix} = \frac{1}{M_0M_2 - M_1^2} \begin{pmatrix} M_1M_3 - M_2^2 \\ M_1M_2 - M_0M_3 \end{pmatrix}. \quad (2.7)$$

Then, we have

$$\begin{aligned} \overline{M_4} &= \rho_1 v_1^4 + \rho_2 v_2^4 \\ &= \rho_1 v_1^3 v_1 + \rho_2 v_2^3 v_2 \\ &= (\rho_1 v_1^3 + \rho_2 v_2^3)(v_1 + v_2) - (\rho_1 v_1^2 + \rho_2 v_2^2)v_1 v_2 \\ &= -M_2 \sigma_0 - M_3 \sigma_1, \end{aligned} \quad (2.8)$$

which finally gives  $\overline{M_4}$  with respect to  $\mathbf{M}$ .  
The Jacobian matrix  $\mathbf{J} = \nabla_{\mathbf{M}} \mathbf{F}$  is given by

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix} \quad \text{with} \quad \begin{cases} a = \partial_{M_0} \overline{M_4}, \\ b = \partial_{M_1} \overline{M_4}, \\ c = \partial_{M_2} \overline{M_4}, \\ d = \partial_{M_3} \overline{M_4}. \end{cases}$$

Using (2.7) and (2.8), the calculations of the last row coefficients eventually lead to

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\sigma_0^2 & -2\sigma_0\sigma_1 & -2\sigma_0 - \sigma_1^2 & -2\sigma_1 \end{pmatrix}.$$

Finally, the characteristic polynomial  $p(\lambda)$  of  $\mathbf{J}$  is easily shown to equal

$$p(\lambda) = (\lambda - v_1)^2 (\lambda - v_2)^2.$$

This concludes the proof.

### 3. Entropy conditions

In this section, we exhibit natural entropy inequalities for the following small viscosity system associated with (2.2) :

$$\begin{cases} \partial_t M_0 + \partial_x M_1 = \varepsilon \partial_{xx} M_0, \\ \partial_t M_1 + \partial_x M_2 = \varepsilon \partial_{xx} M_1, \\ \partial_t M_2 + \partial_x M_3 = \varepsilon \partial_{xx} M_2, \\ \partial_t M_3 + \partial_x M_4 = \varepsilon \partial_{xx} M_3, \end{cases} \quad (3.1)$$

which gives in condensed form

$$\partial_t \mathbf{M} + \partial_x \mathbf{F}(\mathbf{M}) = \varepsilon \partial_{xx} \mathbf{M}. \quad (3.2)$$

Throughout this section, we will consider smooth solutions only. Using clear notations, we thus have

$$\partial_t \mathbf{M} + \mathbf{J} \partial_x \mathbf{M} = \varepsilon \partial_{xx} \mathbf{M} \quad \text{with} \quad \mathbf{J} = \nabla_{\mathbf{M}} \mathbf{F}.$$

Setting  $\mathbf{A} = \nabla_{\mathbf{U}}\mathbf{M}$ , we then get

$$\partial_t \mathbf{U} + \mathbf{A}^{-1} \mathbf{J} \mathbf{A} \partial_x \mathbf{U} = \varepsilon \mathbf{A}^{-1} \partial_x (\mathbf{A} \partial_x \mathbf{U}). \quad (3.3)$$

Our objective is to prove that

$$\partial_t \eta + \partial_x q \leq \varepsilon \partial_{xx} \eta, \quad (3.4)$$

for a natural choice of couple  $(\eta, q)$  given by

$$\begin{cases} \eta = \rho_1 S(v_1) + \rho_2 S(v_2), \\ q = \rho_1 v_1 S(v_1) + \rho_2 v_2 S(v_2). \end{cases} \quad (3.5)$$

Here  $S$  denotes a convex function from  $\mathbb{R}$  to  $\mathbb{R}$ , and we will especially consider the case where  $S(v) = v^{2\alpha}$ ,  $\alpha \geq 0$ . Of course, the densities  $\rho_1$ ,  $\rho_2$  and velocities  $v_1$ ,  $v_2$  involved in (3.5) are naturally defined by means of the one-to-one and onto function  $\mathbf{U} = \mathbf{U}(\mathbf{M})$ . In the following and with a little abuse in the notations, we will consider without distinction  $\eta$  and  $q$  as functions of  $\mathbf{M}$  or  $\mathbf{U}$ .

We first observe

$$\begin{aligned} \partial_t \eta + \partial_x q &= \partial_t \eta(\mathbf{U}) + \partial_x q(\mathbf{U}) \\ &= \nabla_{\mathbf{U}} \eta \partial_t \mathbf{U} + \nabla_{\mathbf{U}} q \partial_x \mathbf{U} \\ &= \nabla_{\mathbf{U}} \eta \{ -\mathbf{A}^{-1} \mathbf{J} \mathbf{A} \partial_x \mathbf{U} + \varepsilon \mathbf{A}^{-1} \partial_x (\mathbf{A} \partial_x \mathbf{U}) \} + \nabla_{\mathbf{U}} q \partial_x \mathbf{U}. \end{aligned}$$

The following two lemmas, the proofs of which are left to the reader, will be useful in order to estimate the entropy dissipation rate  $\mathbf{D}$  defined by

$$\mathbf{D} = \nabla_{\mathbf{U}} \eta \{ -\mathbf{A}^{-1} \mathbf{J} \mathbf{A} \partial_x \mathbf{U} + \varepsilon \mathbf{A}^{-1} \partial_x (\mathbf{A} \partial_x \mathbf{U}) \} + \nabla_{\mathbf{U}} q \partial_x \mathbf{U}.$$

LEMMA 3.1. *The matrices  $\mathbf{J}$  and  $\mathbf{A}$  are given by*

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -v_1^2 v_2^2 & 2v_1 v_2 (v_1 + v_2) & -2v_1 v_2 - (v_1 + v_2)^2 & 2(v_1 + v_2) \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -v_1^2 & -v_2^2 & 2v_1 & 2v_2 \\ -2v_1^3 & -2v_2^3 & 3v_1^2 & 3v_2^2 \end{pmatrix}$$

and we have

$$\mathbf{A}^{-1} \mathbf{J} \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -v_1^2 & 0 & 2v_1 & 0 \\ 0 & -v_2^2 & 0 & 2v_2 \end{pmatrix}.$$

LEMMA 3.2. *The gradients  $\nabla_{\mathbf{U}} \eta$  and  $\nabla_{\mathbf{U}} q$  are given by*

$$\nabla_{\mathbf{U}} \eta = \begin{pmatrix} S(v_1) - v_1 S'(v_1) \\ S(v_2) - v_2 S'(v_2) \\ S'(v_1) \\ S'(v_2) \end{pmatrix}^t, \quad \nabla_{\mathbf{U}} q = \begin{pmatrix} -v_1^2 S'(v_1) \\ -v_2^2 S'(v_2) \\ S(v_1) + v_1 S'(v_1) \\ S(v_2) + v_2 S'(v_2) \end{pmatrix}^t,$$

and we have

$$\nabla_{\mathbf{U}}q = \nabla_{\mathbf{U}}\eta\mathbf{A}^{-1}\mathbf{J}\mathbf{A}.$$

Before going on, let us make the following two remarks. We first note that thanks to the first lemma, (3.3) with  $\varepsilon = 0$  gives :

$$\begin{cases} \partial_t \rho_1 + \partial_x \rho_1 v_1 = 0, \\ \partial_t \rho_2 + \partial_x \rho_2 v_2 = 0, \\ \partial_t \rho_1 v_1 - v_1^2 \partial_x \rho_1 + 2v_1 \partial_x \rho_1 v_1 = 0, \\ \partial_t \rho_2 v_2 - v_2^2 \partial_x \rho_2 + 2v_2 \partial_x \rho_2 v_2 = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \partial_t \rho_1 + \partial_x \rho_1 v_1 = 0, \\ \partial_t \rho_1 v_1 + \partial_x \rho_1 v_1^2 = 0, \\ \partial_t \rho_2 + \partial_x \rho_2 v_2 = 0, \\ \partial_t \rho_2 v_2 + \partial_x \rho_2 v_2^2 = 0. \end{cases}$$

We then observe that for smooth solutions, the system (2.3) is nothing but two decoupled pressureless gas dynamics system of equations. We then observe that still with  $\varepsilon = 0$ ,  $\mathbf{D} = 0$  by lemma 3.2, so that

$$\partial_t \eta + \partial_x q = 0.$$

Let us go back to the case  $\varepsilon > 0$ . We thus have the following equality,

$$\mathbf{D} = \varepsilon \nabla_{\mathbf{U}}\eta\mathbf{A}^{-1}\partial_x(\mathbf{A}\partial_x\mathbf{U}),$$

from which it is natural to isolate  $\varepsilon\partial_{xx}\eta$  :

$$\begin{aligned} \mathbf{D} &= \varepsilon \nabla_{\mathbf{U}}\eta\mathbf{A}^{-1}\partial_x(\mathbf{A}\partial_x\mathbf{U}) \\ &= \varepsilon \nabla_{\mathbf{U}}\eta\partial_{xx}\mathbf{U} + \varepsilon \nabla_{\mathbf{U}}\eta\mathbf{A}^{-1}\partial_x\mathbf{A}\partial_x\mathbf{U} \\ &= \varepsilon \partial_x(\nabla_{\mathbf{U}}\eta\partial_x\mathbf{U}) - \varepsilon \partial_x(\nabla_{\mathbf{U}}\eta)\partial_x\mathbf{U} + \varepsilon \nabla_{\mathbf{U}}\eta\mathbf{A}^{-1}\partial_x\mathbf{A}\partial_x\mathbf{U} \\ &= \varepsilon \partial_{xx}\eta + \varepsilon(\nabla_{\mathbf{U}}\eta\mathbf{A}^{-1}\partial_x\mathbf{A}\partial_x\mathbf{U} - \partial_x(\nabla_{\mathbf{U}}\eta)\partial_x\mathbf{U}). \end{aligned}$$

By lemma 3.2 giving  $\nabla_{\mathbf{U}}\eta$ , we easily get

$$\partial_x(\nabla_{\mathbf{U}}\eta)\partial_x\mathbf{U} = \rho_1(\partial_x v_1)^2 S''(v_1) + \rho_2(\partial_x v_2)^2 S''(v_2).$$

It is now a matter to calculate  $\nabla_{\mathbf{U}}\eta\mathbf{A}^{-1}\partial_x\mathbf{A}\partial_x\mathbf{U} = \mathbf{A}^{-t}(\nabla_{\mathbf{U}}\eta)^t\partial_x\mathbf{A}\partial_x\mathbf{U}$ . We first observe that

$$\partial_x\mathbf{A}\partial_x\mathbf{U} = \begin{pmatrix} 0 \\ 0 \\ 2\rho_1(\partial_x v_1)^2 + 2\rho_2(\partial_x v_2)^2 \\ 6\rho_1 v_1(\partial_x v_1)^2 + 6\rho_2 v_2(\partial_x v_2)^2 \end{pmatrix}$$

so that only the last two components of  $\mathbf{A}^{-t}(\nabla_{\mathbf{U}}\eta)^t$  are actually needed. Finally, easy calculations not reported here lead to

$$\mathbf{A}^{-t}(\nabla_{\mathbf{U}}\eta)^t\partial_x\mathbf{A}\partial_x\mathbf{U} = -\frac{1}{(v_1 - v_2)^4}(2\rho_1(\partial_x v_1)^2 X_1 + 2\rho_2(\partial_x v_2)^2 X_2),$$



where we have set

$$\begin{cases} X_1 = (v_1 - v_2)^2 \left( 3(S(v_1) - S(v_2)) - 2(v_1 - v_2)S'(v_1) - (v_1 - v_2)S'(v_2) \right), \\ X_2 = -(v_1 - v_2)^2 \left( 3(S(v_1) - S(v_2)) - (v_1 - v_2)S'(v_1) - 2(v_1 - v_2)S'(v_2) \right). \end{cases}$$

The entropy inequality (3.4) is then valid if and only if

$$\nabla_{\mathbf{U}} \eta \mathbf{A}^{-1} \partial_x \mathbf{A} \partial_x \mathbf{U} - \partial_x (\nabla_{\mathbf{U}} \eta) \partial_x \mathbf{U} \leq 0,$$

that is, setting  $S_i = S(v_i)$ ,  $S'_i = S'(v_i)$  and  $S''_i = S''(v_i)$ ,  $i = 1, 2$ ,

$$\begin{aligned} & (v_1 - v_2)^2 \rho_1 (\partial_x v_1)^2 \left( 6(S_1 - S_2) - 4(v_1 - v_2)S'_1 - 2(v_1 - v_2)S'_2 + (v_1 - v_2)^2 S''_1 \right) \\ & \quad + \\ & (v_1 - v_2)^2 \rho_2 (\partial_x v_2)^2 \left( -6(S_1 - S_2) + 2(v_1 - v_2)S'_1 + 4(v_1 - v_2)S'_2 + (v_1 - v_2)^2 S''_2 \right) \\ & \geq 0. \end{aligned}$$

A sufficient condition is given by

$$\begin{cases} 6(S_1 - S_2) - 4(v_1 - v_2)S'_1 - 2(v_1 - v_2)S'_2 + (v_1 - v_2)^2 S''_1 \geq 0, \\ -6(S_1 - S_2) + 2(v_1 - v_2)S'_1 + 4(v_1 - v_2)S'_2 + (v_1 - v_2)^2 S''_2 \geq 0. \end{cases} \quad (3.6)$$

Let us focus for instance on the first inequality (the second one is treated in a similar way), and let us consider the left-hand side as a function of  $v_2$ , for any given  $v_1$  :

$$\mathcal{F}_1(v_2) = 6(S_1 - S_2) - 4(v_1 - v_2)S'_1 - 2(v_1 - v_2)S'_2 + (v_1 - v_2)^2 S''_1.$$

Derivating yields

$$\begin{aligned} \mathcal{F}'_1(v_2) &= 4(S'_1 - S'_2) - 2(v_1 - v_2)(S''_1 + S''_2), \\ \mathcal{F}''_1(v_2) &= 2(S''_1 - S''_2) - 2(v_1 - v_2)S'''_2, \\ \mathcal{F}'''_1(v_2) &= 2(v_2 - v_1)S''''_2. \end{aligned}$$

It is then clear that  $\mathcal{F}_1(v_1) = \mathcal{F}'_1(v_1) = \mathcal{F}''_1(v_1) = \mathcal{F}'''_1(v_1) = 0$ . Then, provided that  $S''''(v) \geq 0$ ,  $\forall v$ , we easily get by a chain argument based on the sign of the derivative and the monotonicity property that  $\mathcal{F}_1(v_2) \geq 0$ ,  $\forall v_1, v_2$ . We have thus proved the following proposition :

**PROPOSITION 3.3.** *Smooth solutions of (3.2) satisfy the entropy inequality (3.4) for any entropy-entropy-flux pair  $(\eta, q)$  defined by (3.5) provided that  $v \rightarrow S(v)$  is a smooth function from  $\mathbb{R}$  to  $\mathbb{R}$  with nonnegative fourth-order derivative. In particular, the natural choice  $S(v) = v^{2\alpha}$  with  $\alpha \geq 2$  is suitable.*

**Remark.** Any third-order polynomial may of course be added to the leading term of  $S$ , without changing the sign of the fourth-order derivative. However, if we focus on (strictly) convex functions  $v \rightarrow S(v)$  in order to get a (strictly) convex entropy  $\eta = \eta(\mathbf{U})$ , only first-order polynomials may be added without changing the convexity property.

**Remark.** If we consider  $S(v) = 1, v, v^2, v^3$ , it is easily checked that (3.6) holds true with two equalities. In agreement with (3.1), these choices that lead to the pairs  $(\eta, q) = (M_i, M_{i+1})$ ,  $i = 0, \dots, 3$ , are admissible.

#### 4. Kinetic-macroscopic relation for smooth solutions

For smooth solutions, we established in the previous section that the four-moment model (2.2) is equivalent to the following two decoupled pressureless gas dynamics model

$$\begin{cases} \partial_t \rho_1 + \partial_x \rho_1 v_1 = 0, \\ \partial_t \rho_1 v_1 + \partial_x \rho_1 v_1^2 = 0, \\ \partial_t \rho_2 + \partial_x \rho_2 v_2 = 0, \\ \partial_t \rho_2 v_2 + \partial_x \rho_2 v_2^2 = 0, \end{cases} \quad (4.1)$$

where  $\rho_1, \rho_2, v_1$  and  $v_2$  are defined by the nonlinear system (2.6). The aim of this section is to prove a rigorous equivalence result, still for smooth solutions, between this macroscopic model and the free transport kinetic formulation (2.1) when the velocity distribution is given by a set of two Dirac delta functions. This result is nothing but a generalization of the one given in [Bouchut (2004)] for the usual pressureless gas dynamics model.

**PROPOSITION 4.1.** *Let  $T > 0$  and  $\rho_i(t, x), v_i(t, x)$  in  $\mathcal{C}^1(]0, T[ \times \mathbb{R})$  for  $i = 1, 2$ . Let us define*

$$f(t, x, v) = \sum_{i=1}^2 \rho_i(t, x) \delta(v - v_i(t, x)).$$

*Then,  $\rho_i$  and  $v_i$  solve (2.2), or equivalently (4.1), in  $]0, T[ \times \mathbb{R}$  if and only if*

$$\partial_t f + v \partial_x f = 0, \quad \text{in } ]0, T[ \times \mathbb{R} \times \mathbb{R} \quad (4.2)$$

*in the distributional sense, that is if and only if for all  $\phi \in \mathcal{C}_c^\infty(]0, T[ \times \mathbb{R})$  and  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$*

$$\int_0^T \int_{\mathbb{R}} \sum_{i=1}^2 \rho_i(t, x) \left( \partial_t \phi(t, x) + v_i(t, x) \partial_x \phi(t, x) \right) \chi(v_i(t, x)) = 0. \quad (4.3)$$

*Proof.* Let us first assume that (4.3) holds true. Since the velocity functions  $v_i$  are in particular locally bounded, one can successively choose  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  such that  $\chi(v) = v^k$ ,  $k = 0, \dots, 3$  for all  $v = v_i(t, x)$  and then get

$$\int_0^T \int_{\mathbb{R}} \sum_{i=1}^2 \rho_i(t, x) \left( \partial_t \phi(t, x) + v_i(t, x) \partial_x \phi(t, x) \right) (v_i(t, x))^k = 0$$

for all  $\phi \in \mathcal{C}_c^\infty(]0, T[ \times \mathbb{R})$ . Invoking the closure relation (2.6), this clearly gives the four-moment model (2.2), or equivalently (4.1), since  $\rho_i(t, x)$  and  $v_i(t, x)$  are smooth functions.

Conversely, let us assume that the partial differential equations of (4.1) are satisfied. Using the mass conservation equations associated with  $\rho_i$ , it is then usual to show that for  $i = 1, 2$

$$\rho_i (\partial_t v_i + v_i \partial_x v_i) = 0,$$

and then multiplying by  $\chi'$  for any smooth function  $\chi$ ,

$$\partial_t \rho_i \chi(v_i) + \partial_x \rho_i \chi(v_i) v_i = 0.$$

Summing over  $i = 1, 2$  and integrating past a test function  $\phi \in \mathcal{C}_c^\infty(]0, T[ \times \mathbb{R})$  gives the expected result (4.3). This concludes the proof.  $\square$

### 5. Riemann problems and entropy measure solutions

In this section, we focus on the Riemann problem, which is associated with the initial condition

$$\mathbf{M}(x,0) = \begin{cases} \mathbf{M}_L & \text{if } x < 0, \\ \mathbf{M}_R & \text{if } x > 0, \end{cases} \quad (5.1)$$

for two constant states  $\mathbf{M}_L$  and  $\mathbf{M}_R$  in  $\Omega$ .

The solution of (2.3)-(5.1) is sought in the form

$$\mathbf{M}(x,t) = \begin{cases} \mathbf{M}_L & \text{if } x < \sigma_L t, \\ \mathbf{M}_L^\delta(t)\delta(x - \sigma_L t) & \text{if } x = \sigma_L t, \\ \mathbf{M}_\star & \text{if } \sigma_L t < x < \sigma_R t, \\ \mathbf{M}_R^\delta(t)\delta(x - \sigma_R t) & \text{if } x = \sigma_R t, \\ \mathbf{M}_R & \text{if } x > \sigma_R t, \end{cases} \quad (5.2)$$

which corresponds to the juxtaposition of two Dirac delta functions with mass  $\mathbf{M}_\beta^\delta$  and position  $x - \sigma_\beta t$ ,  $\beta = L, R$ , and separated by a constant state  $\mathbf{M}_\star$  in  $\Omega$ . Here,  $\sigma_\beta$  denotes a real number and  $\mathbf{M}_\beta^\delta(t)$  is defined by

$$\mathbf{M}_\beta^\delta(t) = \begin{pmatrix} m_\beta(t) \\ m_\beta(t)\sigma_\beta \\ m_\beta(t)\sigma_\beta^2 \\ m_\beta(t)\sigma_\beta^3 \end{pmatrix} \quad (5.3)$$

with  $m_\beta(t) \geq 0$ ,  $m_\beta(0) = 0$  and  $\beta = L, R$ .

We introduce the following natural definitions of (entropy) measure solutions.

**DEFINITION 5.1.** *Let  $\eta = \rho_1 S(v_1) + \rho_2 S(v_2)$  and  $q = \rho_1 v_1 S(v_1) + \rho_2 v_2 S(v_2)$  with  $S(v) = v^{2\alpha}$ ,  $\alpha \in \mathbb{N}$ .*

*We say that (5.2) is a measure solution of (2.3)-(5.1) if and only if  $\partial_t \eta + \partial_x q = 0$  in  $\mathcal{D}'([0, \infty[ \times \mathbb{R})$  for  $S(v) = v^{2\alpha}$  with  $\alpha = 0, \frac{1}{2}, 1, \frac{3}{2}$ , that is for  $(\eta, q) = (M_i, M_{i+1})$ ,  $i = 0, 1, 2, 3$ .*

*We say that (5.2) is an entropic measure solution of (2.3)-(5.1) if and only if  $\partial_t \eta + \partial_x q = 0$  in  $\mathcal{D}'([0, \infty[ \times \mathbb{R})$  for  $S(v) = v^{2\alpha}$  with  $\alpha = 0, \frac{1}{2}, 1, \frac{3}{2}$  and  $\alpha \geq 2$ .*

We can prove the following equivalence result.

**THEOREM 5.2.** *We say that (5.2) is a measure solution of (2.3)-(5.1) if and only if*

$$\begin{cases} \sigma_L(\mathbf{M}_L - \mathbf{M}_\star)t - (\mathbf{F}(\mathbf{M}_L) - \mathbf{F}(\mathbf{M}_\star))t + \mathbf{M}_L^\delta(t) = 0 \\ \sigma_R(\mathbf{M}_\star - \mathbf{M}_R)t - (\mathbf{F}(\mathbf{M}_\star) - \mathbf{F}(\mathbf{M}_R))t + \mathbf{M}_R^\delta(t) = 0. \end{cases} \quad (5.4)$$

*We say that (5.2) is an entropic measure solution of (2.3)-(5.1) if and only if in addition*

$$\begin{cases} \sigma_L(\eta(\mathbf{M}_L) - \eta(\mathbf{M}_\star))t - (q(\mathbf{M}_L) - q(\mathbf{M}_\star))t + \eta(\mathbf{M}_L^\delta(t)) \leq 0 \\ \sigma_R(\eta(\mathbf{M}_\star) - \eta(\mathbf{M}_R))t - (q(\mathbf{M}_\star) - q(\mathbf{M}_R))t + \eta(\mathbf{M}_R^\delta(t)) \leq 0, \end{cases} \quad (5.5)$$

for all  $S(v) = v^{2\alpha}$  with  $\alpha \geq 2$ .

**Proof.**

We first recall that  $\partial_t \eta + \partial_x q = 0$  in  $\mathcal{D}'([0, \infty[ \times \mathbb{R})$  means that for any smooth function with compact support  $\varphi \in C_c^\infty([0, \infty[ \times \mathbb{R})$ , we have

$$\langle \partial_t \eta + \partial_x q, \varphi \rangle = - \langle \eta, \partial_t \varphi \rangle - \langle q, \partial_x \varphi \rangle =$$

$$- \int_0^\infty \int_{-\infty}^{+\infty} \eta(x, t) \partial_t \varphi(x, t) dx dt - \int_0^\infty \int_{-\infty}^{+\infty} q(x, t) \partial_x \varphi(x, t) dx dt = 0,$$

where by definition (5.2)

$$\eta(x, t) = \begin{cases} \eta(\mathbf{M}_L) & \text{if } x < \sigma_L t, \\ \eta(\mathbf{M}_L^\delta(t)) \delta(x - \sigma_L t) & \text{if } x = \sigma_L t, \\ \eta(\mathbf{M}_*) & \text{if } \sigma_L t < x < \sigma_R t, \\ \eta(\mathbf{M}_R^\delta(t)) \delta(x - \sigma_R t) & \text{if } x = \sigma_R t, \\ \eta(\mathbf{M}_R) & \text{if } x > \sigma_R t, \end{cases}$$

and

$$q(x, t) = \begin{cases} q(\mathbf{M}_L) & \text{if } x < \sigma_L t, \\ q(\mathbf{M}_L^\delta(t)) \delta(x - \sigma_L t) & \text{if } x = \sigma_L t, \\ q(\mathbf{M}_*) & \text{if } \sigma_L t < x < \sigma_R t, \\ q(\mathbf{M}_R^\delta(t)) \delta(x - \sigma_R t) & \text{if } x = \sigma_R t, \\ q(\mathbf{M}_R) & \text{if } x > \sigma_R t. \end{cases}$$

Here we note that  $q(\mathbf{M}_\beta^\delta(t)) = \sigma_\beta \eta(\mathbf{M}_\beta^\delta(t))$  with  $\eta(\mathbf{M}_\beta^\delta(t)) = m_\beta(t) S(\sigma_\beta)$ ,  $\beta = L, R$ . For all  $\varphi \in C_c^\infty([0, \infty[ \times \mathbb{R})$  we thus have

$$\begin{aligned} \langle \partial_t \eta + \partial_x q, \varphi \rangle = & \\ & - \int_0^\infty dt \int_{-\infty}^{\sigma_L t} \eta(\mathbf{M}_L) \partial_t \varphi(x, t) dx - \int_0^\infty \eta(\mathbf{M}_L^\delta(t)) \partial_t \varphi(\sigma_L t, t) dt \\ & - \int_0^\infty dt \int_{\sigma_R t}^\infty \eta(\mathbf{M}_*) \partial_t \varphi(x, t) dx - \int_0^\infty \eta(\mathbf{M}_R^\delta(t)) \partial_t \varphi(\sigma_R t, t) dt \\ & - \int_0^\infty dt \int_{\sigma_R t}^{\sigma_L t} \eta(\mathbf{M}_R) \partial_t \varphi(x, t) dx - \int_0^\infty dt \int_{-\infty}^{\sigma_L t} q(\mathbf{M}_L) \partial_t \varphi(x, t) dx \\ & - \int_0^\infty q(\mathbf{M}_L^\delta(t)) \partial_t \varphi(\sigma_L t, t) dt - \int_0^\infty dt \int_{\sigma_R t}^{\sigma_L t} q(\mathbf{M}_*) \partial_t \varphi(x, t) dx \\ & - \int_0^\infty q(\mathbf{M}_R^\delta(t)) \partial_t \varphi(\sigma_R t, t) dt - \int_0^\infty dt \int_{\sigma_R t}^{\sigma_L t} q(\mathbf{M}_R) \partial_t \varphi(x, t) dx \end{aligned}$$

that is, using in particular  $q(\mathbf{M}_\beta^\delta(t)) = \sigma_\beta \eta(\mathbf{M}_\beta^\delta(t))$ ,

$$\begin{aligned}
& \langle \partial_t \eta + \partial_x q, \varphi \rangle = \\
& = - \int_0^\infty dt \eta(\mathbf{M}_L) \left( \frac{d}{dt} \int_{-\infty}^{\sigma_L t} \varphi(x, t) dx - \sigma_L \varphi(t, \sigma_L t) \right) - \int_0^\infty q(\mathbf{M}_L) \varphi(\sigma_L t, t) dt \\
& - \int_0^\infty dt \eta(\mathbf{M}_*) \left( \frac{d}{dt} \int_{\sigma_L t}^{\sigma_R t} \varphi(x, t) dx - \sigma_R \varphi(t, \sigma_R t) + \sigma_L \varphi(t, \sigma_L t) \right) \\
& - \int_0^\infty q(\mathbf{M}_*) (\varphi(\sigma_R t, t) - \varphi(\sigma_L t, t)) dt \\
& - \int_0^\infty dt \eta(\mathbf{M}_R) \left( \frac{d}{dt} \int_{\sigma_R t}^\infty \varphi(x, t) dx + \sigma_R \varphi(t, \sigma_R t) \right) + \int_0^\infty q(\mathbf{M}_R) \varphi(\sigma_R t, t) dt \\
& - \int_0^\infty \eta(\mathbf{M}_L^\delta(t)) \frac{d}{dt} [\varphi(\sigma_L t, t)] dt - \int_0^\infty \eta(\mathbf{M}_R^\delta(t)) \frac{d}{dt} [\varphi(\sigma_R t, t)] dt \\
& = - \int_0^\infty \varphi(\sigma_L t, t) (\sigma_L \eta(\mathbf{M}_L) - q(\mathbf{M}_L)) dt + \int_0^\infty \varphi(\sigma_R t, t) (\sigma_R \eta(\mathbf{M}_*) - q(\mathbf{M}_*)) dt \\
& - \int_0^\infty \varphi(\sigma_L t, t) (\sigma_L \eta(\mathbf{M}_*) - q(\mathbf{M}_*)) dt - \int_0^\infty \varphi(\sigma_R t, t) (\sigma_R \eta(\mathbf{M}_R) - q(\mathbf{M}_R)) dt \\
& - \int_0^\infty \eta(\mathbf{M}_L^\delta(t)) \frac{d}{dt} [\varphi(\sigma_L t, t)] dt - \int_0^\infty \eta(\mathbf{M}_R^\delta(t)) \frac{d}{dt} [\varphi(\sigma_R t, t)] dt \\
& = \int_0^\infty \varphi(\sigma_L t, t) (\sigma_L (\eta(\mathbf{M}_L) - \eta(\mathbf{M}_*)) - (q(\mathbf{M}_L) - q(\mathbf{M}_*))) dt \\
& + \int_0^\infty \varphi(\sigma_R t, t) (\sigma_R (\eta(\mathbf{M}_*) - \eta(\mathbf{M}_R)) - (q(\mathbf{M}_*) - q(\mathbf{M}_R))) dt \\
& - \int_0^\infty \eta(\mathbf{M}_L^\delta(t)) \frac{d}{dt} [\varphi(\sigma_L t, t)] dt - \int_0^\infty \eta(\mathbf{M}_R^\delta(t)) \frac{d}{dt} [\varphi(\sigma_R t, t)] dt \\
& = \int_0^\infty \varphi(\sigma_L t, t) \frac{d}{dt} (\sigma_L (\eta(\mathbf{M}_L) - \eta(\mathbf{M}_*)) t - (q(\mathbf{M}_L) - q(\mathbf{M}_*)) t + \eta(\mathbf{M}_L^\delta(t))) dt \\
& + \int_0^\infty \varphi(\sigma_R t, t) \frac{d}{dt} (\sigma_R (\eta(\mathbf{M}_*) - \eta(\mathbf{M}_R)) t - (q(\mathbf{M}_*) - q(\mathbf{M}_R)) t + \eta(\mathbf{M}_R^\delta(t))) dt.
\end{aligned}$$

Then, since  $m_\beta(0) = 0$  and then  $\eta(\mathbf{M}_\beta^\delta(t)) = 0$ ,  $\beta = L, R$ , it is clear that (5.2) is a measure solution of (2.3)-(5.1) if and only if (5.4) is valid for all  $t \geq 0$ , and an entropic measure solution if and only if in addition (5.5) holds true for all  $t \geq 0$  and all  $\eta = \rho_1 S(v_1) + \rho_2 S(v_2)$  and  $q = \rho_1 v_1 S(v_1) + \rho_2 v_2 S(v_2)$  with  $S(v) = v^{2\alpha}$ ,  $\alpha \geq 2$ .

**Remark.** Let us recall that  $\eta(\mathbf{M}_\beta^\delta(t)) = m_\beta(t) S(\sigma_\beta)$ . Since (5.5) is made of equalities when  $S(v) = 1$  and  $S(v) = v$  (we get in these cases the first two components of (5.4)), the validity of (5.5) for all  $S(v) = v^{2\alpha}$ ,  $\alpha \geq 2$ , is equivalent to the validity of

$$\begin{cases} \sigma_L (\eta(\mathbf{M}_L) - \eta(\mathbf{M}_*)) - (q(\mathbf{M}_L) - q(\mathbf{M}_*)) \leq 0 \\ \sigma_R (\eta(\mathbf{M}_*) - \eta(\mathbf{M}_R)) - (q(\mathbf{M}_*) - q(\mathbf{M}_R)) \leq 0, \end{cases} \quad (5.6)$$

for all  $S(v) = v^{2\alpha} - \sigma_L^{2\alpha} \frac{v - \sigma_R}{\sigma_L - \sigma_R} - \sigma_R^{2\alpha} \frac{v - \sigma_L}{\sigma_R - \sigma_L}$ ,  $\alpha \geq 2$ .

## 6. Examples of entropic solutions

In this section, we propose two particular entropic solutions. The first one models the collision of two particles packets. We prove that in such a situation the four-moment model does not develop  $\delta$ -choc Dirac delta functions and is actually able to properly represent the crossing of the two packets. The second one models the collision of four particles packets. In this case and as expected since the number of moments is set to four, the entropic solutions involves two  $\delta$ -choc Dirac delta functions singularities.

### 6.1. Collision of two particles packets

We consider a Riemann initial data (5.1) where  $\mathbf{M}_L = \mathbf{M}(\mathbf{U}_L)$  and  $\mathbf{M}_R = \mathbf{M}(\mathbf{U}_R)$  are such that

$$\mathbf{U}_L = \frac{1}{2} \begin{pmatrix} \rho_L \\ \rho_L \\ \rho_L v_L \\ \rho_L v_L \end{pmatrix} \quad \text{and} \quad \mathbf{U}_R = \frac{1}{2} \begin{pmatrix} \rho_R \\ \rho_R \\ \rho_R v_R \\ \rho_R v_R \end{pmatrix}$$

for two given densities  $\rho_L > 0$  and  $\rho_R > 0$  and velocities  $v_L > 0$  and  $v_R < 0$ . We recall that the function  $\mathbf{M} = \mathbf{M}(\mathbf{U})$  is defined by (2.6). We define

$$\mathbf{M}(x, t) = \begin{cases} \mathbf{M}_L & \text{if } x < v_R t, \\ \mathbf{M}_\star & \text{if } v_R t < x < v_L t, \\ \mathbf{M}_R & \text{if } x > v_L t, \end{cases} \quad (6.1)$$

with  $\mathbf{M}_\star = \mathbf{M}(\mathbf{U}_\star)$  given by

$$\mathbf{U}_\star = \begin{pmatrix} \rho_L \\ \rho_R \\ \rho_L v_L \\ \rho_R v_R \end{pmatrix}.$$

Our objective here is to prove that the following Dirac delta functions free solution is an entropy solution of (2.3)-(5.1). Conditions (5.4) and (5.5) write here

$$\begin{cases} v_R(\mathbf{M}_L - \mathbf{M}_\star) - (\mathbf{F}(\mathbf{M}_L) - \mathbf{F}(\mathbf{M}_\star)) = 0 \\ v_L(\mathbf{M}_\star - \mathbf{M}_R) - (\mathbf{F}(\mathbf{M}_\star) - \mathbf{F}(\mathbf{M}_R)) = 0. \end{cases} \quad (6.2)$$

and

$$\begin{cases} v_R(\eta(\mathbf{M}_L) - \eta(\mathbf{M}_\star)) - (q(\mathbf{M}_L) - q(\mathbf{M}_\star)) \leq 0 \\ v_L(\eta(\mathbf{M}_\star) - \eta(\mathbf{M}_R)) - (q(\mathbf{M}_\star) - q(\mathbf{M}_R)) \leq 0, \end{cases} \quad (6.3)$$

with  $\eta(\mathbf{U}) = \rho_1 S(v_1) + \rho_2 S(v_2)$  and  $q(\mathbf{U}) = \rho_1 v_1 S(v_1) + \rho_2 v_2 S(v_2)$  for all  $S(v) = v^{2\alpha}$  with  $\alpha \geq 2$ . We will focus only on the first equality of (6.2) and the first inequality of (6.3), the second ones being treated similarly. We clearly have

$$\mathbf{M}_L - \mathbf{M}_\star = \begin{pmatrix} \rho_L \\ \rho_L v_L \\ \rho_L v_L^2 \\ \rho_L v_L^3 \end{pmatrix} - \begin{pmatrix} \rho_L + \rho_R \\ \rho_L v_L + \rho_R v_R \\ \rho_L v_L^2 + \rho_R v_R^2 \\ \rho_L v_L^3 + \rho_R v_R^3 \end{pmatrix} = - \begin{pmatrix} \rho_R \\ \rho_R v_R \\ \rho_R v_R^2 \\ \rho_R v_R^3 \end{pmatrix},$$

while

$$\mathbf{F}(\mathbf{M}_L) - \mathbf{F}(\mathbf{M}_\star) = \begin{pmatrix} \rho_L v_L \\ \rho_L v_L^2 \\ \rho_L v_L^3 \\ \rho_L v_L^4 \end{pmatrix} - \begin{pmatrix} \rho_L v_L + \rho_R v_R \\ \rho_L v_L^2 + \rho_R v_R^2 \\ \rho_L v_L^3 + \rho_R v_R^3 \\ \rho_L v_L^4 + \rho_R v_R^4 \end{pmatrix} = - \begin{pmatrix} \rho_R v_R \\ \rho_R v_R^2 \\ \rho_R v_R^3 \\ \rho_R v_R^4 \end{pmatrix}.$$

It is then clear that the first equality of (6.2) holds true. Let us now check that the proposed Riemann solution fulfills the entropy condition. We clearly have

$$\begin{aligned} & v_R(\eta(\mathbf{M}_L) - \eta(\mathbf{M}_\star)) - (q(\mathbf{M}_L) - q(\mathbf{M}_\star)) = \\ & v_R(\rho_L S(v_L) - (\rho_L S(v_L) + \rho_R S(v_R))) - (\rho_L v_L S(v_L) - (\rho_L v_L S(v_L) + \rho_R v_R S(v_R))) \\ & = 0, \end{aligned}$$

which allows to prove that the proposed solution is an entropic smooth solution.

### 6.2. Collision of four particles packets

We consider a Riemann initial data (5.1) where  $\mathbf{M}_L = \mathbf{M}(\mathbf{U}_L)$  and  $\mathbf{M}_R = \mathbf{M}(\mathbf{U}_R)$  are such that

$$\mathbf{U}_L = \frac{1}{2} \begin{pmatrix} \rho \\ \rho \\ \rho v_1 \\ \rho v_2 \end{pmatrix} \quad \text{and} \quad \mathbf{U}_R = \frac{1}{2} \begin{pmatrix} \rho \\ \rho \\ -\rho v_2 \\ -\rho v_1 \end{pmatrix}$$

for a given density  $\rho > 0$  and two velocities  $v_2 > v_1 > 0$ . We have

$$\begin{aligned} \mathbf{M}_L &= \frac{1}{2} \begin{pmatrix} 2\rho \\ \rho(v_1 + v_2) \\ \rho(v_1^2 + v_2^2) \\ \rho(v_1^3 + v_2^3) \end{pmatrix} \quad \text{and} \quad \mathbf{M}_R = \frac{1}{2} \begin{pmatrix} 2\rho \\ -\rho(v_1 + v_2) \\ \rho(v_1^2 + v_2^2) \\ -\rho(v_1^3 + v_2^3) \end{pmatrix} \\ \mathbf{F}(\mathbf{M}_L) &= \frac{1}{2} \begin{pmatrix} \rho(v_1 + v_2) \\ \rho(v_1^2 + v_2^2) \\ \rho(v_1^3 + v_2^3) \\ \rho(v_1^4 + v_2^4) \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{M}_R) = \frac{1}{2} \begin{pmatrix} -\rho(v_1 + v_2) \\ \rho(v_1^2 + v_2^2) \\ -\rho(v_1^3 + v_2^3) \\ \rho(v_1^4 + v_2^4) \end{pmatrix}. \end{aligned}$$

We define

$$\mathbf{M}(x, t) = \begin{cases} \mathbf{M}_L & \text{if } x < -\sigma t, \\ \mathbf{M}_L^\delta(t) \delta(x + \sigma t) & \text{if } x = -\sigma t, \\ \mathbf{M}_\star & \text{if } -\sigma t < x < \sigma t, \\ \mathbf{M}_R^\delta(t) \delta(x - \sigma t) & \text{if } x = \sigma t, \\ \mathbf{M}_R & \text{if } x > \sigma t, \end{cases} \quad (6.4)$$

with  $\sigma > 0$ ,  $\mathbf{M}_\star = \mathbf{M}(\mathbf{U}_\star)$  given by

$$\mathbf{U}_\star = \frac{1}{2} \begin{pmatrix} \rho_\star \\ \rho_\star \\ -\rho_\star v_\star \\ \rho_\star v_\star \end{pmatrix}, \quad \mathbf{M}_\star = \begin{pmatrix} \rho_\star \\ 0 \\ \rho_\star v_\star^2 \\ 0 \end{pmatrix}, \quad \rho_\star > 0, \quad v_\star > 0,$$

and  $\mathbf{M}_L^\delta(t)$ ,  $\mathbf{M}_R^\delta(t)$  given by

$$\mathbf{M}_L^\delta(t) = m(t) \begin{pmatrix} 1 \\ -\sigma \\ \sigma^2 \\ -\sigma^3 \end{pmatrix}, \quad \mathbf{M}_R^\delta(t) = m(t) \begin{pmatrix} 1 \\ \sigma \\ \sigma^2 \\ \sigma^3 \end{pmatrix},$$

with  $m(t) \geq 0$ . We have

$$\mathbf{F}(\mathbf{M}_\star) = \begin{pmatrix} 0 \\ \rho_\star v_\star^2 \\ 0 \\ \rho_\star v_\star^4 \end{pmatrix}.$$

The generalized Rankine-Hugoniot jump conditions (5.4) write here

$$\begin{cases} -\sigma(\mathbf{M}_L - \mathbf{M}_\star)t - (\mathbf{F}(\mathbf{M}_L) - \mathbf{F}(\mathbf{M}_\star))t + \mathbf{M}_L^\delta(t) = 0 \\ \sigma(\mathbf{M}_\star - \mathbf{M}_R)t - (\mathbf{F}(\mathbf{M}_\star) - \mathbf{F}(\mathbf{M}_R))t + \mathbf{M}_R^\delta(t) = 0. \end{cases}$$

that is, equivalently

$$\begin{cases} 2\sigma\mathbf{M}_\star - \sigma(\mathbf{M}_L + \mathbf{M}_R) + (\mathbf{F}(\mathbf{M}_R) - \mathbf{F}(\mathbf{M}_L)) + \frac{m(t)}{t} \begin{pmatrix} 2 \\ 0 \\ 2\sigma^2 \\ 0 \end{pmatrix} = 0, \\ 2\mathbf{F}(\mathbf{M}_\star) + \sigma(\mathbf{M}_R - \mathbf{M}_L) - (\mathbf{F}(\mathbf{M}_L) + \mathbf{F}(\mathbf{M}_L)) - \frac{m(t)}{t} \begin{pmatrix} 0 \\ 2\sigma \\ 0 \\ 2\sigma^3 \end{pmatrix} = 0. \end{cases} \quad (6.5)$$

This is made of eight equalities and we note that four are trivial (zero equals zero), so that four are left to determine the four unknowns  $\rho_\star$ ,  $v_\star$ ,  $\sigma$  and  $m(t)/t$ . We propose below to numerically solve this nonlinear system for a specific set of values for  $\rho$ ,  $v_1$  and  $v_2$ .

Regarding the entropy inequalities (5.5), we first remark that for  $S(v) = v^{2\alpha}$ ,

$$\begin{aligned} \eta(\mathbf{M}_L) &= \eta(\mathbf{M}_R) = \frac{\rho}{2}(S(v_1) + S(v_2)) \\ \eta(\mathbf{M}_\star) &= \rho_\star S(v_\star), \\ \eta(\mathbf{M}_L^\delta(t)) &= \eta(\mathbf{M}_R^\delta(t)) = m(t)S(\sigma), \end{aligned}$$

while

$$\begin{aligned} q(\mathbf{M}_L) &= -q(\mathbf{M}_R) = \frac{\rho}{2}(v_1 S(v_1) + v_2 S(v_2)) \\ q(\mathbf{M}_\star) &= 0. \end{aligned}$$

As an immediate consequence, both inequalities in (5.5) are equivalent and the entropy condition is

$$\sigma(\rho_\star S(v_\star) - \frac{\rho}{2}(S(v_1) + S(v_2))) - \frac{\rho}{2}(v_1 S(v_1) + v_2 S(v_2)) + \frac{m(t)}{t} S(\sigma) \leq 0. \quad (6.6)$$



**A concrete example.** We propose to take  $\rho = 1$ ,  $v_1 = 0.8$  and  $v_2 = 1.2$ . Numerically solving (6.5) gives  $\rho_* = 1.88265$ ,  $v_* = 1.06026$ ,  $\sigma = 0.87983$  and  $m(t)/t = 0.22342$ . If the left-hand side of (6.6), which represents the entropy dissipation rate associated with  $S(v) = v^{2\alpha}$ , is denoted  $D(\alpha)$ , a simple calculation gives for instance  $D(2) = -0.27324$ ,  $D(3) = -0.86854$ ,  $D(4) = -1.88678, \dots$ . The proposed solution (6.4) is then actually an entropy measure solution of the four-moment model. Such an exact entropic solution will be used in the following to prove the relevance of the numerical scheme proposed hereafter with respect to the exact solution when singularities are present.

## 7. Numerical simulations via kinetic schemes

This section is devoted to the discretization of (2.3)-(2.5)-(2.6). As already stated, we use as a building block a natural first-order kinetic scheme already proposed in the literature [Gosse *et al.* (2003), Desjardins *et al.* (2008)] and briefly recalled here for the sake of completeness.

Let us first introduce a time step  $\Delta t > 0$  and a space step  $\Delta x > 0$  that we assume to be constant for simplicity. We set  $\lambda = \Delta t / \Delta x$  and define the mesh interfaces  $x_{j+1/2} = j\Delta x$  for  $j \in \mathbb{Z}$ , and the intermediate times  $t^n = n\Delta t$  for  $n \in \mathbb{N}$ . In the sequel,  $\mathbf{M}_j^n$  denotes the approximate value of  $\mathbf{M}$  at time  $t^n$  and on the cell  $\mathcal{C}_j = [x_{j-1/2}, x_{j+1/2})$ . For  $n = 0$ , we set

$$\mathbf{M}_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{M}_0(x) dx, \quad j \in \mathbb{Z},$$

where  $\mathbf{M}_0(x)$  is the initial condition.

Let us now assume as given  $(\mathbf{M}_j^n)_{j \in \mathbb{Z}}$  the sequence in  $\Omega$  of approximate values at time  $t^n$ . In order to advance it to the next time level  $t^{n+1}$ , the kinetic scheme is decomposed into two steps.

*First step : transport ( $t^n \rightarrow t^{n+1-}$ )*

With clear notations, we first set  $\mathbf{U}_j^n = \mathbf{U}(\mathbf{M}_j^n)$  and define the function  $(x, v) \rightarrow f^n(x, v)$  by

$$f^n(x, v) = (\rho_1)_j^n \delta(v - (v_1)_j^n) + (\rho_2)_j^n \delta(v - (v_2)_j^n),$$

$$\forall (x, v) \in \mathcal{C}_j \times \mathbb{R}, \quad j \in \mathbb{Z}.$$

We then solve the transport equation

$$\begin{cases} \partial_t f + v \partial_x f = 0, & (x, v) \in \mathbb{R} \times \mathbb{R}, \\ f(t=0, x, v) = f^n(x, v), \end{cases} \quad (7.1)$$

the solution of which is given by  $f(t, x, v) = f^n(x - vt, v)$ .

At last, we set  $f^{n+1-}(x, v) = f^n(x - v\Delta t, v)$ .

*Second step : collapse ( $t^{n+1-} \rightarrow t^{n+1}$ )*

The first four moments at time  $t^{n+1}$  are now naturally defined by setting

$$(M_i)_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{-\infty}^{+\infty} v^i f^{n+1-}(x, v) dv dx.$$

Then, we have  $\mathbf{M}_j^{n+1} = ((M_0)_j^{n+1}, (M_1)_j^{n+1}, (M_2)_j^{n+1}, (M_3)_j^{n+1})^t$  for all  $j \in \mathbb{Z}$ , which completes the algorithm description.

**Remark.** It is easy to see that this scheme preserves the moment space  $\Omega$ , see for instance [Desjardins *et al.* (2008)].

**Remark.** Under the natural CFL condition

$$\Delta t \max_{j \in \mathbb{Z}} ((v_1)_j^n, (v_2)_j^n) \leq CFL \Delta x,$$

with  $CFL \leq 1$ , integrating (7.1) over  $(t, x, v) \in (0, \Delta t) \times \mathcal{C}_j \times \mathbb{R}$  and against  $v^i$ ,  $i = 0, \dots, 3$  easily leads to the equivalent update formula

$$\mathbf{M}_j^{n+1} = \mathbf{M}_j^n - \frac{\Delta t}{\Delta x} (\mathbf{F}_{j+1/2}^n - \mathbf{F}_{j-1/2}^n), \quad j \in \mathbb{Z},$$

where we have set  $\mathbf{F}_{j+1/2}^n = ((M_1)_{j+1/2}^n, (M_2)_{j+1/2}^n, (M_3)_{j+1/2}^n, (M_4)_{j+1/2}^n)^t$  and  $(M_i)_{j+1/2}^n = (M_i)_{j+1/2}^{n+} + (M_i)_{j+1/2}^{n-}$ , and with

$$(M_i)_{j+1/2}^{n-} = (\rho_1)_{j+1}^n \min(0, (v_1)_{j+1}^n) ((v_1)_{j+1}^n)^{i-1} + \min(0, (v_2)_{j+1}^n) (\rho_2)_{j+1}^n ((v_2)_{j+1}^n)^{i-1},$$

$$(M_i)_{j+1/2}^{n+} = (\rho_1)_j^n \max(0, (v_1)_{j+1}^n) ((v_1)_j^n)^{i-1} + (\rho_2)_j^n \max(0, (v_2)_{j+1}^n) ((v_2)_j^n)^{i-1}.$$

### 7.1. Numerical results

This section is devoted to numerical illustrations of solutions to the Riemann problems for which we have derived analytical entropic solutions in Section 6, involving two and four particle packets.

In both figures, we choose to represent  $\rho_1$  and  $v_1$  by solid lines, and  $\rho_2$  and  $v_2$  by lines with markers (circles for the analytical solution, and triangles for the numerical solution). In the representation of weights and abscissas, values have to be assigned for  $v_1$  and  $v_2$ : we thus decide that  $v_1$  is the maximum of the relative values of velocity. Figure 7.1 presents the numerical and analytical solutions for the two particle packet case with  $\rho_L = \rho_R = 1$ , and  $v_L = 1$ ,  $v_R = -1$ . The computation is run with a 1000 cell grid on the spatial domain  $[0, 1]$ , with  $CFL = 1$ . The length of each packet is 0.4 ( $\rho_1 = \rho_2 = v_1 = v_2 = 0$  for  $x \leq 0.1$  and  $x \geq 0.9$ ) and the two packets start to collide exactly at time  $t = 0$ . Moving in opposite direction once across the other, with the same speed, they then mix together and we note in particular that  $\rho_1 = \rho_2 = 1$  and  $v_1 = v_2 = 0$  in the mixing zone, see for instance the plots at time  $t = 0.1$ . As expected, they finally get separated again and we note that a perfect agreement is obtained with the exact entropic solution.

Figure 7.2 presents the numerical and analytical solutions for the four particle packet case with  $\rho_L = \rho_R = 1$ , and  $v_1 = 1.2$ ,  $v_2 = 0.8$ . The computation is run with a 1000 cell grid on the spatial domain  $[0, 1]$ , at  $CFL = 1$ . Here, we observe the presence of two Dirac delta functions as already discussed in Section 6. Again, we get a very good agreement between exact and numerical solutions. The disparities encountered between the analytical and numerical are only due to numerical diffusion, which results in a velocity definition in an extended zone of very small density. Concerning the wave propagating at velocity 0.8, the change of weight, theoretically discontinuous, is actually smooth in the numerical case. This is due to the fact that the CFL number is based on the highest value of velocity, which is 1.2 in this studied case. Meanwhile, because of the conservation of the velocity moments, the numerical velocity jump (at

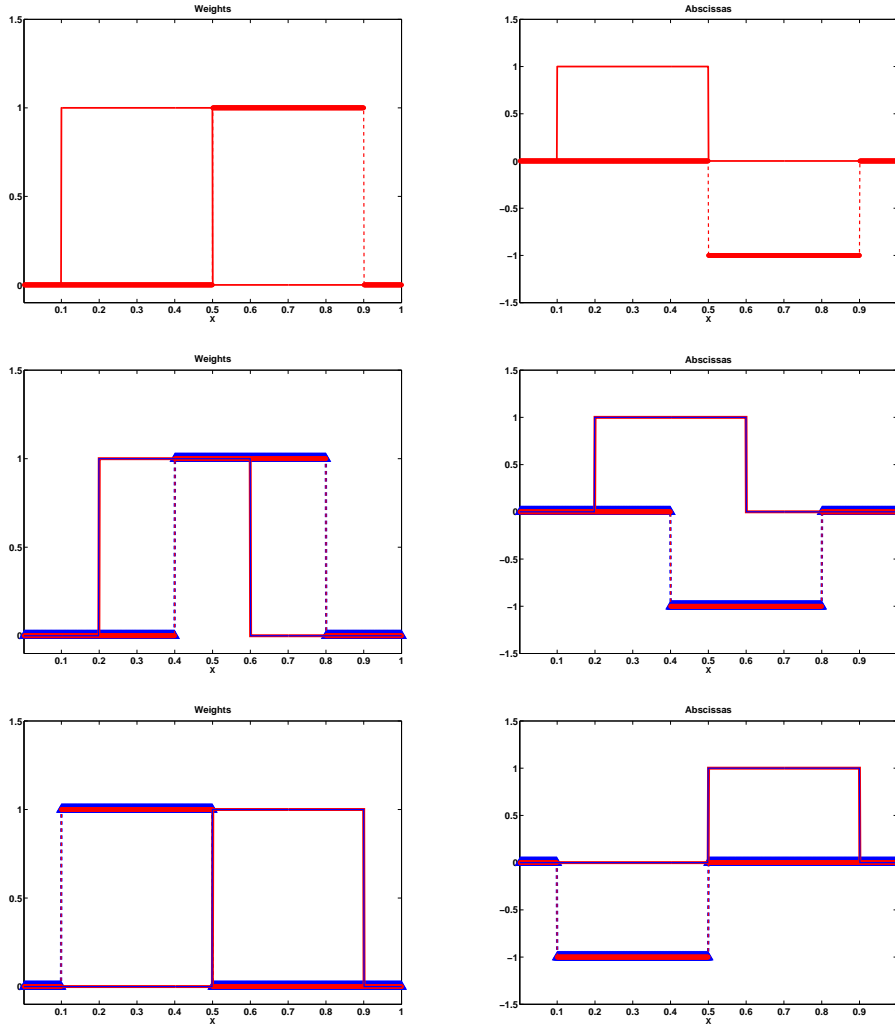


FIG. 7.1. Numerical and analytical solution for the two particle packet case with  $\rho_L = \rho_R = 1$ , and  $v_L = 1$ ,  $v_R = -1$ . The red lines represent the analytical solution, the solid line corresponds to the higher abscissa, the dashed line with circle to the lower one. The blue lines represent the numerical solution. The solid line corresponds to the weight associated with the higher abscissa, and the dashed line with triangle to the one associated with the lower abscissa. Top-left: Initial fields of weights  $\rho_1$  and  $\rho_2$ . Top-right: Initial field of abscissas  $v_1$  and  $v_2$ . Middle: Results at time  $t=0.1$ . Middle-left: Result for weights; Middle-right: Results for abscissas. Bottom: Results at time  $t=0.4$ . Bottom-left: Results for weights; Bottom-right: Results for abscissas. The computation is run in a 1000 cell grid, with  $CFL=1$ .

$x=0.154$ ) happens before the analytical velocity jump (at  $x=0.18$ ). One can here notice that the quadrature method provides the expected value of velocity in the numerical diffusion zones. The same explanation holds for the different velocity jump locations between the analytical and numerical solution at  $x=0.82$  and  $x=0.845$ . The same phenomenon is responsible for the disparities between the analytical and numerical solutions at the  $\delta$ -shocks locations, at  $x=0.4$  and  $x=0.6$ . Finally, the

analytical and numerical waves propagating at velocity 1.2 coincide, since the CFL number, set at 1, is based on this velocity value. The velocity jumps of  $v_1$  and  $v_2$  observed at locations  $x = 0.22$  corresponds to the wave propagating at velocity 1.2 and separating the constants states  $(v_1 = 0.8, v_2 = 0)$  and  $(v_1 = 1.2, v_2 = 0.8)$ . The fact that this wave leads to jump for both the abscissas is purely due to the adopted convention for the representation of  $v_1$  and  $v_2$ . The same explanation holds for the symmetric jump at location  $x = 0.78$ .

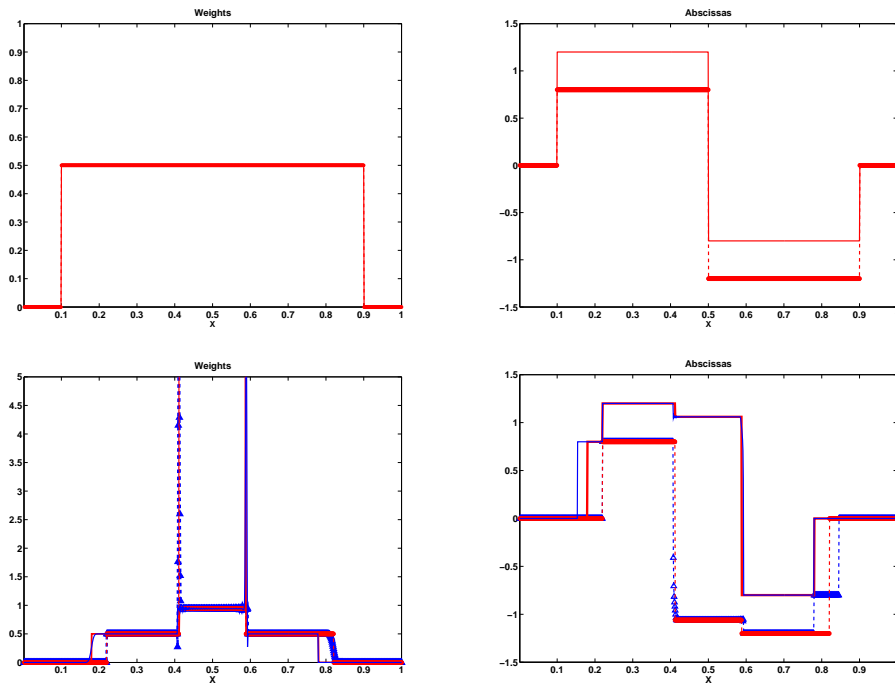


FIG. 7.2. Numerical and analytical solution for the two particle packet case with  $\rho_L = \rho_R = 1$ , and  $v_1 = 1.2$ ,  $v_2 = 0.8$ . The red lines represent the analytical solution, the solid line corresponds to the higher abscissa, the dashed line with circle to the lower one. The blue lines represent the numerical solution. The solid line corresponds to the weight associated with the higher abscissa, and the dashed line with triangle to the one associated with the lower abscissa. Top-left: Initial fields of weights  $\rho_1$  and  $\rho_2$ . Top-right: Initial field of abscissas  $v_1$  and  $v_2$ . Bottom: Results at time  $t = 0.1$ . Bottom-left: Results for weights; Bottom-right: Results for abscissas. The computation is run in a 1000 cell grid, with  $CFL = 1$ .

We have thus provided numerical simulations in the two cases for which we have at our disposal an analytical entropic solution, either in the smooth case, or in the singular case where  $\delta$ -shock measure solutions are present. In the first case, the crossing of the two droplets monokinetic packets is very properly reproduced without numerical diffusion since we work at CFL one, even if this is not symptomatic of the numerical diffusion such methods will encounter with a first order method in realistic configurations [Kah *et al.* (2010)]. In the second case, the numerical method is able to capture the creation of the measure singular solutions associated to the fact that we have limited the number of quadrature node to two and thus, as in the case of pressureless gas dynamics at a lower level, the proper physical solution, in the infinite Knudsen num-

ber limit, where the various droplet packets cross without interacting, differs from the entropic solution of the system of partial differential equations (2.2) obtained through the quadrature-based closure.

## 8. Conclusion

In this paper, we have extended the notion of entropic measure solution of quadrature-based moment method for kinetic equations. Such kinetic equations are frequently encountered in many application fields where a complex dynamics in phase space is involved. Following the contribution of [Bouchut & al (2003)] for the pressureless gas dynamics which is the one-node quadrature version of a more general system of conservation laws for quadrature-based moment models, we have been able to provide a few Riemann problem test-cases showing that the numerical solution of the resulting system of conservation laws through kinetic schemes reproduces the defined entropic solution. It is an important point for the case of PTC where the solution remains smooth and where the scheme allows to describe the phase space dynamics properly as well as for cases where the complexity of the dynamics in phase space leads to generalized  $\delta$ -shocks, as observed for pressureless gas dynamics due to the weakly hyperbolic structure of the system of conservation laws. Two stumbling blocks still remains to be treated. First, we would need a uniqueness theory and a convergence analysis in a general framework in order to fully justify the use of the kinetic schemes for the simulation of such models. However, as explained already in [Bouchut (2004)], the framework of entropic solution is not sufficient in order to provide uniqueness since one can exhibit multiple entropic solutions for measure solutions. Besides, the construction of fully high order methods is still an open question and requires further developments.

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