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On Vertex Partitions of Hypercubes by Isometric Trees

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Abstract

When $n = 2^m - 1$ M.Ramras proved, by a counting argument, that for any isometrically embedded tree T on n edges in Q_n there exists a group of translations G such that $\{g(T); g \in G\}$ is a vertex partition of Q_n . Considering a more general context we are able to give an explicit construction of G and can construct non group vertex partitions by isometric trees. We extend also this problem to vertex partition of $Q_{n'}$ by translates of an isometrically embedded tree on $n = 2^m - 1$ edges for any $n' \geq n$.

Keywords: Graph, Perfect code, Hypercube, Vertex Partition, Tiling.

1 Introduction

Twenty years ago M.Ramras [5] published a paper where he answered the following question of D. Rogers: If $n = 2^m - 1$ does the hypercube Q_n have a vertex partition into antipodal paths? M.Ramras gave an explicit construction of such a partition. For this purpose he exhibited a set of generators of a subgroup G of the group of translations $\Sigma(Q_n) \subset \text{Aut}(Q_n)$, such that the set of translates $\{g(P); g \in G\}$ of the vertex set of P of an antipodal path is the desired vertex partition.

He extended this result, proving, via a counting argument, that for any tree on $n = 2^m - 1$ edges isometrically embedded in Q_n with vertex set P there exists a subgroup G of $\Sigma(Q_n)$ such that the set of translates $\{g(P); g \in G\}$ is a vertex partition. Notice that, in the general case, the author's method does not give an explicit construction of G .

This nice result generalizes the existence of perfect single-error-correcting codes constructed first by R.W.Hamming [3]. In this case, we take as tree the star $K_{1,n}$.

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29 Notice that J.L.Vasiliev [6] constructed, for $n = 2^m - 1, n \geq 15$, perfect codes which
30 are not equivalent to linear codes, i.e. vertex partitions by stars such that the set
31 of translations G is not a subgroup, or the translate of a subgroup, of the group of
32 translations $\Sigma(Q_n)$.

33 In another series of papers M.Ramras considers edges partitions of Q_n into iso-
34 morphic trees. More recent work have been done of this subject but it seems that
35 this is not the case for vertex partitions. Both problems arise in the context of par-
36 allel computing and thus it will be interesting to improve our knowledge of vertex
37 partitions.

38 Our goal at the beginning of this work was to prove the existence of non group
39 partitions into antipodal paths. It seems also interesting, for the general case of
40 trees, to give an explicit construction of a group, or more generally a set, of trans-
41 lations G . It would be also nice, for the case of the path P_n to understand how the
42 group proposed by M.Ramras can be derived from the Hamming code. In fact we
43 found that all these problems can be easily solved, using elementary linear algebra.
44 We arrive at the conclusion that looking for a vertex partition of Q_n by translates
45 of an isometrically embedded tree on n edges is a problem independent, in some
46 sense, of the choice of the tree, thus is equivalent to looking for a perfect code.

47 2 Definitions and main result

48 Let \mathbb{F}^n be the vector space of dimension n over the finite field \mathbb{Z}_2 . The *hypercube*
49 of dimension n is the graph Q_n whose vertices are the vectors of \mathbb{F}^n , and where two
50 vertices are adjacent if they differ in exactly one coordinate.

51 The *Hamming distance* between two vectors $x, y \in \mathbb{F}^n$, $d(x, y)$ is the number of
52 coordinates in which they differ. Notice that Hamming distance is the usual graph
53 distance on Q_n .

54 The *support* of a vector x is the set $\{i \in \{1, 2, \dots, n\}; x_i \neq 0\}$. The *parity function*
55 is the function from \mathbb{F}^n to \mathbb{Z}_2 defined by $\pi(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$.

56 A *perfect code*, or more precisely a *perfect single-error-correcting code of length*
57 n is a set C of vertices of Q_n such that every vertex $x \in V(Q_n)$ is at distance at
58 most 1 of exactly one element of C .

59 Two codes C and C' are called *equivalent* if C' can be obtained from C by an
60 automorphism of Q_n , thus by applying a translation from a fixed vector and a fixed
61 permutation of the coordinates. Using a translation of all the perfect code vectors
62 by one of them we can always assume that the zero vector $\mathbf{0}$ belongs to the code.

63 Let e_1, e_2, \dots, e_n be the standard basis of \mathbb{F}^n , thus e_i denotes the vector with
64 just one single non zero coordinate position i . Let $e_0 = \mathbf{0}$ be the zero vector and let
65 $\mathbf{1} = (11 \dots 1)$. Denote by B_n the set $\{e_0, e_1, e_2, \dots, e_n\}$. The *direction* of an edge
66 xy of Q_n is the integer $i \in \{1, 2, \dots, n, \}$ such that $y = x + e_i$.

67 For a subset A of vectors of \mathbb{F}^n and a vertex x , let $x+A$ be the set $\{x+a; a \in A\}$.
68 By definition of a perfect code $C = \{c_1, c_2, \dots, c_k\}$ is a perfect code if and only if
69 the sets $c_1 + B_n, c_2 + B_n, \dots, c_k + B_n$ define a partition of \mathbb{F}^n i.e.

- 70 • $\mathbb{F}^n = c_1 + B_n \cup c_2 + B_n \cup \dots \cup c_k + B_n$
- 71 • $\forall i, j \in \{1, 2, \dots, k\} \quad c_i + B_n \cap c_j + B_n = \emptyset$

This immediately leads to a necessary condition for the existence of C , the so-called packing condition, $2^n = |C|(n + 1)$ thus $n = 2^m - 1$ for some m .

By analogy, G.Cohen, S.Litsyn, A.Vardy and G.Zémor [2] define a set S as a *tile* of \mathbb{F}^n if there exists a set $C = \{c_1, c_2, \dots, c_k\}$ such that the sets $c_1 + S, c_2 + S, \dots, c_k + S$ define a partition of \mathbb{F}^n . Notice that the definition is symmetric, C is also a tile of \mathbb{F}^n , and they call the pair (C, S) a tiling of \mathbb{F}^n .

R.W.Hamming [3] constructed, for any integer m , a linear subspace of \mathbb{F}^n , where $n = 2^m - 1$, which is a perfect code. It is easy to prove that all linear perfect codes are Hamming codes. In 1961 J.L.Vasiliev [6], and later many authors ([1, 4] for a survey) constructed perfect codes which are not linear codes.

Let $W = (v_1, v_2, \dots, v_n)$ be a basis of \mathbb{F}^n . We will denote by θ_W be the automorphism of \mathbb{F}^n defined by $\theta_W(\sum_{i=1}^n \lambda_i e_i) = (\sum_{i=1}^n \lambda_i v_i)$.

Lemma 1 *Let $V = \{v_0, v_1, \dots, v_n\}$ where $W = (v_1, v_2, \dots, v_n)$ is a a basis of \mathbb{F}^n and $v_0 = \mathbf{0}$. Then $C = \{c_1, c_2, \dots, c_k\}$ is a perfect code if and only if $\theta_W(c_1) + V, \theta_W(c_2) + V, \dots, \theta_W(c_k) + V$ is a partition of \mathbb{F}^n .*

Proof : Notice that for any $j \in \{0, 1, \dots, n\}$ we have $\theta_W(e_j) = v_j$. By linearity for any i, j we have $\theta_W(c_i) + v_j = \theta_W(c_i) + \theta_W(e_j) = \theta_W(c_i + e_j)$. Therefore, because θ_W is an automorphism, $\theta_W(c_i) + v_j = \theta_W(c_{i'}) + v_{j'}$ if and only if $c_i + e_j = c_{i'} + e_{j'}$. Assume that C is a perfect code then the $\theta_W(c_i) + V$ are disjoint. Furthermore for any x of \mathbb{F}^n we knows that there exist $i \in \{1, 2, \dots, k\}$ and $j \in \{0, 1, \dots, n\}$ such that $\theta_W^{-1}(x) = c_i + e_j$, thus $x = \theta_W(c_i) + v_j$ and we have a partition.

Conversely if $\theta_W(c_1) + V, \theta_W(c_2) + V, \dots, \theta_W(c_k) + V$ is a partition of \mathbb{F}^n then the $\theta_W(c_i) + B_n$ are disjoint. Moreover for any x of \mathbb{F}^n there exist i and j such that $\theta_W(x) = \theta_W(c_i) + v_j$ and thus $x = c_i + e_j$. \square

For two graphs G and H an *isometric embedding* of G in H is a map $\alpha : V(G) \mapsto V(H)$ which preserves distance. By extension we will denote by $\alpha(G)$ the subgraph of H induced by $\alpha(V(G))$. If G is injectively embedded in Q_n we will say that there exists a *vertex partition of Q_n by G* if there exists a tiling of $V(Q_n)$ by $\alpha(V(G))$. It is immediate to check, as noticed by M.Ramras [5], that a tree T is isometrically embedded in Q_n if and only if no edges of $\alpha(T)$ use the same direction. If α is an isometric embedding in Q_n then for any translation t the map $\alpha' = \alpha + t$ is also an isometric embedding. Therefore, if a graph G is isometrically embeddable in Q_n then, for any vertex x of G there exists an isometric embedding such that $\alpha(x) = \mathbf{0}$.

Lemma 2 *Let T be any tree on $p \leq n$ edges, and let α be an isometric embedding of T in Q_n . Assume $\alpha(T) = \{\mathbf{0}, v_1, v_2, \dots, v_p\}$ then the vectors v_1, v_2, \dots, v_p are linearly independent.*

Proof : The proof is by induction on p . The result is clearly true when $p = 1$ and assume it holds for any tree on $p - 1$ edges. Let x be a terminal vertex of T and let xy be the edge of T incident to x . Consider the tree T' obtained by deletion of x from T . We can always assume that $\alpha(x) \neq \mathbf{0}$ thus $\alpha(x) = v_i$ for some $i \in \{1, 2, \dots, p\}$. Let $j \in \{1, 2, \dots, n\}$ such that $\alpha(x) = \alpha(y) + e_j$. The restriction to T' of α is an isometric embedding thus the vectors $\{v_k; k \in \{1, 2, \dots, p\}, k \neq i\}$ are independent by induction hypothesis. Notice that e_j does not appear in the

basis decomposition of the $\{v_k; k \in \{1, 2, \dots, p\}, k \neq i\}$. But $\alpha(y)$ belongs to this set and because $v_i = \alpha(y) + e_j$ the vector v_i is also linearly independent of them.

□

Theorem 3 Let T be any tree on $n = 2^m - 1$ edges, and let α be an isometric embedding of T in Q_n . Assume $\alpha(T) = \{\mathbf{0}, v_1, v_2, \dots, v_n\}$. Then the vectors v_1, v_2, \dots, v_n form a basis W of \mathbb{F}^n . Furthermore if $C = \{c_1, c_2, \dots, c_k\}$ is a perfect code of Q_n then $\{\theta_W(c_1), \theta_W(c_2), \dots, \theta_W(c_k)\}$ defines a vertex partition of Q_n by the embedded tree T . All vertex partitions of Q_n by $\alpha(T)$ can be obtained by this way.

Proof : By lemma 2 W is a basis of \mathbb{F}^n and the result follows by lemma 1.

Conversely if $S = \{s_1, s_2, \dots, s_k\}$ defines a vertex partition of Q_n by T then $\{\theta_W^{-1}(s_1), \theta_W^{-1}(s_2), \dots, \theta_W^{-1}(s_k)\}$ is a perfect code and thus all vertex partition of Q_n by T arise in this way.

□

Notice that the set $D = \{\theta_W(c_1), \theta_W(c_2), \dots, \theta_W(c_k)\}$ is a linear subspace if and only if C is linear. Furthermore if b_1, b_2, \dots, b_p is a basis of C then $\theta_W(b_1), \theta_W(b_2), \dots, \theta_W(b_p)$ will be a basis of D .

Consider now a vertex partition of $Q_{n'}$ by translates of an isometrically embedded tree on n edges for some $n' \geq n$. By the packing condition, $2^{n'} = (n + 1)|D|$, thus $n = 2^m - 1$ for some m .

Corollary 4 Let T be any tree on $n = 2^m - 1$ edges, and let α be an isometric embedding of T in $Q_{n'}$, $n' \geq n$. Then there exists a vertex partition of $Q_{n'}$ by translates of the embedded tree $\alpha(T)$.

Proof : The vertices of $\alpha(T) = \{\mathbf{0}, v_1, v_2, \dots, v_n\}$ define a subspace of dimension n . By a permutation of coordinates we can assume that this subspace is $Vect(e_1, e_2, \dots, e_n)$ thus there exist a vertex partition of Q_n with set of translation say D . Then $D \cup \{e_{n+1}, e_{n+2}, \dots, e_{n'}\}$ define a vertex partition of $Q_{n'}$ by translates of $\alpha(T)$.

□

3 An example: antipodal paths

The *antipodal vertex* of a vertex x in Q_n is the unique vertex \bar{x} at distance n of x . Notice that $\bar{x} = x + \mathbf{1}$. An *antipodal path* is a path in Q_n of n edges connecting some pair of antipodal vertices. We will say that an isometric embedding $\alpha(P)$ of an antipodal path in Q_n is *canonical* if $v_0 = \mathbf{0}$ and along the path the directions used are $1, 2, \dots, n$ in this order. We have thus, for any $i \in \{0, 1, \dots, n\}$, $v_i = \sum_{j=0}^i e_j$ and $\theta_W(\sum_{i=1}^n \lambda_i e_i) = (\sum_{i=1}^n \lambda_i \sum_{j=1}^i e_j)$. By a translation and a permutation of the coordinates we can always assume that an isometric embedding of P is canonical.

A vector $u \in \mathbb{F}^n$ is of *type 01*, respectively of *type 10*, if there exists $i_0 \in \{0, 1, \dots, n\}$ such that $u = \sum_{i=i_0+1}^n e_i$, respectively $u = \sum_{i=1}^{i_0} e_i$. Notice that $\mathbf{1}$ and $\mathbf{0}$ are the only vectors of both types.

Lemma 5 If u and v are two distinct vectors both of type 10, or both of type 01, then u and v differ by a set of consecutive coordinates.

155 **Proof :** Assume first $u = \sum_{i=1}^{i_0} e_i$ and $v = \sum_{i=1}^{j_0} e_i$ for some $i_0, j_0 \in \{0, 1, \dots, n\}$
 156 Assume w.l.o.g. that $i_0 < j_0$ we have thus $v = u + \sum_{i=i_0+1}^{j_0} e_i$. For the second case
 157 notice that if u is a vector of type 01 then $u + \mathbf{1}$ is of type 10. \square

158 **Lemma 6** *Let $\alpha(P)$ be an isometrically embedded antipodal path in Q_n such that*
 159 *$\forall i \in \{0, 1, \dots, n\} v_i = \sum_{j=0}^i e_j$. Then a subset C of Q_n define a vertex partition by*
 160 *translates of $\alpha(P)$ if and only if*

161 (i) $2^n = |C|(n+1)$ and

162 (ii) No pair of elements of C differ by a set of consecutive coordinates.

163 **Proof :** Consider two translates of the embedded path say, $x + \alpha(P)$ and $y + \alpha(P)$.
 164 If z is a common vertex of the two paths then for some $i, j \in \{0, 1, \dots, n\}$ we have
 165 $z = x + \sum_{k=1}^i e_k$ and $z = y + \sum_{k=1}^j e_k$. Thus by lemma 5 condition (ii) implies that
 166 the paths $x + \alpha(P)$ and $y + \alpha(P)$ are disjoint. By condition (i) every vertex belongs
 167 to some path. \square

168 If we assume that C is a linear subspace, the last condition is equivalent to the
 169 condition used by Ramras:

170 (ii') No element of C has as support a non-empty set of consecutive integers.

171 Let us recall Vasiliev's construction of perfect codes.

172 **Theorem 7** (Vasiliev [6]) *Let C_n be a perfect code of length n . Assume $\mathbf{0} \in C_n$ and*
 173 *let λ be a function from C_n to \mathbb{Z}_2 such that $\lambda(\mathbf{0}) = 0$ and π be the parity function.*
 174 *Then the set $C_{2n+1} = \{(x, \pi(x) + \lambda(c), x + c); x \in \mathbb{F}^n, c \in C_n\}$ is a perfect code of*
 175 *length $2n + 1$.*

176 Notice that if there exists $u, v \in C_n$ such that $\lambda(u + v) \neq \lambda(u) + \lambda(v)$ then C_{2n+1}
 177 is not equivalent to any linear code. Such a function λ exists when $|C_n| > 2$ thus
 178 there exists non linear codes when $n = 2^m - 1, n \geq 15$. If $\lambda(u) = 0$ for any $u \in C$
 179 we obtain the classical inductive construction of Hamming codes.

180 **Theorem 8** *Let $\alpha(P)$ be a canonical isometric embedding of an antipodal path in*
 181 *Q_n and assume that D_n defines a vertex partition by translates of $\alpha(P)$. Let γ*
 182 *be a function from D_n to \mathbb{Z}_2 such that $\gamma(\mathbf{0}) = 0$. Let Γ be the function from*
 183 *D_n to \mathbb{F}^n defined by $\Gamma(d) = \mathbf{0}$ if $\gamma(d) = 0$ and $\Gamma(d) = \mathbf{1}$ otherwise. Then the set*
 184 *$D_{2n+1} = \{(y, \gamma(d), y + d + \Gamma(d)); y \in \mathbb{F}^n, d \in D_n\}$ defines a vertex partition of \mathbb{F}^{2n+1}*
 185 *by isometrically embedded antipodal paths.*

186 **Proof :** Let us start by a direct proof using lemma 6. Notice first that $|D_{2n+1}|(2n +$
 187 $2) = |D_n|2^n(2n + 2) = |D_n|(n + 1)2^{n+1} = 2^{2n+1}$. Consider two vectors of D_{2n+1} say
 188 $u = (y, \gamma(d), y + d + \Gamma(d))$ and $u' = (y', \gamma(d'), y' + d' + \Gamma(d'))$. Assume that u and
 189 u' differ by a set of consecutive coordinates.

190 • $\gamma(d) = \gamma(d')$. We have $y = y'$ or $y + d = y' + d'$.

191 – If $y = y'$ then $u + u' = (\mathbf{0}, 0, d + d')$. But $d, d' \in D_n$ thus by lemma 6
 192 $d = d'$ and $u = u'$.

194 – If $y + d = y' + d'$ then $u + u' = (y + y', 0, \mathbf{0})$. But $y + y' = d + d'$ and again
 195 by lemma 6 $d = d'$, $y = y'$ and $u = u'$.

196 • $\gamma(d) \neq \gamma(d')$ thus $u + u' = (y + y', 1, d + y + d' + y' + \mathbf{1})$. Therefore $y + y'$ is
 197 of type 01 and $d + y + d' + y' + \mathbf{1}$ must be of type 10, thus $d + y + d' + y'$ of
 198 type 01. But $d + d' = (y + y') + (d + y + d' + y')$ then by lemma 5 d and d'
 199 must differ by a set of consecutive coordinates; a contradiction with lemma 6.

200 □

201 We will now deduce theorem 8 from theorem 7 showing that this construction
 202 is in fact the analogue of Vasiliev's construction.

203

204 **Alternative proof :** Let $W = (v_1, v_2, \dots, v_n)$ and $W' = (v_1, v_2, \dots, v_{2n+1})$,
 205 where $v_i = \sum_{j=1}^i e_j$. By theorem 3 D_n is obtained from a perfect code of length n
 206 $C_n = \{\theta_W^{-1}(d); d \in D_n\}$. Let $\lambda : C_n \mapsto \mathbb{Z}_2$ defined by $\lambda(c) = \gamma(\theta_W(c))$.
 207 Consider $C_{2n+1} = \{(\theta_W^{-1}(y), \pi(\theta_W^{-1}(y)) + \lambda(\theta_W^{-1}(d)), \theta_W^{-1}(y) + \theta_W^{-1}(d)); y \in \mathbb{F}^n, d \in D_n\}$.
 208 By theorem 7 C_{2n+1} is a perfect code of length $2n + 1$ and thus by theorem 3 D_{2n+1}
 209 defines a vertex partition of \mathbb{F}^{2n+1} by isometrically embedded antipodal paths where
 210 $D_{2n+1} = \{\theta_{W'}(c); c \in C_{2n+1}\}$.

211 Notice that, for any $x \in \mathbb{F}^n$ and any $a \in \mathbb{Z}_2$ we have:

212
$$\theta_{W'}(x, 0, \mathbf{0}) = (\theta_W(x), 0, \mathbf{0}) + \pi(x) \cdot (\mathbf{0}, 1, \mathbf{1})$$

213
$$\theta_{W'}(\mathbf{0}, 0, x) = (\mathbf{0}, 0, \theta_W(x))$$

214
$$\theta_{W'}(\mathbf{0}, a, \mathbf{0}) = a \cdot (\mathbf{0}, 1, \mathbf{1}).$$

215 Therefore:

216
$$\theta_{W'}(\theta_W^{-1}(y), 0, \mathbf{0}) = (y, 0, \mathbf{0}) + \pi(\theta_W^{-1}(y)) \cdot (\mathbf{0}, 1, \mathbf{1}),$$

217
$$\theta_{W'}(\theta_W^{-1}(\mathbf{0}, \pi(\theta_W^{-1}(y)) + \lambda(\theta_W^{-1}(d))), \mathbf{0}) = [\pi(\theta_W^{-1}(y)) + \gamma(d)] \cdot (\mathbf{0}, 1, \mathbf{1})$$

218 and $\theta_{W'}(\mathbf{0}, 0, \theta_W^{-1}(y) + \theta_W^{-1}(d)) = (\mathbf{0}, 0, y + d)$.

219 By linearity of $\theta_{W'}$ we obtain the expression of D_{2n+1} .

220 □

221 Here also, if there exists $u, v \in D_n$ such that $\gamma(u + v) \neq \gamma(u) + \gamma(v)$ then D_{2n+1}
 222 is not a linear subspace. Notice that, if we set $\gamma(u) = 0$ for any $u \in D_n$, we obtain
 223 a recursive construction of the linear subspace proposed by M.Ramras [5].

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