On perfect codes in Cartesian product of graphs

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Abstract

Assuming the existence of a partition in perfect codes of the vertex set of a finite or infinite bipartite graph G we give the construction of a perfect code in the Cartesian product $G \square G \square P_2$. Such a partition is easily obtained in the case of perfect codes in Abelian Cayley graphs and we give some example of applications of this result and its generalizations.

Keywords: Graph, Perfect code, Cartesian product.

1 Introduction

Hamming and Golay [9, 13] constructed perfect binary single-error correcting codes of length n where $n = 2^p - 1$ for some integer p. Perfect codes played a central role in the fast growing of error-correcting codes theory.

Later Biggs [3] and Kratochvíl [18] proposed the study of the existence of perfect codes in graphs. From this point of view Hamming codes are perfect codes in the hypercube Q_n.

Infinite classes of graphs with perfect codes have been constructed by Cameron, Thas and Payne [4], Thas [22], Hammond [14] and others. The existence of perfect codes have also been proved in Towers of Hanoi graphs [6] and Sierpinski graph [16].

Dejter and Serra [7] give a construction tool to produce various infinite families of graphs with perfect codes. All graphs constructed this way are, for some chosen n, Cayley graphs of degree n on the symmetric group S_{n+1} thus are of order $(n + 1)!$. These families include star graphs, for which the existence of perfect codes was already proved by Arumugam and Kala [2] and pancake graphs.

Perfect codes have also been studied in infinite graphs. For example Golomb and Welsh [10, 11] considered the multi dimensional rectangular grid $\mathbb{Z}^n$. More

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recently Dorbec and Mollard [8] studied the existence of perfect codes in $\mathbb{Z}^n \Box Q_k$ thus a common generalization of the hypercube Q_k and the grid $\mathbb{Z}^n$.

Recently many authors investigated also perfect codes in direct [17, 24], strong [1] and lexicographic [21] product of graphs.

We will focus on the Cartesian product.

Hamming codes are classically constructed using linear algebra. Vasiliev [23] and later many authors [15, 5] constructed other families of perfect codes in Q_n. Most of these constructions start from a more geometrical point of view, the fact that Hamming codes can also be constructed recursively in multiple ways. Assuming that there exists a perfect code in Q_n they deduce the existence of perfect codes in Q_{2n+1}. We will generalize to regular graphs one of these constructions, the so called doubling construction independently found by Soloveva [20] and Phelps[19].

2 Notations and code-coloring

For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ two graphs, the *Cartesian product* $G_1 \Box G_2$ is the graph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \Box G_2)$ if and only if $x_1 y_1 \in E_1$ and $x_2 = y_2$ or $x_2 y_2 \in E_2$ and $x_1 = y_1$. We will use the notation G^n for the graph $G \Box G \Box \ldots \Box G$ (n times).

The *hypercube* of dimension n is the graph Q_n whose vertices are the words of length n over the alphabet $\{0, 1\}$, and where two vertices are adjacent if they differ in exactly one place. Notice that Q_1 is P_2 the path with 2 vertices and that $Q_{n+1} = Q_n \Box P_2$.

The *infinite grid* $\mathbb{Z}^n$ is the graph whose vertices are the words of length n over the alphabet $\mathbb{Z}$ and where two vertices are adjacent if and only if they differ by 1 in exactly one place. Notice that if we denote by P_∞ the two way infinite path, we have also $\mathbb{Z}^1 = P_\infty$ and $\mathbb{Z}^{n+1} = \mathbb{Z}^n \Box P_\infty$.

For two vertices x and y, we will denote by $d(x, y)$ the classical *distance* on graphs.

The set of *neighbors* of any vertex x in G is $N_G(x) = \{y \in V(G)/xy \in E(G)\}$.

For an integer r and vertex u, we call *ball of radius* r centered on u the set of vertices v such that $d(u, v) \leq r$.

In this paper, we will only consider balls of radius 1 and thus call them simply *balls*.

In a graph, a *single error correcting code* (or *code* for shorter) is a subset C of the set of vertices $V(G)$ such that any two vertices of C are at distance at least 3. This is equivalent to say that the balls centered on these vertices are disjoint sets.

We say that a vertex u *dominates* a vertex v if v belongs to the ball centered on u. A subset S of $V(G)$ is called a *dominating set* if every vertex of G is dominated by at least one vertex of S.

A code is said to be *perfect* if it also a dominating set. It equivalently means that the balls centered on code vertices form a partition of the vertex set.

We will call *code-coloring* of a regular graph of degree n a labeling c of the vertices with $\{0, 1, \ldots, n\}$ such that the neighbors of any vertex u are colored with all distinct colors from $\{0, 1, \ldots, n\} \backslash \{c(u)\}$.

Proposition 1 *For all $i \in \{0, 1, \ldots, n\}$ the set of vertices colored i in a code-coloring of a regular graph is a perfect code.*

Proof : By definition vertices of the same color cannot be at distance 1 or 2. Furthermore if a vertex is not colored i then one of its neighbors is labeled with this color. Thus for every i, the set C_i of vertices colored i is both a code and a dominating set. $\square$

It is clear that conversely from a partition of the vertex set of a graph in perfect codes we obtain a code-coloring of this graph.

We will call *extended code-coloring* of a regular graph of degree n a labeling c of the vertices with $\{0, 1, \ldots, n\}$ such that:

- The neighbors of any vertex u colored 0 are colored with all distinct colors $\{1, \ldots, n\}$.
- The neighbors of any vertex u colored with a color in $\{1, \ldots, n\}$ are colored 0.

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Proposition 2 *Let G be a regular graph of degree n and c be an extended code-coloring of G. Then G is bipartite and for all $i \in \{1, \ldots, n\}$ the set of vertices colored i is a code.*

Proof : By definition of c the set of vertices colored 0 and the set of vertices colored with a color in $\{1, \ldots, n\}$ define a bipartition of G. Let u and v be two vertices colored with colors in $\{1, \ldots, n\}$. Then u and v cannot be adjacent or at distance 3. Assume u and v are at distance 2 and let w be a common neighbor of them. The vertex w must be colored 0 thus $c(u) \neq c(v)$. Therefore if $c(u) = c(v)$ the vertices u and v are at distance at least 4. $\square$

Proposition 3 *Assume that there exists a code-coloring in a bipartite regular graph G then there exists an extended code-coloring in $G \square P_2$.*

Proof : It will be convenient to see the vertices of P_2 as the elements of the set $\{0, 1\}$. Let $c : V(G) \mapsto \{0, \ldots, n\}$ be a code-coloring of G and $P : V(G) \mapsto \{0, 1\}$ be a proper 2-coloring of the graph G. Let $c' : V(G \square P_2) \mapsto \{0, \ldots, n+1\}$ be the labeling defined by:

- If $P(x) = 0$ then $c'((x, 0)) = c(x) + 1$ and $c'((x, 1)) = 0$.
- If $P(x) = 1$ then $c'((x, 0)) = 0$ and $c'((x, 1)) = c(x) + 1$.

We claim that c' is an extended code-coloring of $G \square P_2$.

Indeed if a vertex $(x, 0)$ is labeled 0 then we have $P(x) = 1$. The neighbors of $(x, 0)$ are $(x, 1)$ (labeled $c(x) + 1$) and the $(y, 0)$ with $y \in N_G(x)$. For all these vertices $P(y) = 0$ thus $c'((y, 0)) = c(y) + 1$. Therefore the $(y, 0)$ are labeled with all distinct colors from $\{1, \ldots, n+1\} \backslash \{c(x) + 1\}$.

If a vertex $(x, 0)$ is labeled with a color in $\{1, \ldots, n+1\}$ then $P(x) = 0$. The neighbors of $(x, 0)$ are $(x, 1)$ (labeled 0) and the $(y, 0)$ with $y \in N_G(x)$. For all these vertices $P(y) = 1$ thus $c'((y, 0)) = 0$.

The two remaining cases corresponding to vertices of type $(x,1)$ are similar to the two first ones and left to the reader. $\Box$

In error-correcting codes theory, extended perfect codes are codes of Q_{n+1} formed from a perfect code C of Q_n by adding to every word of C an overall parity check bit. In her doubling construction Soloveva [20] uses partitions of the vertex set of Q_n and Q_{n+1} by perfect codes and extended perfect codes, respectively. Our definition of extended code-coloring generalizes, in some sense, to regular graphs the notion of partition of the set of even vertices of Q_{n+1} by extended perfect codes.

However notice that there exist extended code-colorings for graphs non decomposable as some $G\Box P_2$. For example there exists an extended code-coloring in the bipartite complete graph $K_{n,n}$. However for a bipartite graph G the existence of an extended code-coloring in $G\Box P_2$ is equivalent to the existence of a code-coloring in G.

Proposition 4 *Let G be a regular graph of degree n. Assume that there exists an extended code-coloring in $G\Box P_2$ then G is bipartite and there exists a code-coloring in G.*

Proof : Let $c : V(G\Box P_2) \mapsto \{0,\dots,n+1\}$ be an extended code-coloring of $G\Box P_2$. Notice that for all $x \in V(G)$ we have $c((x,0)) = 0$ if and only if $c((x,1)) \neq 0$.

It is immediate to verify that the sets $\{x \in V(G)/c((x,0)) = 0\}$ and $\{x \in V(G)/c((x,0)) \neq 0\}$ define a bipartition of G.

Let $c' : V(G) \mapsto \{0,\dots,n\}$ be the labeling defined by:

- $c'(x) = c((x,0)) - 1$ if $c((x,0)) \neq 0$
- $c'(x) = c((x,1)) - 1$ if $c((x,0)) = 0$

c' is a code-coloring of G.

Indeed let $x \in V(G)$. Without loose of generality we can assume $c((x,0)) \neq 0$; we will deduce the case $c((x,1)) \neq 0$ by symmetry.

We have then $c((x,1)) = 0$ and $\{c((x',1))/x' \in N_G(x)\} = \{1,\dots,n+1\}\backslash\{c((x,0))\}$.

If $x' \in N_G(x)$ we have $c((x',0)) = 0$ thus from the definition of $c'(x')$ we obtain $\{c'(x')/x' \in N_G(x)\} = \{0,\dots,n\}\backslash\{c((x,0)) - 1\} = \{0,\dots,n\}\backslash\{c'(x)\}$.

$\Box$

Let G be a regular graph and let c be an extended code-coloring of $G\Box P_2$. The *dual coloring* $\tilde{c}$ is defined for any vertex (x,ϵ) with $x \in V(G)$ and $\epsilon \in \{0,1\}$ by $\tilde{c}((x,\epsilon)) = c((x,1-\epsilon))$.

Proposition 5 *Let G be a regular graph. The dual coloring $\tilde{c}$ of an extended code-coloring c of $G\Box P_2$ is also an extended code-coloring .*

Proof : Consider the bijection from $V(G\Box P_2)$ to itself defined for any vertex (x,ϵ), with $x \in V(G)$, by $\theta(x,\epsilon) = (x,1-\epsilon)$. Then θ is the graph automorphism of $G\Box P_2$ corresponding to the exchange of two G-layers. The proposition follows by the fact that $\tilde{c}$ is the composition of θ and c. $\Box$

3 Main results and applications

Theorem 6 *Let G and H be two finite or infinite regular graphs of degree respectively n and $n+1$. If there exist a code-coloring in G and an extended code-coloring in H then there exists a perfect code in $G \square H$.*

Proof : Let $c : V(G) \mapsto \{0, \dots, n\}$ be a code-coloring of G and $c' : V(H) \mapsto \{0, \dots, n+1\}$ be an extended code-coloring of H. Consider the set D of vertices of $G \square H$ defined by $D = \{(x, y) \ / x \in V(G), y \in V(H), c'(y) = c(x) + 1\}$. Notice that if $c'(y) = 0$ there is no vertex x of G such that $(x, y) \in D$.

We will prove first that for any distinct vertices (x, y) and (x', y') of D we have $d((x, y), (x', y')) > 2$.

- If $x = x'$ then $c'(y) = c'(y') = c(x) + 1$. But $c'(y) = c'(y')$ is in $\{1, \dots, n+1\}$ thus by proposition 2 $d(y, y') \geq 3$.
- If $d(x, x') = 1$ then $c(x) \neq c(x')$ thus $c'(y) \neq c'(y')$ and $y \neq y'$. But $c'(y)$ and $c'(y')$ are in $\{1, \dots, n+1\}$ thus $d(y, y') \geq 2$.
- If $d(x, x') = 2$ then again $c(x) \neq c(x')$ thus $y \neq y'$.
- If $d(x, x') \geq 3$ we are done.

Thus D is a code and if G and H are finite graphs, by cardinality arguments, we can prove that D is a perfect code. However we consider also infinite graphs, therefore we will prove directly that D is a dominating set. Let (x, y) be a vertex of $G \square H$ not in D.

- if $c'(y) = 0$. There exists a vertex y' of $N_H(y)$ with color $c'(y') = c(x) + 1$. The vertex (x, y') is in D and dominates (x, y).
- if $c'(y) \in \{1, \dots, n+1\}$ then $c(x) \neq c'(y) - 1$ and thus there exists a vertex x' of $N_G(x)$ with color $c(x') = c'(y) - 1$. The vertex (x', y) is in D and dominates (x, y).

□

Let G and H be two finite or infinite regular graphs of the same degree n. Assume that H is bipartite and that there exists a code-coloring in G and H then, using the previous theorem and proposition 3, there exists a perfect code in the graph $G \square H \square P_2$. In fact we can prove directly the following stronger result.

Theorem 7 *Let G and H be two finite or infinite regular graphs of the same degree n. Assume that H is bipartite and that there exists a code-coloring in G and H then there exists a code-coloring, thus a partition in perfect codes, in the graph $G \square H \square P_2$.*

Proof : Let $c : V(G) \mapsto \{0, \dots, n\}$ be a code-coloring of G and $c' : V(H \square P_2) \mapsto \{0, \dots, n+1\}$ be the extended code-coloring of $H \square P_2$ deduced from the code-coloring of H. Consider $\tilde{c}'$ the dual of c'.

Let (x, y) with $x \in G, y \in H \square P_2$, be a vertex of $G \square H \square P_2$. Let us define the color $e(x, y)$ by

- If $c'(y) \neq 0$ then $e(x, y) = c(x) - c'(y) \ mod(n+1)$
- If $c'(y) = 0$ then $e(x, y) = \tilde{e}(x, y) + n + 1$ where $\tilde{e}(x, y) = c(x) - \tilde{c}'(y) \ mod(n+1)$.

Notice that in the first (respectively in the second) case $e(x,y)$ belongs to $\{0,\ldots,n\}$ (respectively to $\{n+1,\ldots,2n+1\}$). Furthermore the degree of $G\Box H\Box P_2$ is $2n+1$. Let us verify that e is a code-coloring.

- If $e(x,y)$ belongs to $\{0,\ldots,n\}$ then $c'(y)\neq 0$. Consider the possible neighbors of (x,y).

  We have $\{e(x',y)/x'\in N_G(x)\}=\{c(x')-c'(y)\ mod(n+1)/x'\in N_G(x)\}$. But $\{c(x')/x'\in N_G(x)\}=\{0,\ldots,n\}\backslash\{c(x)\}$ thus $\{c(x')-c'(y)\ mod(n+1)/x'\in N_G(x)\}=\{0,\ldots,n\}\backslash\{e(x,y)\}$.

  Consider now a vertex (x,y') such that $y'\in N_{H\Box P_2}(y)$. We have $c'(y')=0$. But $\{\tilde{c}'(y')/y'\in N_{H\Box P_2}(y)\}=\{1,\ldots,n+1\}$ thus $\{\tilde{e}(x,y')/y'\in N_{H\Box P_2}(y)\}=\{0,\ldots,n\}$ and $\{e(x,y')/y'\in N_{H\Box P_2}(y)\}=\{n+1,\ldots,2n+1\}$.

- If $e(x,y)$ belongs to $\{n+1,\ldots,2n+1\}$ then $c'(y)=0$. From $\{c(x')/x'\in N_G(x)\}=\{0,\ldots,n\}\backslash\{c(x)\}$ we deduce $\{c(x')-\tilde{c}'(y)\ mod(n+1)/x'\in N_G(x)\}=\{0,\ldots,n\}\backslash\{c(x)-\tilde{c}'(y)\}$ and thus $\{e(x',y)/x'\in N_G(x)\}=\{n+1,\ldots,2n+1\}\backslash\{e(x,y)\}$.

  We have also $\{c'(y')/y'\in N_{H\Box P_2}(y)\}=\{1,\ldots,n+1\}$ thus $\{e(x,y')/y'\in N_{H\Box P_2}(y)\}=\{0,\ldots,n\}$.

□

It is easy to prove that there is no perfect code in $K_3\Box K_3\Box P_2$. Thus we cannot drop the condition that H is bipartite in theorem 7.

We will often use this theorem in the particular case $G=H$. Using our construction recursively we obtain

Corollary 8 *Let G be a bipartite graph. If there exists a code-coloring in G then for all integer k there exists a code-coloring, thus a partition in perfect codes, in the graph $G^{2^k}\Box P_2^{2^k-1}=G^{2^k}\Box Q_{2^k-1}$.*

Let Γ be a group, S a finite set of elements of Γ such that $1\notin S$ and $S^{-1}=\{s^{-1}/s\in S\}=S$. The undirected *Cayley graph* $G=Cay(\Gamma,S)$ over Γ with *connection set* S has vertex set $V(G)=\Gamma$ and edge set $E(G)=\{\{a,b\}:a^{-1}b\in S\}$. This graph is regular of degree $|S|$. When the group Γ is commutative $Cay(\Gamma,S)$ is called an *Abelian Cayley graph.* Notice that the Cartesian product of two Cayley graphs is a Cayley graph over the direct product of the two groups thus is Abelian if the factors are Abelian. From a perfect code it is easy to construct a code-coloring in the particular case of Abelian Cayley graph.

Lemma 9 *Let G be an Abelian Cayley graph then if there exists a perfect code in G there exists a code-coloring of G.*

Proof :

Let C be a perfect code in $G=Cay(\Gamma,S)$ and $s_1,s_2,\ldots,s_n$ be the elements of S. Consider the coloring c of $V(G)$ define by

- If $x\in C$ then $c(x)=0$.
- If $x\notin C$ then let u be the unique element of C which dominates x. Then $u^{-1}x=s_i$ for some unique i in $\{1,\ldots,n\}$. Let $c(x)=i$.

The coloring c is a code-coloring.

- Assume first that $c(x) = 0$. Then the neighbors of x in G are the xs_i, $i \in \{1, \ldots, n\}$ and are all dominated by x thus are of color i.
- Assume now that $c(x) = j$ for some $j \neq 0$ and let u be the vertex of C which dominates x. We have thus $x = us_j$. Let k be such that $s_k = s_j^{-1}$ and let $K = \{1, \ldots, n\} \backslash \{k\}$. The neighbors of x are u and the $\{xs_i \;/\; i \in K\}$.

  Consider one of these vertices $x' = xs_i$. The color of x' cannot be 0 because x' is not in C. It is not j because this would imply the existence of v in C with $x' = vs_j$ and using $x' = xs_i$, $x = us_j$ this gives $v = us_i$ thus $d(u, v) = 1$.

  Assume now that two neighbors of x say $x' = xs_i$ and $x'' = xs_h$ for some $i, h \in K$ are of the same color d. This would imply the existence of two distinct vertices v and w in C with $s_d = v^{-1}x' = w^{-1}x''$. Then using $x' = xs_i$ and $x'' = xs_h$ this would imply $v^{-1}s_i = w^{-1}s_h$ thus the existence of a vertex at distance 1 of both v and w and this is not possible because C is a code. Thus the neighbors of x are colored with all distinct colors from $\{0, 1, \ldots, n\} \backslash \{j\}$.

□

We will now give some examples of use of our construction.

Consider for $p \geq 3$ the *p-crown graph* $\tilde{K}_{p,p}$ obtained from the bipartite complete graph $K_{p,p}$ after deletion of the edges of a perfect matching. This graph is an Abelian Cayley graph $G = Cay(\Gamma, S)$ with $\Gamma = \mathbb{Z}_p \times \mathbb{Z}_2$ and $S = \{(1, 1), (2, 1), \ldots, (p-1, 1)\}$. By construction each pair of vertices of the removed matching is a perfect code.

The graph $\tilde{K}_{4,4}$ is Q_3 and we obtain the construction of Hamming codes in hypercubes.

The graph $\tilde{K}_{3,3}$ is C_6 and we obtain a construction of perfect codes in $C_6^{2^k} \Box Q_{2^k-1}$. On $\mathbb{Z}^n \Box Q_k$, we will say that a code is *i-periodical* ($i \in \{1, \ldots, n\}$) if there exists a positive integer p_i (called the i-period) such that for any vertex $x = x_1x_2 \ldots x_nv$ ($\forall i, x_i \in \mathbb{Z}, v \in V(Q_k)$), the vertex $x_1x_2 \ldots x_{i-1}(x_i + p_i)x_{i+1} \ldots x_nv$ is in the code if and only if x is in the code. We thus obtain a perfect code on $\mathbb{Z}^{2^k} \Box Q_{2^k-1}$ of i-period 6 for all $i \in \{1, \ldots, 2^k\}$.

Other values of $p \geq 5$ give infinite families of graphs with perfect codes.

Golomb and Welsh [10, 11] proved the existence of perfect codes in the grid $\mathbb{Z}^n$. This is again a bipartite Abelian Cayley graph. Thus for all integer k there exists a perfect code in $\mathbb{Z}^{n2^k} \Box Q_{2^k-1}$. More precisely the codes of Golomb and Welsh are of i-period $2n+1$ thus they induce a code-coloring in the bipartite graph C_{4n+2}^n. Therefore there exists in $\mathbb{Z}^{n2^k} \Box Q_{2^k-1}$ a perfect code of i-period $4n+2$ for all $i \in \{1, \ldots, n2^k\}$.

Our work [8] about the possible parameters values for the existence of perfect codes in $\mathbb{Z}^n \Box Q_k$ can be completed by the following direct consequence of theorem 7.

Corollary 10 *Let a, b, c be integers such that $b \geq 2c$. Then if there exist perfect codes in $\mathbb{Z}^a \Box Q_b$ and in $\mathbb{Z}^{a+c} \Box Q_{b-2c}$ then there exists a perfect code in $\mathbb{Z}^{2a+c} \Box Q_{2b-2c+1}$.*

For any integer $p \geq 1$ consider the bipartite complete graph $K_{p,p}$. Label 0 the p vertices of one of the independent sets, and $\{1, \ldots, p\}$ the p other vertices. We obtain an extended code-coloring of $K_{p,p}$. Assume that $p = 2^q$ for some integer $q \geq 0$. Then by theorem 6 there exists a perfect code in $K_{2^q,2^q} \Box Q_{2^q-1}$. But this is a bipartite Abelian Cayley Graph thus by lemma 9 there exits a code-coloring of $K_{2^q,2^q} \Box Q_{2^q-1}$. Using again corollary 8 we obtain the following result.

Corollary 11 *For any integers $q, k \geq 0$ there exists a perfect code in $K^{2^k}_{2^q,2^q} \Box Q_{2^{q+k}-1}$.*

There exists a trivial code-coloring of the complete graph K_n. Thus by theorem 6 there exists a perfect code in $K_{n,n} \Box K_n$. But this is also an Abelian Cayley Graph thus we obtain a code-coloring of $K_{n,n} \Box K_n$. Using the same idea with $\tilde{K}_{n,n}$ and $\mathbb{Z}^n$ we obtain

Corollary 12 *For any integer $n \geq 1$ there exist code-colorings of $K_{n,n} \Box K_n$, $K_{n,n} \Box \tilde{K}_{n,n}$ and $K_{2n+1,2n+1} \Box \mathbb{Z}^n$.*

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