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# On perfect codes in Cartesian product of graphs

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## Abstract

Assuming the existence of a partition in perfect codes of the vertex set of a finite or infinite bipartite graph  $G$  we give the construction of a perfect code in the Cartesian product  $G \square G \square P_2$ . Such a partition is easily obtained in the case of perfect codes in Abelian Cayley graphs and we give some example of applications of this result and its generalizations.

**Keywords:** Graph, Perfect code, Cartesian product.

## 1 Introduction

Hamming and Golay [9, 13] constructed perfect binary single-error correcting codes of length  $n$  where  $n = 2^p - 1$  for some integer  $p$ . Perfect codes played a central role in the fast growing of error-correcting codes theory.

Later Biggs [3] and Kratochvíl [18] proposed the study of the existence of perfect codes in graphs. From this point of view Hamming codes are perfect codes in the hypercube  $Q_n$ .

Infinite classes of graphs with perfect codes have been constructed by Cameron, Thas and Payne [4], Thas [22], Hammond [14] and others. The existence of perfect codes have also been proved in Towers of Hanoi graphs [6] and Sierpinski graph [16].

Dejter and Serra [7] give a construction tool to produce various infinite families of graphs with perfect codes. All graphs constructed this way are, for some chosen  $n$ , Cayley graphs of degree  $n$  on the symmetric group  $S_{n+1}$  thus are of order  $(n+1)!$ . These families include star graphs, for which the existence of perfect codes was already proved by Arumugam and Kala [2] and pancake graphs.

Perfect codes have also been studied in infinite graphs. For example Golomb and Welsh [10, 11] considered the multi dimensional rectangular grid  $\mathbb{Z}^n$ . More

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recently Dorbec and Mollard [8] studied the existence of perfect codes in  $\mathbb{Z}^n \square Q_k$  thus a common generalization of the hypercube  $Q_k$  and the grid  $\mathbb{Z}^n$ .

Recently many authors investigated also perfect codes in direct [17, 24], strong [1] and lexicographic [21] product of graphs.

We will focus on the Cartesian product.

Hamming codes are classically constructed using linear algebra. Vasiliev [23] and later many authors [15, 5] constructed other families of perfect codes in  $Q_n$ . Most of these constructions start from a more geometrical point of view, the fact that Hamming codes can also be constructed recursively in multiple ways. Assuming that there exists a perfect code in  $Q_n$  they deduce the existence of perfect codes in  $Q_{2n+1}$ . We will generalize to regular graphs one of these constructions, the so called doubling construction independently found by Soloveva [20] and Phelps[19].

## 2 Notations and code-coloring

For  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  two graphs, the *Cartesian product*  $G_1 \square G_2$  is the graph with vertex set  $V_1 \times V_2$  and  $(x_1, x_2)(y_1, y_2) \in E(G_1 \square G_2)$  if and only if  $x_1 y_1 \in E_1$  and  $x_2 = y_2$  or  $x_2 y_2 \in E_2$  and  $x_1 = y_1$ . We will use the notation  $G^n$  for the graph  $G \square G \square \dots \square G$  ( $n$  times).

The *hypercube* of dimension  $n$  is the graph  $Q_n$  whose vertices are the words of length  $n$  over the alphabet  $\{0, 1\}$ , and where two vertices are adjacent if they differ in exactly one place. Notice that  $Q_1$  is  $P_2$  the path with 2 vertices and that  $Q_{n+1} = Q_n \square P_2$ .

The *infinite grid*  $\mathbb{Z}^n$  is the graph whose vertices are the words of length  $n$  over the alphabet  $\mathbb{Z}$  and where two vertices are adjacent if and only if they differ by 1 in exactly one place. Notice that if we denote by  $P_\infty$  the two way infinite path, we have also  $\mathbb{Z}^1 = P_\infty$  and  $\mathbb{Z}^{n+1} = \mathbb{Z}^n \square P_\infty$ .

For two vertices  $x$  and  $y$ , we will denote by  $d(x, y)$  the classical *distance* on graphs.

The set of *neighbors* of any vertex  $x$  in  $G$  is  $N_G(x) = \{y \in V(G) / xy \in E(G)\}$ .

For an integer  $r$  and vertex  $u$ , we call *ball of radius  $r$*  centered on  $u$  the set of vertices  $v$  such that  $d(u, v) \leq r$ .

In this paper, we will only consider balls of radius 1 and thus call them simply *balls*.

In a graph, a *single error correcting code* (or *code* for shorter) is a subset  $C$  of the set of vertices  $V(G)$  such that any two vertices of  $C$  are at distance at least 3. This is equivalent to say that the balls centered on these vertices are disjoint sets.

We say that a vertex  $u$  *dominates* a vertex  $v$  if  $v$  belongs to the ball centered on  $u$ . A subset  $S$  of  $V(G)$  is called a *dominating set* if every vertex of  $G$  is dominated by at least one vertex of  $S$ .

A code is said to be *perfect* if it also a dominating set. It equivalently means that the balls centered on code vertices form a partition of the vertex set.

We will call *code-coloring* of a regular graph of degree  $n$  a labeling  $c$  of the vertices with  $\{0, 1, \dots, n\}$  such that the neighbors of any vertex  $u$  are colored with all distinct colors from  $\{0, 1, \dots, n\} \setminus \{c(u)\}$ .

**Proposition 1** For all  $i \in \{0, 1, \dots, n\}$  the set of vertices colored  $i$  in a code-coloring of a regular graph is a perfect code.

**Proof :** By definition vertices of the same color cannot be at distance 1 or 2. Furthermore if a vertex is not colored  $i$  then one of its neighbors is labeled with this color. Thus for every  $i$ , the set  $C_i$  of vertices colored  $i$  is both a code and a dominating set.  $\square$

It is clear that conversely from a partition of the vertex set of a graph in perfect codes we obtain a code-coloring of this graph.

We will call *extended code-coloring* of a regular graph of degree  $n$  a labeling  $c$  of the vertices with  $\{0, 1, \dots, n\}$  such that:

- The neighbors of any vertex  $u$  colored 0 are colored with all distinct colors  $\{1, \dots, n\}$ .
- The neighbors of any vertex  $u$  colored with a color in  $\{1, \dots, n\}$  are colored 0.

**Proposition 2** Let  $G$  be a regular graph of degree  $n$  and  $c$  be an extended code-coloring of  $G$ . Then  $G$  is bipartite and for all  $i \in \{1, \dots, n\}$  the set of vertices colored  $i$  is a code.

**Proof :** By definition of  $c$  the set of vertices colored 0 and the set of vertices colored with a color in  $\{1, \dots, n\}$  define a bipartition of  $G$ . Let  $u$  and  $v$  be two vertices colored with colors in  $\{1, \dots, n\}$ . Then  $u$  and  $v$  cannot be adjacent or at distance 3. Assume  $u$  and  $v$  are at distance 2 and let  $w$  be a common neighbor of them. The vertex  $w$  must be colored 0 thus  $c(u) \neq c(v)$ . Therefore if  $c(u) = c(v)$  the vertices  $u$  and  $v$  are at distance at least 4.  $\square$

**Proposition 3** Assume that there exists a code-coloring in a bipartite regular graph  $G$  then there exists an extended code-coloring in  $G \square P_2$ .

**Proof :** It will be convenient to see the vertices of  $P_2$  as the elements of the set  $\{0, 1\}$ . Let  $c : V(G) \mapsto \{0, \dots, n\}$  be a code-coloring of  $G$  and  $P : V(G) \mapsto \{0, 1\}$  be a proper 2-coloring of the graph  $G$ . Let  $c' : V(G \square P_2) \mapsto \{0, \dots, n+1\}$  be the labeling defined by:

- If  $P(x) = 0$  then  $c'((x, 0)) = c(x) + 1$  and  $c'((x, 1)) = 0$ .
- If  $P(x) = 1$  then  $c'((x, 0)) = 0$  and  $c'((x, 1)) = c(x) + 1$ .

We claim that  $c'$  is an extended code-coloring of  $G \square P_2$ .

Indeed if a vertex  $(x, 0)$  is labeled 0 then we have  $P(x) = 1$ . The neighbors of  $(x, 0)$  are  $(x, 1)$  (labeled  $c(x) + 1$ ) and the  $(y, 0)$  with  $y \in N_G(x)$ . For all these vertices  $P(y) = 0$  thus  $c'((y, 0)) = c(y) + 1$ . Therefore the  $(y, 0)$  are labeled with all distinct colors from  $\{1, \dots, n+1\} \setminus \{c(x) + 1\}$ .

If a vertex  $(x, 0)$  is labeled with a color in  $\{1, \dots, n+1\}$  then  $P(x) = 0$ . The neighbors of  $(x, 0)$  are  $(x, 1)$  (labeled 0) and the  $(y, 0)$  with  $y \in N_G(x)$ . For all these vertices  $P(y) = 1$  thus  $c'((y, 0)) = 0$ .

The two remaining cases corresponding to vertices of type  $(x, 1)$  are similar to the two first ones and left to the reader.  $\square$

In error-correcting codes theory, extended perfect codes are codes of  $Q_{n+1}$  formed from a perfect code  $C$  of  $Q_n$  by adding to every word of  $C$  an overall parity check bit. In her doubling construction Soloveva [20] uses partitions of the vertex set of  $Q_n$  and  $Q_{n+1}$  by perfect codes and extended perfect codes, respectively. Our definition of extended code-coloring generalizes, in some sense, to regular graphs the notion of partition of the set of even vertices of  $Q_{n+1}$  by extended perfect codes.

However notice that there exist extended code-colorings for graphs non decomposable as some  $G \square P_2$ . For example there exists an extended code-coloring in the bipartite complete graph  $K_{n,n}$ . However for a bipartite graph  $G$  the existence of an extended code-coloring in  $G \square P_2$  is equivalent to the existence of a code-coloring in  $G$ .

**Proposition 4** *Let  $G$  be a regular graph of degree  $n$ . Assume that there exists an extended code-coloring in  $G \square P_2$  then  $G$  is bipartite and there exists a code-coloring in  $G$ .*

**Proof :** Let  $c : V(G \square P_2) \mapsto \{0, \dots, n+1\}$  be an extended code-coloring of  $G \square P_2$ . Notice that for all  $x \in V(G)$  we have  $c((x, 0)) = 0$  if and only if  $c((x, 1)) \neq 0$ .

It is immediate to verify that the sets  $\{x \in V(G) / c((x, 0)) = 0\}$  and  $\{x \in V(G) / c((x, 0)) \neq 0\}$  define a bipartition of  $G$ .

Let  $c' : V(G) \mapsto \{0, \dots, n\}$  be the labeling defined by:

- $c'(x) = c((x, 0)) - 1$  if  $c((x, 0)) \neq 0$
- $c'(x) = c((x, 1)) - 1$  if  $c((x, 0)) = 0$

$c'$  is a code-coloring of  $G$ .

Indeed let  $x \in V(G)$ . Without loose of generality we can assume  $c((x, 0)) \neq 0$ ; we will deduce the case  $c((x, 1)) \neq 0$  by symmetry.

We have then  $c((x, 1)) = 0$  and  $\{c((x', 1)) / x' \in N_G(x)\} = \{1, \dots, n+1\} \setminus \{c((x, 0))\}$ .

If  $x' \in N_G(x)$  we have  $c((x', 0)) = 0$  thus from the definition of  $c'(x')$  we obtain  $\{c'(x') / x' \in N_G(x)\} = \{0, \dots, n\} \setminus \{c((x, 0)) - 1\} = \{0, \dots, n\} \setminus \{c'(x)\}$ .  $\square$

Let  $G$  be a regular graph and let  $c$  be an extended code-coloring of  $G \square P_2$ . The dual coloring  $\tilde{c}$  is defined for any vertex  $(x, \epsilon)$  with  $x \in V(G)$  and  $\epsilon \in \{0, 1\}$  by  $\tilde{c}((x, \epsilon)) = c((x, 1 - \epsilon))$ .

**Proposition 5** *Let  $G$  be a regular graph. The dual coloring  $\tilde{c}$  of an extended code-coloring  $c$  of  $G \square P_2$  is also an extended code-coloring .*

**Proof :** Consider the bijection from  $V(G \square P_2)$  to itself defined for any vertex  $(x, \epsilon)$ , with  $x \in V(G)$ , by  $\theta(x, \epsilon) = (x, 1 - \epsilon)$ . Then  $\theta$  is the graph automorphism of  $G \square P_2$  corresponding to the exchange of two G-layers. The proposition follows by the fact that  $\tilde{c}$  is the composition of  $\theta$  and  $c$ .  $\square$

### 3 Main results and applications

**Theorem 6** *Let  $G$  and  $H$  be two finite or infinite regular graphs of degree respectively  $n$  and  $n+1$ . If there exist a code-coloring in  $G$  and an extended code-coloring in  $H$  then there exists a perfect code in  $G \square H$ .*

**Proof :** Let  $c : V(G) \mapsto \{0, \dots, n\}$  be a code-coloring of  $G$  and  $c' : V(H) \mapsto \{0, \dots, n+1\}$  be an extended code-coloring of  $H$ . Consider the set  $D$  of vertices of  $G \square H$  defined by  $D = \{(x, y) \mid x \in V(G), y \in V(H), c'(y) = c(x) + 1\}$ . Notice that if  $c'(y) = 0$  there is no vertex  $x$  of  $G$  such that  $(x, y) \in D$ .

We will prove first that for any distinct vertices  $(x, y)$  and  $(x', y')$  of  $D$  we have  $d((x, y), (x', y')) > 2$ .

- If  $x = x'$  then  $c'(y) = c'(y') = c(x) + 1$ . But  $c'(y) = c'(y')$  is in  $\{1, \dots, n+1\}$  thus by proposition 2  $d(y, y') \geq 3$ .
- If  $d(x, x') = 1$  then  $c(x) \neq c(x')$  thus  $c'(y) \neq c'(y')$  and  $y \neq y'$ . But  $c'(y)$  and  $c'(y')$  are in  $\{1, \dots, n+1\}$  thus  $d(y, y') \geq 2$ .
- If  $d(x, x') = 2$  then again  $c(x) \neq c(x')$  thus  $y \neq y'$ .
- If  $d(x, x') \geq 3$  we are done.

Thus  $D$  is a code and if  $G$  and  $H$  are finite graphs, by cardinality arguments, we can prove that  $D$  is a perfect code. However we consider also infinite graphs, therefore we will prove directly that  $D$  is a dominating set. Let  $(x, y)$  be a vertex of  $G \square H$  not in  $D$ .

- if  $c'(y) = 0$ . There exists a vertex  $y'$  of  $N_H(y)$  with color  $c'(y') = c(x) + 1$ . The vertex  $(x, y')$  is in  $D$  and dominates  $(x, y)$ .
- if  $c'(y) \in \{1, \dots, n+1\}$  then  $c(x) \neq c'(y) - 1$  and thus there exists a vertex  $x'$  of  $N_G(x)$  with color  $c(x') = c'(y) - 1$ . The vertex  $(x', y)$  is in  $D$  and dominates  $(x, y)$ .

□

Let  $G$  and  $H$  be two finite or infinite regular graphs of the same degree  $n$ . Assume that  $H$  is bipartite and that there exists a code-coloring in  $G$  and  $H$  then, using the previous theorem and proposition 3, there exists a perfect code in the graph  $G \square H \square P_2$ . In fact we can prove directly the following stronger result.

**Theorem 7** *Let  $G$  and  $H$  be two finite or infinite regular graphs of the same degree  $n$ . Assume that  $H$  is bipartite and that there exists a code-coloring in  $G$  and  $H$  then there exists a code-coloring, thus a partition in perfect codes, in the graph  $G \square H \square P_2$ .*

**Proof :** Let  $c : V(G) \mapsto \{0, \dots, n\}$  be a code-coloring of  $G$  and  $c' : V(H \square P_2) \mapsto \{0, \dots, n+1\}$  be the extended code-coloring of  $H \square P_2$  deduced from the code-coloring of  $H$ . Consider  $\check{c}'$  the dual of  $c'$ .

Let  $(x, y)$  with  $x \in G, y \in H \square P_2$ , be a vertex of  $G \square H \square P_2$ . Let us define the color  $e(x, y)$  by

- If  $c'(y) \neq 0$  then  $e(x, y) = c(x) - c'(y) \pmod{n+1}$
- If  $c'(y) = 0$  then  $e(x, y) = \tilde{e}(x, y) + n + 1$  where  $\tilde{e}(x, y) = c(x) - \check{c}'(y) \pmod{n+1}$ .

Notice that in the first (respectively in the second) case  $e(x, y)$  belongs to  $\{0, \dots, n\}$  (respectively to  $\{n+1, \dots, 2n+1\}$ ). Furthermore the degree of  $G \square H \square P_2$  is  $2n+1$ . Let us verify that  $e$  is a code-coloring.

- If  $e(x, y)$  belongs to  $\{0, \dots, n\}$  then  $c'(y) \neq 0$ . Consider the possible neighbors of  $(x, y)$ .

We have  $\{e(x', y)/x' \in N_G(x)\} = \{c(x') - c'(y) \bmod(n+1)/x' \in N_G(x)\}$ . But  $\{c(x')/x' \in N_G(x)\} = \{0, \dots, n\} \setminus \{c(x)\}$  thus  $\{c(x') - c'(y) \bmod(n+1)/x' \in N_G(x)\} = \{0, \dots, n\} \setminus \{e(x, y)\}$ .

Consider now a vertex  $(x, y')$  such that  $y' \in N_{H \square P_2}(y)$ . We have  $c'(y') = 0$ . But  $\{c'(y')/y' \in N_{H \square P_2}(y)\} = \{1, \dots, n+1\}$  thus  $\{\tilde{e}(x, y')/y' \in N_{H \square P_2}(y)\} = \{0, \dots, n\}$  and  $\{e(x, y')/y' \in N_{H \square P_2}(y)\} = \{n+1, \dots, 2n+1\}$ .

- If  $e(x, y)$  belongs to  $\{n+1, \dots, 2n+1\}$  then  $c'(y) = 0$ . From  $\{c(x')/x' \in N_G(x)\} = \{0, \dots, n\} \setminus \{c(x)\}$  we deduce  $\{c(x') - c'(y) \bmod(n+1)/x' \in N_G(x)\} = \{0, \dots, n\} \setminus \{c(x) - c'(y)\}$  and thus  $\{e(x', y)/x' \in N_G(x)\} = \{n+1, \dots, 2n+1\} \setminus \{e(x, y)\}$ .

We have also  $\{c'(y')/y' \in N_{H \square P_2}(y)\} = \{1, \dots, n+1\}$  thus  $\{e(x, y')/y' \in N_{H \square P_2}(y)\} = \{0, \dots, n\}$ .

□

It is easy to prove that there is no perfect code in  $K_3 \square K_3 \square P_2$ . Thus we cannot drop the condition that  $H$  is bipartite in theorem 7.

We will often use this theorem in the particular case  $G = H$ . Using our construction recursively we obtain

**Corollary 8** *Let  $G$  be a bipartite graph. If there exists a code-coloring in  $G$  then for all integer  $k$  there exists a code-coloring, thus a partition in perfect codes, in the graph  $G^{2^k} \square P_2^{2^k-1} = G^{2^k} \square Q_{2^k-1}$ .*

Let  $\Gamma$  be a group,  $S$  a finite set of elements of  $\Gamma$  such that  $1 \notin S$  and  $S^{-1} = \{s^{-1}/s \in S\} = S$ . The undirected *Cayley graph*  $G = \text{Cay}(\Gamma, S)$  over  $\Gamma$  with *connection set*  $S$  has vertex set  $V(G) = \Gamma$  and edge set  $E(G) = \{\{a, b\} : a^{-1}b \in S\}$ . This graph is regular of degree  $|S|$ . When the group  $\Gamma$  is commutative  $\text{Cay}(\Gamma, S)$  is called an *Abelian Cayley graph*. Notice that the Cartesian product of two Cayley graphs is a Cayley graph over the direct product of the two groups thus is Abelian if the factors are Abelian. From a perfect code it is easy to construct a code-coloring in the particular case of Abelian Cayley graph.

**Lemma 9** *Let  $G$  be an Abelian Cayley graph then if there exists a perfect code in  $G$  there exists a code-coloring of  $G$ .*

**Proof :**

Let  $C$  be a perfect code in  $G = \text{Cay}(\Gamma, S)$  and  $s_1, s_2, \dots, s_n$  be the elements of  $S$ . Consider the coloring  $c$  of  $V(G)$  define by

- If  $x \in C$  then  $c(x) = 0$ .
- If  $x \notin C$  then let  $u$  be the unique element of  $C$  which dominates  $x$ . Then  $u^{-1}x = s_i$  for some unique  $i$  in  $\{1, \dots, n\}$ . Let  $c(x) = i$ .

The coloring  $c$  is a code-coloring.

- Assume first that  $c(x) = 0$ . Then the neighbors of  $x$  in  $G$  are the  $xs_i$ ,  $i \in \{1, \dots, n\}$  and are all dominated by  $x$  thus are of color  $i$ .
- Assume now that  $c(x) = j$  for some  $j \neq 0$  and let  $u$  be the vertex of  $C$  which dominates  $x$ . We have thus  $x = us_j$ . Let  $k$  be such that  $s_k = s_j^{-1}$  and let  $K = \{1, \dots, n\} \setminus \{k\}$ . The neighbors of  $x$  are  $u$  and the  $\{xs_i / i \in K\}$ .

Consider one of these vertices  $x' = xs_i$ . The color of  $x'$  cannot be 0 because  $x'$  is not in  $C$ . It is not  $j$  because this would imply the existence of  $v$  in  $C$  with  $x' = vs_j$  and using  $x' = xs_i$ ,  $x = us_j$  this gives  $v = us_i$  thus  $d(u, v) = 1$ .

Assume now that two neighbors of  $x$  say  $x' = xs_i$  and  $x'' = xs_h$  for some  $i, h \in K$  are of the same color  $d$ . This would imply the existence of two distinct vertices  $v$  and  $w$  in  $C$  with  $s_d = v^{-1}x' = w^{-1}x''$ . Then using  $x' = xs_i$  and  $x'' = xs_h$  this would imply  $v^{-1}s_i = w^{-1}s_h$  thus the existence of a vertex at distance 1 of both  $v$  and  $w$  and this is not possible because  $C$  is a code. Thus the neighbors of  $x$  are colored with all distinct colors from  $\{0, 1, \dots, n\} \setminus \{j\}$ .

□

We will now give some examples of use of our construction.

Consider for  $p \geq 3$  the  $p$ -crown graph  $\tilde{K}_{p,p}$  obtained from the bipartite complete graph  $K_{p,p}$  after deletion of the edges of a perfect matching. This graph is an Abelian Cayley graph  $G = \text{Cay}(\Gamma, S)$  with  $\Gamma = \mathbb{Z}_p \times \mathbb{Z}_2$  and  $S = \{(1, 1), (2, 1), \dots, (p-1, 1)\}$ . By construction each pair of vertices of the removed matching is a perfect code.

The graph  $\tilde{K}_{4,4}$  is  $Q_3$  and we obtain the construction of Hamming codes in hypercubes.

The graph  $\tilde{K}_{3,3}$  is  $C_6$  and we obtain a construction of perfect codes in  $C_6^{2^k} \square Q_{2^k-1}$ . On  $\mathbb{Z}^n \square Q_k$ , we will say that a code is  $i$ -periodical ( $i \in \{1, \dots, n\}$ ) if there exists a positive integer  $p_i$  (called the  $i$ -period) such that for any vertex  $x = x_1x_2 \dots x_nv$  ( $\forall i, x_i \in \mathbb{Z}, v \in V(Q_k)$ ), the vertex  $x_1x_2 \dots x_{i-1}(x_i + p_i)x_{i+1} \dots x_nv$  is in the code if and only if  $x$  is in the code. We thus obtain a perfect code on  $\mathbb{Z}^{2^k} \square Q_{2^k-1}$  of  $i$ -period 6 for all  $i \in \{1, \dots, 2^k\}$ .

Other values of  $p \geq 5$  give infinite families of graphs with perfect codes.

Golomb and Welsh [10, 11] proved the existence of perfect codes in the grid  $\mathbb{Z}^n$ . This is again a bipartite Abelian Cayley graph. Thus for all integer  $k$  there exists a perfect code in  $\mathbb{Z}^{n2^k} \square Q_{2^k-1}$ . More precisely the codes of Golomb and Welsh are of  $i$ -period  $2n + 1$  thus they induce a code-coloring in the bipartite graph  $C_{4n+2}^n$ . Therefore there exists in  $\mathbb{Z}^{n2^k} \square Q_{2^k-1}$  a perfect code of  $i$ -period  $4n + 2$  for all  $i \in \{1, \dots, n2^k\}$ .

Our work [8] about the possible parameters values for the existence of perfect codes in  $\mathbb{Z}^n \square Q_k$  can be completed by the following direct consequence of theorem 7.

**Corollary 10** *Let  $a, b, c$  be integers such that  $b \geq 2c$ . Then if there exist perfect codes in  $\mathbb{Z}^a \square Q_b$  and in  $\mathbb{Z}^{a+c} \square Q_{b-2c}$  then there exists a perfect code in  $\mathbb{Z}^{2a+c} \square Q_{2b-2c+1}$ .*

For any integer  $p \geq 1$  consider the bipartite complete graph  $K_{p,p}$ . Label 0 the  $p$  vertices of one of the independent sets, and  $\{1, \dots, p\}$  the  $p$  other vertices. We



obtain an extended code-coloring of  $K_{p,p}$ . Assume that  $p = 2^q$  for some integer  $q \geq 0$ . Then by theorem 6 there exists a perfect code in  $K_{2^q,2^q} \square Q_{2^q-1}$ . But this is a bipartite Abelian Cayley Graph thus by lemma 9 there exists a code-coloring of  $K_{2^q,2^q} \square Q_{2^q-1}$ . Using again corollary 8 we obtain the following result.

**Corollary 11** *For any integers  $q, k \geq 0$  there exists a perfect code in  $K_{2^q,2^q}^{2^k} \square Q_{2^{q+k}-1}$ .*

There exists a trivial code-coloring of the complete graph  $K_n$ . Thus by theorem 6 there exists a perfect code in  $K_{n,n} \square K_n$ . But this is also an Abelian Cayley Graph thus we obtain a code-coloring of  $K_{n,n} \square K_n$ . Using the same idea with  $\tilde{K}_{n,n}$  and  $\mathbb{Z}^n$  we obtain

**Corollary 12** *For any integer  $n \geq 1$  there exist code-colorings of  $K_{n,n} \square K_n$ ,  $K_{n,n} \square \tilde{K}_{n,n}$  and  $K_{2n+1,2n+1} \square \mathbb{Z}^n$ .*

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