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Contents

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1 Introduction

A successful strategy for solving the Navier-Stokes system

$$\left\{\begin{array}{rcll} \frac{\partial u}{\partial t}-\Delta_x u+\nabla_x\pi+(u\cdot\nabla_x)u &=& 0 & \text{in} \quad]0,T]\times\Omega,\\ \operatorname{div}_x u &=& 0 & \text{in} \quad [0,T]\times\Omega,\\ \operatorname{Tr}_x u &=& 0 & \text{on} \quad [0,T]\times\partial\Omega,\\ u(0) &=& u_0 & \text{in} \quad \Omega, \end{array}\right. \tag{1.1}$$

brought to prominence by the pioneering work of H. Fujita, and T. Kato in the 60's, entails the following three steps:

(i) recast (1.1) in the form of an abstract initial value problem:

$$\left\{\begin{array}{l} u'(t)+(Au)(t)=f(t) \qquad t\in]0,T[,\\ f(t):=-\mathbb{P}\big[(u(t)\cdot\nabla_x)u(t)\big],\\ u(0)=u_0; \end{array}\right. \tag{1.2}$$

(ii) convert (1.2) into the integral equation

$$u(t)=e^{-tA}u_0-\int_0^t e^{-(t-s)A}\mathbb{P}\Big[(u(s)\cdot\nabla_x)u(s)\Big]\,ds, \qquad 0<t<T; \tag{1.3}$$

(iii) solve (1.3) via fixed point methods (typically, a Picard iterative scheme).

Above, $\mathbb{P}$ is the Leray projection of $L^2(\Omega,\mathbb{R}^3)$ onto $\mathcal{H}:=\{u\in L^2(\Omega,\mathbb{R}^3):\operatorname{div}u=0,\ \nu\cdot u=0\}$, where ν is the outward unit normal to $\partial\Omega$, and A is the Stokes operator, i.e. the Friedrichs extension of the symmetric operator $\mathbb{P}\circ(-\Delta)$, originally defined on $\mathcal{H}\cap\mathscr{C}^\infty_c(\Omega,\mathbb{R}^3)$, to $\mathcal{H}$.

By relying on the theory of analytic semigroups generated by self-adjoint operators, Fujita and Kato have proved in [22] short time existence of strong solutions for (1.1) when $\Omega\subset\mathbb{R}^3$ is bounded and sufficiently smooth. Somewhat more specifically, they have shown that if Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^3$, and if the initial datum u_0 belongs to $D(A^{\frac14})$, then a strong solution can be found for which $u(t)\in D(A^{\frac34})$ for $t\in]0,T[$, granted that T is small. Hereafter, $D(A^\alpha)$, $\alpha>0$, stands for the domain of the fractional power A^α of A.

An important aspect of this analysis is the ability to describe the size/smoothness of vector fields belonging to $D(A^\alpha)$ in terms of more familiar spaces. For example, the estimates (1.18) and (2.17) in [22] amount to

$$D(A^\gamma)\subset\mathscr{C}^\alpha(\overline{\Omega},\mathbb{R}^3) \quad \text{ if } \ \tfrac34<\gamma<1 \ \text{ and } \ 0<\alpha<2(\gamma-\tfrac34), \tag{1.4}$$

which plays a key role in [22]. Although Fujita and Kato have proved (1.4) via *ad hoc* methods, it was later realized that a more resourceful and elegant approach to such regularity results is to view them as corollaries of optimal embeddings for $D(A^\alpha)$, $\alpha>0$, into the scale of vector-valued Sobolev (potential) spaces of fractional order, $L^p_s(\Omega,\mathbb{R}^3)$, $1<p<\infty$, $s\in\mathbb{R}$. This latter

issue turned out to be intimately linked to the smoothness assumptions made on the boundary of the domain Ω. For example, Fujita and Morimoto have proved in [23] that

$$\partial\Omega\in\mathscr{C}^\infty\Longrightarrow D(A^\alpha)\subset L^2_{2\alpha}(\Omega,\mathbb{R}^3),\qquad 0\leq\alpha\leq 1, \tag{1.5}$$

whereas the presence of a single conical singularity on $\partial\Omega$ may result in the failure of $D(A)$ to be included in $L^2_2(\Omega,\mathbb{R}^3)$.

Another property of the Stokes operator which is heavily influenced by the smoothness of $\partial\Omega$ is whether e^{-tA}, originally considered on $\mathcal{H}$, extends to a bounded analytic semigroup of operators in $\mathcal{H}_p$, where

$$\mathcal{H}_p:=\{u\in L^p(\Omega,\mathbb{R}^3):\,\operatorname{div}u=0,\ \nu\cdot u=0\}. \tag{1.6}$$

When $\partial\Omega\in\mathscr{C}^\infty$, this is indeed the case for all $p\in]1,\infty[$ (cf. [24], [51]) but matters are considerably more subtle in the case when $\partial\Omega$ is only Lipschitz. For example, Taylor has conjectured in [55] that for a given bounded Lipschitz domain $\Omega\subset\mathbb{R}^3$ there exists $\varepsilon=\varepsilon(\Omega)>0$ such that e^{-tA} extends to an analytic semigroup on $\mathcal{H}_p$ provided $\frac{3}{2}-\varepsilon<p<3+\varepsilon$. This range of p's is naturally dictated by the mapping properties of the Leray projection. Indeed, it has been proved by Fabes, Mendez and Mitrea in [19] that in the case when Ω is a bounded, Lipschitz domain in $\mathbb{R}^3$,

$$\mathbb{P}:L^p(\Omega,\mathbb{R}^3)\longrightarrow\mathcal{H}_p \tag{1.7}$$

boundedly, precisely when $\frac{3}{2}-\varepsilon<p<3+\varepsilon$ for some $\varepsilon=\varepsilon(\Omega)>0$. That this range of p's in Taylor's conjecture is in the nature of best possible is also supported by the counterexamples constructed by Deuring in [15], where he shows that, contrary to the case of smooth domains, the Stokes operator in a cone-like domain in $\mathbb{R}^3$ may fail to be sectorial in L^p for some $p>3$. Quite recently, the version of Taylor's conjecture corresponding to the Stokes operator with boundary conditions of Neumann type has been proved by Mitrea and Monniaux in [40].

The main goal of the present paper is to continue this line of work and extend the Fujita-Kato program outlined above to the case when the underlying Euclidean domain Ω has only a Lipschitz boundary. An earlier attempt in this regard is in [16], where Deuring and von Wahl have established the local existence of strong solutions for the Navier-Stokes equations in a bounded Lipschitz domain $\Omega\subset\mathbb{R}^3$ with a connected boundary, when the initial data satisfies

$$u_0\in D(A^{\frac{1}{4}+\varepsilon})\quad\text{ for some }\ \varepsilon>0. \tag{1.8}$$

At the core of their analysis is the fact that, for any bounded Lipschitz domain $\Omega\subset\mathbb{R}^3$,

$$D(A^{\frac{3}{4}+\varepsilon})\subset L^2_{\frac{3}{2}-\delta}(\Omega,\mathbb{R}^3),\qquad\forall\,\varepsilon,\delta>0, \tag{1.9}$$

which they proved by relying on the work of Fabes, Kenig and Verchota [18], as well as Shen [48]. On p. 114 of [16] the authors also raise the question of describing $D(A)$ is terms of Sobolev spaces. Shortly thereafter, by relying on the progress made by Shen in [49], Brown and Shen have obtained in [9] certain related regularity results, including

$$D(A)\subset L^p_1(\Omega,\mathbb{R}^3)\quad\text{ for some }\ p=p(\Omega)>3, \tag{1.10}$$

for any bounded, Lipschitz domain $\Omega \subset \mathbb{R}^3$.

Here we shall refine this analysis and improve upon these results in several important respects. First, in Corollary 5.5 and Theorem 5.3, we shall show that (1.9) is still valid if either $\varepsilon = 0$ or $\delta = 0$. In the class of Lipschitz domains, this is in the nature of best possible since, in the critical case,

$$D(A^{\frac{3}{4}}) = \Big\{ u \in L^2_{1,z}(\Omega, \mathbb{R}^n) : \operatorname{div} u = 0 \ \& \ \Delta u \in L^2_{-\frac{1}{2}}(\Omega, \mathbb{R}^3) + \nabla L^2(\Omega) \Big\}. \tag{1.11}$$

Thus, there is no reason to expect that $D(A^{\frac{3}{4}}) \subset L^2_{\frac{3}{2}}(\Omega, \mathbb{R}^3)$ for every bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. In fact, Jerison and Kenig have constructed a bounded $\mathscr{C}^1$ domain $\Omega \subset \mathbb{R}^n$ and $f \in L^2_{-\frac{1}{2}}(\Omega)$ such that the unique solution $u \in L^2_{1,z}(\Omega)$ of $\Delta u = f$ does not belong to $L^2_{\frac{3}{2}}(\Omega)$; see Theorem 0.4 on p. 164 in [29]. We will, nonetheless, prove that for any bounded Lipschitz domain Ω in $\mathbb{R}^3$,

$$D(A^{\frac{3}{4}}) \subset L^p_{\frac{3}{p}}(\Omega, \mathbb{R}^3) \quad \forall\, p > 2. \tag{1.12}$$

Moreover, the limiting case $\varepsilon = \delta = 0$ of (1.9) holds as well when the bounded Lipschitz domain Ω satisfies a uniform exterior ball condition; cf. Theorem 5.3. In Corollary 5.5, we are also able to sharpen (1.10) to

$$\forall\, \alpha > \tfrac{3}{4} \quad \exists\, p > 3 \ \text{ such that } \ D(A^\alpha) \subset L^p_1(\Omega, \mathbb{R}^3). \tag{1.13}$$

In fact, we shall prove more general results, of the following nature. First, for $0 < \gamma < \frac{3}{2}$, we show in Theorem 5.1 that $D(A^\gamma) = L^2_{2\gamma,z}(\Omega, \mathbb{R}^3) \cap \mathcal{H}$. Second, when $\frac{3}{2} \leq \gamma < \frac{3}{2} + \varepsilon$ where $\varepsilon = \varepsilon(\Omega) > 0$, we give sufficient conditions on the indices p, θ so that $D(A^\gamma) \subset L^p_\theta(\Omega, \mathbb{R}^3)$; see Theorem 5.4. In particular, this analysis shows that for any bounded, Lipschitz domain Ω in $\mathbb{R}^3$, there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that

$$\tfrac{3}{4} < \gamma < \tfrac{3}{4} + \varepsilon \Longrightarrow D(A^\gamma) \subset \mathscr{C}^{2\gamma - 3/2}(\overline{\Omega}, \mathbb{R}^3). \tag{1.14}$$

This is in agreement with the Fujita-Kato regularity result (1.4), which is thereby extended from domains of class $\mathscr{C}^3$ to the class of Lipschitz domains. Compared with the setting in [16], all of our results are proved without the artificial assumption that the boundary of the Lipschitz domain is connected.

More could be said if extra information about the geometry of $\partial\Omega$ is available. For example, if $\Omega \subset \mathbb{R}^3$ is a convex polyhedron, then the results proved by Dauge in [11] imply that $D(A) = L^2_2(\Omega, \mathbb{R}^3) \cap L^2_{1,z}(\Omega, \mathbb{R}^3) \cap \mathcal{H}$. Nonetheless, the goal here is to determine the maximal (Sobolev) regularity exhibited by the elements in $D(A^\alpha)$ without assuming that the Lipschitz surface $\partial\Omega$ has any particular structure.

Consider now the initial problem for the Navier-Stokes system (1.1) when $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. As in [22], we shall work with initial data u_0 belonging to the critical space $D(A^{\frac{1}{4}})$, a membership which we prove to be equivalent to

$$\begin{aligned} &u_0 \in L^2_{\frac{1}{2}}(\Omega, \mathbb{R}^3), \qquad \int_\Omega |u_0(x)|^2 \operatorname{dist}(x, \partial\Omega)^{-1} dx < \infty, \\ &\nu \cdot u_0 = 0 \ \text{ on } \ \partial\Omega, \ \text{ and } \ \operatorname{div} u_0 = 0 \ \text{ in } \ \Omega. \end{aligned} \tag{1.15}$$

The description (1.15) is particularly satisfactory in the light of the comment made by Fujita and Kato on p. 313 of their seminal paper [22], where they note that "*in principle, it is desirable to have existence theorems in which the assumption on the initial velocity is not only sufficiently weak but easy to verify.*"

For u_0 as in (1.15), we then prove the local existence of a strong solution for (1.1) satisfying

$$u \in \mathscr{C}([0,T];D(A^{\frac{1}{4}}))\cap\mathscr{C}^1(]0,T];D(A^{\frac{3}{4}})), \tag{1.16}$$

$$u \in L^p_1(]0,T[;\mathcal{H})\cap L^p(]0,T[;D(A)), \qquad 1<p<\tfrac{4}{3}, \tag{1.17}$$

plus naturally accompanying estimates. See Theorem 6.4 and (6.1). Furthermore, uniqueness holds in the space $\mathscr{C}([0,T];D(A^{\frac{1}{4}}))$; cf. Theorem 6.7 where this issue is addressed.

We now wish to comment on a couple of key ingredients used in the proofs of our main results. First, central to our approach to the regularity of the Stokes operator are the well-posedness results for the homogeneous and inhomogeneous Dirichlet problems for the Stokes system in Lipschitz domains from [17] and [42]. These results are reviewed in §3.1 and §3.2, where precise statements are given.

Second, we shall make essential use of an interpolation result of the following nature. For $1<p<\infty$ and $s>-1+1/p$, denote by $V^{s,p}(\Omega)$ the closure of $\{u\in\mathscr{C}^\infty_c(\Omega,\mathbb{R}^n):\operatorname{div}u=0\}$ in $L^p_{s,z}(\Omega,\mathbb{R}^n)$. These smoothness spaces are well-adapted to the particular nature of the Stokes operator and the question arises whether this scale is stable under complex interpolation ([8]). More specifically, it is of interest to establish the identity

$$\Big[V^{s_0,p_0}(\Omega)\,,\,V^{s_1,p_1}(\Omega)\Big]_\theta = V^{s,p}(\Omega) \tag{1.18}$$

whenever $1<p_j<\infty$, $-1+\frac{1}{p_j}<s_j$, $j=0,1$, and $\theta\in[0,1]$, granted that $\frac{1}{p}:=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $s:=(1-\theta)s_0+\theta s_1$. When the domain Ω is smooth, the Leray projection $\mathbb{P}$ extends to a bounded operator from $L^p_s(\Omega,\mathbb{R}^n)$ onto $V^{s,p}(\Omega)$ for any $1<p<\infty$ and $s\in\mathbb{R}$ and this readily yields (1.18) for all indices (cf. Exercise 4 on p. 492 in Vol. III of [54], at least for the case when $p_0=p_1=2$). For a general Lipschitz domain, the fact that $\mathbb{P}:L^p_s(\Omega,\mathbb{R}^n)\to V^{s,p}(\Omega)$ is bounded imposes strong limitations on the indices p, s (see Proposition 2.16 in the body of the paper for a precise formulation), thus a new approach had to be devised. We were able to overcome this difficulty by relying on an abstract subspace interpolation scheme due to Lions and Magenes [36] which, in turn, requires the existence of a linear right-inverse for the divergence operator, with adequate mapping properties. A prototype of such an operator has been first constructed by Bogovskiĭ in [4]-[5], although the mapping properties established there are not strong enough for our present purposes. Instead, here we make use of a refined version of this construction, recently carried out in [38], where finer mapping properties have been proved.

A remarkable feature of our analysis of the Stokes operator and the Navier-Stokes system is that all our main results can be formulated in the context of Lipschitz subdomains of a smooth, compact, Riemannian manifold $\mathscr{M}$. We elaborate on this point more fully in Section 7. In particular, our results further refine and strengthen those in §8-§9 of [42].

Let us now survey more literature dealing with issues pertaining to the regularity of the Stokes operator and generalizations of the Fujita-Kato approach for the Navier-Stokes problem.

An excellent account of recent progress in the entire Euclidean space is Lemarié-Rieusset's monograph [33] (in particular, §1 of Chapter 35 contains a nice, brief survey of work in this area). In [26], Y. Giga and T. Miyakawa have employed the Fujita-Kato approach in order to prove existence and uniqueness of strong solutions for (1.1) when the initial data is in $L^p(\Omega)$, $1 < p < \infty$, provided $\partial\Omega \in \mathscr{C}^\infty$. In [10], R. Brown, P. Perry and Z. Shen have studied the domains of the fractional powers of the Stokes operator in two-dimensional Lipschitz domains. That the Stokes operator on L^p spaces, $1 < p < \infty$, in a smooth domain has bounded imaginary powers is contained in [25], where Y. Giga also identifies the domains of its fractional powers in this context. A conceptually simple variant of Fujita-Kato's approach in two and three dimensional Lipschitz domains was suggested by M. Taylor in [55]. In [28], Grubb has treated the Navier-Stokes problem in smooth domains with data from anisotropic L^p Sobolev spaces of Besov and Bessel-potential type. The earliest proof of the fact that, on a smooth domain Ω, the Stokes operator generates an analytic semigroup on the space $\mathcal{H}_p$, introduced in (1.6), is due to V.A. Solonnikov in [51]. New proofs and extensions of this result have been proved by Giga [24], Borchers and Sohr in [6], and by Borchers and Varnhorn in [7], among others. Let us also note that a version of the regularity result (1.5) already appears in the pioneering work of P.E. Sobolevskiĭ in [50]. The strong solvability of the system (1.1) in smooth domains for rough initial data has been investigated by Amann in [1]. A more up-to-date account of the Fujita-Kato method, as well as other pertinent bibliographical references can be found in, e.g., von Wahl's book [58].

We conclude this introduction with a brief discussion of a number of notational conventions used throughout the paper. We denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N} = \{1, 2, ...\}$ the subset of $\mathbb{Z}$ consisting of positive numbers. Also, we set $\mathbb{N}_o := \mathbb{N} \cup \{0\}$. By $\mathscr{C}^\infty(\Omega)$ we shall denote the space of infinitely differentiable functions in the open set $\Omega \subset \mathbb{R}$, by $\mathscr{C}^\infty(\bar{\Omega})$ the restrictions of $\mathscr{C}^\infty(\mathbb{R}^n)$ to Ω, and by $\mathscr{C}^\infty_c(\Omega)$ the subspace of $\mathscr{C}^\infty(\Omega)$ consisting of compactly supported functions. When viewed as a topological space, the latter is equipped with the usual inductive limit topology and its dual, i.e. the space of distributions in Ω, is denoted by $\mathscr{D}'(\Omega) := \Big(\mathscr{C}^\infty_c(\Omega)\Big)'$. Generally speaking, if $\mathscr{X}(\Omega)$ is a space of distributions in Ω, we set $\mathscr{X}(\Omega, \mathbb{R}^n) := \mathscr{X}(\Omega) \otimes \mathbb{R}^n$, i.e., the space of vector-valued distributions with coefficients in $\mathscr{X}(\Omega)$. Throughout the paper, we make the convention that

$$1 < p < \infty \Longrightarrow p' := \tfrac{p}{p-1} \tag{1.19}$$

denotes the Hölder conjugate exponent of p. Finally, $\langle \cdot, \cdot \rangle$ will stand for various duality brackets between a topological space $\mathscr{X}$ and its dual $\mathscr{X}^*$ (in each case, the space $\mathscr{X}$ should be clear from the context). We shall occasionally write ${}_{\mathscr{X}^*}\langle \cdot, \cdot \rangle_{\mathscr{X}}$ in order to stress the dependence of the pairing $\langle \cdot, \cdot \rangle$ on the space $\mathscr{X}$.

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2 The Stokes scale

The main aim of this section is to introduce and study spaces measuring smoothness which are also algebraically well-adapted to the particular form of the Stokes operator.

2.1 Potential spaces in Lipschitz domains

General references for the well-understood aspects of the material presented in this section are [3], [27], [29], [30], [46], [47], [57]. In the interest of brevity, we shall refer the reader to these references and only sketch the proofs of seemingly less known results.

We begin by reviewing the Sobolev (or potential) class $L^p_s(\mathbb{R}^n)$ defined for $1 < p < \infty$ and $s \in \mathbb{R}$ by

$$L^p_s(\mathbb{R}^n) := \{(I-\Delta)^{-s/2} f : \, f \in L^p(\mathbb{R}^n)\}. \tag{2.1}$$

As is well-known,

$$\Big(L^p_s(\mathbb{R}^n)\Big)^* = L^{p'}_{-s}(\mathbb{R}^n), \qquad 1 < p < \infty, \;\; s \in \mathbb{R}. \tag{2.2}$$

Given an open subset Ω of $\mathbb{R}^n$, we shall denote by $\mathscr{R}_\Omega$ the operator of restriction to Ω of distributions in $\mathbb{R}^n$ and, for $1 < p < \infty$ and $s \in \mathbb{R}$ we introduce the following families of spaces:

$$L^p_s(\Omega) := \{\mathscr{R}_\Omega u : \, u \in L^p_s(\mathbb{R}^n)\}, \tag{2.3}$$

equipped with the natural infimum norm,

$$L^p_{s,0}(\Omega) := \{u \in L^p_s(\mathbb{R}^n) : \, \operatorname{supp} u \subseteq \bar{\Omega}\} \tag{2.4}$$

with the norm inherited from $L^p_s(\mathbb{R}^n)$, as well as

$$L^p_{s,z}(\Omega) := \{\mathscr{R}_\Omega u : \, u \in L^p_{s,0}(\Omega)\}, \qquad p \in]1,\infty[, \; s \in \mathbb{R}, \tag{2.5}$$

equipped with the natural infimum norm. One property readily seen to be inherited from their counterparts in $\mathbb{R}^n$ is that, for $1 < p < \infty$,

$$L^p_{s_1}(\Omega) \hookrightarrow L^p_{s_2}(\Omega), \quad -\infty < s_2 < s_1 < +\infty. \tag{2.6}$$

For the remainder of this section we shall assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ which means that $\Omega \subset \mathbb{R}^n$ is open and bounded, and there exists a finite open covering $\{\mathcal{O}_j\}_{1\leq j\leq N}$ of $\partial\Omega$ with the property that, for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a Lipschitz function $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ (where $\mathbb{R}^{n-1} \times \mathbb{R}$ is a new system of coordinates obtained from the original one via a rigid motion). Such domains are referred to as *minimally smooth* in E. Stein's book [52] (cf. p. 189 loc. cit.). It is a classical result that, for a Lipschitz domain Ω, the surface measure $d\sigma$ is well-defined on $\partial\Omega$ and that there exists an outward pointing normal vector ν at almost every (with respect to $d\sigma$) point on $\partial\Omega$.

In [46] it has been proved that there exists a universal linear extension operator mapping potential spaces from a Lipschitz domain to the entire Euclidean space with preservation of smoothness. More specifically, we have the following.

Proposition 2.1. *For each Lipschitz domain Ω in $\mathbb{R}^n$ there exists a linear operator $\mathscr{E}$ mapping $\mathscr{C}^\infty_c(\Omega)$ into tempered distributions in $\mathbb{R}^n$, and such that*

$$\mathscr{E} : L^p_s(\Omega) \longrightarrow L^p_s(\mathbb{R}^n), \tag{2.7}$$

$$\mathscr{R}_\Omega \circ \mathscr{E} = I, \quad \text{the identity operator on } \ L^p_s(\Omega), \tag{2.8}$$

whenever $1 < p < \infty$ and $s \in \mathbb{R}$.

Next, it is known that the following inclusions

$$\widetilde{\mathscr{C}^\infty_c(\Omega)} \hookrightarrow L^p_{s,0}(\Omega), \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{2.9}$$

$$\mathscr{C}^\infty(\overline{\Omega}) \hookrightarrow L^p_s(\Omega), \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{2.10}$$

$$\mathscr{C}^\infty_c(\Omega) \hookrightarrow L^p_s(\Omega), \quad 1 < p < \infty, \ s \leq 1/p, \tag{2.11}$$

$$\mathscr{C}^\infty_c(\Omega) \hookrightarrow L^p_{s,z}(\Omega), \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{2.12}$$

$$\mathscr{C}^\infty_c(\Omega) \hookrightarrow \Big(L^p_s(\Omega)\Big)^*, \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{2.13}$$

have dense ranges for the indicated values of p and s. In (2.9), tilde denotes the extension by zero outside Ω. Note the inclusion of the critical value $s = \frac{1}{p}$ in (2.11); cf. p. 180 in [29]. In this regard, we would also like to point out that

$$L^2_{\frac{1}{2},z}(\Omega, \mathbb{R}^n) = \Big\{u \in L^2_{\frac{1}{2}}(\Omega, \mathbb{R}^n) : \int_\Omega |u(x)|^2 \text{dist}\,(x, \partial\Omega)^{-1} dx < \infty\Big\}, \tag{2.14}$$

plus a natural equivalence of norms.

For later reference, let us also point out that $\mathcal{R}_\Omega$, the restriction to Ω,

$$\mathscr{R}_\Omega : L^p_{s,0}(\Omega) \longrightarrow L^p_{s,z}(\Omega), \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{2.15}$$

is a linear, bounded, onto operator. It admits the factorization

$$L^p_{s,0}(\Omega) \xrightarrow{\text{pr}} \frac{L^p_{s,0}(\Omega)}{\{u \in L^p_s(\mathbb{R}^n) : \text{supp}\, u \subseteq \partial\Omega\}} \xrightarrow{\mathscr{R}_\Omega} L^p_{s,z}(\Omega), \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{2.16}$$

where the first arrow is the canonical projection onto the factor space, and the second arrow is an isomorphism. Moreover, since

$$1 < p < \infty, \ -1 + 1/p < s \Longrightarrow \{u \in L^p_s(\mathbb{R}^n) : \text{supp}\, u \subseteq \partial\Omega\} = 0 \tag{2.17}$$

then

$$\mathscr{R}_\Omega : L^p_{s,0}(\Omega) \longrightarrow L^p_{s,z}(\Omega) \quad \text{ isomorphically if } \ 1 < p < \infty, \ s > -1 + 1/p. \tag{2.18}$$

In this latter case, its inverse is the operator of extension by zero outside Ω, denoted throughout the paper by tilde, i.e.,

$$L^p_{s,z}(\Omega) \ni u \longmapsto \widetilde{u} \in L^p_{s,0}(\Omega), \quad 1 < p < \infty, \ -1 + 1/p < s. \tag{2.19}$$

In particular, this allows the identification

$$L^p_{s,0}(\Omega) \equiv L^p_{s,z}(\Omega), \qquad \forall\, p \in (1, \infty), \ \forall\, s > -1 + 1/p. \tag{2.20}$$

Lemma 2.2. *The restriction operator satisfies*

$$s > -1 + 1/p \Longrightarrow \langle u, v\rangle = \langle \mathscr{R}_\Omega u\,,\, \mathscr{R}_\Omega v\rangle, \quad \forall\, u \in L^p_{s,0}(\Omega),\ \ \forall\, v \in L^{p'}_{-s}(\mathbb{R}^n). \tag{2.21}$$

As a corollary,

$$s > -1 + 1/p \Longrightarrow \langle \tilde{u}, w\rangle = \langle u, \mathscr{R}_\Omega w\rangle, \qquad \forall\, u \in L^p_{s,z}(\Omega), \quad \forall\, w \in L^{p'}_{-s}(\mathbb{R}^n), \tag{2.22}$$

$$-1 + 1/p < s < 1/p \Longrightarrow \langle \tilde{u}, w\rangle = \langle u, \mathscr{R}_\Omega w\rangle, \qquad \forall\, u \in L^p_{s}(\Omega), \quad \forall\, w \in L^{p'}_{-s}(\mathbb{R}^n). \tag{2.23}$$

Proof. Let $u \in L^p_{s,0}(\Omega)$ and $v \in L^{p'}_{-s}(\mathbb{R}^n)$ be arbitrary and consider a sequence $\varphi_j \in \mathscr{C}^\infty_c(\Omega)$, $j \in \mathbb{N}$, such that $\tilde{\varphi}_j \to u$ in $L^p_{s,0}(\Omega)$ (and, hence, in $L^p_s(\mathbb{R}^n) = \Big(L^{p'}_{-s}(\mathbb{R}^n)\Big)^*$ also) as $j \to \infty$. Thus, as $j \to \infty$, we have that $\varphi_j \to \mathscr{R}_\Omega u$ in $L^p_{s,z}(\Omega) = \Big(L^{p'}_{-s}(\Omega)\Big)^*$ as $j \to \infty$, since $s > -1 + 1/p$. Based on this analysis we may then conclude that

$$\langle u, v\rangle = \lim_{j\to\infty} \langle \tilde{\varphi}_j, v\rangle = \lim_{j\to\infty} \langle \varphi_j, \mathscr{R}_\Omega v\rangle = \langle \mathscr{R}_\Omega u, \mathscr{R}_\Omega v\rangle, \tag{2.24}$$

which justifies (2.21).

With this in hand, (2.22) follows by observing that if $u \in L^p_{s,z}(\Omega)$ then $\tilde{u} \in L^p_{s,0}(\Omega)$ and $\mathscr{R}_\Omega(\tilde{u}) = u$. Finally, (2.23) is a consequence of (2.22) and the fact that $L^p_{s,z}(\Omega) = L^p_s(\Omega)$ if $-1 + 1/p < s < 1/p$; cf. (2.30) below. □

Next, assume that $1 < p_j < \infty$, $s_j \in \mathbb{R}$, $j \in \{1, 2\}$, $\theta \in (0, 1)$ and that $1/p = (1-\theta)/p_1 + \theta/p_2$, $s = (1-\theta)s_1 + \theta s_2$. Then

$$[L^{p_1}_{s_1}(\Omega), L^{p_2}_{s_2}(\Omega)]_\theta \;=\; L^p_s(\Omega), \tag{2.25}$$

$$[L^{p_1}_{s_1,0}(\Omega), L^{p_2}_{s_2,0}(\Omega)]_\theta \;=\; L^p_{s,0}(\Omega), \tag{2.26}$$

where $[\cdot\,,\,\cdot]_\theta$ stands for the complex interpolation bracket.

Going further, let us also consider the space

$$\overset{\circ}{L}{}^p_s(\Omega) := \text{ the closure of } \mathscr{C}^\infty_c(\Omega) \text{ in } L^p_s(\Omega), \qquad 1 < p < \infty,\ s \in \mathbb{R}. \tag{2.27}$$

Then we have the continuous embeddings

$$L^p_{s,z}(\Omega) \hookrightarrow \overset{\circ}{L}{}^p_s(\Omega) \hookrightarrow L^p_s(\Omega) \tag{2.28}$$

and, furthermore,

$$\overset{\circ}{L}{}^p_s(\Omega) = L^p_{s,z}(\Omega) \ \text{ if } \ \tfrac{1}{p} - s \notin \mathbb{Z} \quad \text{and} \quad \overset{\circ}{L}{}^p_s(\Omega) = L^p_s(\Omega) \ \text{ if } \ s < \tfrac{1}{p}. \tag{2.29}$$

In particular,

$$\overset{\circ}{L}{}^p_s\,(\Omega) = L^p_s(\Omega) = L^p_{s,z}(\Omega) \ \text{ if } \ s < \tfrac{1}{p} \ \text{ and } \ \tfrac{1}{p} - s \notin \mathbb{N}. \tag{2.30}$$

Moreover, for every $j=\{1,...,n\}$ and $1<p<\infty$, $s\in\mathbb{R}$,

$$\begin{aligned}&\partial_j: L^p_s(\Omega)\longrightarrow L^p_{s-1}(\Omega), && \partial_j: \overset{\circ}{L}{}^p_s(\Omega)\longrightarrow \overset{\circ}{L}{}^p_{s-1}(\Omega),\\ &\partial_j: L^p_{s,0}(\Omega)\longrightarrow L^p_{s-1,0}(\Omega), && \partial_j: L^p_{s,z}(\Omega)\longrightarrow L^p_{s-1,z}(\Omega),\end{aligned} \tag{2.31}$$

are well-defined, linear, bounded operators. Let us also record here a useful lifting result for Sobolev spaces on Lipschitz domains, which has been proved in [38].

Proposition 2.3. *Let $1<p<\infty$ and $s\in\mathbb{R}$. Then for any distribution u in the bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$, the following implication holds:*

$$\nabla u\in L^p_{s-1}(\Omega,\mathbb{R}^n)\implies u\in L^p_s(\Omega). \tag{2.32}$$

Throughout the paper, all duality pairings on Ω are extensions of the natural pairing between test functions and distributions on Ω. Our next result elaborates on the nature of the dual scale for $L^p_s(\Omega)$ when $1<p<\infty$ and $s\in\mathbb{R}$ are arbitrary.

Lemma 2.4. *For every $1<p<\infty$ and $s\in\mathbb{R}$, the application*

$$\begin{aligned}&\widetilde{\mathscr{C}^\infty_c(\Omega)}\ni\tilde{\varphi}\longmapsto\varphi\in\mathscr{C}^\infty_c(\Omega)\ \textit{extends to}\\ &\textit{an isomorphism}\ \Psi: L^p_{s,0}(\Omega)\longrightarrow\Big(L^{p'}_{-s}(\Omega)\Big)^*.\end{aligned} \tag{2.33}$$

Proof. Let us define

$$\Psi: L^p_{s,0}(\Omega)\ni u\longmapsto\Lambda_u\in(L^{p'}_{-s}(\Omega))^*, \tag{2.34}$$

where

$$\langle\Lambda_u,v\rangle:=\langle u,V\rangle\ \text{for every}\ v\in L^{p'}_{-s}(\Omega)\ \text{and}\ V\in L^{p'}_{-s}(\mathbb{R}^n)\ \text{such that}\ \mathscr{R}_\Omega V=v. \tag{2.35}$$

We claim that above definition does not depend on the particular extension V of a given v. To justify this claim, let $V_1,V_2\in L^{p'}_{-s}(\mathbb{R}^n)$ be such that $\mathscr{R}_\Omega V_1=\mathscr{R}_\Omega V_2$. In particular, $V_1-V_2\in L^{p'}_{-s,0}(\Omega)$. Then, for a sequence $\{u_j\}_{j\in\mathbb{N}}$ of functions from $\mathscr{C}^\infty_c(\Omega)$ with the property that $\tilde{u}_j\to u$ as $j\to\infty$ in $L^p_s(\mathbb{R}^n)=\Big(L^{p'}_{-s}(\mathbb{R}^n)\Big)^*$, we may write

$$\langle u,V_1-V_2\rangle=\lim_{j\to\infty}\langle\tilde{u}_j,V_1-V_2\rangle=\lim_{j\to\infty}\langle u_j,\mathscr{R}_\Omega(V_1-V_2)\rangle=0, \tag{2.36}$$

which proves the claim. It follows that the map (2.34)-(2.35) is well-defined, linear and bounded. Note that if $\varphi\in\mathscr{C}^\infty_c(\Omega)$ and $v\in L^{p'}_{-s}(\Omega)$, $V\in L^{p'}_{-s}(\mathbb{R}^n)$ are such that $\mathscr{R}_\Omega V=v$, then

$$\langle\Lambda_{\tilde{\varphi}},v\rangle=\langle\tilde{\varphi},V\rangle=\langle\varphi,\mathscr{R}_\Omega V\rangle=\langle\varphi,v\rangle. \tag{2.37}$$

Consequently, the map (2.34)-(2.35) satisfies $\Psi(\tilde{\varphi})=\varphi$, for each $\varphi\in\mathscr{C}^\infty_c(\Omega)$. Thus, the lemma is proved as soon as we show that Ψ in (2.34)-(2.35) is an isomorphism.

We shall do so by constructing an explicit inverse for Ψ. Concretely, for every functional $\Lambda\in\Big(L^{p'}_{-s}(\Omega)\Big)^*$ we set $\Psi^{-1}(\Lambda):=\Lambda\circ\mathscr{R}_\Omega$, where $\mathscr{R}_\Omega: L^{p'}_{-s}(\mathbb{R}^n)\to L^{p'}_{-s}(\Omega)$. It follows that

$\Psi^{-1}(\Lambda) \in \Big(L^{p'}_{-s}(\mathbb{R}^n)\Big)^* = L^p_s(\mathbb{R}^n)$ and we claim that, in fact, $\Psi^{-1}(\Lambda)$ belongs to $L^p_{s,0}(\Omega)$ for each $\Lambda \in \Big(L^{p'}_{-s}(\Omega)\Big)^*$. Indeed, to check that $\operatorname{supp}\Psi^{-1}(\Lambda) \subset \bar{\Omega}$ it suffices to observe that $\langle \Psi^{-1}(\Lambda), \varphi\rangle = \langle \Lambda, \mathscr{R}_\Omega(\varphi)\rangle = \langle \Lambda, 0\rangle = 0$ for each $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^n \setminus \bar{\Omega})$. Hence, all in all,

$$\Psi^{-1} : \Big(L^{p'}_{-s}(\Omega)\Big)^* \ni \Lambda \longmapsto \Lambda \circ \mathscr{R}_\Omega \in L^p_{s,0}(\Omega), \tag{2.38}$$

is well-defined, linear and bounded.

There remains to check that Ψ and Ψ^{-1} are inverse of each other. With this in mind, for an arbitrary $\Lambda \in \Big(L^{p'}_{-s}(\Omega)\Big)^*$, $u \in L^{p'}_{-s}(\Omega)$ and $U \in L^{p'}_{-s}(\mathbb{R}^n)$ such that $\mathscr{R}_\Omega U = u$, we have

$$\langle (\Psi \circ \Psi^{-1})\Lambda, u\rangle = \langle \Psi^{-1}(\Lambda), U\rangle = \langle \Lambda, \mathscr{R}_\Omega U\rangle = \langle \Lambda, u\rangle, \tag{2.39}$$

i.e., $(\Psi \circ \Psi^{-1})\Lambda = \Lambda$, as desired. Conversely, if $u \in L^p_{s,0}(\Omega)$ and $V \in L^{p'}_{-s}(\mathbb{R}^n) = (L^p_s(\mathbb{R}^n))^*$, we may write

$$\langle (\Psi^{-1} \circ \Psi)u, V\rangle = \langle \Psi u, \mathscr{R}_\Omega V\rangle = \langle u, V\rangle, \tag{2.40}$$

which shows that $(\Psi^{-1} \circ \Psi)u = u$ and finishes the proof. □

Here we also want to note that there is a natural inclusion

$$\Big(L^p_{s,z}(\Omega)\Big)^* \hookrightarrow L^{p'}_{-s}(\Omega) \quad \text{whenever} \quad s \in \mathbb{R} \quad \text{and} \quad 1 < p < \infty, \tag{2.41}$$

described as follows. If $\xi \in \Big(L^p_{s,z}(\Omega)\Big)^*$ then $\xi \circ \mathscr{R}_\Omega \in \Big(L^p_{s,0}(\Omega)\Big)^*$ and, by the Hahn-Banach Theorem, it can be be extended to some $\widehat{\xi \circ \mathscr{R}_\Omega} \in \Big(L^p_s(\mathbb{R}^n)\Big)^* = L^{p'}_{-s}(\mathbb{R}^n)$. Then (2.41) is simply the assignment $\xi \mapsto \mathscr{R}_\Omega(\widehat{\xi \circ \mathscr{R}_\Omega})$, which can be seen to be well-defined, linear, bounded and one-to-one. In the case when $s > -1 + \frac{1}{p}$ it can be shown that this assignment becomes onto as well, leading to the identification

$$\Big(L^p_{s,z}(\Omega)\Big)^* = L^{p'}_{-s}(\Omega) \quad \text{if} \quad 1 < p < \infty \quad \text{and} \quad s > -1 + \tfrac{1}{p}. \tag{2.42}$$

Since for each $s \in \mathbb{R}$ and $1 < p < \infty$ the space $L^p_{s,0}(\Omega)$ is reflexive (cf. (2.2)), we have, as a consequence of Lemma 2.4, that $L^p_s(\Omega)$ is reflexive as well. Furthermore, by (2.18), so is $L^p_{s,z}(\Omega)$ if $s > -1 + 1/p$. Consequently, taking the dual of (2.42) then yields

$$\Big(L^p_s(\Omega)\Big)^* = L^{p'}_{-s,z}(\Omega) \quad \text{if} \quad 1 < p < \infty \quad \text{and} \quad s < \tfrac{1}{p}. \tag{2.43}$$

In particular,

$$\Big(L^p_s(\Omega)\Big)^* = L^{p'}_{-s}(\Omega), \qquad \forall\, s \in (-1 + 1/p, 1/p). \tag{2.44}$$

Next, denote by $L^p_1(\partial\Omega)$ the Sobolev space of functions in $L^p(\partial\Omega)$ with tangential gradients in $L^p(\partial\Omega)$, $1 < p < \infty$. Besov spaces on $\partial\Omega$ can then be introduced via real interpolation and duality, i.e.

$$B^{p,q}_s(\partial\Omega) := \big(L^p(\partial\Omega), L^p_1(\partial\Omega)\big)_{s,q}, \quad \text{with } 0 < s < 1,\ 1 < p, q < \infty, \tag{2.45}$$

and if $0<s<1$, $1<p,q<\infty$,

$$B^{p,q}_{-s}(\partial\Omega):=\Big(B^{p',q'}_{s}(\partial\Omega)\Big)^{*}. \tag{2.46}$$

Recall (cf. [30], [29]) that the trace operator

$$\mathrm{Tr}: L^p_s(\Omega)\longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \tag{2.47}$$

is well-defined, bounded and onto if $1<p<\infty$ and $\frac{1}{p}<s<1+\frac{1}{p}$. Furthermore, for this range of indices, the trace operator (2.47) has a bounded, linear right inverse and its kernel is $L^p_{s,z}(\Omega)$.

Proposition 2.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists a linear operator K such that*

$$K:\Big(L^{p'}_{1-s}(\Omega)\Big)^{*}\longrightarrow L^p_{s,z}(\Omega,\mathbb{R}^n),\qquad 1<p<\infty,\quad s>-1+1/p, \tag{2.48}$$

boundedly, and which satisfies the following additional properties:

$$f\in\mathscr{C}^\infty_c(\Omega)\Longrightarrow Kf\in\mathscr{C}^\infty_c(\Omega,\mathbb{R}^n), \tag{2.49}$$

$$f\in\mathscr{C}^\infty_c(\Omega)\ \text{with}\ \langle f,1\rangle=0\Longrightarrow \mathrm{div}\,Kf=f. \tag{2.50}$$

Proof. When the Lipschitz domain Ω is star-like with respect to a ball, an operator K satisfying the properties (2.49)-(2.50) and such that

$$K:\Big(L^{p'}_{1-s}(\Omega)\Big)^{*}\longrightarrow \overset{\circ}{L}{}^p_s(\Omega,\mathbb{R}^n),\qquad 1<p<\infty,\quad s\in\mathbb{R}, \tag{2.51}$$

is bounded, has been constructed in [38]. Note that (2.48) follows from (2.51), first when $s-1/p\notin\mathbb{Z}$ by virtue of (2.29), then when $s>-1+1/p$ with the help of (2.20) and interpolation; cf. (2.26). Thus, we only need to explain how this construction can be further refined in order to apply to an arbitrary bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$.

Given that any Lipschitz domain is locally star-like, it suffices to prove the following claim. Let $\Omega\subset\mathbb{R}^n$ be an open, bounded, connected set such that

$$\bar\Omega\subset\bigcup_{j=1}^N D_j,\qquad \Omega\cap D_j\neq\emptyset,\quad 1\leq j\leq N, \tag{2.52}$$

for a finite collection of open, bounded sets $D_j\subset\mathbb{R}^n$, $j=1,...,N$. Then there exists a family of linear operators $\{P_j\}_{1\leq j\leq N}$ satisfying the following properties:

$$P_j\Big[\widetilde{\mathscr{C}^\infty_c(\Omega)}\Big]\subseteq\widetilde{\mathscr{C}^\infty_c(\Omega\cap D_j)},\qquad 1\leq j\leq N, \tag{2.53}$$

$$\int P_jf\,dx=0,\quad\forall\,f\in\widetilde{\mathscr{C}^\infty_c(\Omega)},\qquad 1\leq j\leq N, \tag{2.54}$$

$$\sum_{j=1}^N P_jf=f,\quad\forall\,f\in\widetilde{\mathscr{C}^\infty_c(\Omega)}\ \text{with}\ \int f\,dx=0, \tag{2.55}$$

$$\begin{aligned}&\forall\,j\in\{1,...,N\}\Rightarrow\exists\,\xi_j\in\widetilde{\mathscr{C}^\infty_c(D_j)},\ \exists\,k_j\in\widetilde{\mathscr{C}^\infty_c(\Omega)}\otimes\widetilde{\mathscr{C}^\infty_c(D_j)}\\&\text{such that}\ \ P_jf=\xi_jf+\int k_j(\cdot,y)f(y)\,dy\quad\forall\,f\in\widetilde{\mathscr{C}^\infty_c(\Omega)},\end{aligned} \tag{2.56}$$

where tilde denotes the extension to $\mathbb{R}^n$ by setting zero outside of the support. Below, we shall call such a family $\{P_j\}_{1\leq j\leq N}$ a partition of test functions (with vanishing moment), subordinate to the cover $\{D_j\}_{1\leq j\leq N}$.

Let us now explain how the existence of such a partition of test functions, $\{P_j\}_{1\leq j\leq N}$, can be used to conclude the proof of the proposition. Given a connected Lipschitz domain Ω, cover its closure as in (2.52) in such a way that each $\Omega\cap D_j$ is a Lipschitz domain which is star-like with respect to a ball. Now, for each $j=1,...,N$, let K_j be an operator which satisfies (2.51), (2.49), (2.50) with Ω replaced by $\Omega\cap D_j$. Then, granted the existence of a family $\{P_j\}_{1\leq j\leq N}$ satisfying (2.53)-(2.56), set

$$Q_j:=\Psi_j P_j\Psi^{-1},\qquad 1\leq j\leq N, \tag{2.57}$$

where

$$\Psi_j: L^p_{s-1,0}(\Omega\cap D_j)\longrightarrow\Big(L^{p'}_{1-s}(\Omega\cap D_j)\Big)^*,\qquad 1\leq j\leq N, \tag{2.58}$$

$$\Psi^{-1}:\Big(L^{p'}_{1-s}(\Omega)\Big)^*\longrightarrow L^p_{s-1,0}(\Omega) \tag{2.59}$$

are applications of the sort introduced in Lemma 2.4. In particular,

$$\Psi_j(\tilde{\varphi})=\varphi,\quad\forall\,\varphi\in\mathscr{C}^\infty_c(\Omega\cap D_j),\quad 1\leq j\leq N, \tag{2.60}$$

$$\Psi^{-1}(\varphi)=\tilde{\varphi},\quad\forall\,\varphi\in\mathscr{C}^\infty_c(\Omega). \tag{2.61}$$

Since the representation formula (2.56) guarantees that each P_j, $1\leq j\leq N$, extends to a bounded operator

$$P_j: L^p_{s-1,0}(\Omega)\longrightarrow L^p_{s-1,0}(\Omega\cap D_j), \tag{2.62}$$

it follows that

$$Q_j:\Big(L^{p'}_{1-s}(\Omega)\Big)^*\longrightarrow\Big(L^{p'}_{1-s}(\Omega\cap D_j)\Big)^*,\qquad 1\leq j\leq N, \tag{2.63}$$

are well-defined, linear and bounded operators which, in light of (2.60)-(2.61) and (2.53)-(2.55), satisfy

$$Q_j\Big[\mathscr{C}^\infty_c(\Omega)\Big]\subseteq\mathscr{C}^\infty_c(\Omega\cap D_j)\text{ and }\int Q_jf\,dx=0\ \ \forall\,f\in\mathscr{C}^\infty_c(\Omega),\ 1\leq j\leq N, \tag{2.64}$$

$$\sum_{j=1}^N\mathscr{R}_\Omega[\widetilde{Q_jf}]=f\ \text{ in }\ \Omega,\quad\forall\,f\in\mathscr{C}^\infty_c(\Omega)\text{ with }\int f\,dx=0. \tag{2.65}$$

Finally, we introduce

$$Kf:=\sum_{j=1}^N\mathscr{R}_\Omega[\widetilde{K_j(Q_jf)}],\qquad\forall\,f\in\mathscr{C}^\infty_c(\Omega). \tag{2.66}$$

By virtue of (2.63) and the fact that each K_j maps $\Big(L^{p'}_{1-s}(\Omega\cap D_j)\Big)^*$ boundedly into $L^p_{s,z}(\Omega\cap D_j)$, we may conclude that (2.48) holds (here (2.15) and (2.19) are also used). Going further, (2.49)

is a direct consequence of (2.64). To justify (2.50), for an arbitrary $f\in\mathscr{C}^\infty_c(\Omega)$ with $\int f\,dx=0$ we write

$$\operatorname{div} Kf=\sum_{j=1}^N\mathscr{R}_\Omega[\operatorname{div}\widetilde{K_j(Q_jf)}]=\sum_{j=1}^N\mathscr{R}_\Omega[\widetilde{Q_jf}]=f, \tag{2.67}$$

by (2.65) and the properties of K_j.

There remains to justify the claim made in the second paragraph of the proof, i.e. prove the existence of a partition of test functions subordinate to a given cover, of cardinality N. We shall proceed inductively, starting with the case $N=2$. In this scenario, we have $\bar\Omega\subset D_1\cup D_2$ and $D_j\cap\Omega\neq\emptyset$, $j=1,2$. Since Ω is connected, it follows that $D_1\cap D_2\cap\Omega\neq\emptyset$ and we pick some function $\psi\in\mathscr{C}^\infty_c(D_1\cap D_2\cap\Omega)$ with $\int\psi\,dx=1$. If for some fixed $\varphi_j\in\mathscr{C}^\infty_c(D_j)$, $j=1,2$, such that $\varphi_1+\varphi_2=1$ on $\bar\Omega$, we now set

$$(P_jf)(x):=\varphi_j(x)f(x)-\Big(\int\varphi_j(y)f(y)\,dy\Big)\psi(x),\quad f\in\mathscr{C}^\infty_c(\Omega),\ x\in D_j\cap\Omega,\ j=1,2, \tag{2.68}$$

then the properties (2.53)-(2.56) are readily verified when $N=2$ for the operators (2.68).

Assuming that a partition of test functions always exists whenever the cardinality of the cover is $N-1$, consider now the case when (2.52) holds. In a first stage, viewing $\cup_{1\leq j\leq N}D_j$ as $\Big(\cup_{1\leq j\leq N-1}D_j\Big)\cup D_N$ and invoking the case $N=2$ yields a partition of test functions, $\{P,P'\}$, subordinate to this cover, with the property that there exists an open, relatively compact subset $\mathcal{O}$ of $\cup_{1\leq j\leq N-1}D_j$ such that $\operatorname{supp}Pf\subset\mathcal{O}\cap\Omega$ for every $f\in\mathscr{C}^\infty_c(\Omega)$. Next, the induction's hypothesis yields yet another partition of test functions subordinate to the cover $\cup_{1\leq j\leq N-1}D_j$ of $\bar{\mathcal{O}}\cap\bar\Omega$, which we shall denote by $\{\mathcal{P}_j\}_{1\leq j\leq N-1}$. Then the desired partition of test functions subordinate to the cover $\{D_j\}_{1\leq j\leq N}$ of $\bar\Omega$ is $\{\mathcal{P}_1\circ P,\,\mathcal{P}_2\circ P,...,\mathcal{P}_{N-1}\circ P,\,P'\}$.

This concludes the proof of the existence of a family $\{P_j\}_{1\leq j\leq N}$ satisfying (2.53)-(2.56), and finishes the proof of the proposition. □

2.2 Sobolev spaces of vector fields

We debut by defining the normal component of a field in a suitable, weak sense.

Lemma 2.6. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ whose outward unit normal is denoted by ν and for $1<p<\infty$, $-1+\frac{1}{p}<s<\frac{1}{p}$, define*

$$\nu\odot:\Big\{(u,\eta)\in L^p_s(\Omega,\mathbb{R}^n)\oplus\Big(L^{p'}_{1-s}(\Omega)\Big)^*:\operatorname{div}u=\eta\ \text{as distributions in }\Omega\Big\}\longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \tag{2.69}$$

by setting

$$\langle\nu\odot(u,\eta),\phi\rangle:=\langle\eta,\Phi\rangle+\langle u,\nabla\Phi\rangle \tag{2.70}$$

for each $\phi\in\Big(B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)\Big)^*=B^{p',p'}_{-s+\frac{1}{p}}(\partial\Omega)$, where $\Phi\in L^{p'}_{1-s}(\Omega)$ is such that $\operatorname{Tr}\Phi=\phi$. Then the above definition is meaningful and the operator (2.69) is bounded in the sense that

$$\|\nu\odot(u,\eta)\|_{B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)}\leq C\Big(\|u\|_{L^p_s(\Omega,\mathbb{R}^n)}+\|\eta\|_{(L^{p'}_{1-s}(\Omega))^*}\Big). \tag{2.71}$$

Proof. Note that the second pairing in (2.70) is well-defined, thanks to (2.44).

We now prove that the definition (2.70) is independent of the choice of Φ. By linearity, this comes down to checking the following claim. If $u \in L^p_s(\Omega,\mathbb{R}^n)$ and $\eta \in \Big(L^{p'}_{1-s}(\Omega)\Big)^*$ are such that $\operatorname{div} u = \eta$ as distributions in Ω, and if $\Phi \in L^{p'}_{1-s}(\Omega)$ has $\operatorname{Tr}\Phi = 0$, then

$$\langle \eta, \Phi\rangle + \langle u, \nabla\Phi\rangle = 0. \tag{2.72}$$

To this end, we note that $-1+\frac{1}{p} < s < \frac{1}{p}$ entails $\frac{1}{p'} < 1-s < 1+\frac{1}{p'}$. Thus $\Phi \in L^{p'}_{1-s,z}(\Omega)$ and, by (2.29), there exists a sequence $\Phi_j \in \mathscr{C}^\infty_c(\Omega)$, $j\in\mathbb{N}$, such that $\Phi_j \to \Phi$ in $L^{p'}_{1-s}(\Omega)$. Now, since $\langle \eta, \Phi_j\rangle = \langle \operatorname{div} u, \Phi_j\rangle = -\langle u, \nabla\Phi_j\rangle$ for each j, (2.72) follows by letting $j\to\infty$ in this identity.

Finally, the estimate (2.71) follows from (2.72), (2.46), (2.44) and the fact that there exists $C>0$ such that any $\phi \in B^{p',p'}_{-s+1/p}(\partial\Omega)$ can be extended to a function $\Phi \in L^{p'}_{1-s}(\Omega)$ such that $\|\Phi\|_{L^{p'}_{1-s}(\Omega)} \leq C\|\phi\|_{B^{p',p'}_{-s+1/p}(\partial\Omega)}$; cf. the discussion pertaining to the properties of (2.47). □

A comment is in order here. The condition that there exists some $\eta \in (L^{p'}_{1-s}(\Omega))^*$ such that $\operatorname{div} u = \eta$ as distributions in Ω is a genuine demand. Indeed, if $u \in L^p_s(\Omega,\mathbb{R}^n)$ then $\operatorname{div} u$ belongs to $L^p_{s-1}(\Omega)$ but this membership does not guarantee that this distribution extends to a functional in $(L^{p'}_{1-s}(\Omega))^*$. Nonetheless, there are cases when such an extension naturally presents itself. For example, since

$$\tfrac{1}{p} > s > -1+\tfrac{1}{p} \Longrightarrow L^p_s(\Omega) = \Big(L^{p'}_{-s}(\Omega)\Big)^* \hookrightarrow \Big(L^{p'}_{1-s}(\Omega)\Big)^* \tag{2.73}$$

it follows that if $-1+\frac{1}{p} < s < \frac{1}{p}$ then any distribution in $L^p_s(\Omega)\Big(\hookrightarrow L^p_{s-1}(\Omega)\Big)$ canonically extends to a functional in $\Big(L^{p'}_{1-s}(\Omega)\Big)^*$. This observation suggests the following result.

Proposition 2.7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with outward unit normal ν and assume that $1<p<\infty$, $-1+\frac{1}{p} < s < \frac{1}{p}$. Define the mapping*

$$\nu\cdot : \Big\{u \in L^p_s(\Omega,\mathbb{R}^n) : \operatorname{div} u \in L^p_s(\Omega)\Big\} \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \tag{2.74}$$

by setting

$$\langle \nu\cdot u, \phi\rangle := \langle \operatorname{div} u, \Phi\rangle + \langle u, \nabla\Phi\rangle \tag{2.75}$$

for each $\phi \in \Big(B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)\Big)^* = B^{p',p'}_{-s+\frac{1}{p}}(\partial\Omega)$, where $\Phi \in L^{p'}_{1-s}(\Omega)$ is such that $\operatorname{Tr}\Phi = \phi$. Then the above definition is meaningful and the operator (2.74) is bounded in the sense that

$$\|\nu\cdot u\|_{B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)} \leq C\Big(\|u\|_{L^p_s(\Omega,\mathbb{R}^n)} + \|\operatorname{div} u\|_{L^p_s(\Omega)}\Big). \tag{2.76}$$

Proof. It suffices to observe that we have the following commutative diagram

$$\begin{array}{ccc} \Big\{(u,\eta)\in L^p_s(\Omega,\mathbb{R}^n)\oplus\Big(L^{p'}_{1-s}(\Omega)\Big)^*:\ \operatorname{div}u=\eta\ \text{in}\ \mathscr{D}'(\Omega)\Big\} & & \\ \Big\uparrow j & \searrow^{\nu\odot} & \\ \Big\{u\in L^p_s(\Omega,\mathbb{R}^n):\ \operatorname{div}u\in L^p_s(\Omega)\Big\} & \xrightarrow{\quad\nu\cdot\quad} & B^{p,p}_{s-\frac1p}(\partial\Omega) \end{array} \tag{2.77}$$

where $j(u):=(u,\operatorname{div}u)$. Then everything follows from Lemma 2.6. □

Remark I. When $1<p<\infty$, $s\geq 1/p$ and $u\in L^p_s(\Omega,\mathbb{R}^n)$ has $\operatorname{div}u\in L^p_s(\Omega)$, Proposition 2.7 yields

$$\nu\cdot u\in\bigcap_{-1+\frac1p<\alpha<\frac1p}B^{p,p}_{\alpha-\frac1p}(\partial\Omega), \tag{2.78}$$

and

$$\nu\cdot u=0\Longleftrightarrow\langle u,\nabla\Phi\rangle=-\langle\operatorname{div}u,\Phi\rangle,\quad\forall\,\Phi\in\mathscr{C}^\infty(\bar\Omega). \tag{2.79}$$

Remark II. If $1<p<\infty$, $s>1/p$ and $u\in L^p_{s,z}(\Omega,\mathbb{R}^n)$ has $\operatorname{div}u\in L^p_s(\Omega)$ then, necessarily, $\nu\cdot u=0$.

Proposition 2.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $1<p<\infty$ and $s>-1+\frac1p$. Then, for each vector field $u\in L^p_{s,z}(\Omega,\mathbb{R}^n)$ with $\operatorname{div}u\in L^p_{s,z}(\Omega)$, the following equivalences hold:*

$$\operatorname{div}\tilde u\in L^p_s(\mathbb{R}^n)\Longleftrightarrow\widetilde{\operatorname{div}u}=\operatorname{div}\tilde u\ \ \text{in}\ \ \mathbb{R}^n\Longleftrightarrow\nu\cdot u=0\ \ \text{on}\ \ \partial\Omega. \tag{2.80}$$

There are two consequences of this result which we wish to single out before going any further. First, if u is as in the statement of the proposition and $s>1/p$, Remark II above shows that $\widetilde{\operatorname{div}u}=\operatorname{div}\tilde u$ in $\mathbb{R}^n$. Second, when $-1+1/p<s<1/p$, then the equivalences (2.80) hold for any vector field $u\in L^p_s(\Omega,\mathbb{R}^n)$ such that $\operatorname{div}u\in L^p_s(\Omega)$.

Proof of Proposition 2.8. In general, given a field u as in the statement of the proposition, the distribution $\operatorname{div}\tilde u$ satisfies

$$\begin{aligned}\langle\operatorname{div}\tilde u,\Phi\rangle&=-\langle\tilde u,\nabla\Phi\rangle=-\langle u,\mathscr{R}_\Omega(\nabla\Phi)\rangle=-\langle u,\nabla(\mathscr{R}_\Omega\Phi)\rangle\\&=\langle\operatorname{div}u,\mathscr{R}_\Omega\Phi\rangle-\langle\nu\cdot u,\operatorname{Tr}\Phi\rangle,\quad\forall\,\Phi\in\mathscr{C}^\infty_c(\mathbb{R}^n).\end{aligned} \tag{2.81}$$

On the one hand, if $\operatorname{div}\tilde u\in L^p_s(\mathbb{R}^n)$ then $\operatorname{div}\tilde u-\widetilde{\operatorname{div}u}$ belongs to $\{w\in L^p_s(\mathbb{R}^n):\operatorname{supp}w\subset\partial\Omega\}$. Thanks to (2.17) and the current assumptions on our indices s, p, the latter space is trivial, which proves that $\widetilde{\operatorname{div}u}=\operatorname{div}\tilde u$ in $\mathbb{R}^n$. Armed with this, the fact that $\widetilde{\operatorname{div}u}\in L^p_{s,0}(\Omega)$, and (2.23), we can further transform the penultimate pairing in (2.81) by writing

$$\langle\operatorname{div}u,\mathscr{R}_\Omega\Phi\rangle=\langle\widetilde{\operatorname{div}u},\Phi\rangle=\langle\operatorname{div}\tilde u,\Phi\rangle. \tag{2.82}$$

In concert, (2.81) and (2.82) then force $\langle \nu \cdot u, \operatorname{Tr}\Phi\rangle = 0$ for each $\Phi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$. Since $\operatorname{Tr}\mathscr{C}^\infty(\bar\Omega) \hookrightarrow B^{p',p'}_{-\alpha+1/p}(\partial\Omega)$ densely whenever $-1+\frac{1}{p} < \alpha < \frac{1}{p}$, we ultimately obtain $\nu \cdot u = 0$ on $\partial\Omega$; cf. Remark I above.

On the other hand, if $\nu \cdot u = 0$ on $\partial\Omega$, we may write, based on (2.81) and (2.23),

$$\langle \operatorname{div}\tilde{u}, \Phi\rangle = \langle \operatorname{div} u, \mathscr{R}_\Omega\Phi\rangle = \langle \widetilde{\operatorname{div} u}, \Phi\rangle, \qquad \forall\, \Phi \in \mathscr{C}^\infty_c(\mathbb{R}^n). \tag{2.83}$$

Thus, $\operatorname{div}\tilde{u} = \widetilde{\operatorname{div} u}$ as distributions in $\mathbb{R}^n$. This concludes the proof of the proposition. □

Corollary 2.9. *Let Ω be a bounded Lipschitz domain and consider $p \in (1,\infty)$, $s > -1+1/p$. Then*

$$w \in L^p_{s,0}(\Omega,\mathbb{R}^n) \ \text{ and } \ \operatorname{div} w \in L^p_{s,0}(\Omega) \Longrightarrow \nu \cdot \mathscr{R}_\Omega w = 0 \ \text{ on } \ \partial\Omega. \tag{2.84}$$

Proof. Let w be as in the right-hand side of the implication (2.84) and set $u := \mathscr{R}_\Omega w \in L^p_{s,z}(\Omega,\mathbb{R}^n)$. In particular, $\operatorname{div} u = \mathscr{R}_\Omega(\operatorname{div} w) \in L^p_{s,z}(\Omega)$ and $\operatorname{div}\tilde{u} = \operatorname{div} w \in L^p_{s,0}(\Omega)$. Thus, by Proposition 2.8, $\nu \cdot u = 0$ on $\partial\Omega$ and the desired conclusion follows. □

We continue by discussing a basic density result.

Proposition 2.10. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $1 < p < \infty$, $s > -1+\frac{1}{p}$. Then the closure of*

$$\mathcal{D} := \{u \in \mathscr{C}^\infty_c(\Omega,\mathbb{R}^n) : \operatorname{div} u = 0 \text{ in } \Omega\} \tag{2.85}$$

in $L^p_{s,z}(\Omega,\mathbb{R}^n)$ is the space

$$V^{s,p}(\Omega) := \Big\{u \in L^p_{s,z}(\Omega,\mathbb{R}^n) : \operatorname{div} u = 0 \text{ in } \Omega \text{ and } \nu\cdot u = 0 \text{ on } \partial\Omega\Big\}. \tag{2.86}$$

We shall refer to the family $V^{s,p}(\Omega)$, $1<p<\infty$, $s > -1+1/p$ as the *Stokes scale* associated with the Lipschitz domain Ω. Since, by the first equality in (2.29), $L^p_{s,z}(\Omega,\mathbb{R}^n)$ is a closed subspace of $L^p_s(\Omega,\mathbb{R}^n)$ for $s-\frac{1}{p} \notin \mathbb{Z}$, it follows that $V^{s,p}(\Omega)$ can also be viewed as the closure of (2.85) in $L^p_s(\Omega,\mathbb{R}^n)$ whenever $1<p<\infty$, $s > -1+\frac{1}{p}$ and $s-\frac{1}{p} \notin \mathbb{N}_o$.

Proof of Proposition 2.10. From Proposition 2.7, we see that $V^{s,p}(\Omega)$ is a closed subspace of $L^p_{s,z}(\Omega,\mathbb{R}^n)$, hence the closure of $\{u \in \mathscr{C}^\infty_c(\Omega,\mathbb{R}^n) : \operatorname{div} u = 0 \text{ in } \Omega\}$ in $L^p_{s,z}(\Omega,\mathbb{R}^n)$ is included in $V^{s,p}(\Omega)$. Conversely, consider now an arbitrary field $u \in V^{s,p}(\Omega)$. Hence $u \in L^p_{s,z}(\Omega,\mathbb{R}^n)$ and, as the inclusion in (2.12) has dense range, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ of vector fields in $\mathscr{C}^\infty_c(\Omega,\mathbb{R}^n)$ converging to u in $L^p_{s,z}(\Omega,\mathbb{R}^n)$. In particular, since $s > -1+\frac{1}{p}$, we have $\tilde{u} \in L^p_{s,0}(\Omega,\mathbb{R}^n)$, $\operatorname{div}\tilde{u} = 0$ in $\mathbb{R}^n$, and

$$\tilde{u_j} \to \tilde{u} \ \text{ in } \ L^p_{s,0}(\Omega,\mathbb{R}^n) \quad \text{and} \quad \widetilde{\operatorname{div} u_j} = \operatorname{div}\tilde{u_j} \to 0 \ \text{ in } \ L^p_{s-1,0}(\Omega) \ \text{ as } \ j\to\infty. \tag{2.87}$$

Consequently, by Proposition 2.5, the sequence $v_j := u_j - K(\operatorname{div} u_j)$, $j \in \mathbb{N}$, consists of divergence-free vector fields in $\mathscr{C}^\infty_c(\Omega,\mathbb{R}^n)$ and

$$v_j = u_j - K(\Psi(\widetilde{\operatorname{div} u_j})) \to u \quad \text{in} \quad L^p_{s,z}(\Omega,\mathbb{R}^n) \quad \text{as } \ j\to\infty, \tag{2.88}$$

thanks to (2.87), (2.33) and (2.48). This finishes the proof of the proposition. □

As an immediate consequence of (2.30) and Remark II above, we have the following.

Corollary 2.11. *The Stokes scale introduced in (2.86) satisfies*

$$V^{s,p}(\Omega)=\begin{cases}\{u\in L^p_s(\Omega,\mathbb{R}^n):\,\operatorname{div}u=0\ \text{in}\ \Omega\ \ \text{and}\ \nu\cdot u=0\} & \text{if}\ -1+\frac{1}{p}<s<\frac{1}{p},\\ \Big\{u\in L^p_{s,z}(\Omega,\mathbb{R}^n):\,\operatorname{div}u=0\ \text{in}\ \Omega\Big\} & \text{if}\ s>\frac{1}{p}.\end{cases}\tag{2.89}$$

We are now ready to state and prove the main result in this subsection.

Theorem 2.12. *For each bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$, the Stokes scale*

$$\Big\{V^{s,p}(\Omega):\,1<p<\infty,\ \ s>-1+\tfrac{1}{p}\Big\}\tag{2.90}$$

is a complex interpolation scale. In other words, if $[\cdot,\cdot]_\theta$ stands for the usual complex interpolation bracket, then

$$\Big[V^{s_0,p_0}(\Omega)\,,\,V^{s_1,p_1}(\Omega)\Big]_\theta=V^{s,p}(\Omega)\tag{2.91}$$

whenever $1<p_i<\infty$, $-1+\frac{1}{p_i}<s_i$, $i=0,1$, $\theta\in[0,1]$, $\frac{1}{p}:=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $s:=(1-\theta)s_0+\theta s_1$.

Before turning to the proof of Theorem 2.12, we recall a version of an abstract interpolation result from [36].

Lemma 2.13. *Let X_i, Y_i, $i=0,1$, be two pairs of Banach spaces such that $X_0\cap X_1$ is dense in both X_0 and X_1, and similarly for Y_0, Y_1. Let D be a linear operator such that $D:X_i\to Y_i$ boundedly for $i=0,1$, and consider the following closed subspaces of X_i, $i=0,1$:*

$$\operatorname{Ker}(D;X_i):=\{u\in X_i:\,Du=0\},\quad i=0,1.\tag{2.92}$$

Finally, suppose that there exists a continuous linear mapping $G:Y_i\to X_i$ with the property $D\circ G=I$, the identity on Y_i for $i=0,1$. Then, for each $0<\theta<1$,

$$[\operatorname{Ker}(D;X_0),\operatorname{Ker}(D;X_1)]_\theta=\{u\in[X_0,X_1]_\theta:\,Du=0\},\qquad\theta\in(0,1).\tag{2.93}$$

Proof of Theorem 2.12. Fix $p_0,p_1,p,s_0,s_1,s,\theta$ as in the statement of the theorem. In a first stage, we attempt to implement Lemma 2.13 in which we take

$$X_i:=L^{p_i}_{s_i,0}(\Omega,\mathbb{R}^n)\quad\text{and}\quad Y_i:=\Big\{f\in L^{p_i}_{s_i-1,0}(\Omega):\,\langle f,1\rangle=0\Big\},\ i=0,1,\tag{2.94}$$

as well as

$$Du:=\operatorname{div}u\quad\text{and}\quad Gf:=\widetilde{K(\Psi f)},\tag{2.95}$$

where tilde is the extension by zero, K is the operator constructed in Proposition 2.5 and Ψ has been introduced in (2.33). In particular,

$$\Psi:L^{p_i}_{s_i-1,0}(\Omega)\longrightarrow\Big(L^{p_i'}_{1-s_i}(\Omega)\Big)^*,\ \ i\in\{0,1\},\tag{2.96}$$

$$\text{and}\ \ \Psi(\tilde\varphi)=\varphi\ \ \text{for every}\ \ \varphi\in\mathscr{C}^\infty_c(\Omega).\tag{2.97}$$

From (2.31) it is then immediate that

$$D : X_i \longrightarrow Y_i \quad \text{is well-defined, linear and bounded, for } i = 0, 1. \tag{2.98}$$

Furthermore, from (2.96), (2.48) and (2.19), the operator

$$G = \widetilde{K \circ \Psi} : L^{p_i}_{s_i-1,0}(\Omega) \longrightarrow L^{p_i}_{s_i,0}(\Omega, \mathbb{R}^n), \quad i = 0, 1, \tag{2.99}$$

is well-defined, linear and bounded. There remains to check that

$$D \circ G = I \quad \text{ on } \quad Y_i \quad \text{ for } \ i = 0, 1. \tag{2.100}$$

To see this, fix $f \in Y_i$ and pick $\varphi_j \in C^\infty_c(\Omega)$ such that $\langle \varphi_j, 1\rangle = 0$ for each j and $\tilde{\varphi}_j \to f$ in $L^{p_i}_{s_i-1,0}(\Omega)$ as $j \to \infty$ (the fact that the inclusion in (2.9) has dense range guarantees that this is possible). Then, based on (2.50), (2.49) and (2.97) we may write

$$\begin{aligned} \tilde{\varphi}_j &= \widetilde{\operatorname{div} K\varphi_j} = \operatorname{div}(\widetilde{K\varphi_j}) \\ &= \operatorname{div}(\widetilde{K\Psi\tilde{\varphi}_j}) = \operatorname{div}(G\tilde{\varphi}_j), \qquad \forall\, j. \end{aligned} \tag{2.101}$$

Passing to the limit $j \to \infty$ in the identity (2.101) finally yields, on account of (2.99) and (2.31), that $f = \operatorname{div}(Gf)$ in $L^{p_i}_{s_i-1,0}(\Omega)$.

This proves (2.100). Thus, Lemma 2.13 in concert with (2.26) then gives

$$\Big[\operatorname{Ker}(D; X_0), \operatorname{Ker}(D; X_1)\Big]_\theta = \Big\{u \in L^p_{s,0}(\Omega, \mathbb{R}^n) : \operatorname{div} u = 0 \text{ in } \mathbb{R}^n\Big\}, \tag{2.102}$$

where

$$\operatorname{Ker}(D; X_i) = \Big\{u \in L^{p_i}_{s_i,0}(\Omega, \mathbb{R}^n) : \operatorname{div} u = 0 \text{ in } \mathbb{R}^n\Big\}, \qquad i = 0, 1. \tag{2.103}$$

Our next step is to prove that the restriction operator induces an isomorphism

$$\mathscr{R}_\Omega : \Big\{u \in L^p_{s,0}(\Omega, \mathbb{R}^n) : \operatorname{div} u = 0 \text{ in } \mathbb{R}^n\Big\} \longrightarrow V^{s,p}(\Omega) \tag{2.104}$$

for each $1 < p < \infty$ and $s > -1 + 1/p$. Indeed, in order to show that $\mathscr{R}_\Omega$ in (2.104) is well-defined, we need to check that $\nu \cdot \mathscr{R}_\Omega u = 0$ for each divergence-free field $u \in L^p_{s,0}(\Omega, \mathbb{R}^n)$. This, however, is a consequence of Corollary 2.9.

Hence, in order to prove that (2.104) is an isomorphism, it suffices to check that the extension by zero operator

$$\widetilde{\cdot} : V^{s,p}(\Omega) \longrightarrow \Big\{u \in L^p_{s,0}(\Omega, \mathbb{R}^n) : \operatorname{div} u = 0 \text{ in } \mathbb{R}^n\Big\} \tag{2.105}$$

is well-defined and bounded. To this end, we notice that if $u \in V^{s,p}(\Omega)$ then Proposition 2.8 yields $\tilde{u} \in L^p_{s,0}(\Omega, \mathbb{R}^n)$ and $\operatorname{div} \tilde{u} = \widetilde{\operatorname{div} u} = 0$, as desired. All in all, the operator (2.104) is an isomorphism, whose inverse is (2.105).

The endgame in the proof of (2.91) is as follows. First, since

$$\mathscr{R}_\Omega : \operatorname{Ker}(D; X_i) \longrightarrow V^{s_i,p_i}(\Omega), \qquad i = 0, 1, \tag{2.106}$$

isomorphically (and with compatible inverses), it follows that

$$\mathscr{R}_{\Omega}:\Big[\mathrm{Ker}\,(D;X_0)\,,\,\mathrm{Ker}\,(D;X_1)\Big]_{\theta}\longrightarrow[V^{s_0,p_0}(\Omega),V^{s_1,p_1}(\Omega)]_{\theta}\quad\text{ isomorphically.}\tag{2.107}$$

Now (2.91) is a direct consequence of this, (2.102) and the fact that (2.104) is an isomorphism. □

Our next goal is to identify the duals of the spaces in the Stokes scale introduced in (2.86). To set the stage, we first recall the following particular case of a more general result due to G. De Rham [13].

Proposition 2.14. *For an arbitrary open set $\Omega\subseteq\mathbb{R}^n$ define the space $\mathcal{D}$ as in (2.85). Then, for each $u\in\Big(\mathscr{C}^{\infty}_{c}(\Omega,\mathbb{R}^n)\Big)'$, the following equivalence holds:*

$$\langle u,v\rangle=0\ \text{for each}\ v\in\mathcal{D}\iff\exists\,\Phi\in(\mathscr{C}^{\infty}_{c}(\Omega))'\ \text{such that}\ u=\nabla\Phi\ \text{in}\ \Omega.\tag{2.108}$$

We are now ready to state the following.

Theorem 2.15. *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain and fix $1<p<\infty$. Next, for each $s>-1+1/p$, let*

$$J_{s,p}:V^{s,p}(\Omega)\hookrightarrow L^{p}_{s,z}(\Omega,\mathbb{R}^n)\tag{2.109}$$

be the canonical inclusion, and consider its dual

$$J^{*}_{s,p}:\Big(L^{p}_{s,z}(\Omega,\mathbb{R}^n)\Big)^{*}\longrightarrow\Big(V^{s,p}(\Omega)\Big)^{*}.\tag{2.110}$$

Then the mapping (2.110) is onto and its kernel is precisely $\nabla[L^{p'}_{1-s}(\Omega)]$. In particular,

$$J^{*}_{s,p}:\frac{L^{p'}_{-s}(\Omega,\mathbb{R}^n)}{\nabla\Big[L^{p'}_{1-s}(\Omega)\Big]}\longrightarrow\Big(V^{s,p}(\Omega)\Big)^{*}\tag{2.111}$$

is an isomorphism.

Proof. Since $V^{s,p}(\Omega)$ is a closed subspace of $L^{p}_{s,z}(\Omega)$, Hahn-Banach's theorem immediately gives that the mapping (2.110) is onto. That (2.111) is an isomorphism will then follow as soon as we show that $\mathrm{Ker}\,J^{*}_{s,p}$, the null-space of the application (2.110), coincides with $\nabla[L^{p'}_{1-s}(\Omega)]$.

In one direction, if $u\in\Big(L^{p}_{s,z}(\Omega,\mathbb{R}^n)\Big)^{*}=L^{p'}_{-s}(\Omega,\mathbb{R}^n)$ is such that $J^{*}_{s,p}(u)=0$, then $\langle u,v\rangle=0$ for each $v\in\mathcal{D}$. In particular, by virtue of Proposition 2.14, there exists $\Phi\in\Big(\mathscr{C}^{\infty}_{c}(\Omega)\Big)^{*}$ such that $\nabla\Phi=u$. Proposition 2.3 then ensures that $\Phi\in L^{p'}_{1-s}(\Omega)$, so that $u\in\nabla\Big[L^{p'}_{1-s}(\Omega)\Big]$, as desired.

Conversely, if $u=\nabla\Phi\in L^{p'}_{-s}(\Omega,\mathbb{R}^n)$ for some $\Phi\in L^{p'}_{1-s}(\Omega)$ then Proposition 2.7 (cf. also the comments following its proof) allows us to write

$$\langle J^{*}_{s,p}(u),v\rangle=\langle\nabla\Phi,v\rangle=-\langle\Phi,\mathrm{div}\,v\rangle+\langle\mathrm{Tr}\,\Phi,\nu\cdot v\rangle=0,\tag{2.112}$$

for every $v\in V^{s,p}(\Omega)$. Thus, $J^{*}_{s,p}(u)=0$, finishing the proof of the theorem. □

Proposition 2.16. *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ there exists $\varepsilon = \varepsilon(\Omega) \in]0,1]$ with the following significance. Assume that $1 < p < \infty$, $-1 + 1/p < s < 1/p$ and that the pair $(s, 1/p)$ satisfies either of the following three conditions:*

$$\begin{aligned} &(I): && 0 < \tfrac{1}{p} < \tfrac{1-\varepsilon}{2} \quad \text{and} \quad -1 + \tfrac{1}{p} < s < \tfrac{3}{p} - 1 + \varepsilon; \\ &(II): && \tfrac{1-\varepsilon}{2} \leq \tfrac{1}{p} \leq \tfrac{1+\varepsilon}{2} \quad \text{and} \quad -1 + \tfrac{1}{p} < s < \tfrac{1}{p}; \\ &(III): && \tfrac{1+\varepsilon}{2} < \tfrac{1}{p} < 1 \ \text{ and } \ -2 + \tfrac{3}{p} - \varepsilon < s < \tfrac{1}{p}. \end{aligned} \tag{2.113}$$

Then

$$L^p_s(\Omega, \mathbb{R}^n) = V^{s,p}(\Omega) \oplus \nabla\Big[L^p_{s+1}(\Omega)\Big], \tag{2.114}$$

where the direct sum is topological (in fact, orthogonal when $s = 0$ and $p = 2$). Furthermore, if

$$\mathbb{P} : L^p_s(\Omega, \mathbb{R}^n) \longrightarrow V^{s,p}(\Omega) \tag{2.115}$$

denotes the projection onto the first summand in the decomposition (2.114), then its kernel is $\nabla\Big[L^p_{s+1}(\Omega)\Big]$. In particular,

$$\mathbb{P} : \frac{L^p_s(\Omega, \mathbb{R}^n)}{\nabla\Big[L^p_{s+1}(\Omega)\Big]} \longrightarrow V^{s,p}(\Omega) \tag{2.116}$$

is an isomorphism. Also, the adjoint of the operator

$$\mathbb{P}_{p,s} : L^p_s(\Omega, \mathbb{R}^n) \xrightarrow{\mathbb{P}} V^{s,p}(\Omega) \xrightarrow{J_{s,p}} L^p_s(\Omega, \mathbb{R}^n) \tag{2.117}$$

is the operator $\mathbb{P}_{p',-s}$, and

$$\Big(V^{s,p}(\Omega)\Big)^* = V^{-s,p'}(\Omega). \tag{2.118}$$

Proof. The decomposition (2.114) corresponding to the case when $s = 0$ has been established in [19] via an approach which reduces matters to the well-posedness of the inhomogeneous Neumann problem for the Laplacian in the Lipschitz domain Ω. The more general case considered here can be proved in an analogous fashion. With (2.114) in hand, the claims about the projection (2.115) are straightforward.

Finally, (2.118) is a consequence of the fact that $V^{s,p}(\Omega)$ is a closed subspace of $L^p_s(\Omega, \mathbb{R}^n)$, the Hahn-Banach theorem, (2.44), (2.114) and the identity $\mathbb{P}^*_{p,s} = \mathbb{P}_{p',-s}$. We omit the details. □

Remark. When $\partial\Omega \in \mathscr{C}^1$, one can choose $\varepsilon = 1$ in Proposition 2.16.

3 Boundary value problems for the Stokes system

For the reader's convenience, in this section we shall briefly review the solution of the Poisson and Dirichlet problems for the Stokes system.

3.1 The Poisson problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain and consider

$$\left\{\begin{array}{rcl} -\Delta u + \nabla \pi &=& f \in L^p_{s+\frac{1}{p}-2}(\Omega, \mathbb{R}^n), \\ \operatorname{div} u &=& 0 \text{ in } \Omega, \\ u &\in& L^p_{s+\frac{1}{p},z}(\Omega, \mathbb{R}^n), \\ \pi &\in& L^p_{s+\frac{1}{p}-1}(\Omega), \quad \langle \pi, 1\rangle = 0. \end{array}\right. \tag{3.1}$$

According to Theorem 5.6 in [17], the boundary value problem (3.1) has a unique solution in each of the following scenarios:

(i) $n \geq 2$, $\partial\Omega \in \mathscr{C}^1$, and $p \in]1, \infty[$, $s \in]0, 1[$ arbitrary;

(ii) $n \geq 2$, $s \in]0, 1[$, and $|\frac{1}{p} - \frac{1}{2}| < \varepsilon$ for some $\varepsilon = \varepsilon(\Omega) \in]0, \frac{1}{2}]$;

(iii) $n = 3$ and, for some $\varepsilon = \varepsilon(\Omega) \in]0, 1]$, the pair $(s, 1/p)$ satisfies either one of the following three conditions:

$$\begin{aligned} &(I): && \tfrac{1+\varepsilon}{2} \leq \tfrac{1}{p} < 1 \quad \text{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1; \\ &(II): && \tfrac{1-\varepsilon}{2} \leq \tfrac{1}{p} \leq \tfrac{1+\varepsilon}{2} \quad \text{and} \quad 0 < s < 1; \\ &(III): && 0 < \tfrac{1}{p} \leq \tfrac{1-\varepsilon}{2} \text{ and } 0 < s < \tfrac{2}{p} + \varepsilon; \end{aligned} \tag{3.2}$$

(iv) $n = 2$ and, for some $\varepsilon = \varepsilon(\Omega) \in]\frac{1}{2}, 1]$,

$$0 < s < 1, \quad 1 < p < \infty, \qquad \left|\tfrac{1}{p} - s\right| < \varepsilon. \tag{3.3}$$

3.2 The Dirichlet problem

Recall that by interpolating Sobolev (potential) spaces by the real method yields Besov spaces. More specifically,

$$\Big(L^p(\Omega), L^p_k(\Omega)\Big)_{s,q} = B^{p,q}_{sk}(\Omega), \tag{3.4}$$

if $1 < p, q < \infty$, $k > 0$, $0 < s < 1$.

It has been proved in [18] and [42] that for each bounded, connected, Lipschitz domain Ω in $\mathbb{R}^n$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the following property. If $2 - \varepsilon < p < 2 + \varepsilon$, the boundary value problem

$$\left\{\begin{array}{rcl} -\Delta u + \nabla \pi &=& 0 \text{ in } \Omega, \\ \operatorname{div} u &=& 0 \text{ in } \Omega, \\ \operatorname{Tr} u &=& g \in L^p_1(\partial\Omega, \mathbb{R}^n), \qquad \int_{\partial\Omega} \nu \cdot g \, d\sigma = 0, \end{array}\right. \tag{3.5}$$

has a solution which satisfies

$$u \in B^{p,p^\#}_{1+\frac{1}{p}}(\Omega, \mathbb{R}^n), \qquad \pi \in B^{p,p^\#}_{\frac{1}{p}}(\Omega), \qquad \text{where} \quad p^\# := \max\{p, 2\}. \tag{3.6}$$

Moreover, when $n = 3$, the above range of p's extends to $1 < p < 2 + \varepsilon$. See [43] for this last statement.

4 The Stokes operator on Lipschitz domains

Starting from an abstract setting, here we define the Stokes operator on a Lipschitz domain and study its properties.

4.1 The $\{\mathcal{H},\mathcal{V},a\}$ formalism

Let $\mathcal{V}$ be a reflexive Banach space continuously and densely embedded into a Hilbert space $\mathcal{H}$ so that, in particular,

$$\mathcal{V}\hookrightarrow\mathcal{H}\hookrightarrow\mathcal{V}^* \tag{4.1}$$

and assume that

$$a(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\longrightarrow\mathbb{C} \tag{4.2}$$

is a sesqui-linear, bounded form. Then

$$A_o:\mathcal{V}\longrightarrow\mathcal{V}^*,\qquad A_ou:=a(u,\cdot)\in\mathcal{V}^*,\quad\forall\,u\in\mathcal{V}, \tag{4.3}$$

is a linear, bounded operator satisfying

$$_{\mathcal{V}^*}\langle A_ou,v\rangle_{\mathcal{V}}=a(u,v),\quad\forall\,u,v\in\mathcal{V}. \tag{4.4}$$

Assume further that $a(\cdot,\cdot)$ is symmetric and coercive, in the sense that there exists $\kappa>0$ such that

$$\operatorname{Re}a(u,u)\geq\kappa\|u\|^2_{\mathcal{V}},\qquad\forall\,u\in\mathcal{V}. \tag{4.5}$$

Then

$$A_o:\mathcal{V}\longrightarrow\mathcal{V}^*\quad\text{ is bounded, self-adjoint and invertible.} \tag{4.6}$$

Going further, take A to be the *part of* A_o *in* $\mathcal{H}$, i.e., the unbounded operator

$$A:=A_o\Big|_{D(A)}:\mathcal{H}\longrightarrow\mathcal{H}, \tag{4.7}$$

where

$$D(A):=\{u\in\mathcal{V}:\,A_ou\in\mathcal{H}\}. \tag{4.8}$$

Then the unbounded operator (4.7)-(4.8) is self-adjoint and invertible on $\mathcal{H}$. Furthermore, there exists $\theta\in(0,\pi/2)$ such that

$$\|(\lambda I-A)^{-1}\|\leq\frac{C}{|\lambda|},\qquad\theta<|\arg(\lambda)|\leq\pi, \tag{4.9}$$

i.e., A is *sectorial*; cf., e.g., [12]. In particular, the operator $-A$ generates an analytic semigroup on $\mathcal{H}$ according to the formula

$$e^{-zA}u:=\frac{1}{2\pi i}\int_{\Gamma_{\theta'}}e^{-\lambda z}(\lambda I-A)^{-1}u\,d\lambda,\qquad|\arg(z)|<\pi/2-\theta', \tag{4.10}$$

where $\theta'\in(\theta,\pi/2)$ and $\Gamma_{\theta'}:=\{re^{i\theta'}:\,r>0\}$. Furthermore, since A is invertible in our case, the semigroup $\left(e^{-tA}\right)_{t>0}$ is bounded. See, e.g., [12], [45], [54] (let us also point out that the above

formalism – discussed in detail in, e.g., [12] – is closely related to K.O. Friedrichs' extension method, as described on p. 325 of [32], and on p. 514 of Vol. I of [54]).

Since A satisfies (4.9), we can also define *fractional powers* of A. Specifically, for $z \in \mathbb{C}$ with $\operatorname{Re} z \in [0,1)$ and $u \in D(A) \cap AD(A)$ (which is a dense subset of $\mathcal{H}$), we set

$$A^z u := \frac{\sin(\pi z)}{\pi} \int_0^\infty t^z (t+A)^{-1} Au \, \frac{dt}{t}. \tag{4.11}$$

More generally, for $z \in \mathbb{C}$ with $\operatorname{Re} z \in (-1,1)$ and $u \in D(A) \cap AD(A)$,

$$A^z u := \frac{\sin(\pi z)}{\pi} \left(\frac{u}{z} - \frac{A^{-1}u}{1+z} + \int_0^1 t^{z+1} (t+A)^{-1} Au \, dt + \int_1^\infty t^{z-1} (t+A)^{-1} Au \, dt \right). \tag{4.12}$$

The above formula reduces, for $-1 < \operatorname{Re} z < 0$, to

$$A^z u := \frac{\sin(\pi z)}{\pi} \int_0^\infty t^z (t+A)^{-1} u \, dt, \qquad \forall\, u \in \mathcal{H}. \tag{4.13}$$

In this case, the integral is absolutely convergent and A^z is bounded on $\mathcal{H}$.

Another useful representation of $A^{-\alpha}$ as a bounded operator on $\mathcal{H}$, whose validity extends to any $\alpha > 0$, is

$$A^{-\alpha} u := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-tA} u \, \frac{dt}{t}, \qquad \forall\, u \in \mathcal{H}, \tag{4.14}$$

where Γ is the classical Gamma function.

Then $A^{-\alpha}$ is one-to-one for every $\alpha > 0$ and one convenient way to introduce the domain of (positive) fractional powers of the unbounded operator A is

$$D(A^\alpha) := A^{-\alpha}\mathcal{H}, \quad \text{the range of } A^{-\alpha} \text{ acting on } \mathcal{H}, \qquad \alpha > 0, \tag{4.15}$$

which becomes a Banach space when equipped with the graph norm

$$\|u\|_{D(A^\alpha)} := \|u\|_{\mathcal{H}} + \|A^\alpha u\|_{\mathcal{H}}. \tag{4.16}$$

In this connection, it is useful to note that since $A^{-\alpha}$ is bounded on $\mathcal{H}$, we have $\|u\|_{\mathcal{H}} = \|A^{-\alpha}(A^\alpha u)\|_{\mathcal{H}} \leq C\|A^\alpha u\|_{\mathcal{H}}$ for every $u \in \mathcal{H}$, $\alpha > 0$, and hence

$$\|u\|_{D(A^\alpha)} \approx \|A^\alpha u\|_{\mathcal{H}}, \quad \text{uniformly for } \ u \in D(A^\alpha), \tag{4.17}$$

for every $\alpha > 0$.

Later on, we shall make frequent use of the fact that

$$\|A^\alpha e^{-tA}\|_{\mathscr{L}(\mathcal{H})} \leq C_\alpha t^{-\alpha}, \qquad \alpha > 0, \tag{4.18}$$

where $\mathscr{L}(\mathcal{H})$ denotes the Banach space of linear, bounded operators mapping $\mathcal{H}$ into itself. In turn, (4.18) can be used to show that, for each $\alpha > 0$ and $u \in \mathcal{H}$, the mapping

$$]0,\infty[\ni t \longmapsto e^{-tA} u \in D(A^\alpha) \tag{4.19}$$

is continuous. Furthermore,

$$\text{the map (4.19) extends continuously to } [0,\infty[\iff u \in D(A^\alpha). \tag{4.20}$$

Other properties are discussed in, e.g., Pazy's book [45], to which we refer the interested reader. Here we only wish to summarize the properties of fractional powers which are relevant for our work.

Proposition 4.1. *For the operator A associated with the triplet $\{\mathcal{V},\mathcal{H},a\}$ as before, the following hold:*

(i) For each $\alpha \geq 0$, $A^{-\alpha}$ is one-to-one, bounded and with dense range on $\mathcal{H}$. Furthermore, there exists $C>0$ such that $\|A^{-\alpha}u\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{H}}$ for every $\alpha \in [0,1]$.

(ii) For every $\alpha,\beta \geq 0$, $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$, and $\lim_{\alpha\to 0^+} A^{-\alpha}u = u$ for every $u \in \mathcal{H}$.

Furthermore, if $A^\alpha := (A^{-\alpha})^{-1}$ for $\alpha > 0$, then also:

(iii) For each $\alpha > 0$, A^α is an unbounded, self-adjoint operator on $\mathcal{H}$, whose domain is $A^{-\alpha}\mathcal{H}$, the range of $A^{-\alpha}$. In particular, $u = A^{-\alpha}(A^\alpha u)$ for every $u \in D(A^\alpha)$.

(iv) If $\alpha \geq \beta \geq 0$, then $D(A^\alpha) \subset D(A^\beta)$.

(v) For each $\alpha \in]0,1[$ there exists $C>0$ such that $\|A^\alpha u\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{H}}^{1-\alpha}\|Au\|_{\mathcal{H}}^{\alpha}$ for every $u \in D(A)$.

(vi) If $\alpha,\beta \in \mathbb{R}$ then $A^\alpha(A^\beta u) = A^{\alpha+\beta}u$ for every $u \in D(A^\gamma)$, where $\gamma := \max\{\alpha,\beta,\alpha+\beta\}$.

(vii) For every $\alpha \in]0,1[$ and $u \in \mathcal{H}$, there holds $A^\alpha\Big(\int_0^t e^{-sA}u\,ds\Big) = \int_0^t A^\alpha e^{-sA}u\,ds$.

We continue by recording some well-known results of Kato and Lions (see [31], [35]).

Proposition 4.2. *For A as above, there holds*

$$D(A^{1/2}) = \mathcal{V} \tag{4.21}$$

and

$$D(A^\theta) = [\mathcal{H}, D(A)]_\theta, \quad 0 \leq \theta \leq 1. \tag{4.22}$$

Hence, by the reiteration theorem for the complex method,

$$\Big\{D(A^{\frac{s}{2}}) : 0 \leq s \leq 2\Big\} \quad \text{is a complex interpolation scale.} \tag{4.23}$$

In particular,

$$D(A^{\theta/2}) = [\mathcal{H},\mathcal{V}]_\theta, \quad 0 \leq \theta \leq 1. \tag{4.24}$$

Corollary 4.3. *Under the assumptions (4.1)-(4.5) and with A_o as in (4.6)-(4.7),*

$$D(A^{\frac{1+\theta}{2}}) = A_o^{-1}\Big(D(A^{\frac{1-\theta}{2}})\Big)^* \tag{4.25}$$

for every $0 \leq \theta \leq 1$.

Proof. As already observed above, the fact that (4.5) holds entails that $A_o : \mathcal{V} \to \mathcal{V}^*$ is an isomorphism. Based on this and the definition of $D(A)$, it is then immediate that $A_o : D(A) \to \mathcal{H}$ is an isomorphism as well. Interpolating between these two cases then proves (with the help of (4.21)-(4.22), and the duality theorem for the complex method) that the operator

$$A_o : D(A^{\frac{1+\theta}{2}}) = [\mathcal{V}, D(A)]_\theta \to [\mathcal{V}^*, \mathcal{H}]_\theta = [\mathcal{H}, \mathcal{V}]^*_{1-\theta} = \Big(D(A^{\frac{1-\theta}{2}})\Big)^* \tag{4.26}$$

is an isomorphism, for every $0 \leq \theta \leq 1$. From this, (4.25) readily follows. □

We conclude with a brief discussion of the abstract Cauchy problem

$$\left\{ \begin{array}{rcll} u' + Au & = & f, & \text{on } (0,T), \\ u(0) & = & u_0, & \end{array} \right. \tag{4.27}$$

for some given

$$u_0 \in \mathcal{H} \quad \text{and} \quad f \in L^p([0,T];\mathcal{H}), \ \ 1 \leq p \leq \infty. \tag{4.28}$$

Call $u \in \mathscr{C}([0,T];\mathcal{H})$ a *mild solution* of (4.27) if it satisfies an integrated version of this problem, i.e.

$$\int_0^t u(s)\,ds \in D(A) \quad \text{and} \quad u(t) = u_0 + A\Big(\int_0^t u(s)\,ds\Big) + \int_0^t f(s)\,ds, \qquad \forall\, t \in [0,T]. \tag{4.29}$$

As is well-known (cf., e.g., the discussion on p. 9 in [2]), given that A is the generator of a C^0-semigroup, (4.27) has a unique mild solution given by

$$u(t) = e^{-tA}u_0 + (e^{-\cdot A} * f)(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)\,ds, \qquad t \in [0,T]. \tag{4.30}$$

In order to be able to discuss the concept of *strong solution* of the Cauchy problem (4.27), consider

$$L^p_1(]0,T[;\mathcal{H}) := \{u \in L^p(]0,T[;\mathcal{H}) : u' \in L^p(]0,T[;\mathcal{H})\}, \tag{4.31}$$

where the time-derivative is taken in the sense of distributions. Sobolev's embedding theorem yields

$$L^p_1(]0,T[;\mathcal{H}) \hookrightarrow \mathscr{C}([0,T];\mathcal{H}) \tag{4.32}$$

for each $p \in [1,\infty]$. In particular, the Fundamental Theorem of Calculus holds for functions $u \in L^p_1(]0,T[;\mathcal{H})$, i.e., $u(t_1) - u(t_o) = \int_{t_o}^{t_1} u'(s)\,ds$ if $t_o, t_1 \in [0,T]$. Consequently, this and Lebesgue's Differentiation Theorem then prove that the *pointwise derivative* $u'(t)$ exists at almost every $t \in [0,T]$.

With this preamble out of the way, call u a *strong solution* of the abstract Cauchy problem (4.27)-(4.28) if $u \in L^p_1(]0,T[;\mathcal{H})$, $u(t) \in D(A)$ for a.e. $t \in [0,T]$, and $u'(t) + (Au)(t) = f(t)$ for a.e. $t \in [0,T]$. A few remarks are in order here. First, for a strong solution, the condition $u(t) \in D(A)$ for a.e. $t \in [0,T]$ self-improves *a posteriori* to $u \in L^p([0,T];D(A))$. Second, since A is the infinitesimal generator of a C^0-semigroup on $\mathcal{H}$, it follows that the (unique) mild solution of (4.27)-(4.28) is a strong solution if and only if $u \in L^p_1(]0,T[;\mathcal{H})$. Third,

$$\alpha \in]0,1[\text{ and } u_0 \in D(A^\alpha) \Longrightarrow e^{-tA}u_0 \in L^p_1(]0,T[;\mathcal{H}) \ \text{ whenever } \ 1 < p < (1-\alpha)^{-1}. \tag{4.33}$$

According to a fundamental result, originally due to L. de Simon [14], for each $p \in (1,\infty)$,

$$e^{-\cdot A} * f \in L^p_1(]0,T[;\mathcal{H}) \quad \text{for all} \quad f \in L^p([0,T];\mathcal{H}), \tag{4.34}$$

given that A generates an analytic semigroup on the Hilbert space $\mathcal{H}$. As a consequence of this discussion we can now state the following.

Corollary 4.4. *Let A, $\mathcal{H}$ be as before, and fix $T>0$, $\alpha\in(0,1)$, and $1<p<(1-\alpha)^{-1}$. Then the (unique) mild solution of the Cauchy problem (4.29)-(4.28) is in fact a strong solution whenever $u_0\in D(A^\alpha)$.*

4.2 The Stokes operator

We continue to assume that $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain and denote by ν its unit normal. Set

$$\mathcal{H} := V^{0,2}(\Omega) = \Big\{u\in L^2(\Omega,\mathbb{R}^n):\ \operatorname{div} u=0 \text{ in } \Omega,\ \nu\cdot u=0 \text{ on } \partial\Omega\Big\}, \tag{4.35}$$

$$\mathcal{V} := V^{1,2}(\Omega) = \mathcal{H}\cap L^2_{1,z}(\Omega,\mathbb{R}^n) = \Big\{u\in L^2_{1,z}(\Omega,\mathbb{R}^n):\ \operatorname{div} u=0 \text{ in } \Omega\Big\}, \tag{4.36}$$

and note that, by Proposition 2.10, the spaces $\mathcal{H}$, $\mathcal{V}$ are the closure of $\mathcal{D}$ in the norm of $L^2(\Omega,\mathbb{R}^n)$ and $L^2_{1,z}(\Omega,\mathbb{R}^n)$, respectively. In particular, the canonical injection $\mathcal{V}\hookrightarrow\mathcal{H}$ has dense range.

Going further, on $\mathcal{V}\times\mathcal{V}$ we define the form $a(\cdot,\cdot)$ by

$$a(u,v) := \int_\Omega \langle\nabla u,\nabla v\rangle\,dx, \qquad u,v\in\mathcal{V}, \tag{4.37}$$

and note that this is a bilinear, symmetric and coercive, thanks to Poincaré's inequality. The goal is to identify the unbounded operator A canonically induced by the triplet $\{V,\mathcal{H},a\}$ just introduced. To this end, we bring in the Leray projection

$$\mathbb{P}: L^2(\Omega,\mathbb{R}^n)\longrightarrow\mathcal{H}, \tag{4.38}$$

i.e., the orthogonal projection from $L^2(\Omega,\mathbb{R}^n)$ onto the closed subspace $\mathcal{H}$. The operator $\mathbb{P}$, originally defined as in (4.38), can then be naturally extended to other settings. First, it is clear that (4.38) is compatible with (2.115), defined under the assumption that one of the three conditions in (2.113) holds.

Second, with $\mathcal{D}$ as in (2.85), we let

$$J:\mathcal{D}\hookrightarrow\mathscr{C}^\infty_c(\Omega,\mathbb{R}^n), \tag{4.39}$$

stand for the canonical inclusion and note that the diagram

$$\begin{array}{ccc} (\mathscr{C}^\infty_c(\Omega,\mathbb{R}^n))' & \xrightarrow{J^*} & \mathcal{D}' \\ \uparrow & & \uparrow \\ L^2(\Omega,\mathbb{R}^n) & \xrightarrow{\mathbb{P}} & \mathcal{H} \end{array} \tag{4.40}$$

in which the vertical arrows are natural inclusions, is commutative. Consequently, we may extend (4.38) to

$$\mathbb{P} = J^* : (\mathscr{C}^\infty_c(\Omega, \mathbb{R}^n))' \longrightarrow \mathcal{D}'. \tag{4.41}$$

Next, for $1 < p < \infty$ and $s > -1 + 1/p$, recall the injections $J_{s,p} : V^{s,p}(\Omega) \hookrightarrow L^p_{s,z}(\Omega, \mathbb{R}^n)$ from (2.109) and consider the commutative diagram

$$\begin{array}{ccc} (\mathscr{C}^\infty_c(\Omega, \mathbb{R}^n))' & \xrightarrow{J^*} & \mathcal{D}' \\ \uparrow & & \uparrow \\ L^{p'}_{-s}(\Omega, \mathbb{R}^n) & \xrightarrow{J^*_{s,p}} & \Big(V^{s,p}(\Omega)\Big)^* \end{array} \tag{4.42}$$

in which, once again, the two vertical arrows are natural inclusions. Thus, the operator (4.41) can be further viewed as

$$\mathbb{P} = J^*_{s,p} : L^{p'}_{-s}(\Omega, \mathbb{R}^n) \longrightarrow \Big(V^{s,p}(\Omega)\Big)^*, \qquad s > -1 + \tfrac{1}{p}. \tag{4.43}$$

In particular, corresponding to $p = 2$ and $s = 1$,

$$\mathbb{P} : L^2_{-1}(\Omega, \mathbb{R}^n) \longrightarrow \mathcal{V}^*. \tag{4.44}$$

For further use, let us also point out that the operator (4.43) factors as

$$\mathbb{P} : L^{p'}_{-s}(\Omega, \mathbb{R}^n) \xrightarrow{\text{pr}} \frac{L^{p'}_{-s}(\Omega, \mathbb{R}^n)}{\nabla[L^{p'}_{1-s}(\Omega)]} \xrightarrow{J^*_{s,p}} \Big(V^{s,p}(\Omega)\Big)^*, \tag{4.45}$$

where the first arrow is the canonical projection onto the quotient space, and the second arrow is the isomorphism (2.111). In addition, as a corollary of Proposition 2.14 and Theorem 2.15, the null-spaces of these operators are

$$\text{Ker}\Big[\mathbb{P} : (\mathscr{C}^\infty_c(\Omega, \mathbb{R}^n))' \longrightarrow \mathcal{D}'\Big] = \nabla\Big(\mathscr{C}^\infty_c(\Omega)\Big)', \tag{4.46}$$

$$\text{Ker}\Big[\mathbb{P} : L^{p'}_{-s}(\Omega, \mathbb{R}^n) \longrightarrow \Big(V^{s,p}(\Omega)\Big)^*\Big] = \nabla\Big[L^{p'}_{1-s}(\Omega)\Big], \tag{4.47}$$

whenever $1 < p < \infty$ and $s > -1 + 1/p$.

In summary, we shall continue to denote by $\mathbb{P}$ the extension of the Leray projection (4.38) to any of the situations (4.41), (4.43). Moreover, since for each $s > -1 + 1/p$ the diagram

$$\begin{array}{ccc} V^{s,p}(\Omega) & \xrightarrow{J_{s,p}} & L^p_{s,z}(\Omega, \mathbb{R}^n) \\ \uparrow & & \uparrow \\ \mathcal{D} & \xrightarrow{J} & \mathscr{C}^\infty_c(\Omega, \mathbb{R}^n) \end{array} \tag{4.48}$$

is also commutative, in an effort to streamline notation we also agree to drop the subscripts s, p when referring to the operator $J_{s,p}$ and simply write J (as in (4.39)).

After this preamble, we return to the task of defining the Stokes operator. As before, the form (4.37) gives rise to a bounded, invertible, self-adjoint operator A_o mapping $\mathcal{V}$ onto its dual, and such that $A_o u := a(u,\cdot)$ for every $u \in \mathcal{V}$.

Proposition 4.5. *With $\Delta_D : L^2_{1,z}(\Omega,\mathbb{R}^n) \to L^2_{-1}(\Omega,\mathbb{R}^n)$, $\Delta_D u := \Delta u$, denoting the Dirichlet-Laplacian acting componentwise on vector fields in Ω, there holds*

$$A_o = \mathbb{P} \circ (-\Delta_D) \circ J : \mathcal{V} \longrightarrow \mathcal{V}^*. \tag{4.49}$$

Proof. Let $u, v \in \mathcal{V}$ be arbitrary. By unraveling definitions, we may write

$$\begin{aligned} {}_{\mathcal{V}^*}\langle A_o u, v\rangle_{\mathcal{V}} &= a(u,v) = \sum_{i=1}^{n} \langle \partial_i Ju, \partial_i Jv\rangle \\ &= {}_{L^2_{-1}(\Omega,\mathbb{R}^n)}\big\langle (-\Delta_D)Ju, Jv\big\rangle_{L^2_{1,z}(\Omega,\mathbb{R}^n)} \\ &= {}_{\mathcal{V}^*}\langle \mathbb{P}(-\Delta_D)u, v\rangle_{\mathcal{V}}, \end{aligned} \tag{4.50}$$

where the last equality uses the fact that

$${}_{L^2_{-1}(\Omega,\mathbb{R}^n)}\big\langle w, Jv\big\rangle_{L^2_{1,z}(\Omega,\mathbb{R}^n)} = {}_{\mathcal{V}^*}\langle \mathbb{P}w, v\rangle_{\mathcal{V}}, \tag{4.51}$$

for each $w \in L^2_{-1}(\Omega,\mathbb{R}^n)$ and $v \in \mathcal{V}$. □

Definition 4.6. Assume that Ω is a connected, bounded Lipschitz domain in $\mathbb{R}^n$. The unbounded operator $A : \mathcal{H} \to \mathcal{H}$ defined on its domain $D(A) := \{u \in \mathcal{V} : A_o u \in \mathcal{H}\}$ by $Au := A_o u$, is called the Stokes operator (associated with the domain Ω).

We are finally ready to describe the Stokes operator associated with a Lipschitz domain Ω.

Theorem 4.7. *The Stokes operator is characterized by*

$$\begin{aligned} D(A) &= \Big\{ u \in L^2_{1,z}(\Omega,\mathbb{R}^n) : \operatorname{div} u = 0 \text{ and } \exists\, \pi \in L^2(\Omega) \text{ such that } -\Delta u + \nabla\pi \in \mathcal{H} \Big\}, \\ Au &= -\Delta u + \nabla\pi, \quad \forall\, u \in D(A) \text{ and } \pi \in L^2(\Omega) \text{ such that } -\Delta u + \nabla\pi \in \mathcal{H}. \end{aligned}$$

Proof. From (4.7)-(4.8) we know that a vector field $u \in \mathcal{V}$ belongs to $D(A)$ if and only if $A_o u \in \mathcal{H}$, in which case $Au = A_o u$. Thus, if $u \in D(A)$ then (4.49) yields

$$\mathbb{P}(Au - (-\Delta_D)Ju) = \mathbb{P}Au - \mathbb{P}((-\Delta_D)Ju) = Au - Au = 0, \tag{4.52}$$

since $\mathbb{P}$ leaves each vector in $\mathcal{H}$ invariant. Hence, by (4.47), there exists a unique scalar function $\pi \in L^2(\Omega)$ such that $\int_\Omega \pi\, dx = 0$ and $Au - (-\Delta_D)Ju = \nabla\pi$.

Conversely, if $u \in \mathcal{V}$ is such that there exists $\pi \in L^2(\Omega)$ for which $-\Delta u + \nabla\pi \in \mathcal{H}$, then

$$A_o u = \mathbb{P}[(-\Delta_D)Ju] = \mathbb{P}[(-\Delta_D)Ju + \nabla\pi] = -\Delta u + \nabla\pi \in \mathcal{H} \tag{4.53}$$

thanks to (4.47) and the fact that $\mathbb{P}$ leaves $\mathcal{H}$ invariant. □

Remark. Note that the unbounded operator

$$\begin{aligned} &B:\mathcal{H}\longrightarrow\mathcal{H},\quad D(B):=\mathcal{D}=\mathscr{C}_c^\infty(\Omega,\mathbb{R}^n)\cap\mathcal{H},\\ &Bu:=-\mathbb{P}(\Delta u),\qquad \forall\, u\in D(B), \end{aligned} \tag{4.54}$$

is densely defined, symmetric and positive. The Friedrichs extension of B is then given by

$$\begin{aligned} &A:\mathcal{H}\longrightarrow\mathcal{H},\qquad Au:=B^*u,\quad \forall\, u\in D(A),\\ &D(A):=\Big\{u\in D(B^*):\,\exists\, u_j\in D(B)\text{ such that}\\ &\qquad\qquad u_j\to u\text{ in }\mathcal{H}\text{ and }\langle B(u_j-u_k),u_j-u_k\rangle\to 0\Big\} \end{aligned} \tag{4.55}$$

(cf., e.g., the discussion on p. 194 of [53], and pp. 325-326 in [32]). It is then straightforward to check that (4.55) is precisely the Stokes operator described in Theorem 4.7.

Corollary 4.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then the Stokes operator A described in Theorem 4.7 is self-adjoint and generates an analytic semigroup $(e^{-tA})_{t\geq 0}$ in $\mathcal{H}$. In addition, (4.21)-(4.24) and (4.25) hold as well.*

Proof. All claims are direct consequences of the discussion in §3.1. □

Corollary 4.9. *The Stokes operator A satisfies the maximal L^p-regularity condition for all $p\in]1,\infty[$. That is, for each $1<p<\infty$ there exists a constant $C_p>0$ such that for all $0<T\leq\infty$, for all $f\in L^p([0,T];\mathcal{H})$ the inhomogenous Cauchy problem*

$$\left\{\begin{array}{rcll} u'(t)+Au(t) &=& f(t) & \text{a.e. } t\in(0,T),\\ u(0) &=& 0 & \end{array}\right. \tag{4.56}$$

has a unique strong solution $u\in L^p_1(]0,T[;\mathcal{H})\cap L^p(]0,T[;D(A))$ given by the convolution formula

$$u(t)=(e^{-\cdot A}*f)(t):=\int_0^t e^{-(t-s)A}f(s)\,ds,\qquad t\in(0,T), \tag{4.57}$$

and for which

$$\|u'\|_{L^p([0,T];\mathcal{H})}+\|Au\|_{L^p([0,T];\mathcal{H})}\leq C_p\|f\|_{L^p([0,T];\mathcal{H})}. \tag{4.58}$$

Proof. That the Stokes operator satisfies the maximal L^p-regularity condition whenever $1<p<\infty$, follows from Corollary 4.8 and the fact that the space $\mathcal{H}$ is Hilbert; cf, e.g., [14]. □

As discussed in Proposition 4.5, the operator $A_o:\mathcal{V}\to\mathcal{V}^*$ factors as

$$\begin{array}{ccccc} V^{1,2}(\Omega) & & \xrightarrow{\qquad A_o\qquad} & & \\ \cap\,\downarrow J & & & & \\ L^2_{1,z}(\Omega,\mathbb{R}^n) & \xrightarrow{-\Delta_D} & L^2_{-1}(\Omega,\mathbb{R}^n) & \xrightarrow{\mathbb{P}} & \Big(V^{1,2}(\Omega)\Big)^* \end{array} \tag{4.59}$$

The diagram (4.59) suggests the possibility of extending the action of the operator (4.49) according to

$$\begin{array}{ccccc} V^{s+\frac{1}{p},p}(\Omega) & & & & \\ \big\downarrow J & & \overset{A_o}{\searrow} & & \\ L^p_{s+\frac{1}{p},z}(\Omega,\mathbb{R}^n) & \xrightarrow{-\Delta_D} & L^p_{s+\frac{1}{p}-2}(\Omega,\mathbb{R}^n) & \xrightarrow{\mathbb{P}} & \Big(V^{1-s+\frac{1}{p'},p'}(\Omega)\Big)^* \end{array} \tag{4.60}$$

whenever $1<p<\infty$ and $-1<s<2$.

Proposition 4.10. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then for every $1<p<\infty$ and $-1<s<2$, the operator*

$$A_o : V^{s+\frac{1}{p},p}(\Omega) \longrightarrow \Big(V^{1-s+\frac{1}{p'},p'}(\Omega)\Big)^* \tag{4.61}$$

which makes the diagram (4.60) commutative, is well-defined, linear, bounded and compatible with (4.59). Furthermore, the adjoint of (4.61) is $A_o : V^{1-s+\frac{1}{p'},p'}(\Omega) \longrightarrow \Big(V^{s+\frac{1}{p},p}(\Omega)\Big)^*$.

Finally, the operator (4.61) is an isomorphism in either of the cases $(i)-(iv)$ listed in §3.1.

Proof. The claims made in the first part of the statement are clear from definitions. As for the last claim in the proposition, we first note that the operator (4.61) is injective if and only if the problem (3.1) has at most one solution. Likewise, (4.61) is surjective if and only if (3.1) has at least one solution. All in all, the operator (4.61) is invertible if and only if the problem (3.1) is well-posed. Then the desired conclusion follows from the discussion in §3.1. □

5 Domains of fractional powers of the Stokes operator

Here, the goal is to determine the domains of fractional powers of the Stokes operator, $D(A^\alpha)$ for $\alpha\in[0,1]$, in Lipschitz domains.

5.1 The n-dimensional case

Retain the notation and conventions used in the previous section; in particular, Ω is a bounded Lipschitz domain in $\mathbb{R}^n$.

Theorem 5.1. *For $s\in[0,2]$, the domain of the fractional power of the Stokes operator $A^{\frac{s}{2}}$ is given by*

$$D(A^{\frac{s}{2}}) = \begin{cases} L^2_s(\Omega,\mathbb{R}^n)\cap\mathcal{H} & \text{if} \quad 0\le s<\frac{1}{2}, \\ \Big\{u\in L^2_{\frac{1}{2}}(\Omega,\mathbb{R}^n)\cap\mathcal{H}:\ \displaystyle\int_\Omega |u(x)|^2 \mathrm{dist}\,(x,\partial\Omega)^{-1}dx<\infty\Big\} & \text{if} \quad s=\frac{1}{2}, \\ \Big\{u\in L^2_{s,z}(\Omega,\mathbb{R}^n):\ div\, u=0\Big\} & \text{if} \quad \frac{1}{2}<s<\frac{3}{2}, \\ \Big\{u\in L^2_{1,z}(\Omega,\mathbb{R}^n):\ div\, u=0\ \&\ \Delta\, u\in L^2_{s-2}(\Omega,\mathbb{R}^n)+\nabla[L^2(\Omega)]\Big\} & \text{if} \quad \frac{3}{2}\le s\le 2. \end{cases}$$

Proof. Consider the families of spaces $\Big\{V^{s,2}(\Omega):\ s>-\frac{1}{2}\Big\}$ and $\Big\{D(A^{\frac{s}{2}}):\ 0\leq s\leq 2\Big\}$. From Theorem 2.12 and Proposition 4.2 we know that both are complex interpolation scales, and

$$D(A^0)=\mathcal{H}=V^{0,2}(\Omega),\qquad D(A^{\frac{1}{2}})=\mathcal{V}=V^{1,2}(\Omega). \tag{5.1}$$

Thus, by interpolation,

$$D(A^{\frac{s}{2}})=V^{s,2}(\Omega),\qquad 0\leq s\leq 1. \tag{5.2}$$

With this in hand, the description of $D(A^{\frac{s}{2}})$ stated in the theorem for $s\in[0,1]$ follows from Corollary 2.11 and (2.14).

Consider next the case when $s\in]1,2]$. From (4.25)-(5.2) we obtain

$$D(A^{\frac{s}{2}})=A_o^{-1}\Big(V^{2-s,2}(\Omega)\Big)^*,\qquad 1\leq s\leq 2. \tag{5.3}$$

Equivalently, for each $s\in[1,2]$,

$$u\in D(A^{\frac{s}{2}})\Longleftrightarrow u\in\mathcal{V}\text{ and }A_ou\in\Big(V^{2-s,2}(\Omega)\Big)^*\hookrightarrow\mathcal{V}^*. \tag{5.4}$$

Next, we note that (4.45) implies that the operator $\mathbb{P}:L^2_{s-2}(\Omega,\mathbb{R}^n)\to\Big(V^{2-s,2}(\Omega)\Big)^*$ is onto and has $\nabla[L^2_{s-1}(\Omega)]$ as null-space. In concert with (4.49) this readily gives that, for each $s\in[1,2]$,

$$u\in D(A^{\frac{s}{2}})\Longleftrightarrow u\in\mathcal{V}\text{ and }\exists\,\pi\in L^2(\Omega)\text{ such that }f:=\Delta u-\nabla\pi\in L^2_{s-2}(\Omega,\mathbb{R}^n). \tag{5.5}$$

On the other hand, in the case $s\in]1,\frac{3}{2}[$, the discussion in §3.1 gives

$$\left.\begin{array}{r}u\in\mathcal{V},\ \ \pi\in L^2(\Omega)\\ -\Delta u+\nabla\pi\in L^2_{s-2}(\Omega,\mathbb{R}^n)\end{array}\right\}\Longleftrightarrow\left\{\begin{array}{l}u\in L^2_{s,z}(\Omega,\mathbb{R}^n),\ \operatorname{div}u=0,\\ \pi\in L^2_{s-1}(\Omega).\end{array}\right. \tag{5.6}$$

With this in hand, the remaining claims in the statement of the theorem follow easily. □

As is customary, by $X(u)\approx Y(u)$ we shall mean that there exists a finite positive constant κ such that $\kappa^{-1}X(u)\leq Y(u)\leq\kappa\,X(u)$ for every u.

Corollary 5.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then for each $0<s<\frac{3}{2}$, $s\neq\frac{1}{2}$,*

$$\|A^{s/2}u\|_{L^2(\Omega,\mathbb{R}^n)}\approx\|u\|_{L^2_s(\Omega,\mathbb{R}^n)},\qquad\text{uniformly for }\ u\in D(A^{s/2}). \tag{5.7}$$

Moreover, corresponding to $s=\frac{1}{2}$,

$$\|A^{1/4}u\|_{L^2(\Omega,\mathbb{R}^n)}\approx\|u\|_{L^2_{1/2}(\Omega,\mathbb{R}^n)}+\Big(\int_\Omega|u(x)|^2\operatorname{dist}(x,\partial\Omega)^{-1}\,dx\Big)^{1/2}, \tag{5.8}$$

uniformly for $u\in D(A^{1/4})$.

Proof. The equivalence (5.7) is a consequence of Theorem 5.1 and the fact that for $0<s<\frac{3}{2}$, $s\neq\frac{1}{2}$, $L^2_{s,z}(\Omega,\mathbb{R}^n)$ is a closed subspace of $L^2_s(\Omega,\mathbb{R}^n)$; cf. the first identity in (2.29). Likewise, (5.8) is a consequence of Theorem 5.1 and the equivalence of norms implicit in (2.14). □

Next, denote by $B(x,r)$ the ball of center $x\in\mathbb{R}^n$ and radius $r>0$. Recall that a domain $\Omega\subset\mathbb{R}^n$ is said to satisfy a *uniform exterior ball condition* if there exists $r>0$ with the property that for every $x\in\partial\Omega$ one can pick $y=y(x)\in\mathbb{R}^n$ such that

$$x\in\partial B(y,r) \quad \text{and} \quad \overline{B(y,r)}\setminus\{x\}\subseteq\mathbb{R}^n\setminus\Omega. \tag{5.9}$$

Informally speaking, the family just described models the class of domains for which the boundary singularities are directed outwardly. In particular, if Ω is convex, or if $\partial\Omega\in\mathscr{C}^{1,1}$, then Ω does satisfy a uniform exterior ball condition.

Theorem 5.3. *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$D(A^{s/2})\subset L^2_{3/2,z}(\Omega,\mathbb{R}^n) \quad \text{if} \quad s>\tfrac{3}{2}. \tag{5.10}$$

Furthermore, if Ω satisfies a uniform exterior ball condition, then

$$D(A^{\frac{3}{4}})=\Big\{u\in L^2_{3/2}(\Omega,\mathbb{R}^n):\,\operatorname{div}u=0 \text{ and } \operatorname{Tr}u=0\Big\}. \tag{5.11}$$

Proof. For starters, we note that if $s\in[\frac{3}{2},2]$ then (5.5) holds. Assume now that u, π, f are as in the right-hand side of (5.5) and use Proposition 2.1 in order to extended f to a compactly supported vector-valued distribution $F\in L^2_{s-2}(\mathbb{R}^n,\mathbb{R}^n)$. Based on standard Fourier analysis and classical Calderón-Zygmund theory, one can then find a solution to the problem

$$\left\{\begin{array}{rcl} -\Delta U+\nabla\Pi &=& F \text{ in } \mathbb{R}^n,\\ \operatorname{div}U &=& 0 \text{ in } \mathbb{R}^n,\\ \mathscr{R}_\Omega U &\in& L^2_s(\Omega,\mathbb{R}^n), \quad \mathscr{R}_\Omega\Pi\in L^2_{s-1}(\Omega). \end{array}\right. \tag{5.12}$$

At this point, the discussion branches out and there are two cases to consider, depending on whether $s>\frac{3}{2}$, or $s=\frac{3}{2}$. In the first case, we note that since

$$L^2_s(\Omega)\xrightarrow{\partial_j}L^2_{s-1}(\Omega)\xrightarrow{\operatorname{Tr}}B^{2,2}_{s-\frac{3}{2}}(\partial\Omega)\hookrightarrow L^p(\partial\Omega) \quad \text{if} \quad 1<p\leq\tfrac{2(n-1)}{n+2-2s}, \quad 1\leq j\leq n, \tag{5.13}$$

it follows that

$$\operatorname{Tr}U\in\Big\{g\in L^p_1(\partial\Omega,\mathbb{R}^n):\,\textstyle\int_{\partial\Omega}\nu\cdot g\,d\sigma=0\Big\}, \qquad \forall\,p\in\,]1,\tfrac{2(n-1)}{n+2-2s}]. \tag{5.14}$$

Thus, by the discussion in §3.2, the boundary-value problem

$$\left\{\begin{array}{rcl} -\Delta w+\nabla\eta &=& 0 \text{ in } \Omega,\\ \operatorname{div}w &=& 0 \text{ in } \Omega,\\ \operatorname{Tr}w &=& \operatorname{Tr}U, \end{array}\right. \tag{5.15}$$

has a solution satisfying $w\in L^2_{\frac{3}{2}}(\Omega,\mathbb{R}^n)$, $\eta\in L^2_{\frac{1}{2}}(\Omega)$. In particular, the pair $u-\mathscr{R}_\Omega U+w\in\mathcal{V}$ and $\pi-\mathscr{R}_\Omega\Pi+\eta\in L^2(\Omega)$ is a null-solution for the Stokes system in Ω. By uniqueness, this forces $u=\mathscr{R}_\Omega U-w\in L^2_{3/2,z}(\Omega,\mathbb{R}^n)$, as desired.

Finally, in the case when $s=\frac{3}{2}$ and Ω satisfies a uniform exterior ball condition, we proceed as before with the main significant difference being that, this time,

$$\mathrm{Tr}\left[L^2_{\frac{3}{2}}(\mathbb{R}^n)\right]=L^2_1(\partial\Omega). \tag{5.16}$$

See [20]. This proves the left-to-right inclusion in (5.11). Finally, with the help of Theorem 5.1, it is easy to check that the right-to-left inclusion in (5.11) holds under the mere assumption that Ω is a bounded, Lipschitz domain. This finishes the proof of the theorem. □

Remark. The same proof shows that for any bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$ there exists some small $\varepsilon>0$ such that

$$\tfrac{3}{2}<s<\tfrac{3}{2}+\varepsilon\Longrightarrow D(A^{s/2})\subset B^{p,p}_{1+\frac{1}{p}}(\Omega,\mathbb{R}^n)\quad\text{where}\quad p=\tfrac{2(n-1)}{n+2-2s}. \tag{5.17}$$

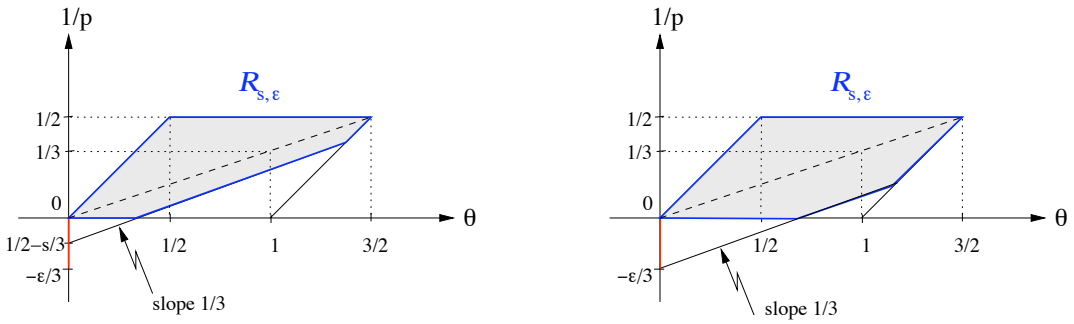
The remarkable feature of (5.17) is that even though we do not expect $D(A^{s/2})$ to be a subspace of $L^2_s(\Omega,\mathbb{R}^n)$ for any $s>\frac{3}{2}$, the embedding $L^2_s(\Omega,\mathbb{R}^n)\subset B^{p,p}_{1+\frac{1}{p}}(\Omega,\mathbb{R}^n)$ is sharp precisely when $p=\frac{2(n-1)}{n+2-2s}$.

5.2 The three-dimensional case

Theorem 5.1 describes $D(A^{\frac{s}{2}})$ completely if $s\in[0,\frac{3}{2}[$ in all space-dimensions. Nonetheless, for bounded Lipschitz domains in $\mathbb{R}^3$, it is possible to further extend the scope of this analysis. To state our main result in this regard, for each $\varepsilon\in]0,1]$ and $s\in[\frac{3}{2},2]$ define the two dimensional region

$$R_{s,\varepsilon}:=\begin{cases}(\theta,\frac{1}{p}):\ 0<\frac{1}{p}<\theta<1+\frac{1}{p}\leq\frac{3}{2}\quad\text{and}\\ \frac{1}{p}-\frac{\theta}{3}\geq\frac{1}{2}-\frac{s}{3}\quad\text{if}\quad\frac{3}{2}\leq s<\frac{3}{2}+\varepsilon,\\ \frac{1}{p}-\frac{\theta}{3}>-\frac{\varepsilon}{3}\quad\text{if}\quad\frac{3}{2}+\varepsilon\leq s\leq 2.\end{cases} \tag{5.18}$$

The figures below depict the region $R_{s,\varepsilon}$ in the case when $\frac{3}{2}\leq s<\frac{3}{2}+\varepsilon$ and when $\frac{3}{2}+\varepsilon\leq s\leq 2$, respectively:



Theorem 5.4. *For every bounded Lipschitz domain $\Omega\subset\mathbb{R}^3$ there exists $\varepsilon=\varepsilon(\partial\Omega)>0$ with the property that for every $3/2\leq s\leq 2$ the following implication holds:*

$$(\theta,1/p)\in R_{s,\varepsilon}\Longrightarrow D(A^{s/2})\subset L^p_\theta(\Omega,\mathbb{R}^3). \tag{5.19}$$

Proof. The strategy is to combine the characterization proved in Theorem 5.1, i.e. that

$$D(A^{\frac{\alpha}{2}})=\Big\{u\in L^2_{1,z}(\Omega,\mathbb{R}^3):\,\mathrm{div}\,u=0\ \&\ \Delta\,u\in L^2_{\alpha-2}(\Omega,\mathbb{R}^3)+\nabla L^2(\Omega)\Big\}\ \text{if}\ \tfrac{3}{2}\leq\alpha\leq 2,\tag{5.20}$$

with the well-posedness result for the Poisson problem for the Stokes system (3.1). In concert, these two results show that $D(A^{\alpha/2})\subset L^p_\theta(\Omega,\mathbb{R}^3)$ provided

$$\exists\,(s,1/p)\ \text{as in (3.2) such that}\ \theta=s+1/p\ \text{and}\ L^2_{\alpha-2}(\Omega)\hookrightarrow L^p_{\theta-2}(\Omega).\tag{5.21}$$

Now, elementary algebra shows that, given $\alpha\in[3/2,2]$, the condition (5.21) holds if and only if $(\theta,1/p)\in R_{\alpha,\varepsilon}$. Clearly, this proves (5.19), after re-adjusting notation. □

Corollary 5.5. *For an arbitrary bounded Lipschitz domain Ω in $\mathbb{R}^3$, the following hold:*

$$D(A^{\frac{3}{4}})\subset\bigcap_{p>2}L^p_{\frac{3}{p},z}(\Omega,\mathbb{R}^3);\tag{5.22}$$

$$D(A^{\frac{3}{4}})\subset L^3_{1,z}(\Omega,\mathbb{R}^3);\tag{5.23}$$

$$D(A^{\alpha})\subset\bigcup_{p>3}L^p_{1,z}(\Omega,\mathbb{R}^3)\quad\ \textit{if}\ \ \alpha>\tfrac{3}{4},\tag{5.24}$$

$$D(A^{\alpha})\subset\mathscr{C}^{2\alpha-3/2}(\bar\Omega,\mathbb{R}^3)\quad\ \textit{if}\ \ \tfrac{3}{4}<\alpha<\tfrac{3}{4}+\varepsilon,\tag{5.25}$$

for some small $\varepsilon=\varepsilon(\Omega)>0$.

Proof. These are all immediate consequences of Theorem 5.4 and classical embeddings. □

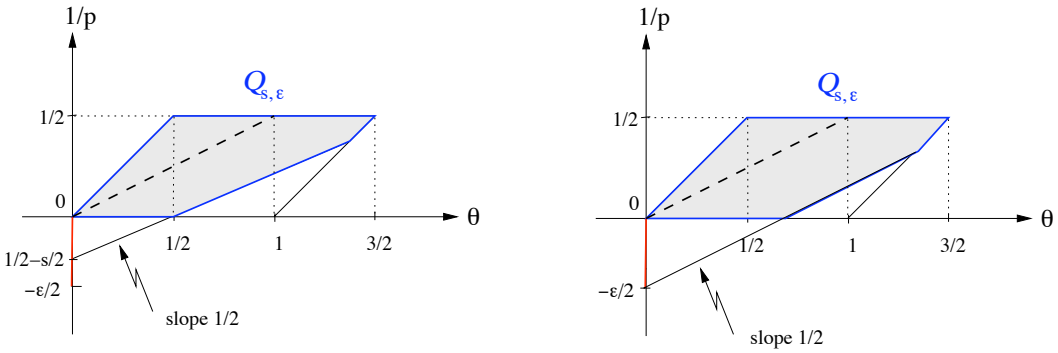
Let us remark that, by relying on the cases (i), (ii) in §3.1, the same strategy employed in the proof of Theorem 5.4 can be used to derive certain regularity results which are similar in spirit to (5.19), in the n-dimensional case. We leave the formulation of these results to the interested reader and, instead, choose to focus on bounded Lipschitz domains in $\mathbb{R}^2$.

5.3 The two-dimensional case

Here we complement Theorem 5.1 by further refining the description of $D(A^{\frac{s}{2}})$ for $s\in[\frac{3}{2},2]$ in the case when Ω is a bounded Lipschitz domain in $\mathbb{R}^2$. To set the stage, for each $\varepsilon\in]\frac{1}{2},1]$ and $s\in[\frac{3}{2},2]$ define the two dimensional region

$$Q_{s,\varepsilon}:=\begin{cases}(\theta,\frac{1}{p}):\ 0<\frac{1}{p}<\theta<1+\frac{1}{p}\leq\frac{3}{2}\quad\text{and}\\ \frac{1}{p}-\frac{\theta}{2}\geq\frac{1-s}{2}\quad\text{if}\quad\frac{3}{2}\leq s<1+\varepsilon,\\ \frac{1}{p}-\frac{\theta}{2}>-\frac{\varepsilon}{2}\quad\text{if}\quad 1+\varepsilon\leq s\leq 2.\end{cases}\tag{5.26}$$

Given some $\varepsilon\in]\frac{1}{2},1]$, here is how the region $Q_{s,\varepsilon}$ looks in the case when $\frac{3}{2}\leq s<1+\varepsilon$ and when $1+\varepsilon\leq s\leq 2$, respectively:



Theorem 5.6. *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ there exists $\varepsilon = \varepsilon(\partial\Omega) \in]\frac{1}{2}, 1]$ with the property that the implication*

$$(\theta, 1/p) \in Q_{s,\varepsilon} \Longrightarrow D(A^{s/2}) \subset L^p_\theta(\Omega, \mathbb{R}^2). \tag{5.27}$$

holds for every $3/2 \leq s \leq 2$.

Proof. The proof parallels that of Theorem 5.4, the only major difference being that, instead of (3.2), the *two-dimensional* version of (3.1) is well-posed whenever the conditions (3.3) hold for some $\varepsilon = \varepsilon(\partial\Omega) \in]\frac{1}{2}, 1]$. □

A simple consequence of Theorem 5.6 and embeddings is as follows.

Corollary 5.7. *For each bounded Lipschitz domain Ω in $\mathbb{R}^2$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that*

$$\tfrac{3}{4} < \gamma < \tfrac{3}{4} + \varepsilon \Longrightarrow D(A^\gamma) \subset \mathscr{C}^{2\gamma-1}(\bar{\Omega}, \mathbb{R}^2). \tag{5.28}$$

6 Navier-Stokes equations

In this section, we make use of our earlier analysis of the fractional powers of the Stokes system in order to study issues such as existence, uniqueness and regularity for the Navier-Stokes system in bounded Lipschitz subdomains of $\mathbb{R}^3$.

6.1 Existence

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and, for each $T > 0$, define the following Banach space:

$$\begin{aligned} \mathcal{F}_T \;:=\; \Big\{ & u \in \mathscr{C}([0,T]; D(A^{\frac{1}{4}})) \cap \mathscr{C}^1(]0,T]; D(A^{\frac{3}{4}})) \;: \\ & \sup_{0<s<T} \|s^{\frac{1}{2}} A^{\frac{3}{4}} u(s)\|_{\mathcal{H}} + \sup_{0<s<T} \|s^{\frac{3}{4}} u'(s)\|_{\mathcal{H}} + \sup_{0<s<T} \|s^{\frac{3}{2}} A^{\frac{3}{4}} u'(s)\|_{\mathcal{H}} < \infty \Big\} \end{aligned} \tag{6.1}$$

endowed with the norm

$$\begin{aligned} \|u\|_{\mathcal{F}_T} := & \sup_{0<s<T} \|A^{\frac{1}{4}} u(s)\|_{\mathcal{H}} + \sup_{0<s<T} \|s^{\frac{1}{2}} A^{\frac{3}{4}} u(s)\|_{\mathcal{H}} \\ & + \sup_{0<s<T} \|s^{\frac{3}{4}} u'(s)\|_{\mathcal{H}} + \sup_{0<s<T} \|s^{\frac{3}{2}} A^{\frac{3}{4}} u'(s)\|_{\mathcal{H}}. \end{aligned} \tag{6.2}$$

For the convenience of notation, let us also denote the Stokes semigroup by

$$(Su)(t) := e^{-tA}u, \qquad u \in \mathcal{H}, \quad t \geq 0. \tag{6.3}$$

Lemma 6.1. *If $u \in D(A^{\frac{1}{4}})$ then $Su \in \mathcal{F}_T$ for each $T > 0$ and*

$$\|Su\|_{\mathcal{F}_T} \leq C\|A^{\frac{1}{4}}u\|_{\mathcal{H}}, \tag{6.4}$$

where $C > 0$ is a finite constant independent of $T > 0$.

Proof. Fix some number $T > 0$, as well as a field $u \in D(A^{\frac{1}{4}})$. Since $(Su)'(t) = -Ae^{-tA}u$ for $t > 0$, it follows from (4.19)-(4.20) that $Su \in \mathscr{C}([0,T]; D(A^{\frac{1}{4}})) \cap \mathscr{C}^1(]0,T]; D(A^{\frac{3}{4}}))$. We also have that $t^{\frac{1}{2}}A^{\frac{3}{4}}(Su)(t) = t^{\frac{1}{2}}A^{\frac{1}{2}}e^{-tA}A^{\frac{1}{4}}u$ is bounded from $]0,T[$ into $\mathcal{H}$, thanks to (4.18). Likewise, the functions $t^{\frac{3}{2}}A^{\frac{3}{4}}(Su)'(t) = -t^{\frac{3}{2}}A^{\frac{3}{2}}e^{-tA}A^{\frac{1}{4}}u$ and $t^{\frac{3}{4}}(Su)'(t) = -t^{\frac{3}{4}}A^{\frac{3}{4}}e^{-tA}A^{\frac{1}{4}}u$ are bounded from $]0,T[$ into $\mathcal{H}$. This proves that $w \in \mathcal{F}_T$. Now, (6.4) is implicit in the above analysis. □

Recall the operator $\mathbb{P}$ from (4.43) and, for each $u, v \in \mathcal{F}_T$, introduce

$$\Phi(u,v)(t) := \int_0^t e^{-(t-s)A}(-\tfrac{1}{2}\mathbb{P})((u(s)\cdot\nabla)v(s) + (v(s)\cdot\nabla)u(s))\,ds, \quad 0 < t < T. \tag{6.5}$$

Proposition 6.2. *The application*

$$\Phi : \mathcal{F}_T \times \mathcal{F}_T \longrightarrow \mathcal{F}_T \tag{6.6}$$

is well-defined, bilinear, symmetric and continuous. Furthermore,

$$\|\Phi(u,v)\|_{\mathcal{F}_T} \leq \kappa\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}, \qquad u, v \in \mathcal{F}_T, \tag{6.7}$$

where $\kappa = \kappa(\Omega) > 0$ is a finite constant, independent of T.

Proof. The fact that Φ is bilinear and symmetric is clear. Moreover, $\Phi(u,v) = e^{-\cdot A} * f$, where f is defined by

$$f(s) := (-\tfrac{1}{2}\mathbb{P})((u(s)\cdot\nabla)v(s) + (v(s)\cdot\nabla)u(s)), \quad 0 < s < T. \tag{6.8}$$

We have $D(A^{\frac{3}{4}}) \subset L^3_1(\Omega,\mathbb{R}^3)$ by Corollary 5.5 and $[D(A^{\frac{1}{4}}), D(A^{\frac{3}{4}})]_{\frac{1}{2}} = D(A^{\frac{1}{2}}) \subset L^6(\Omega,\mathbb{R}^3)$. Thus, by Hölder's inequality, $(u(s)\cdot\nabla)v(s) + (v(s)\cdot\nabla)u(s) \in L^2(\Omega,\mathbb{R}^3)$ for each $u, v \in \mathcal{F}_T$ and, therefore, $f(s) \in \mathcal{H}$ for $s \in]0,T[$, with

$$\begin{aligned}
\sup_{0<s<T} s^{\frac{3}{4}}\|f(s)\|_{\mathcal{H}} &\leq \sup_{0<s<T} s^{\frac{3}{4}}\Big(\|u(s)\|_{L^3_1(\Omega,\mathbb{R}^n)}\|v(s)\|_{L^6(\Omega,\mathbb{R}^n)} + \|v(s)\|_{L^3_1(\Omega,\mathbb{R}^n)}\|u(s)\|_{L^6(\Omega,\mathbb{R}^n)}\Big)\\
&\leq C \sup_{0<s<T} s^{\frac{3}{4}}\Big(\|u(s)\|_{D(A^{\frac{3}{4}})}\|v(s)\|^{1/2}_{D(A^{\frac{1}{4}})}\|v(s)\|^{1/2}_{D(A^{\frac{3}{4}})} + \|v(s)\|_{D(A^{\frac{3}{4}})}\|u\|^{1/2}_{D(A^{\frac{1}{4}})}\|u(s)\|^{1/2}_{D(A^{\frac{3}{4}})}\Big)\\
&\leq C \sup_{0<s<T} s^{\frac{3}{4}}\Big(\|A^{\frac{3}{4}}u(s)\|_{\mathcal{H}}\|A^{\frac{1}{4}}v(s)\|^{1/2}_{\mathcal{H}}\|A^{\frac{3}{4}}v(s)\|^{1/2}_{\mathcal{H}} + \|A^{\frac{3}{4}}v(s)\|_{\mathcal{H}}\|A^{\frac{1}{4}}u(s)\|^{1/2}_{\mathcal{H}}\|A^{\frac{3}{4}}u(s)\|^{1/2}_{\mathcal{H}}\Big)\\
&\leq C\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}.
\end{aligned} \tag{6.9}$$

Based on (6.9) and (4.18) we may then estimate

$$\begin{aligned}\|A^{\frac{1}{4}}\Phi(u,v)(t)\|_{\mathcal{H}} &\leq \int_0^t \|A^{\frac{1}{4}}e^{-(t-s)A}\|_{\mathscr{L}(\mathcal{H})}\|f(s)\|_{\mathcal{H}}\,ds\\ &\leq C\Big(\int_0^t (t-s)^{-\frac{1}{4}}s^{-\frac{3}{4}}\,ds\Big)\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\ &\leq C\Big(\int_0^1 (1-\sigma)^{-\frac{1}{4}}\sigma^{-\frac{3}{4}}\,ds\Big)\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\ &\leq C\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}. \end{aligned}\tag{6.10}$$

In order to check that the application $[0,T]\ni t\mapsto \Phi(u,v)(t)\in D(A^{\frac{1}{4}})$ is continuous, fix an arbitrary $t_o\in[0,T]$ and estimate $\|A^{\frac{1}{4}}\Phi(u,v)(t)-A^{\frac{1}{4}}\Phi(u,v)(t_o)\|_{\mathcal{H}}$ by distinguishing two scenarios: $0\leq t\leq t_o$, and $t_o\leq t\leq T$. In the first case, we recall a general identity to the effect that

$$e^{-t_oA}w-e^{-tA}w=A\left(\int_t^{t_o}e^{-\tau A}w\,d\tau\right),\qquad \forall\, w\in\mathcal{H}.\tag{6.11}$$

Cf. (2.4) on p. 5 of [45]. Formula (6.11) allows us to write

$$\begin{aligned}&A^{\frac{1}{4}}\Phi(u,v)(t)-A^{\frac{1}{4}}\Phi(u,v)(t_o)\\ &\quad=-A^{\frac{1}{4}}\int_0^t A\left(\int_t^{t_o}e^{-(\tau-s)A}f(s)\,d\tau\right)ds-\int_t^{t_o}A^{\frac{1}{4}}e^{-(t-s)A}f(s)\,ds\\ &\quad=:\mathcal{I}_1+\mathcal{I}_2.\end{aligned}\tag{6.12}$$

Now,

$$\begin{aligned}\|\mathcal{I}_1\|_{\mathcal{H}} &\leq C\sup_{0<s<T}\Big[s^{\frac{3}{4}}\|f(s)\|_{\mathcal{H}}\Big]\left[\int_0^t\left(\int_t^{t_o}\frac{d\tau}{(\tau-s)^{5/4}}\right)s^{-\frac{3}{4}}\,ds\right]\\ &\leq C\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\int_0^t\Big[(t_o-s)^{-\frac{1}{4}}-(t-s)^{-\frac{1}{4}}\Big]s^{-\frac{3}{4}}\,ds\to 0\ \text{ as }\ t\nearrow t_o,\end{aligned}\tag{6.13}$$

and

$$\|\mathcal{I}_2\|_{\mathcal{H}}\leq C\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\Big(\int_t^{t_o}(t-s)^{-\frac{1}{4}}s^{-\frac{3}{4}}\,ds\Big)\to 0\quad\text{as}\quad t\nearrow t_o.\tag{6.14}$$

Thus, altogether, $\|A^{\frac{1}{4}}\Phi(u,v)(t)-A^{\frac{1}{4}}\Phi(u,v)(t_o)\|_{\mathcal{H}}\to 0$ as $t\nearrow t_o$. In fact, the same is true when $t\searrow t_o$ and this ultimately shows that

$$\Phi(u,v)\in\mathscr{C}([0,T];D(A^{\frac{1}{4}}))\quad\text{and}\quad\sup_{0<t<T}\|A^{\frac{1}{4}}\Phi(u,v)(t)\|_{\mathcal{H}}\leq C\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\tag{6.15}$$

for every $u,v\in\mathcal{F}_T$, where $C>0$ is a finite constant, independent of $T>0$.

Going further, we estimate

$$\begin{aligned}\|A^{\frac{3}{4}}\Phi(u,v)(t)\|_{\mathcal{H}} &\leq \int_0^t \|A^{\frac{3}{4}}e^{-(t-s)A}\|_{\mathscr{L}(\mathcal{H})}\|f(s)\|_{\mathcal{H}}\,ds\\ &\leq C\left(\int_0^t (t-s)^{-\frac{3}{4}}s^{-\frac{3}{4}}\,ds\right)\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\ &\leq C\,t^{-\frac{1}{2}}\Big(\int_0^1 (1-\sigma)^{-\frac{3}{4}}\sigma^{-\frac{3}{4}}\,d\sigma\Big)\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\ &\leq C\,t^{-\frac{1}{2}}\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}. \end{aligned}\tag{6.16}$$

The continuity of the map $]0,T]\ni t\mapsto A^{\frac{3}{4}}\Phi(u,v)(t)\in\mathcal{H}$ can then be established as before.

In order to estimate the derivative in time of $\Phi(u,v)(t)$, we first note that for each $s\in]0,T[$

$$f'(s)=(-\tfrac{1}{2}\mathbb{P})((u'(s)\cdot\nabla)v(s)+(u(s)\cdot\nabla)v'(s)+(v'(s)\cdot\nabla)u(s)+(v(s)\cdot\nabla)u'(s)).\tag{6.17}$$

In particular, much as in (6.9),

$$\sup_{0<s<T} s^{\frac{7}{4}}\|f'(s)\|_{\mathcal{H}}\leq C\,\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T},\tag{6.18}$$

where $C>0$ is independent of T. After this preamble we write

$$\Phi(u,v)(t)=\int_0^{\frac{t}{2}} e^{-sA}f(t-s)\,ds+\int_0^{\frac{t}{2}} e^{-(t-s)A}f(s)\,ds,\quad t\in]0,T[,\tag{6.19}$$

and, therefore,

$$\Phi(u,v)'(t)=e^{-\frac{t}{2}A}f(\tfrac{t}{2})+\int_0^{\frac{t}{2}} e^{-sA}f'(t-s)\,ds+\int_0^{\frac{t}{2}} -Ae^{-(t-s)A}f(s)\,ds.\tag{6.20}$$

In concert with (6.9) and (6.18), this allows us to estimate

$$\begin{aligned}\|\Phi(u,v)'(t)\|_{\mathcal{H}} &\leq C\,\|f(\tfrac{t}{2})\|_{\mathcal{H}}+C\int_0^{\frac{t}{2}}\|-Ae^{-(t-s)A}\|_{\mathscr{L}(\mathcal{H})}\|f(s)\|_{\mathcal{H}}\,ds\\ &\quad +C\int_0^{\frac{t}{2}}\|e^{-sA}\|_{\mathscr{L}(\mathcal{H})}\|f'(t-s)\|_{\mathcal{H}}\,ds\\ &\leq C\,t^{-\frac{3}{4}}\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}+C\int_0^{\frac{t}{2}}(t-s)^{-1}s^{-\frac{3}{4}}\,ds\,\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\ &\quad +C\int_0^{\frac{t}{2}}(t-s)^{-\frac{7}{4}}\,ds\,\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\ &\leq C\,t^{-\frac{3}{4}}\Big(1+\int_0^{\frac{1}{2}}(1-\sigma)^{-\frac{7}{4}}\,d\sigma+\int_0^{\frac{1}{2}}(1-\sigma)^{-1}\sigma^{-\frac{3}{4}}\,d\sigma\Big)\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\ &\leq C\,t^{-\frac{3}{4}}\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}, \end{aligned}\tag{6.21}$$

where $C>0$ is independent of T. Furthermore, by reasoning as before, one can show that the application $]0,T]\ni t\mapsto\Phi(u,v)'(t)\in D(A^{\frac{3}{4}})$ is continuous.

Finally,

$$\begin{aligned}
\|A^{\frac{3}{4}}\Phi(u,v)'(t)\|_{\mathcal{H}} &\leq C\|A^{\frac{3}{4}}e^{-\frac{t}{2}A}\|_{\mathscr{L}(\mathcal{H})}\|f(\tfrac{t}{2})\|_{\mathcal{H}}+C\int_0^{\frac{t}{2}}\|-A^{\frac{7}{4}}e^{-(t-s)A}\|_{\mathscr{L}(\mathcal{H})}\|f(s)\|_{\mathcal{H}}\,ds\\
&\quad +C\int_0^{\frac{t}{2}}\|A^{\frac{3}{4}}e^{-sA}\|_{\mathscr{L}(\mathcal{H})}\|f'(t-s)\|_{\mathcal{H}}\,ds\\
&\leq C\,t^{-\frac{3}{2}}\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}+C\int_0^{\frac{t}{2}}(t-s)^{-\frac{7}{4}}s^{-\frac{3}{4}}\,ds\,\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\
&\quad +C\int_0^{\frac{t}{2}}(t-s)^{-\frac{7}{4}}s^{-\frac{3}{4}}\,ds\,\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\
&\leq C\,t^{-\frac{3}{2}}\Big(1+\int_0^{\frac{1}{2}}(1-\sigma)^{-\frac{7}{4}}\sigma^{-\frac{3}{4}}\,d\sigma+\int_0^{\frac{1}{2}}(1-\sigma)^{-\frac{7}{4}}\sigma^{-\frac{3}{4}}\,d\sigma\Big)\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T}\\
&\leq C\,t^{-\frac{3}{2}}\|u\|_{\mathcal{F}_T}\|v\|_{\mathcal{F}_T},
\end{aligned}\tag{6.22}$$

where, once again, the constant C does not depend on T.

The above analysis ensures that $\Phi(u,v)\in\mathcal{F}_T$ whenever $u,v\in\mathcal{F}_T$. Moreover, from (6.15), (6.16), (6.21) and (6.22), there exists a constant $\kappa>0$ independent of $T>0$ such that (6.7) holds. □

We are now ready to discuss the existence of *mild solutions* for the Navier-Stokes system.

Theorem 6.3. *Given $u_0\in D(A^{\frac{1}{4}})$ and $T>0$, the equation*

$$u(t)=e^{-tA}u_0+\Phi(u,u)(t),\qquad 0<t<T,\tag{6.23}$$

has a unique solution $u\in\mathcal{F}_T$, if either $\|u_0\|_{D(A^{\frac{1}{4}})}$ or T are sufficiently small.

Proof. Let $T>0$ be given and consider the bilinear, continuous mapping $\Phi:\mathcal{F}_T\times\mathcal{F}_T\to\mathcal{F}_T$ defined as in (6.5). As in [22], a solution of (6.23) will be found implementing Picard's fixed point theorem. That is, consider the sequence in $\{v_j\}_j$ of functions in $\mathcal{F}_T$ defined by $v_0:=Su_0$ and

$$v_{j+1}:=v_0+\Phi(v_j,v_j),\qquad j\in\mathbb{N}.\tag{6.24}$$

As is well-known (cf., e.g., Lemma 20 on p. 157 of [37]), this sequence converges to the unique solution $u\in\mathcal{F}_T$ of (6.23) provided

$$\|v_0\|_{\mathcal{F}_T}<\frac{1}{4\kappa},\tag{6.25}$$

where κ is the constant appearing in (6.7). In turn, since $\|v_0\|_{\mathcal{F}_T}\leq C\|A^{\frac{1}{4}}u_0\|_{\mathcal{H}}$, the estimate (6.25) is satisfied granted that $\|u_0\|_{D(A^{\frac{1}{4}})}$ is small enough.

To finish the proof, it suffices to show that, irrespective of the size of $\|u_0\|_{D(A^{\frac{1}{4}})}$, matters can be arranged so that (6.25) holds by taking T small enough (relative to $\|u_0\|_{D(A^{\frac{1}{4}})}$). To

see this, we shall make use of the fact that for each $\varepsilon > 0$ there exists $u_{0,\varepsilon} \in D(A)$ such that $\|A^{\frac{1}{4}}(u_0 - u_{0,\varepsilon})\|_{\mathcal{H}} \leq \varepsilon$. If we now consider $v_{0,\varepsilon}(t) := Su_{0,\varepsilon}$ for $0 < t < T$, then

$$\|v_0 - v_{0,\varepsilon}\|_{\mathcal{F}_T} \leq C\|A^{\frac{1}{4}}(u_0 - u_{0,\varepsilon})\|_{\mathcal{H}} \leq C\varepsilon, \tag{6.26}$$

by (6.4) and, for each fixed ε,

$$\|v_{0,\varepsilon}\|_{\mathcal{F}_T} \leq C\, T^{\frac{3}{4}}\|Au_{0,\varepsilon}\|_{\mathcal{H}} \xrightarrow[T\to 0^+]{} 0. \tag{6.27}$$

By first choosing $\varepsilon > 0$ small enough, we can therefore find $T > 0$ such that (6.25) is valid. This concludes the proof of the theorem. □

Remark. A somewhat smaller space for which the analogues of (6.4) and (6.6) hold is as follows

$$\mathcal{F}_T^0 := \{u \in \mathcal{F}_T : \lim_{\tau\to 0^+} \|u\|_{\mathcal{F}_\tau} = 0\}. \tag{6.28}$$

6.2 Regularity

Here, we shall prove that the solution $u \in \mathcal{F}_T$ of the fixed point problem (6.23) is actually a solution of the Navier-Stokes system

$$\left\{\begin{array}{rcll} \frac{\partial u}{\partial t} - \Delta_x u + \nabla_x \pi + (u\cdot\nabla_x)u &=& 0 \quad \text{in} &]0,T]\times\Omega, \\ \operatorname{div}_x u &=& 0 \quad \text{in} & [0,T]\times\Omega, \\ \operatorname{Tr}_x u &=& 0 \quad \text{on} & [0,T]\times\partial\Omega, \\ u(0) &=& u_0 \quad \text{in} & \Omega, \end{array}\right. \tag{6.29}$$

in the suitable sense, made precise in the theorem below.

Theorem 6.4. *Any solution $u \in \mathcal{F}_T$ of the problem (6.23) satisfies $u(0) = u_0$ in Ω and, in addition, has the following properties. For every $t \in [0,T]$, the field $u(t,\cdot)$ is divergence free in Ω and of vanishing trace on $\partial\Omega$. Also, there exists a scalar function $\pi \in \mathscr{C}(]0,T];L^2(\Omega))$ such that $-\Delta_x u + \nabla_x \pi \in L^2(\Omega,\mathbb{R}^3)$ and for which the first equation in (6.29) is satisfied everywhere in the time variable $t \in]0,T]$ and almost everywhere in the space variable $x \in \Omega$. Furthermore,*

$$u \in L_1^p(]0,T[;\mathcal{H}) \cap L^p(]0,T[;D(A)), \qquad 1 < p < \tfrac{4}{3}, \tag{6.30}$$

and matters can be arranged so that

$$\lim_{\tau\to 0^+} \|u\|_{\mathcal{F}_\tau} = 0. \tag{6.31}$$

Proof. Assume that $u \in \mathcal{F}_T$ solves (6.23) and introduce

$$f(s) := -\mathbb{P}\big[(u(s)\cdot\nabla_x)u(s)\big], \qquad s \in [0,T]. \tag{6.32}$$

From (6.9) we may conclude that $f \in L^p([0,T];\mathcal{H})$ whenever $1 < p < \frac{4}{3}$ and, from (6.23), that $u = e^{-\cdot A}u_0 + e^{-\cdot A} * f$. Now, Corollary 4.9 and the fact that $u_0 \in D(A^{\frac{1}{4}})$ entail $Au \in L^p([0,T];\mathcal{H})$ and that u solves

$$u'(t) + (Au)(t) = f(t) \text{ for a.e. } t \in]0,T[, \text{ and } u(0) = u_0. \tag{6.33}$$

Thus, since the definition of the space $\mathcal{F}_T$ implies $u' \in \mathscr{C}(]0,T];\mathcal{H})$, it follows that $\mathbb{P}u' = u'$ and, further,

$$\mathbb{P}\Big(\frac{\partial u}{\partial t} - \Delta_x u + (u\cdot\nabla_x)u\Big) = 0 \quad \text{ in } \mathscr{C}(]0,T];\mathcal{V}^*), \tag{6.34}$$

thanks to (4.49) and (6.32). With the help of (4.44), it now follows from (6.32) that there exists a unique scalar function $\pi \in \mathscr{C}(]0,T], L^2(\Omega))$ such that

$$\frac{\partial u}{\partial t} - \Delta_x u + (u\cdot\nabla_x)u = -\nabla_x \pi \quad \text{ in } \mathscr{C}(]0,T];L^2_{-1}(\Omega;\mathbb{R}^3)). \tag{6.35}$$

Moreover, since $u' \in \mathscr{C}(]0,T];\mathcal{H})$ and $f \in \mathscr{C}(]0,T];\mathcal{H})$, we may finally conclude from (6.35) that $-\Delta_x u + \nabla_x \pi \in \mathscr{C}(]0,T];L^2(\Omega,\mathbb{R}^3))$. Thus, the Navier-Stokes system (6.29) holds as mentioned. Finally, (6.30) follows from Corollary 4.9 and (4.33), whereas (6.31) is a consequence of the remark made at the end of §6.1. □

6.3 Uniqueness

We have already proved that there exists a local mild solution to the Navier-Stokes system which is unique in the space $\mathcal{F}_T$. Following [44], here we shall prove that, in fact, uniqueness holds in the larger space $\mathscr{C}([0,T];D(A^{\frac{1}{4}}))$.

Prior to formally stating this as a theorem, we need to make sense of the non-linearity $\Phi(u,u)$ for fields $u \in \mathscr{C}([0,T];D(A^{\frac{1}{4}}))$. To this end, for $u,v \in \mathscr{C}([0,T];D(A^{\frac{1}{4}}))$ consider

$$f(s) := \Big(-\tfrac{1}{2}\mathbb{P}\nabla\cdot\Big)\Big(u(s)\otimes v(s) + v(s)\otimes u(s)\Big), \quad s \in]0,T[, \tag{6.36}$$

where, generally speaking, $a\otimes b$ denotes the matrix $(a_i b_j)_{1\le i,j\le 3}$ for any $a = (a_1,a_2,a_3)$ and $b = (b_1,b_2,b_3) \in \mathbb{R}^3$. In this connection, let us also note that if a and b are smooth vector fields then

$$\nabla\cdot(a\otimes b) = (a\cdot\nabla)b + (\nabla\cdot a)b. \tag{6.37}$$

This elementary identity allows us to extend the bilinear form Φ, originally defined on $\mathcal{F}_T\times\mathcal{F}_T$, to the larger space $\mathscr{C}([0,T];D(A^{\frac{1}{4}}))$ in the following sense. First, if $u,v \in \mathscr{C}([0,T];D(A^{\frac{1}{4}}))$ are arbitrary then both $u\otimes v$ and $v\otimes u$ belong to $\mathscr{C}([0,T];L^{\frac{3}{2}}(\Omega,\mathbb{R}^{3\times 3}))$, since $D(A^{\frac{1}{4}}) \subset L^3(\Omega,\mathbb{R}^3)$. In particular,

$$\nabla\cdot(u\otimes v + v\otimes u) \in \mathscr{C}([0,T];L^{\frac{3}{2}}_{-1}(\Omega,\mathbb{R}^3)). \tag{6.38}$$

We now digress momentarily in order to establish a useful auxiliary result.

Lemma 6.5. *The operator $\mathbb{P}$, originally introduced in (4.41), has the property that*

$$A^{-\frac{3}{4}}\mathbb{P} : L^{\frac{3}{2}}_{-1}(\Omega,\mathbb{R}^3) \longrightarrow \mathcal{H} \tag{6.39}$$

in a bounded fashion.

Proof. From (4.45) we know that $\mathbb{P}$ maps $L^{\frac{3}{2}}_{-1}(\Omega,\mathbb{R}^3)$ boundedly into the space $\big(V^{1,3}(\Omega)\big)^*$ which, in turn, embeds continuously into $D(A^{\frac{3}{4}})^*$ by (5.23). Since A is self-adjoint, we also have $A^{-\frac{3}{4}}\big[D(A^{\frac{3}{4}})^*\big] = \mathcal{H}$, and (6.39) follows. □

Returning to the mainstream discussion, we note that $A^{-\frac{3}{4}} f \in \mathscr{C}([0,T];\mathcal{H})$, by (6.38) and Lemma 6.5. Therefore, writing

$$\Phi(u,v)(t) = \int_0^t A^{\frac{3}{4}} e^{-(t-s)A} A^{-\frac{3}{4}} f(s)\,ds, \qquad t\in[0,T], \tag{6.40}$$

it follows that

$$\Phi : \mathscr{C}([0,T];D(A^{\frac{1}{4}})) \times \mathscr{C}([0,T];D(A^{\frac{1}{4}})) \longrightarrow \mathscr{C}([0,T],\mathcal{H}) \tag{6.41}$$

in a bilinear, bounded fashion. Another useful property of this map is as follows.

Proposition 6.6. *For each $p\in(1,\infty)$ the mapping (6.41) further extends to a bounded bilinear application*

$$\Phi : L^p([0,T];D(A^{\frac{1}{4}})) \times L^\infty([0,T];D(A^{\frac{1}{4}})) \longrightarrow L^p([0,T];D(A^{\frac{1}{4}})). \tag{6.42}$$

Furthermore, the norm of (6.42) is bounded by a constant which depends exclusively on p.

Proof. For $u\in L^p([0,T];D(A^{\frac{1}{4}}))$ and $v\in L^\infty([0,T];D(A^{\frac{1}{4}}))$, the function f defined in (6.36) satisfies the estimate

$$\|A^{-\frac{3}{4}} f\|_{L^p([0,T];\mathcal{H})} \leq C_p \|A^{\frac{1}{4}} u\|_{L^p([0,T];\mathcal{H})} \|A^{\frac{1}{4}} v\|_{L^\infty([0,T];\mathcal{H})} \tag{6.43}$$

for a finite constant $C_p>0$. Then, according to Corollary 4.9, we have $A^{\frac{1}{4}}\Phi(u,v) = A(e^{-\cdot A} * f) \in L^p([0,T];\mathcal{H})$ and

$$\|A^{\frac{1}{4}}\Phi(u,v)\|_{L^p([0,T];\mathcal{H})} \leq C_p \|A^{\frac{1}{4}} u\|_{L^p([0,T];\mathcal{H})} \|A^{\frac{1}{4}} v\|_{L^\infty([0,T];\mathcal{H})}, \tag{6.44}$$

as desired. □

We are now in a position to discuss the uniqueness of mild solutions for the Navier-Stokes system, which is the main result of this subsection. To state it formally, for a measurable set $E\subset\mathbb{R}$ and a Banach space $\mathscr{X}$, we set $\mathscr{C}_b(E;\mathscr{X}) := \mathscr{C}(E;\mathscr{X}) \cap L^\infty(E;\mathscr{X})$.

Theorem 6.7. *For each $u_0\in D(A^{\frac{1}{4}})$, there is at most one field $u\in\mathscr{C}_b([0,T[;D(A^{\frac{1}{4}}))$ which satisfies (6.23).*

Proof. Assume that for some $u_0 \in D(A^{\frac{1}{4}})$ there exist two vector fields u_1, u_2 which belong to $\mathscr{C}_b([0,T[;D(A^{\frac{1}{4}}))$ and which solve (6.23). Then $w := u_1 - u_2$ also belongs to $\mathscr{C}_b([0,T[;D(A^{\frac{1}{4}}))$ and, in addition, satisfies

$$w = \Phi(u_1,u_1) - \Phi(u_2,u_2) = \Phi(w,u_1+u_2) = \Phi(w,u_1+u_2-2Su_0) + 2\Phi(w,Su_0), \tag{6.45}$$

where S is the Stokes semigroup (cf. (6.3)).

The traditional strategy (cf., e.g., [44] and the references therein) is to prove that, for a fixed $p \in]1,\infty[$, there exists $\tau \in]0,T]$ such that

$$\|w\|_{L^p([0,\tau];D(A^{\frac{1}{4}}))} \leq \frac{\|w\|_{L^p([0,\tau];D(A^{\frac{1}{4}}))}}{2}. \tag{6.46}$$

Granted this estimate, we may conclude that w vanishes on $[0,\tau[$ which, in turn, proves that $\{\tau \in]0,T] : w(t) = 0 \text{ for } 0 \leq t < \tau\}$ is nonempty. Let us denote its supremum by τ_{max}. If $\tau_{max} < T$, the continuity of w entails $w(\tau_{max}) = 0$. In this scenario, the above scheme can be reiterated, taking τ_{max} as the initial time, and we eventually conclude that there exists some $\delta > 0$ such that $w = 0$ on $[0,\tau_{max}+\delta[$. This contradicts the maximality of τ_{max} and proves that $\tau_{max} = T$. Thus $w = 0$ on $[0,T[$, as wanted.

There remains to establish (6.46). For starters, we note that for any $p \in (1,\infty)$, Proposition 6.6 gives

$$\begin{aligned}\|\Phi(w,u_1+u_2-2Su_0)\|_{L^p([0,\tau];D(A^{\frac{1}{4}}))} &\leq C_p\|w\|_{L^p([0,\tau];D(A^{\frac{1}{4}}))} \times \\ &\times\Big(\|u_1-Su_0\|_{L^\infty([0,\tau];D(A^{\frac{1}{4}}))} + \|u_2-Su_0\|_{L^\infty([0,\tau];D(A^{\frac{1}{4}}))}\Big).\end{aligned} \tag{6.47}$$

Since

$$\|u_j - Su_0\|_{L^\infty([0,\tau];D(A^{\frac{1}{4}}))} \xrightarrow[\tau\to 0^+]{} 0, \qquad j=1,2, \tag{6.48}$$

it follows that (6.47) is useful for the purpose of establishing (6.46).

There remains to handle the term $2\Phi(w,Su_0)$. To this end, for an arbitrary $\varepsilon > 0$, to be specified later, pick $u_{0,\varepsilon} \in D(A)$ such that $\|u_0 - u_{0,\varepsilon}\|_{D(A^{\frac{1}{4}})} < \varepsilon$ and then write

$$\begin{aligned}\|\Phi(w,Su_0)\|_{L^p([0,\tau];D(A^{\frac{1}{4}}))} &\leq C_p\|w\|_{L^p([0,\tau];D(A^{\frac{1}{4}}))} \times \\ &\times\Big(\|S(u_0-u_{0,\varepsilon})\|_{L^\infty([0,\tau];D(A^{\frac{1}{4}}))} + \|Su_{0,\varepsilon}\|_{L^\infty([0,\tau];D(A^{\frac{1}{4}}))}\Big).\end{aligned} \tag{6.49}$$

Next,

$$\|S(u_0-u_{0,\varepsilon})\|_{L^\infty([0,\tau];D(A^{\frac{1}{4}}))} \leq \|u_0-u_{0,\varepsilon}\|_{D(A^{\frac{1}{4}})} < \varepsilon \tag{6.50}$$

Finally, much as with (6.27),

$$\|Su_{0,\varepsilon}\|_{L^\infty([0,\tau];D(A^{\frac{1}{4}}))} \leq C\,\tau^{\frac{3}{4}}\|Au_{0,\varepsilon}\|_{\mathcal{H}} \xrightarrow[\tau\to 0^+]{} 0. \tag{6.51}$$

In summary, by first choosing $\varepsilon > 0$ small enough (relative to the constant C_p in (6.49)) it is then possible to ensure that (6.46) holds provided $\tau > 0$ is sufficiently small. This justifies (6.46) and concludes the proof of the theorem. □

7 The case of domains on manifolds

7.1 Geometrical preliminaries

Let $\mathscr{M}$ be a smooth, compact, boundaryless manifold of (real) dimension n. As usual, by $T\mathscr{M}$ and $T^*\mathscr{M}$ we denote, respectively, the tangent and cotangent bundle on $\mathscr{M}$. Also, we shall let Λ^ℓ stand for the corresponding (exterior) power of the tangent bundle $T\mathscr{M}$. We assume that $\mathscr{M}$ is equipped with a smooth Riemannian metric tensor $g = g_{jk}dx_j \otimes dx_k$, denote by $(g^{jk})_{jk}$ the inverse matrix to (g_{jk}) and set $g := \det(g_{jk})_{jk}$. Thus, in local coordinates, the volume element is given by $dV = \sqrt{g}\, dx_1...dx_n$. The pairing $\langle dx_j, dx_k\rangle := g^{jk}$ defines an inner product in Λ^1. As it is customary, we may identify vector fields with one-forms (i.e., $T\mathscr{M} \cong T^*\mathscr{M} = \Lambda^1$) via $\partial_j \mapsto g_{jk}dx_k$ (lowering indices). This mapping is an isometry whose inverse (raising indices) is given by $dx_j \mapsto g^{jk}\partial_k$. In the sequel, we shall not make any notational distinction between a vector field and its associated one-form. Under this identification, we have grad $\equiv d$ and div $\equiv -\delta$. Hereafter, we let d and δ stand, respectively, for the exterior derivative and exterior co-derivative operators. The Hodge Laplacian is then given by

$$\Delta := -d\delta - \delta d. \tag{7.1}$$

Furthermore, if ∇ is the Levi-Civita connection and Ric is the Ricci tensor on $\mathscr{M}$ then, under the above identification, the Bochner Laplacian and the Hodge Laplacian are related by

$$-\nabla^*\nabla \equiv \Delta + \mathrm{Ric}, \tag{7.2}$$

a special case of the Weitzenbock identity.

The deformation tensor $\mathscr{D}ef X$ of a field $X \in T\mathscr{M}$ is given by

$$(\mathscr{D}ef\, X)(Y,Z) = \tfrac{1}{2}\{\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle\}, \quad \forall\, X, Y, Z \in T^*\mathscr{M}. \tag{7.3}$$

Thus, $\mathscr{D}ef : \mathscr{C}^\infty(\mathscr{M}, T\mathscr{M}) \to \mathscr{C}^\infty(\mathscr{M}, S^2T^*\mathscr{M})$, where $S^2T^*\mathscr{M}$ stands for the bundle of symmetric tensor fields of type $(0,2)$ on $\mathscr{M}$. In coordinate notation,

$$(\mathscr{D}ef\, X)_{jk} = (\mathscr{D}ef\, X)(\partial_j, \partial_k) = \tfrac{1}{2}(X_{j;k} + X_{k;j}), \quad \forall\, j,k. \tag{7.4}$$

Here, for a vector field $X = X^j\partial_j$ it is customary to set $X_{k;j} := \partial_j X_k - \Gamma^l_{kj}X_l$, where Γ^l_{kj} are the Christoffel symbols associated with the metric. In the sequel, we shall find it convenient to denote $T\mathscr{M} \ni Z \mapsto (\mathscr{D}ef\, X)(Y,Z) \in \mathbb{R}$ by $(\mathscr{D}ef\, X)Y \in \Lambda^1$. In local coordinates, the adjoint of the operator $\mathscr{D}ef$ is $(\mathscr{D}ef^* v)^j = -v^{jk}{}_{;k}$ for each $v \in S^2T^*\mathscr{M}$ and each j. For an arbitrary $u \in T\mathscr{M}$, we also compute

$$\begin{aligned} \langle \mathscr{D}ef^*(u\otimes u), X\rangle &= \langle u\otimes u, \mathscr{D}ef X\rangle = \langle \nabla_u X, u\rangle \\ &= \langle X, \nabla^*_u u\rangle = -\langle X, \nabla_u u + (\operatorname{div} u)u\rangle, \qquad \forall\, X \in T\mathscr{M}, \end{aligned} \tag{7.5}$$

which proves that

$$\mathscr{D}ef^*(u\otimes u) = -\nabla_u u + (\operatorname{div} u)u, \qquad \forall\, u \in T\mathscr{M}. \tag{7.6}$$

Another operator which is going to play an important role is

$$L := 2\,\mathscr{D}ef^*\,\mathscr{D}ef = \nabla^*\nabla - \operatorname{grad}\operatorname{div} - \mathrm{Ric} \equiv -\Delta + d\delta - 2\,\mathrm{Ric}. \tag{7.7}$$

Clearly, (7.7) is a second-order, symmetric, partial differential operator and a symbol calculation reveals that L is strongly elliptic as well.

7.2 Outline of results

The Navier-Stokes equations, modeling the flow of a viscous, incompressible fluid occupying the subdomain Ω of the manifold $\mathscr{M}$ read

$$\left\{\begin{array}{rcll} \frac{\partial u}{\partial t} - L_x u + \mathrm{grad}_x \pi + \nabla_u u &=& 0 & \text{in} \quad]0,T]\times\Omega, \\ \mathrm{div}_x\, u &=& 0 & \text{in} \quad [0,T]\times\Omega, \\ \mathrm{Tr}_x\, u &=& 0 & \text{on} \quad [0,T]\times\partial\Omega, \\ u(0) &=& u_0 & \text{in} \quad \Omega, \end{array}\right. \tag{7.8}$$

where L is the operator introduced in (7.7), the velocity u is a time-dependent section in $T\mathscr{M}\Big|_{\Omega}$, and the pressure π is a scalar function defined in $]0,T[\times\Omega$.

Sobolev (potential) spaces on $\mathscr{M}$ can be lifted from $\mathbb{R}^n$ via smooth local coordinate charts and a standard localization argument involving a smooth (finite) partition of unity. Next, assuming that an arbitrary Lipschitz domain $\Omega \subset \mathscr{M}$ has been fixed, define $L^p_s(\Omega)$ as the restriction of distributions from $L^p_s(\mathscr{M})$ to Ω, and set $L^p_s(\Omega, T\mathscr{M}) := L^p_s(\Omega)\otimes T\mathscr{M}$ for the space of vector fields with components from $L^p_s(\Omega)$. Starting from these, all the other smoothness spaces considered in §2.1 can be defined in an analogous fashion and all the results stated there hold with virtually identical proofs. Here we only wish to point out that Proposition 2.5 naturally extends to Lipschitz subdomains of smooth manifolds since the results from [38] on which its proof is based have been originally derived in the manifold setting to begin with. Going further, the Stokes scale can be introduced as in (2.86) and all its properties discussed in §2.2 continue to hold in this more general setting. In this connection, we would like to mention that Proposition 2.14 is known to hold on arbitrary open subdomains of manifolds, and that Hodge decompositions analogous to (2.114) have been proved in [41] and [39].

Next, the boundary value problems (3.1), (3.5), have been studied for Lipschitz subdomains of Riemannian manifolds in [17] and [42], respectively, where well-posedness statements analogous to those in §3.1-§3.2 have been established.

As regards the Stokes operator introduced in §4.2, in the current setting we shall take

$$a(u,v) := \int_\Omega \langle \mathscr{D}ef u, \mathscr{D}ef v\rangle\, dV, \qquad u,v\in\mathcal{V}, \tag{7.9}$$

and note that matters can be arranged so that this form continues to be coercive. More specifically, by eventually altering $\mathscr{M}$ away from $\bar\Omega$, we can henceforth ensure that:

$$\mathscr{M} \text{ has no global nontrivial Killing fields, and } \mathscr{M}\setminus\bar\Omega \text{ is connected.} \tag{7.10}$$

See [42] for more details. Now (7.10) guarantees that $\mathrm{Ker}\,\mathscr{D}ef = \{0\}$. In particular, the Korn type estimate

$$\|u\|_{L^2_1(\Omega,T\mathscr{M})} \approx \|\mathscr{D}ef\, u\|_{L^2(\Omega,S^2T^*\mathscr{M})} \tag{7.11}$$

holds uniformly for $u\in L^2_{1,z}(\Omega,T\mathscr{M})$. With this as a substitute for Poincaré's inequality (used in the flat, Euclidean setting), it follows that (7.9) is indeed coercive. The construction in §4.2 then eventually leads to the identification

$$A_o = \mathbb{P}\circ L\circ J : \mathcal{V}\longrightarrow\mathcal{V}^* \tag{7.12}$$

in place of (4.49), with the operator L from (7.7) playing the role of $-\Delta_D$.

Finally, after this preamble, results analogous to those proved in §5-§6 follow based on similar considerations. Here we only want to remark that the manifold version of the identity (6.37) is (7.6), which shows that

$$\mathscr{D}ef^*(u\otimes u) = -\nabla_u u, \qquad \forall\, u\in T\mathscr{M} \quad \text{with} \ \ \operatorname{div} u = 0. \tag{7.13}$$

References

[1] H. Amann. On the strong solvability of the Navier-Stokes equations. *J. Math. Fluid Mech.*, 2 (2000), no. 1, 16–98.

[2] W. Arendt. Semigroups and evolution equations: functional calculus, regularity and kernel estimates, pp. 1–85 in *Evolutionary Equations*, Vol. I, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[3] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction.* Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[4] M.E. Bogovskiĭ. Solution of the first boundary value problem for an equation of continuity of an incompressible medium. *Dokl. Akad. Nauk SSSR*, 248 (1979), no. 5, 1037–1040.

[5] M. E. Bogovskiĭ. Solution of some vector analysis problems connected with operators div and grad. Trudy Seminar N. L. Sobolev, Akademia Nauk SSSR, Sibirskoe Otdelenie Matematiki, Nowosibirsk, 80 (1980), no. 1, 5–40.

[6] W. Borchers and H. Sohr. On the semigroup of the Stokes operator for exterior domains in L^q-spaces, *Math. Z.*, 196 (1987), no. 3, 415–425.

[7] W. Borchers and W. Varnhorn. On the boundedness of the Stokes semigroup in two-dimensional exterior domains, *Math. Z.*, 213 (1993), no. 2, 275–299.

[8] R. Brown. Personal communication.

[9] R. Brown and Z. Shen. Estimates for the Stokes operator in Lipschitz domains. *Indiana Univ. J.*, 44 (1995), 1183–1206.

[10] R. Brown, P. Perry and Z. Shen. On the dimension of the attractor for the non-homogeneous Navier-Stokes equations in non-smooth domains. *Indiana Univ. Math. J.*, 49 (2000), no. 1, 81–112.

[11] M. Dauge. Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. I. Linearized equations, *SIAM J. Math. Anal.*, 20 (1989), no. 1, 74–97.

[12] R. Dautray and J.-L. Lions. *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, Vol. 8, Évolution: Semi-groupe, Variationnel. Masson, Paris, 1988.

[13] G. de Rham. *Differentiable Manifolds. Forms, Currents, Harmonic Forms*, Springer-Verlag, Berlin, 1984.

[14] L. de Simon. Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. (Italian) *Rend. Sem. Mat. Univ. Padova*, 34 (1964), 205–223.

[15] P. Deuring. The Stokes resolvent in 3D domains with conical boundary points: nonregularity in L_p-spaces, *Adv. Differential Equations*, 6 (2001), 175–228.

[16] P. Deuring, W. von Wahl. Strong solutions of the Navier-Stokes system in Lipschitz bounded domains. *Math. Nachr.*, 171 (1995), 111–148.

[17] M. Dindoş and M. Mitrea. The stationary Navier-Stokes system in nonsmooth manifolds: the Poisson problem in Lipschitz and C^1 domains. *Arch. Ration. Mech. Anal.*, 174 (2004), 1–47.

[18] E.B. Fabes, C.E. Kenig and G.C. Verchota. The Dirichlet problem for the Stokes system on Lipschitz domains, *Duke Math. J.*, 57 (1988), no. 3, 769–793.

[19] E. Fabes, O. Mendez and M. Mitrea. Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains, *J. Funct. Anal.*, 159 (1998), 323–368.

[20] S. Fromm. Potential space estimates for Green potentials in convex domains, *Proc. Amer. Math. Soc.*, 119 (1993), no. 1, 225–233.

[21] H. Fujita and T. Kato. On the nonstationary Navier-Stokes system, *Rend. Sem. Mat. Univ. Padova*, 32 (1962) 243–260.

[22] H. Fujita and T. Kato. On the Navier-Stokes initial value problem I. *Arch. Rational Mech. Anal.*, 16 (1964), 269–315.

[23] H. Fujita and H. Morimoto. On fractional powers of the Stokes operator. *Proc. Japan Acad.*, 46 (1970), 1141–1143.

[24] Y. Giga. Analyticity of the semigroup generated by the Stokes operator in L_r spaces, *Math. Z.*, 178 (1981), no. 3, 297–329.

[25] Y. Giga. The Stokes operator in L_r spaces. *Proc. Japan Acad. Ser. A Math. Sci.*, 57 (1981), no. 2, 85–89.

[26] Y. Giga and T. Miyakawa. Solutions in L_r of the Navier-Stokes initial value problem. *Arch. Rational Mech. Anal.*, 89 (1985), no. 3, 267–281.

[27] P. Grisvard. *Elliptic problems in nonsmooth domains.* Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[28] G. Grubb. Nonhomogeneous Dirichlet Navier-Stokes problems in low regularity L_p Sobolev spaces, *J. Math. Fluid Mech.*, 3 (2001), no. 1, 57–81.

[29] D. Jerison and C. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains, *J. Funct. Anal.*, 130 (1995), 161–219.

[30] A. Jonsson and H. Wallin. *Function Spaces on Subsets of* $\mathbb{R}^n$. Harwood Academic, New York, 1984.

[31] T. Kato. Fractional powers of dissipative operators. II. *J. Math. Soc. Japan*, 14 (1962), 242–248.

[32] T. Kato. *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.

[33] P.G. Lemarié-Rieusset. *Recent Developments in the Navier-Stokes Problem*, Chapman & Hall/CRC Research Notes in Mathematics, 431. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[34] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math. J.*, 63 (1934), 193–248.

[35] J.-L. Lions. Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs, (French) *J. Math. Soc. Japan*, 14 (1962), 233–241.

[36] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications*, Vol. 1. (French) Travaux et Recherches Mathématiques, No. 17 Dunod, Paris 1968.

[37] Y. Meyer. Wavelets, paraproducts, and Navier-Stokes equations. pp. 105–212 in *Current Developments in Mathematics*, Int. Press, Boston, MA, 1997.

[38] D. Mitrea, M. Mitrea and S. Monniaux. *The Poisson problem for the exterior derivative operator with Dirichlet boundary condition on nonsmooth domains*, preprint (2005).

[39] M. Mitrea. Sharp Hodge decompositions, Maxwell's equations, and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds, *Duke Math. J.*, 125 (2004), no. 3, 467–547.

[40] M. Mitrea and S. Monniaux. *On the analyticity of the semigroup generated by the Stokes operator with Neumann type boundary conditions on Lipschitz subdomains of Riemannian manifolds*, preprint (2005).

[41] M. Mitrea and M. Taylor. Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev-Besov space results and the Poisson problem, *J. Funct. Anal.*, 176 (2000), 1–79.

[42] M. Mitrea and M. Taylor. Navier-Stokes Equations on Lipschitz Domains in Riemannian Manifolds. *Math. Annalen*, 321 (2001), no. 4, 955–987.

[43] M. Mitrea and M. Wright. Sharp L^p and Hardy space estimates for the Stokes system in Lipschitz domains, *preprint*, (2006).

[44] S. Monniaux. Uniqueness of mild solutions of the Navier-Stokes equation and maximal L^p-regularity. *C. R. Acad. Sci. Paris Sér. I Math.*, 328 (1999), 663–668.

[45] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, 1983.

[46] V. Rychkov. On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains. *J. London Math. Soc.*, 60 (1999), 237–257.

[47] T. Runst and W. Sickel. *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Operators*, de Gruyter, Berlin, New York, 1996.

[48] Z. Shen. Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders. *Amer. J. Math.*, 113 (1991), 293–373.

[49] Z. Shen. A note on the Dirichelt problem for the Stokes systems in Lipschitz domains. *Proc. Amer. Math. Soc.*, 123, (1995), no.3, 801–811.

[50] P.E. Sobolevskiĭ. Non-stationary equations of viscous fluid dynamics. *Dokl. Akad. Nauk SSSR*, 128 (1959) 45–48.

[51] V.A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes equations, *J. Sov. Math.*, 8 (1997) 467–529.

[52] E.M. Stein. *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[53] S. Strătilă and L. Zsidó. *Lectures on von Neumann Algebras*, Editura Academiei, Bucharest, Abacus Press, Tunbridge Wells, 1979.

[54] M.E. Taylor. *Partial Differential Equations*, Vol. I-III, Springer-Verlag, 1996.

[55] M.E. Taylor. Incompressible fluid flows on rough domains, pp. 320–334 in *Semigroups of Operators: Theory and Applications*, Progr. Nonlinear Differential Equations Appl., Vol. 42, Birkhäuser, Basel, 2000.

[56] R. Temam. *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland Publishing Co., Amsterdam, 1984.

[57] H. Triebel. Function spaces on Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers. *Rev. Mat. Complut.*, 15 (2002), 475–524.

[58] W. von Wahl. *The Equations of Navier-Stokes and Abstract Parabolic Equations*, Aspects of Mathematics, E8. Friedr. Vieweg & Sohn, Braunschweig, 1985.

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