ON MEASURES OF THE EDGE UNCOLORABILITY OF GRAPHS WITH MAXIMUM DEGREE THREE

J.L. FOUQUET

Abstract. In [5] Kochol study three invariants of graphs measuring how far a graph is from having a proper 3-edge-coloring. We show here how to get his main result easily.

1. Introduction

Throughout this note we shall be concerned with connected graphs with maximum degree 3.

Let $\phi : E(G) \to \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-coloring of G. It is often of interest to try to use one color (say δ) as few as possible. When an edge coloring is optimal, following this constraint, we shall say that ϕ is $\delta - minimum$. Since any two $\delta-$minimum edge-coloring of G have the same number of edges colored δ we shall denote by $s(G)$ this number (the *color number* as defined in [8]).

In [1] we gave without proof (in French, see [4] for a translation) results on δ-minimum edge-colorings of graphs with maximum degree three.

An edge coloring of G with $\{\alpha, \beta, \gamma, \delta\}$ is said to be δ*-improper* whenever we only allow edges colored with δ to be incident. It must be clear that a proper edge coloring (and hence a $\delta-$minimum edge-coloring) of G is a particular δ-improper edge coloring. For a proper or δ-improper edge coloring ϕ of G, it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges colored with x by ϕ.

A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2. Results

The proof of the following lemma is given in [3] in the context of simple graphs. We extend this result here to graphs with multiple edges (without loops).

Lemma 2.1. [2] *Let ϕ be $\delta-$improper coloring of G. Suppose that $E_\phi(\delta)$ contains two edges uv and uw. Then there exists a δ-improper edge coloring ϕ' of G such that $E_{\phi'}(\delta) = E_\phi(\delta) - uv$ or $E_{\phi'}(\delta) = E_\phi(\delta) - uw$*

Proof If $v = w$, u and v are joined by two edges and one color, α, β or γ, at least is not incident to both vertices. By giving this color to one of the edges joining u and v we get a new δ-improper edge coloring ϕ' satisfying our conclusion.

We can thus assume that $v \neq w$. If some color in $\{\alpha, \beta, \gamma\}$ is missing in u and v, we can give this color to uv and the result follows. For the same reason, we can

Date: January 24, 2011.
1991 *Mathematics Subject Classification.* 05C15.
Key words and phrases. Cubic graph; Edge-coloring;

assume that the 3 above colors are incident to u and w. Without loss of generality, we can consider that the two colors α and β are incident to v (as well as to w) while u is incident to γ.

Let P be the path alternately colored with α and γ with u as end. If P does not end with v, then we can exchange the colors α and γ on this path leading to a δ-improper edge coloring such that the color γ is missing in u and v. The color γ could be now given to the edge uv leading to a δ-improper edge coloring $\phi^{'}$ satisfying our conclusion. If P ends with v, let $P^{'}$ be the path alternately colored with α and γ with w as end. This path does not end with u and an exchange of colors on this path leads to a a δ-improper edge coloring such that the color α is missing in u and w. The color α could be now given to the edge uw leading to a δ-improper edge coloring $\phi^{'}$ satisfying our conclusion. □

As in [5] denote by $\rho(G)$ the minimum number of vertices that must be deleted from G so that the resulting graph is $3-$edge colorable. We give here a short proof of the main result of Kochol [5].

Theorem 2.2. [5] *Let G be a graph with maximum degree 3 then $s(G) = \rho(G)$*

Proof Let $X \subseteq V(G)$ such that $G-X$ is $3-$edge colorable and let $\phi: E(G-X) \rightarrow \{\alpha, \beta, \gamma\}$ be a $3-$edge coloring. We can extend ϕ in a δ-improper edge coloring $\phi^{'}$ of the edge set of G by giving the color δ to the edges incident to the vertices of X. By repeated application of Lemma 2.1, we can find a proper edge coloring of G $\phi^{''}$ such that $E_{\phi^{''}}(\delta) \subseteq E_{\phi^{'}}(\delta)$. Hence $|X| \geq |E_{\phi^{''}}(\delta)| \geq s(G)$.

Conversely let ϕ be a $\delta-$minimum edge-coloring of G. By Lemma 2.1 $E_{\phi}(\delta)$ is a matching. Let $X \subseteq V(G)$ be a minimal set of vertices intersecting each edge of $E_{\phi}(\delta)$. Then $G - X$ is $3-$edge colorable and $\rho(G) \leq |X| \leq s(G)$. □

The following theorem was first proved by Payan in [6].

Theorem 2.3. [2] [6] *Let G be a graph with maximum degree at most 3. Then G has a $\delta-$minimum edge-coloring ϕ where $E_{\phi}(\delta)$ is a strong matching and, moreover, any edge in $E_{\phi}(\delta)$ has its two ends of degree 3 in G.*

In [7] Steffen showed that it is always possible to find an independent set of size at most $s(G)$ in a graph with maximum degree 3 whose deletion leaves a $3-$edge colorable graph. It can be noticed that this is an immediate corollary of Theorem 2.3. Denote by $\rho_{\alpha}(G)$ the minimum number of vertices of an independent set that must be deleted from G so that the resulting graph is $3-$edge colorable.

Theorem 2.4. *Let G be a graph with maximum degree 3 then $s(G) = \rho_{\alpha}(G)$*

Proof Obviously we have $\rho(G) \leq \rho_{\alpha}(G)$ and hence $s(G) \leq \rho_{\alpha}(G)$ by Theorem 2.2. Conversely let ϕ be a $\delta-$minimum edge-coloring of G such that $E_{\phi}(\delta)$ is a strong matching (Theorem 2.3) and let $X \subseteq V(G)$ be a minimal set of vertices intersecting each edge of $E_{\phi}(\delta)$. Then X is an independent set and $G - X$ is $3-$edge colorable. Hence $\rho_{\alpha}(G) \leq |X| \leq s(G)$. □

By Theorem 2.3 an optimal set satisfying Theorem 2.4 can be found with all vertices of degree 3. In fact Theorem 2.4 says that we can find at least $2^{s(G)}$ such independent sets (when $s(G) > 0$).

In [8] Steffen showed that $s(G) \leq \frac{n}{8}$ when G is a bridgeless cubic graph with at least 16 vertices. In fact this result can be easily extended to a subclass of graphs with maximum degree 3.

Proposition 2.5. *Let G be a bridgeless graph with maximum degree 3 and $k \neq 1$ vertices of degree 2 with at least 16 vertices. Then $s(G) \leq \frac{n}{8}$.*

Proof By the result of Steffen [8], we can assume that G has $k \geq 2$ vertices of degree 2. Let $v_1 \dots v_k$ be these vertices. Let $G^{'}$ be a copy of G and $v_1^{'} \dots v_k^{'}$ be the corresponding vertices of degree 2. Let H be the graph obtained from these two copies by joining the vertices v_i and $v_i^{'}$ $(i = 1 \dots k)$ by an edge. It is an easy matter to see that H is a bridgeless cubic graph. By the result of Steffen [8], we have $s(H) \leq \frac{2n}{8}$.

Let p (q respectively) be the number of edges colored with δ in the copy of G ($G^{'}$ respectively). We obviously have $p \leq q$ and thus $p \leq \frac{2n}{16}$. That is $s(G) \leq \frac{n}{8}$ as claimed.

□

Corollary 2.6. *Let G be a bridgeless graph with maximum degree 3 and at least 16 vertices. Assume that G has $k \neq 1$ vertices of degree 2. Then $s(G) = \rho(G) = \rho_\alpha(G) \leq \frac{n}{8}$*

References

1. J-L Fouquet, *Graphes cubiques d'indice chromatique quatre*, Annals of Discrete Mathematics **9** (1980), 23–28.
2. J-L Fouquet, *Contribution à l' étude des graphes cubiques et problèmes hamiltoniens dans les graphes orientés.*, Ph.D. thesis, Université Paris SUD, 1981.
3. J-L Fouquet and J-M Vanherpe, *On parsimonious edge-colouring of graphs with maximum degree three*, submitted (2009).
4. ______, *Tools for parsimonious edge-colouring of graphs with maximum degree three*, http://hal.archives-ouvertes.fr/hal-00502201/PDF/ToolsForParcimoniousColouring.pdf, 2010.
5. M. Kochol, *Three measures of edge-uncolorability*, Discrete Mathematics **311** (2011), 106–108.
6. C. Payan, *Sur quelques problèmes de couverture et de couplage en combinatoire*, Thèse d'état, 1977.
7. E. Steffen, *Classifications and characterizations of snarks*, Discrete Mathematics **188** (1998), 183–203.
8. ______, *Measurements of edge-uncolorability*, Discrete Mathematics **280** (2004), 191–214.

L.I.F.O., Faculté des Sciences, B.P. 6759, Université d'Orléans, 45067 Orléans Cedex 2, FR