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ON MEASURES OF THE EDGE UNCOLORABILITY OF GRAPHS WITH MAXIMUM DEGREE THREE

J.L. FOUQUET

ABSTRACT. In [5] Kochol study three invariants of graphs measuring how far a graph is from having a proper 3-edge-coloring. We show here how to get his main result.

1. INTRODUCTION

Throughout this note we shall be concerned with connected graphs with maximum degree 3.

Let $\phi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-coloring of G . It is often of interest to try to use one color (say δ) as few as possible. When an edge coloring is optimal, following this constraint, we shall say that ϕ is δ -*minimum*. Since any two δ -minimum edge-coloring of G have the same number of edges colored δ we shall denote by $s(G)$ this number (the *color number* as defined in [7]).

In [1] we gave without proof (in French, see [4] for a translation) results on δ -minimum edge-colorings of graphs with maximum degree three.

An edge coloring of G with $\{\alpha, \beta, \gamma, \delta\}$ is said to be δ -*improper* whenever we only allow edges colored with δ to be incident. It must be clear that a proper edge coloring (and hence a δ -minimum edge-coloring) of G is a particular δ -improper edge coloring. For a proper or δ -improper edge coloring ϕ of G , it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges colored with x by ϕ .

A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C , or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2. RESULTS

The proof of the following lemma is given in [3] in the context of simple graphs. We extend this result here to graphs with multiple edges (without loops).

Lemma 2.1. [2] *Let ϕ be δ -improper coloring of G . Suppose that $E_\phi(\delta)$ contains two edges uv and wv . Then there exists a δ -improper edge coloring ϕ' of G such that $E_{\phi'}(\delta) = E_\phi(\delta) - uv$ or $E_{\phi'}(\delta) = E_\phi(\delta) - wv$*

Proof If $v = w$, u and v are joined by two edges and one color, α, β or γ , at least is not incident to both vertices. By giving this color to one of the edges joining u and v we get a new δ -improper edge coloring ϕ' satisfying our conclusion.

We can thus assume that $v \neq w$. If some color in $\{\alpha, \beta, \gamma\}$ is missing in u and v , we can give this color to uv and the result follows. For the same reason, we can

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assume that the 3 above colors are incident to u and w . Without loss of generality, we can consider that the two colors α and β are incident to v (as well as to w) while u is incident to γ .

Let P be the path alternately colored with α and γ with u as end. If P does not end with v , then we can exchange the colors α and γ on this path leading to a δ -improper edge coloring such that the color γ is missing in u and v . The color γ could be now given to the edge uv leading to a δ -improper edge coloring ϕ' satisfying our conclusion. If P ends with v , let P' be the path alternately colored with α and γ with w as end. This path does not end with u and an exchange of colors on this path leads to a δ -improper edge coloring such that the color α is missing in u and w . The color α could be now given to the edge uw leading to a δ -improper edge coloring ϕ' satisfying our conclusion. \square

As in [5] denote by $\rho(G)$ the minimum number of vertices that must be deleted from G so that the resulting graph is 3-edge colorable. We give here a short proof of the main result of Kochol [5].

Theorem 2.2. [5] *Let G be a graph with maximum degree 3 then $s(G) = \rho(G)$*

Proof Let $X \subseteq V(G)$ such that $G - X$ is 3-edge colorable and let $\phi : E(G - X) \rightarrow \{\alpha, \beta, \gamma\}$ be a 3-edge coloring. We can extend ϕ in a δ -improper edge coloring ϕ' of the edge set of G by giving the color δ to the edges incident to the vertices of X . By repeated application of Lemma 2.1, we can find a proper edge coloring of G ϕ'' such that $E_{\phi''}(\delta) \subseteq E_{\phi'}(\delta)$. Hence $|X| \geq |E_{\phi''}(\delta)| \geq s(G)$.

Conversely let ϕ be a δ -minimum edge-coloring of G and let $X \subseteq V(G)$ be a minimal set of vertices intersecting each edge of $E_{\phi}(\delta)$. Then $G - X$ is 3-edge colorable and $\rho(G) \leq |X| \leq s(G)$. \square

The following theorem was first proved by Payan in [6].

Theorem 2.3. [2] [6] *Let G be a graph with maximum degree at most 3. Then G has a δ -minimum edge-coloring ϕ where $E_{\phi}(\delta)$ is a strong matching and, moreover, any edge in $E_{\phi}(\delta)$ has its two ends of degree 3 in G .*

In [7] Steffen showed that it is always possible to find an independent set in a graph with maximum degree 3 whose deletion leaves a 3-edge colorable graph. It can be noticed that this is an immediate corollary of Theorem 2.3 (see the proof below). Denote by $\rho_{\alpha}(G)$ the minimum number of vertices of an independent set that must be deleted from G so that the resulting graph is 3-edge colorable.

Theorem 2.4. *Let G be a graph with maximum degree 3 then $s(G) = \rho_{\alpha}(G)$*

Proof Obviously we have $\rho(G) \leq \rho_{\alpha}(G)$ and hence $s(G) \leq \rho_{\alpha}(G)$ by Theorem 2.2. Conversely let ϕ be a δ -minimum edge-coloring of G such that $E_{\phi}(\delta)$ is a strong matching (Theorem 2.3) and let $X \subseteq V(G)$ be a minimal set of vertices intersecting each edge of $E_{\phi}(\delta)$. Then $G - X$ is 3-edge colorable and $\rho_{\alpha}(G) \leq |X| \leq s(G)$. \square

By Theorem 2.3 an optimal set satisfying Theorem 2.4 can be found with all vertices of degree 3. In fact Theorem 2.4 says that we can find at least $2^{s(G)}$ such independent sets (when $s(G) > 0$).

REFERENCES

1. J-L Fouquet, *Graphes cubiques d'indice chromatique quatre*, Annals of Discrete Mathematics **9** (1980), 23–28.
2. J-L Fouquet, *Contribution à l'étude des graphes cubiques et problèmes hamiltoniens dans les graphes orientés.*, Ph.D. thesis, Université Paris SUD, 1981.
3. J-L Fouquet and J-M Vanherpe, *On parsimonious edge-colouring of graphs with maximum degree three*, submitted (2009).
4. ———, *Tools for parsimonious edge-colouring of graphs with maximum degree three*, <http://hal.archives-ouvertes.fr/hal-00502201/PDF/ToolsForParsimoniousColouring.pdf>, 2010.
5. M. Kochol, *Three measures of edge-uncolorability*, Discrete Mathematics **311** (2011), 106–108.
6. C. Payan, *Sur quelques problèmes de couverture et de couplage en combinatoire*, Thèse d'état, 1977.
7. E. Steffen, *Measurements of edge-uncolorability*, Discrete Mathematics **280** (2004), 191–214.

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