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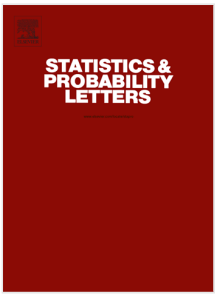

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Analysis of the virtual waiting time in multiphase queues

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Abstract

This paper presents a heavy traffic limit theorem for the virtual waiting time of a customer in multiphase queues. In this work, functional limit theorem is proved for values of important probability characteristics of the queueing system investigated as well as the virtual waiting time of a customer.

Keywords: queueing systems, multiphase queues, heavy traffic, probability limit theorem, virtual waiting time of a customer.

1 Introduction

One of the main research trends in the theory of queues corresponds to the asymptotic analysis of formulas or equations, describing the distribution of one or another probabilistic characteristic of the queue. Kingman (1961, 1962)) achieved the first results on the limiting behaviour of single channel queues in heavy traffic. The single-phase case, where the intervals of time between the arrival of customers are independent identically distributed random variables and there is one single device, has been completely investigated in the well known papers (see, for example, Borovkov (1972, 1980)). Functional limit theorems for a virtual waiting time of a customer in a single queue are proven under various conditions of heavy traffic (see Kyprianou (1971)). Functional limit theorems for the waiting time of a customer and

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queue length of customers in a multiphase queue are also proved under various heavy traffic conditions (see Minkevičius (1986, 1991)). In this work, we will prove a functional limit theorem for another important probabilistic characteristic of multiphase queues (virtual waiting time of a customer in conditions of heavy traffic).

The virtual waiting time of a customer in every phase of the queue is unlimited, and the service discipline is "first come, first served". Denote $\Rightarrow$ as a weak convergence of probability measures (see Billingsley (1968)). The main tool for the analysis of a multiphase queue in heavy traffic is functional limit theorems for renewal and compound renewal processes (the proof can also be found in Billingsley (1968)).

Let us consider a k-phase queue. When a customer is served in the j-st phase of the queue, he goes to the $j+1$-st phase of the queue. When a customer is served in the k-st phase of the queue, he leaves the queue. Let us define t_n as the time of arrival of the n-th customer at a multiphase queue, $S_n^{(j)}$ as the service time of the n-th customer in the j-th phase of the queue, $z_n = t_{n+1} - t_n$, $x_j(t) = \max\{l : \sum_{i=1}^{l} S_i^{(j)} \leq t\}$ (such a total number of customers can be served in the j-th phase of the queue if devices in the queue are working without time out), denote $e(t) = \max\{l : \sum_{i=1}^{l} z_i \leq t\}$ as the total number of customers arriving at a multiphase queue until time t, and $\tau_j(t)$ as the total number of customers that depart after service at the j-th phase of the queue until time t, $x_0(t) = \tau_0(t) = e(t)$, $j = 1, 2, \ldots, k$. $x_j(t), j = 1, 2, \ldots, k$ and $e(t)$ are counting processes (see Borovkov (1980)).

Also, let us define $W_j(t)$ as a virtual waiting time of a customer in the j-th phase of the queue for time t (i. e., time which must waiting until service customer, arrive to j-th phase of the queue at time t), $S_j(t) = \sum_{i=1}^{\tau_{j-1}(t)} S_i^{(j)}$ as the time that can got, if we summarize service times of customers, arriving at the j-th phase of the queue until time t;

$$y_j(t) = S_j(t) - t, \;\; \tau_0(t) = e(t), \;\; f_t(y(\cdot)) = y(t) - \inf_{0 \leq s \leq t} y(s), \;\; j = 1, 2, \ldots, k.$$

In particular, the j-th phase can be considered as a queue $< G, G, G, 1 >$ (see definition in Borovkov (1972)). Hence, if $S_j(0) = W_j(0) = 0$ (also see Borovkov (1972)),

$$W_j(t) = f_t(y_j(\cdot)), \;\; j = 1, 2, \ldots, k. \tag{1}$$

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Let us consider, as in Minkevičius (1986), multiphase queues in the series: $S_{i,n}^{(j)}$ are independent identically distributed random variables in the n-th multiphase queue, $S_{i,n}^{(0)} = z_{i,n},\ i \geq 1,\ n \geq 1,\ j = 1, 2, \ldots, k$.

Also let us define $\alpha_{j,n} = MS_{1,n}^{(j)},\ \alpha_{0,n} = Mz_{1,n},\ \sigma_{j,n} = DS_{1,n}^{(j)} > 0,$

$$\sigma_{0,n} = Dz_{1,n} > 0,\ \gamma_{j,n} = \sigma_{j,n} \cdot (\alpha_{j,n})^{-1},\ \sigma_{j,n}^2 = \hat{\sigma}_{j,n} \cdot (\alpha_{j,n})^{-1} +$$
$$\hat{\sigma}_{j-1,n} \cdot (\alpha_{j-1,n})^{-3} \cdot (\alpha_{j,n})^2 > 0,\ \beta_{j,n} = \frac{\alpha_{j,n}}{\alpha_{j-1,n}} - 1,\ j = 1, 2, \ldots, k,\ n \geq 1.$$

Throughout this work, we assume that the following conditions hold

$$\alpha = \max_{0 \leq j \leq k} \sup_{n \geq 1} \alpha_{j,n} < \infty, \tag{2}$$

$$\sigma_{j,n}^2 \to \sigma_j^2 > 0 \tag{3}$$

and

$$\gamma_{j,n} \to \hat{\sigma}_j^2 > 0,\ j = 1, 2, \ldots, k. \tag{4}$$

2 The main result

Our primary focus is on the case

$$\beta_{j,n} \cdot \sqrt{n} \to B_j < \infty,\ j = 1, 2, \ldots, k. \tag{5}$$

So we will prove such a theorem.

Theorem 2.1. *If (2) - (5) hold, then*

$$\left(\frac{W_1(nt)}{\sqrt{n}}; \ldots; \frac{W_k(nt)}{\sqrt{n}} \right) \Rightarrow (f_t(\hat{y}_1(\cdot)); \ldots; f_t(\hat{y}_k(\cdot))).$$

where $\hat{y}_j(t) = \hat{\sigma}_j \cdot z_j(t) + \hat{\tau}_{j-1}(t) + B_j \cdot t\ ,\ j = 1, 2, \ldots, k$ *and* $\hat{\tau}_{j-1}(t)$ *satisfy the functional recurrent stochastic equation*

$$\hat{\tau}_{j-1}(t) = \hat{\sigma}_{j-1} \cdot z_{j-1}(t) - \sup_{0 \leq s \leq t} (\hat{\sigma}_{j-1} \cdot z_{j-1}(s) - \hat{\tau}_{j-2}(s)),\ \hat{\tau}_0(\cdot) = 0, \text{ and}$$

$z_j(t)$ *are independent standard Wiener processes,* $j = 2, \ldots, k,\ 0 \leq t \leq 1.$

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Proof. Note that

$$\frac{y_j(nt)}{\sqrt{n}} = \frac{\sum_{i=1}^{\tau_{j-1}(nt)} S_i^{(j)} - nt}{\sqrt{n}} = \frac{\sum_{i=1}^{\tau_{j-1}(nt)} (S_i^{(j)} - \alpha_{j,n})}{\sqrt{n}} + \frac{\alpha_{j,n} \cdot \tau_{j-1}(nt) - nt}{\sqrt{n}}$$
$$= \frac{\{\sum_{i=1}^{\tau_{j-1}(nt)} (S_i^{(j)} - \alpha_{j,n})\}}{\sqrt{n}} - \frac{\{\sum_{i=1}^{x_{j-1}(nt)} (S_i^{(j)} - \alpha_{j,n})\}}{\sqrt{n}} + \frac{\{\sum_{i=1}^{x_{j-1}(nt)} (S_i^{(j)} - \alpha_{j,n})\}}{\sqrt{n}}$$
$$+ \frac{\alpha_{j,n} \cdot \left\{\tau_{j-1}(nt) - \frac{nt}{\alpha_{j-1,n}}\right\}}{\sqrt{n}} + \beta_{j,n} \cdot \sqrt{n} \cdot t,\ j = 1, 2, \ldots, k,\ 0 \le t \le 1. \tag{6}$$

Hence, for the difference between the first and second terms on the right in (6) to converge in probability to zero, it suffices that (see Iglehart (1973))

$$\sup_{0 \le t \le 1} \frac{x_j(nt) - \tau_j(nt)}{n} \Rightarrow 0,\ j = 1, 2, \ldots, k. \tag{7}$$

Also note that for $j = 1, 2, \ldots, k$ (see Borovkov (1972))

$$\sup_{0 \le t \le 1} \frac{x_j(nt) - \tau_j(nt)}{n} \le \sum_{i=1}^{k} \sup_{0 \le t \le 1} \frac{x_i(nt) - x_{i-1}(nt)}{n}$$
$$\le \sum_{i=1}^{k} \{ \sup_{0 \le t \le 1} |\frac{x_i(nt)}{n} - \frac{t}{\alpha_{i,n}}| + \sup_{0 \le t \le 1} |\frac{x_{i-1}(nt)}{n} - \frac{t}{\alpha_{i-1,n}}| + |\frac{1}{\alpha_{i,n}} - \frac{1}{\alpha_{i-1,n}}|\}$$
$$\le 2 \cdot \{\sum_{i=1}^{k} \sup_{0 \le t \le 1} |\frac{x_i(nt)}{n} - \frac{t}{\alpha_{i,n}}| + \hat{\alpha} \cdot \beta_{i,n}\} \Rightarrow 0, \tag{8}$$

where $\hat{\alpha} = \max_{0 \le i \le k} \max_{n \ge 1} \frac{1}{\alpha_{i,n}} < \infty$.

From (7) and (8) it follows that the difference between the first and second terms on the right of (6) converges in probability to zero.

The third term on the right of (6) is a complex counting process and

$$\frac{\sum_{i=1}^{x_{j-1}(nt)} (S_i^{(j)} - \alpha_{j,n})}{\sqrt{n}} \Rightarrow \sigma_j \cdot z_j(t),\ j = 1, 2, \ldots, k \tag{9}$$

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where $z_j(t)$ are independent standard Wiener processes, $j = 1, 2, \ldots, k$ (see Borovkov (1980)).

Let us investigate the last term in (6). Note that

$$\begin{aligned}
&\sup_{0\le t\le 1} \frac{|\alpha_{j,n} - \alpha_{j-1,n}|}{n} \cdot \tau_j(nt) \le \sup_{0\le t\le 1} |\alpha_{j-1,n}| \cdot |\beta_{j,n}| \cdot \frac{\tau_j(nt)}{n} \\
&\le |\alpha_{j-1,n}| \cdot |\beta_{j,n}| \cdot \sup_{0\le t\le 1} \frac{e_j(nt)}{n} \le |\alpha_{j-1,n}| \cdot |\beta_{j,n}| \cdot \sup_{0\le t\le 1} |\frac{e_j(nt)}{n} - \frac{t}{\alpha_{0,n}}| \\
&+ \frac{|\alpha_{j-1,n}|}{|\alpha_{0,n}|} \cdot |\beta_{j,n}| \le \alpha \cdot |\beta_{j,n}| \cdot \sup_{0\le t\le 1} |\frac{e_j(nt)}{n} - \frac{t}{\alpha_{0,n}}| + \alpha \cdot \hat{\alpha} \cdot |\beta_{j,n}| \Rightarrow 0, \\
&j = 1, 2, \ldots, k.
\end{aligned} \tag{10}$$

So from the formula (see Minkevičius (1991))

$$\tau_j(t) = x_j(t) - \sup_{0\le s\le t} (x_j(s) - \tau_{j-1}(s))$$

it follows that

$$\begin{aligned}
&\frac{\alpha_{j,n} \cdot \left\{\tau_j(nt) - \dfrac{nt}{\alpha_{j,n}}\right\}}{\sqrt{n}} = \frac{\alpha_{j,n} \cdot \left\{x_j(nt) - \dfrac{nt}{\alpha_{j,n}}\right\}}{\sqrt{n}} \\
&- \sup_{0\le s\le t} \left\{\alpha_{j,n} \cdot \left(\frac{x_j(ns) - \dfrac{ns}{\alpha_{j,n}}}{\sqrt{n}}\right) - \alpha_{j,n} \cdot \left(\frac{\tau_{j-1}(ns) - \dfrac{ns}{\alpha_{j-1,n}}}{\sqrt{n}}\right)\right\}, \\
&j = 1, 2, \ldots, k,\ 0 \le t \le 1.
\end{aligned} \tag{11}$$

However,

$$\alpha_{j,n} \cdot \left(\frac{x_j(nt) - \dfrac{ns}{\alpha_{j,n}}}{\sqrt{n}}\right) \Rightarrow \hat{\sigma}_j \cdot z_j(t), \tag{12}$$

where $z_j(t)$ are independent standard Wiener processes, $j = 1, 2, \ldots, k,\ 0 \le t \le 1$.

Define

$$\hat{\tau}_{j,n}(t) = \alpha_{j,n} \cdot \left(\frac{x_j(ns) - \dfrac{ns}{\alpha_{j,n}}}{\sqrt{n}}\right),\ j = 1, 2, \ldots, k,\ 0 \le t \le 1.$$

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Hence, from (11) and (12) we get

$$\hat{\tau}_{j,n}(t) \Rightarrow \hat{\tau}_j(t),$$

where $\hat{\tau}_j(t)$ are fulfilled functional recurrent stochastic equation

$$\hat{\tau}_j(t) = \hat{\sigma}_j \cdot z_j(t) - \sup_{0 \le s \le t} (\hat{\sigma}_j \cdot z_j(s) - \hat{\tau}_{j-1}(s)), \tag{13}$$

$j = 1, 2, \ldots, k,\ \tau_0 \equiv 0,\ 0 \le t \le 1.$

Thus, (6) – (12) imply that we have proved the theorem. The proof is complete.

□

Concluding remarks

1. A single phase case of Theorem 2.1 is investigated in Kyprianou (1971).

2. Theorem 2.1 of this paper has been proved for a class of multiphase queueing systems in heavy traffic with the service discipline "first come, first served", endless waiting time of customers in each phase of the queue, and where the times between the arriving customers to queue are independent identically distributed random variables. However, Theorem 2.1 can be applied to a wider class of multiphase queueing systems in heavy traffic: when arrival and service of customers in a queue is by group, when times between the arriving customers to the multiphase queueing systems are independent and weakly dependent random variables, etc.

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