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# Finite $\mathcal{L}_2$ -induced gain and $\lambda$ -contractivity of discrete-time switching systems including modal nonlinearities and actuator saturations<sup>☆</sup>

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## Abstract

The paper addresses two control problems of discrete-time switching systems subject to modal nonlinearities satisfying sector conditions, actuator saturations and additive  $\mathcal{L}_2$ -bounded disturbances. Such switching systems consist of a finite family of discrete-time processes that are supervised by logical decision-making algorithms. Conditions based on LMIs are provided for synthesizing the control gains and for obtaining regions of absolute  $\lambda$ -contractivity as large as possible or for obtaining a  $\mathcal{L}_2$ -gain as small as possible. A numerical example illustrates the proposed approaches.

*Keywords:* Switching systems, Nonlinear systems, Saturations, Switching Lyapunov functions, LMIs,  $\mathcal{L}_2$ -induced gain.

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Hybrid Systems are dynamical systems exhibiting heterogeneous interactions between logic and differential or difference dynamics [1]. The classical

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case is given by a finite family of differential or difference processes supervised by logical decision-making algorithms. Among this generic class of systems, two important and wide classes could be pointed out: switched systems and switching systems. In switched systems, the control inputs include (at least partially) the switching rule  $\sigma(\cdot)$ . In switching systems, the switching rule  $\sigma(\cdot)$  is not a manipulable variable and is often generated by a complex logical decision-making algorithm, which could be state-, or output- or heterogenous input dependent. Such a switching rule is then modeled by an *a priori* unknown and induced signal, whose current value could be available. The properties of the designed control law should be thus satisfied for any arbitrary switching rule. One can also consider only switching sequences which are admissible for the physical process [2, 3]. This paper focuses on switching systems, which are an elegant framework to deal with a large number of applications as, for example, power converters, automotive domain or air traffic control.

Stability and stabilization of such a general switching system is nowadays a theoretical challenge, which has attracted growing attention in the literature. Lyapunov-like functions and multiple Lyapunov functions introduced in [4] offer adapted tools to deal with the stability of hybrid systems. Considering a common Lyapunov function is known to be highly conservative. For switching linear systems, a switching Lyapunov function approach has been provided in [5, 6, 7] (see also references therein). Nevertheless this technique has been developed for systems switching between only linear modes.

Depending on the physical application, the concept of linear system is valid only in a limited area, due to the fact that actuators cannot provide unbounded magnitude signals. Thus some nonlinearity with respect to the state or nonlinearity with respect to the control input, as saturation, should be taken into account to improve the modeling step and make it more realistic. Control design for classical systems including a saturation has been tackled in the literature, mainly by using a polytopic or a sector-bounded nonlinearity representation for the saturation phenomenon.

The polytopic representation of the saturation requires important computational capabilities [8]. This is why a sector bounded approach is preferred to cope with saturated control input (see for instance [9, 10, 11, 12]). The case of nonlinear systems, with a dynamic being a sum of an actuator saturation, a sector bounded nonlinear term and a linear term with respect to the state as in [13, 14], is treated in [15, 12] for continuous- and discrete-time systems, respectively.

The input-output properties, and particularly  $\mathcal{L}_2$ -induced gains, play an essential role in control theory [16, 17] for both linear and nonlinear (especially for saturated systems [18, 19, 20, 21]) systems. The extension of such methods for switching systems implies some difficulties and is still an open problem [22]. Some results should be noticed on this topic [23, 24] for continuous-time linear switching systems. To the best of our knowledge, the problem of  $\mathcal{L}_2$ -induced gain for discrete-time nonlinear switching systems has not been addressed in literature. Neural network approaches have been proposed for discrete-time systems [25]. However, the proposed methods do not take into account switching phenomena which is the main purpose here.

This paper aims at investigating discrete-time switching systems including nonlinearities with respect to the state for each mode, actuator saturations and additive  $\mathcal{L}_2$ -bounded disturbances. Two problems are then addressed in this context. The first one is to design a switching feedback control ensuring that the controlled system in the disturbance-free case is locally  $\lambda$ -contractive at the origin with a basin of attraction as large as possible. The second one is to design a switching feedback control guaranteeing the boundedness of the closed-loop trajectories and minimizing the  $\mathcal{L}_2$ -gain from the disturbance to the output. Sufficient conditions, associated with both these problems, are provided by using a classical and a modified sector condition for taking into account the nonlinearity of each mode and the saturation of control inputs, respectively. These conditions, involving Linear Matrix Inequalities (LMI) [26], allow to pose the control design as a convex programming problem.

It should be emphasized that switched linear continuous-time systems including saturated actuators have been already the objective of investigation, dealt with using a polytopic representation of the saturation [27, 28]. In these papers, the considered systems are however not affected by modal nonlinearities.

The paper is organized as follows: Section 1 is devoted to the description of the system, while Section 2 gathers some preliminary results and definitions and Section 3 the statement of considered problems. The main results, formalized as convex programming problems, are proposed respectively in Subsections 4.1 and 4.2. Sections 5 and 6 contain respectively simulation results and a brief conclusion.

**Notation.** Relative to a matrix  $A \in \mathbb{R}^{m \times n}$ ,  $A'$  denotes its transpose, and  $A_{(i)}$ ,  $i = 1, \dots, m$ , denotes its  $i$ th row. If  $A = A' \in \mathbb{R}^{n \times n}$ , then  $A < 0$  ( $A \leq 0$ )

means that  $A$  is negative- (semi-)definite. The components of any vector  $x \in \mathbb{R}^n$  are denoted by  $x_{(i)}$ ,  $\forall i = 1, \dots, n$ . Inequalities between vectors are component-wise:  $x \leq 0$  means that  $x_{(i)} \leq 0$  and  $x \leq y$  means that  $x_{(i)} - y_{(i)} \leq 0$ .  $I_n$  (resp.  $0_n$ ) denotes the  $n \times n$  identity (resp. null) matrix. The symbol  $\star$  stands for symmetric blocks in matrices. For a symmetric and positive-definite matrix  $M \in \mathbb{R}^{n \times n}$ , the ellipsoidal set  $\mathcal{E}(M, \alpha)$  associated with the matrix  $M$  and the positive scalar  $\alpha$  is given by  $\{x \in \mathbb{R}^n; x'Mx \leq \alpha\}$ . We will use the shortcut  $\mathcal{E}(M) = \mathcal{E}(M, 1)$ .

## 1. System Description

Consider the following discrete-time switching nonlinear system ( $\Sigma$ ):

$$(\Sigma) : \begin{cases} x_{k+1} &= A_{\sigma(k)}x_k + G_{\sigma(k)}\varphi_{\sigma(k)}(y_k) + B_{\sigma(k)}\text{sat}(u_k) + E_{\sigma(k)}^x w_k, \\ z_k &= C_{\sigma(k)}^z x_k + L_{\sigma(k)}\varphi_{\sigma(k)}(y_k) + D_{\sigma(k)}\text{sat}(u_k) + E_{\sigma(k)}^z w_k, \\ y_k &= C_{\sigma(k)}^y x_k + E_{\sigma(k)}^y w_k, \end{cases} \quad (1)$$

where  $x_k \in \mathbb{R}^n$ ,  $u_k \in \mathbb{R}^m$ ,  $z_k \in \mathbb{R}^{p_z}$ ,  $y_k \in \mathbb{R}^{p_y}$  and  $w_k \in \mathbb{R}^r$  are respectively the state, the control input, the controlled output, the output involved in the nonlinearities  $\varphi_{\sigma(k)}(\cdot)$  and the disturbance. The assumptions satisfied by system ( $\Sigma$ ) are now specified.

**Remark 1.** *The output  $y_k$  is independent with respect to  $\varphi_{\sigma(k)}(y_k)$  and to  $\text{sat}(u_k)$  in order to guarantee the well-posedness of this nonlinearity. It is possible to consider  $y_k = C_{\sigma(k)}^y x_k + D_{\sigma(k)}^y \text{sat}(u_k) + E_{\sigma(k)}^y w_k + L_{\sigma(k)}^y \varphi_{\sigma(k)}(y_k)$  if it is assumed that this relation defines one and only one value  $y_k$ , for all possible  $x_k$ ,  $u_k$  and  $w_k$ .*

**Assumption 1.** *The disturbance  $\{w_k\}_{k \in \mathbb{N}}$  is assumed to have a bounded energy, less than or equal to  $\frac{1}{\delta} \in \mathbb{R}^+$ , that is it belongs to the set*

$$\mathcal{W}_\delta = \left\{ w : \mathbb{N} \rightarrow \mathbb{R}^r ; \sum_{k \in \mathbb{N}} w_k' w_k \leq \frac{1}{\delta} \right\}. \quad (2)$$

**Assumption 2.** *The switching rule  $\sigma(\cdot)$  takes its value in the finite set  $\mathcal{I}_N = \{1, \dots, N\}$ . In order to avoid a complex description of the decision-making automaton,  $\sigma(\cdot)$  is assumed to be not known a priori, but its current value  $\sigma(k)$ , the output of the automaton, is assumed to be available in real-time. The evolution of the switching rule is assumed to be not governed by the user.*

The mode characterized by  $\sigma(\cdot)$  will be called the *active* mode of the switching system. In other words, the notation  $M_{\sigma(k)}$  means that at each time  $k$ ,  $M_{\sigma(k)}$  takes the matrix value in the set  $\{M_1, \dots, M_N\}$  indexed by  $\sigma(k)$ .

**Assumption 3.** *The  $N$  nonlinearities  $\varphi_i(\cdot) : \mathbb{R}^{p_y} \rightarrow \mathbb{R}^{p_y}$  associated with each mode  $i \in \mathcal{I}_N$  are assumed to be decentralized [29].*

Thus they verify the cone bounded sector condition  $\varphi_i(\cdot) \in [0_{p_y}, \Omega_i]$ , [29], *i.e.*,  $\varphi_i(0) = 0$  and there exist  $N$  diagonal positive definite matrices  $\Omega_i = \Omega'_i \in \mathbb{R}^{p_y \times p_y}$  such that independently,  $\forall y \in \mathbb{R}^{p_y}$  and  $\forall l = 1, \dots, p_y$ :

$$\varphi_{i,(l)}(y) [\varphi_i(y) - \Omega_i y]_{(l)} \leq 0. \quad (3)$$

Thus, we have the following inequality,  $\forall i \in \mathcal{I}_N$ :

$$\varphi'_i(y) (\Delta_i)^{-1} [\varphi_i(y) - \Omega_i y] \leq 0, \quad (4)$$

where  $\Delta_i \triangleq \text{diag}\{d_{q,i}\}_{q=1;\dots;p_y} \in \mathbb{R}^{p_y \times p_y}$  is any diagonal and positive matrix. Thus,  $\Delta_i$  represents a degree-of-freedom and is an optimization variable. Notice, however, that in a more general case where there may exist dependencies among different components of  $\varphi_i(\cdot)$ , it could be possible to consider only the sector condition provided in [29], by restricting the degree-of-freedom  $\Delta_i$  to  $\Delta_i = d_i I_{p_y}$ :  $\varphi'_i(y) [\varphi_i(y) - \Omega_i y] \leq 0$ . Note that  $\Omega_i$  is given by the designer and assumed to be known in the sequel for each mode  $i \in \mathcal{I}_N$ .

The control inputs are bounded in magnitude and the standard saturation function is considered:

$$\text{sat}(u(k))_{(\ell)} = \text{sign}(u_{(\ell)}(k)) \min(\rho_{(\ell)}, |u_{(\ell)}(k)|) \quad (5)$$

$\forall \ell = 1, \dots, m$ , where  $\rho_{(\ell)} > 0$  denotes the symmetric saturation level relative to the  $\ell$ -th control. Throughout this work, the vector  $\rho$  is supposed to be given.

By extending the kind of control law provided in [12], we consider:

$$u_k = K_{\sigma(k)} x_k + \Gamma_{\sigma(k)} \varphi_{\sigma(k)}(y_k) \quad (6)$$

where the  $m \times n$ -matrix  $K_{\sigma(k)}$  is a switching state feedback gain and the  $m \times p_y$ -matrix  $\Gamma_{\sigma(k)}$  is a switching feedback gain associated to the active nonlinearity  $\varphi_{\sigma(k)}(\cdot)$ . Thus, with non trivial  $\Gamma_{\sigma(k)}$ , this feedback control law requires the following assumption.

**Assumption 4.** *The nonlinearity  $\varphi_{\sigma(k)}(\cdot)$  is assumed to be known. This assumption could be weakened by assuming only the availability of  $\varphi_{\sigma(k)}(y_k)$  as a signal [14].*

## 2. Preliminaries

It is helpful to introduce the indicator function  $\xi(\cdot)$  mapping  $\mathbb{N}$  into  $\mathbb{R}^N$ , defined by

$$\xi_{(i)}(k) \triangleq \begin{cases} 1, & \text{if } i = \sigma(k); \\ 0, & \text{if } i \neq \sigma(k); \end{cases} \quad \forall i \in \mathcal{I}_N. \quad (7)$$

Note that the indicator function satisfies

$$\sum_{i=1}^N \xi_{(i)}(k) = 1; \quad \forall i \in \mathcal{I}_N, \quad \xi_{(i)}(k) \in \{0, 1\}, \quad (8)$$

furthermore,  $\forall (i, j) \in \mathcal{I}_N \times \mathcal{I}_N, \quad i \neq j,$

$$(\xi_{(i)}(k))^2 = \xi_{(i)}(k), \quad \xi_{(i)}(k)\xi_{(j)}(k) = 0. \quad (9)$$

It is then possible to rewrite the switching matrices involved in the system (1) as matrices that are linear with respect to the indicator function:

$$M_{\sigma(k)} = \sum_{i=1}^N \xi_{(i)}(k) M_i. \quad (10)$$

Consider in addition the generic dead-zone nonlinearity:

$$\Psi(u_k) = u_k - \text{sat}(u_k). \quad (11)$$

By considering  $u_k$  given by (6), the closed-loop system (12) can be written under the form

$$x_{k+1} = A_{\sigma(k)}^{\text{cl}} x_k + G_{\sigma(k)}^{\text{cl}} \varphi_{\sigma(k)}(y_k) - B_{\sigma(k)} \Psi(u_k) + E_{\sigma(k)}^x w_k, \quad (12)$$

$$z_k = C_{\sigma(k)}^{z, \text{cl}} x_k + L_{\sigma(k)}^{\text{cl}} \varphi_{\sigma(k)}(y_k) - D_{\sigma(k)} \Psi(u_k) + E_{\sigma(k)}^z w_k, \quad (13)$$

$$y_k = C_{\sigma(k)}^y x_k + E_{\sigma(k)}^y w_k, \quad (14)$$

$$u_k = K_{\sigma(k)} x_k + \Gamma_{\sigma(k)} \varphi_{\sigma(k)}(y_k), \quad (15)$$

where

$$A_{\sigma(k)}^{\text{cl}} = A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma(k)}, \quad (16)$$

$$G_{\sigma(k)}^{\text{cl}} = G_{\sigma(k)} + B_{\sigma(k)}\Gamma_{\sigma(k)}, \quad (17)$$

$$C_{\sigma(k)}^{z,\text{cl}} = C_{\sigma(k)}^z + D_{\sigma(k)}K_{\sigma(k)}, \quad (18)$$

$$L_{\sigma(k)}^{\text{cl}} = L_{\sigma(k)} + D_{\sigma(k)}\Gamma_{\sigma(k)}. \quad (19)$$

The following lemma will be used to consider the dead-zone as a non-linearity belonging to a generalized sector condition. For given switching matrices  $H_i \in \mathbb{R}^{m \times n}$ ,  $i \in \mathcal{I}_N$  consider the set  $\mathcal{S}(H_i, \rho)$  defined by

$$\mathcal{S}(H_i, \rho) = \{x \in \mathbb{R}^n; -\rho \leq H_i x \leq \rho; \forall i \in \mathcal{I}_N\}. \quad (20)$$

**Lemma 1.** *Consider switching  $m \times n$ -matrices  $K_i, J_i$ ,  $i \in \mathcal{I}_N$ . If  $x_k$  is an element of  $\mathcal{S}((K_i - J_i), \rho)$ , then with  $u_k$  defined by (6), the nonlinearity  $\Psi(u_k)$  satisfies the following inequality*

$$\Psi(u_k)'(T_{\sigma(k)})^{-1}[\Psi(u_k) - J_{\sigma(k)}x_k - \Gamma_{\sigma(k)}\varphi_{\sigma(k)}(z_k)] \leq 0 \quad (21)$$

for any diagonal positive definite switching matrix  $T_i \in \mathbb{R}^{m \times m}$ ,  $i \in \mathcal{I}_N$ .

**Proof:** It follows the same lines as the proof of Lemma 1 in [9] (see also [30] or [15]).  $\square$

Let us consider a switching Lyapunov Function (SLF) defined by

$$V : \begin{cases} \mathbb{R}^n \times \mathbb{N} & \longrightarrow \mathbb{R}^+, \\ (x_k, k) & \longmapsto V_{\sigma(k)}(x_k) = \sum_{i=1}^N \xi_{(i)}(k) V_i(x_k). \end{cases} \quad (22)$$

The Switching Level Set (SLS) associated to  $V$  and  $\delta$  is given by

$$L_V(\delta) \triangleq \left\{ x \in \mathbb{R}^n; V_i(x) \leq \frac{1}{\delta}, \forall i \in \mathcal{I}_N \right\}. \quad (23)$$

The notion of contractive sets is basic for determining regions of asymptotic stability for the saturating closed-loop system (12). The following definition of  $\lambda$ -contractivity is adapted to consider the sector bounded nonlinearity  $\varphi_i(\cdot)$  and to consider some time-domain performance associated with the coefficient  $\lambda$ .



**Definition 1.** Consider a non-negative scalar  $0 < \lambda \leq 1$ . The SLS  $L_V(\delta)$  is absolutely  $\lambda$ -contractive with respect to the trajectories of system (12), with  $w_k = 0$ , if  $\forall x_k \in L_V(\delta)$ ,  $\forall \sigma(k) \in \mathcal{I}_N$  and  $\forall \varphi_\sigma(\cdot) \in [0_{p_y}, \Omega_\sigma]$ ,

$$\Delta^\lambda V = V_{\sigma(k+1)}(x_{k+1}) - \lambda V_{\sigma(k)}(x_k) < 0. \quad (24)$$

To provide the desired  $\lambda$ -contractivity conditions, the class of SLF of the form

$$V_{\sigma(k)}(x_k) = x_k' P_{\sigma(k)} x_k, \quad (25)$$

with  $P_{\sigma(k)} = \sum_{i=1}^N \xi_{(i)}(k) P_i$  and  $P_i = P_i' > 0$  is considered throughout the paper. It is noteworthy that by invoking the property (9),  $P_{\sigma(k)}$  and its inverse are linear with respect to the indicator function  $\xi(k)$ :

$$P_{\sigma(k)}^{-1} = \sum_{i=1}^N \xi_{(i)}(k) P_i^{-1} = \sum_{i=1}^N \xi_{(i)}(k) Q_i, \quad (26)$$

by noting  $Q_i = P_i^{-1}$ ,  $\forall i \in \mathcal{I}_N$ . The property of  $\xi(k)$  allows obviously to rewrite the SLS  $L_V(\delta)$ , defined by (23), as

$$L_V(\delta) = \bigcap_{\sigma(k) \in \mathcal{I}_N} \mathcal{E}(P_{\sigma(k)}, \frac{1}{\delta}) = \bigcap_{i \in \mathcal{I}_N} \mathcal{E}(P_i, \frac{1}{\delta}) = \bigcap_{i \in \mathcal{I}_N} \mathcal{E}(Q_i^{-1}, \frac{1}{\delta}). \quad (27)$$

**Definition 2.** The  $\mathcal{L}_2$ -induced gain [16],  $\|\Sigma\|_{\mathcal{L}_2}$ , associated to the closed-loop system (12) of the system  $\Sigma$ , is defined by

$$\|\Sigma\|_{\mathcal{L}_2}^2 = \max_{x_0=0; w \in \mathcal{W}_\delta} \frac{\|z\|^2}{\|w\|^2}. \quad (28)$$

The  $\mathcal{L}_2$ -induced gain of the closed-loop system (12) is equal or less to  $\sqrt{\gamma}$  if and only if

$$\frac{1}{\gamma} \sum_{k \in \mathbb{N}} z_k' z_k \leq \sum_{k \in \mathbb{N}} w_k' w_k. \quad (29)$$

### 3. Formulation of the problems

The goal of the paper is twofold and corresponds to provide a solution to the two following problems.

**Problem 1 ( $\lambda$ -contractivity).** For  $w_k = 0, \forall k \in \mathbb{N}$ , a fixed  $\delta > 0$  and  $0 < \lambda \leq 1$ , design a control law in the class (6) that allows the  $\lambda$ -contractivity of some  $L_V(\delta)$ , verifying (27), so that the closed-loop system is locally asymptotically stable to the origin with a basin of attraction as large as possible.

**Problem 2 (Disturbance rejection).** The objective in this case consists in ensuring that the trajectories, with  $x_0 = 0$ , of the system (1) are bounded for any disturbance belonging to  $\mathcal{W}_\delta$ , as defined in (2) and, in addition, in minimizing the  $\mathcal{L}_2$ -induced gain  $\|\Sigma\|_{\mathcal{L}_2}$  from the disturbance  $w_k$  to the regulated output  $z_k$ .

Let us consider the following technical essential lemmas, which are required to derive the main results.

**Lemma 2.** For a fixed  $0 < \lambda \leq 1$ , consider for  $i \in \mathcal{I}_N$ , the existence of symmetric positive definite matrices  $Q_i \in \mathbb{R}^{n \times n}$ , positive diagonal matrices  $\Delta_i \in \mathbb{R}^{p_y \times p_y}$  and  $T_i \in \mathbb{R}^{m \times m}$ , matrices  $U_i \in \mathbb{R}^{n \times n}$ ,  $Y_{1,i} \in \mathbb{R}^{m \times n}$ ,  $Z_{1,i} \in \mathbb{R}^{m \times n}$ ,  $Y_{2,i} \in \mathbb{R}^{m \times p_y}$  and a scalar  $\gamma$  such that

$$\mathcal{M}_{ij} < 0, \quad \forall (i, j) \in \mathcal{I}_N^2, \quad (30)$$

where  $\mathcal{M}_{ij} =$

$$\begin{bmatrix} -Q_j & \star & \star & \star & \star & \star \\ 0 & -\gamma I_{p_z} & \star & \star & \star & \star \\ (A_i U_i + B_i Y_{1,i})' & (C_i^z U_i + D_i Y_{1,i})' & \lambda(Q_i - U_i - U_i') & \star & \star & \star \\ (G_i \Delta_i + B_i Y_{2,i})' & (L_i \Delta_i + D_i Y_{2,i})' & \Omega(C_i^z U_i + D_i Y_{1,i}) & -2\Delta_i & \star & \star \\ -(B_i T_i)' & -(D_i T_i)' & Z_{1,i} & Y_{2,i} & -2T_i & \star \\ (E_i^x)' & (E_i^z)' & 0 & -(E_i^y)' \Omega_i' & 0 & -I_r \end{bmatrix}. \quad (31)$$

Then, applying the control law

$$\begin{aligned} u_k &= K_{\sigma(k)} x_k + \Gamma_{\sigma(k)} \varphi_{\sigma(k)}(z_k), \\ &= (Y_{1,\sigma(k)} U_{\sigma(k)}^{-1}) x_k + (Y_{2,\sigma(k)} \Delta_{\sigma(k)}^{-1}) \varphi_{\sigma(k)}(z_k), \end{aligned} \quad (32)$$

implies that

$$\begin{aligned} V_{\sigma(k+1)}(x_{k+1}) - \lambda V_{\sigma(k)}(x_k) + \frac{1}{\gamma} z_k' z_k - w_k' w_k \\ - 2\psi'(u_k) T_{\sigma(k)}^{-1} (\psi(u_k) - J_{\sigma(k)} x_k - \Gamma_{\sigma(k)} \varphi_{\sigma(k)}) \\ - 2\varphi_{\sigma(k)}' \Delta_{\sigma(k)}^{-1} (\varphi_{\sigma(k)} - \Omega_{\sigma(k)} y_k) < 0. \end{aligned} \quad (33)$$

**Proof:** Due to the properties of  $\xi(k)$  given by (9) and LMIs (30), the sum

$$\mathcal{M}(\xi(k), \xi(k+1)) = \sum_{i \in \mathcal{I}_N} \sum_{j \in \mathcal{I}_N} \xi_{(i)}(k+1) \xi_{(j)}(k) \mathcal{M}_{ij} \quad (34)$$

is negative definite. For a sake of clarity, the dependency with respect to  $\xi(k)$  will be avoided and the dependency with respect to  $\xi(k+1)$  will be denoted with a '+' in the matrix index *i.e.*  $Q(\xi(k+1)) = Q_+$ . Thus rewriting Inequality (34) leads to

$$\begin{bmatrix} -Q_+ & \star & \star & \star & \star & \star \\ 0 & -\gamma I_{p_z} & \star & \star & \star & \star \\ (AU + BY_1)' & (C^z U + DY_1)' & \lambda(Q - U - U') & \star & \star & \star \\ (G\Delta + BY_2)' & (L\Delta + DY_2)' & \Omega(C^z U + DY_1) & -2\Delta & \star & \star \\ -(BT)' & -(DT)' & Z_1 & Y_2 & -2T & \star \\ (E^x)' & (E^z)' & 0 & -(\Omega E^y)' & 0 & -I_r \end{bmatrix} < 0. \quad (35)$$

It follows that  $Q > 0$  and  $U' + U - Q > 0$ , which imply that  $U$  has full rank. Then  $(Q - U)' Q^{-1} (Q - U) \geq 0$  and

$$\lambda(Q - U - U') \geq -\lambda U' Q^{-1} U. \quad (36)$$

Combining the change of basis  $\text{diag} [ I; I; U^{-1}; \Delta^{-1}; T^{-1}; I ]$  and Inequality (36) leads, by introducing  $K_{\sigma(k)} = Y_{1,\sigma(k)} U_{\sigma(k)}^{-1}$ ,  $\Gamma_{\sigma(k)} = Y_{2,\sigma(k)} \Delta_{\sigma(k)}^{-1}$ ,  $J_{\sigma(k)} = Z_{1,\sigma(k)} U_{\sigma(k)}^{-1}$  and by using a Schur complement, to inequality

$$\begin{bmatrix} (A^{\text{cl}})' \\ (G^{\text{cl}})' \\ -B' \\ (E^x)' \end{bmatrix} Q_+^{-1} \begin{bmatrix} (A^{\text{cl}})' \\ (G^{\text{cl}})' \\ -B' \\ (E^x)' \end{bmatrix}' + \begin{bmatrix} (C^{z,\text{cl}})' \\ (L^{\text{cl}})' \\ -D' \\ (E^z)' \end{bmatrix} \frac{1}{\gamma} \begin{bmatrix} (C^{z,\text{cl}})' \\ (L^{\text{cl}})' \\ -D' \\ (E^z)' \end{bmatrix}' + \begin{bmatrix} -\lambda Q^{-1} & \star & \star & \star \\ \Delta^{-1} \Omega C^{z,\text{cl}} & -2\Delta^{-1} & \star & \star \\ T^{-1} J & T^{-1} \Gamma & -2T^{-1} & \star \\ 0 & -(\Omega E^y)' \Delta^{-1} & 0 & -I_r \end{bmatrix} < 0. \quad (37)$$

By multiplying this last inequality at left by  $[ x'_k \ \varphi'_{\sigma(k)}(y_k) \ \psi'(u_k) \ w'_k ]$  and at right by its transpose, one has the inequality (33).  $\square$

**Lemma 3.** For a fixed  $0 < \lambda \leq 1$ , consider for  $i \in \mathcal{I}_N$ , the existence of symmetric positive definite matrices  $Q_i \in \mathbb{R}^{n \times n}$ , positive diagonal matrices  $\Delta_i \in \mathbb{R}^{p_y \times p_y}$  and  $T_i \in \mathbb{R}^{m \times m}$ , matrices  $U_i \in \mathbb{R}^{n \times n}$ ,  $Y_{1,i} \in \mathbb{R}^{m \times n}$ ,  $Z_{1,i} \in \mathbb{R}^{m \times n}$  and  $Y_{2,i} \in \mathbb{R}^{m \times p_y}$  such that

$$\mathcal{L}_{ij} < 0, \quad \forall (i, j) \in \mathcal{I}_N^2, \quad (38)$$

where

$$\mathcal{L}_{ij} = \begin{bmatrix} -Q_j & \star & \star & \star \\ (A_i U_i + B_i Y_{1,i})' & \lambda(Q_i - U_i - U_i') & \star & \star \\ (G_i \Delta_i + B_i Y_{2,i})' & \Omega(C_i^z U_i + D_i^z Y_{1,i}) & -2\Delta_i & \star \\ -(B_i T_i)' & Z_{1,i} & Y_{2,i} & -2T_i \end{bmatrix}, \quad (39)$$

then applying the control law defined by (32) implies that

$$\begin{aligned} V_{\sigma(k+1)}(x_{k+1}) - \lambda V_{\sigma(k)}(x_k) - 2\varphi'_{\sigma(k)} \Delta_{\sigma(k)}^{-1} (\varphi_{\sigma(k)} - \Omega_{\sigma(k)} y_k) \\ - 2\psi'(u_k) T_{\sigma(k)}^{-1} (\psi(u_k) - J_{\sigma(k)} x_k - \Gamma_{\sigma(k)} \varphi_{\sigma(k)}) < 0. \end{aligned} \quad (40)$$

**Proof:** The proof of Lemma 3 follows the one of Lemma 2 by noticing that  $\mathcal{L}_{ij}$  is an restriction of matrix  $\mathcal{M}_{ij}$ . It could also be found in [31].  $\square$

**Lemma 4.** Consider, for  $i \in \mathcal{I}_N$ , that there exist matrices  $U_i \in \mathbb{R}^{n \times n}$ , symmetric positive definite matrices  $Q_i \in \mathbb{R}^{n \times n}$ , matrices  $Y_{1,i} \in \mathbb{R}^{m \times n}$ ,  $Z_{1,i} \in \mathbb{R}^{m \times n}$  and a scalar  $\delta$  such that  $\forall i \in \mathcal{I}_N$  and  $\forall \ell = 1, \dots, m$ :

$$\mathcal{N}_{i,\ell} = \begin{bmatrix} -Q_i + U_i + U_i' & \star \\ (Y_{1,i} - Z_{1,i})_{(\ell)} & \delta \rho_{(\ell)}^2 \end{bmatrix} > 0. \quad (41)$$

Then

$$L_V(\delta) \subset \mathcal{S}(K(\cdot) - J(\cdot), \rho), \quad (42)$$

with  $J_{\sigma(k)} = Z_{1,\sigma(k)} U_{\sigma(k)}^{-1}$  and definition (32).

**Proof:** The proof is obtained from derivation of classical conditions of inclusion (see [26]). By noting

$$\mathcal{N}_\ell(\xi(k)) = \begin{bmatrix} -Q(\xi(k)) + U(\xi(k)) + U'(\xi(k)) & \star \\ (Y_1(\xi(k)) - Z_1(\xi(k)))_{(\ell)} & \delta \rho_{(\ell)}^2 \end{bmatrix}, \quad (43)$$

and due to the property (9), we have

$$\mathcal{N}_\ell(\xi(k)) = \sum_{i \in \mathcal{I}_N} \xi_{(i)}(k) \mathcal{N}_{i,\ell}. \quad (44)$$

It implies that the inequality  $\mathcal{N}_\ell(\xi(k)) > 0$  is equivalent to  $\forall i \in \mathcal{I}_N$ ,  $\ell = 1, \dots, m$ ,  $\mathcal{N}_{i,\ell} > 0$ . By using the change of basis  $\text{diag}[U^{-1}(\xi(k)); 1]$  and Inequality (36),  $\mathcal{N}_\ell(\xi(k)) > 0$  leads to

$$\begin{bmatrix} Q^{-1}(\xi(k)) & \star \\ (K(\xi(k)) - J(\xi(k)))_{(\ell)} & \delta \rho_{(\ell)}^2 \end{bmatrix} > 0. \quad (45)$$

Schur complement induces

$$\delta Q^{-1}(\xi(k)) > \frac{1}{\rho_{(\ell)}^2} (K(\xi(k)) - J(\xi(k)))'_{(\ell)} \times (K(\xi(k)) - J(\xi(k)))_{(\ell)}. \quad (46)$$

By multiplying this last inequality at left by  $x'_k$  and at right by its transpose, one has

$$\delta V_{\sigma(k)}(x_k) \geq \frac{1}{\rho_{(\ell)}^2} \left\| (K_{\sigma(k)} - J_{\sigma(k)})_{(\ell)} x_k \right\|^2. \quad (47)$$

For all  $x_k \in L_V(\delta)$ ,  $V_{\sigma(k)}(x_k) < \frac{1}{\delta}$ , which implies that  $x_k \in \mathcal{S}((K_i - J_i), \rho)$ . We obtain the relation (42).  $\square$

## 4. Main results

### 4.1. Control design solving $\lambda$ -contractivity problem

This subsection is dedicated to provide an optimization problem which solves Problem 1 related to the  $\lambda$ -contractivity of the system (12) without disturbance. Since no disturbance is considered, the value  $\delta$  has not anymore physical sense and could be arbitrarily fixed. To simplify,  $\delta$  is normalized in the following by  $\delta = 1$ , as in [31]. The SLS  $L_V(1)$ , defined by (23) is then the set  $\mathcal{S}_0$  of initial condition  $x_0$  of switching closed-loop system which is  $\lambda$ -contractive for any nonlinearities  $\varphi_i(\cdot)$  ( $i \in \mathcal{I}_N$ ) verifying the sector condition (4), without disturbance ( $w_k = 0$ ). This set  $L_V(1)$  is convex, since it is the intersection of convex sets.

The optimization problem consists of determining a control defined by (32), with the largest set  $L_V(1) = \mathcal{S}_0$ , under the constraints (30) and (41). For obtaining the largest set  $L_V(1)$ , let us consider, as in [12], a given and scaled polyhedral shape set included in  $L_V(1)$ . The goal is to maximize the scaling factor of this set. However this approach depends on the direction of the vectors of the polyhedral set. In order to avoid this dependency, a  $\sqrt{\alpha}$ -radius ball included into  $L_V(1)$  is considered:

$$\mathcal{E}(I_n, \alpha) = \{x \in \mathbb{R}^n; x'x \leq \alpha\} \subset L_V(1) = \mathcal{S}_0. \quad (48)$$

Being the SLS  $L_V(1)$  the intersection of the  $N$  ellipsoidal sets  $\mathcal{E}(Q_i^{-1})$ , Inclusion (48) is equivalent [26] to:

$$\begin{bmatrix} \frac{1}{\alpha} I_n & I_n \\ I_n & Q_i \end{bmatrix} > 0, \quad \forall i \in \mathcal{I}_N. \quad (49)$$

An important property is exposed in the following Proposition.

**Proposition 1.** *Given  $0 < \lambda \leq 1$  and  $\delta > 0$ , by considering symmetric positive definite matrices  $Q_i \in \mathbb{R}^{n \times n}$ , positive diagonal matrices  $\Delta_i \in \mathbb{R}^{p_y \times p_y}$  and  $T_i \in \mathbb{R}^{m \times m}$ , matrices  $U_i \in \mathbb{R}^{n \times n}$ ,  $Y_{1,i} \in \mathbb{R}^{m \times n}$ ,  $Z_{1,i} \in \mathbb{R}^{m \times n}$  and  $Y_{2,i} \in \mathbb{R}^{m \times p_y}$ , for  $i \in \mathcal{I}_N$  and a scalar  $\mu$ , the convex optimization problem*

$$\begin{aligned} & \min_{Q_i, U_i, \Delta_i, T_i, Z_{1,i}, Y_{1,i}, Y_{2,i}, \mu} \mu \\ & \text{subject to LMIs (38), (41) and } \begin{bmatrix} \mu I_n & I_n \\ I_n & Q_i \end{bmatrix} > 0, \quad \forall i \in \mathcal{I}_N, \end{aligned}$$

*leads to a switching control law represented by (32), solution of Problem 1.*

**Proof:** Being LMIs (41) satisfied, Lemma 4 implies that the modified bounded sector condition for the dead-zone function (21) is verified on  $L_V(1)$ . LMIs (38) being satisfied, Lemma 3 allows to write Inequality (33). Since there is no disturbance and the modified bounded conditions (21) and (4) are verified, we obtain  $\Delta^\lambda V < 0$ , which proves the  $\lambda$ -contractivity of the set. The proof of the proposition is obtained by denoting  $\mu = 1/\alpha$ .  $\square$

#### 4.2. Control design solving disturbance rejection problem

This subsection presents the solution of Problem 2, dealing with the best  $\mathcal{L}_2$ -induced gain for a null initial condition ( $x_0 = 0$ ) and for *a priori* given

upper bound of the energy of the admissible disturbances  $\frac{1}{\delta}$ . The idea consists in minimizing the upper bound  $\sqrt{\gamma}$  for the  $\mathcal{L}_2$ -induced gain  $\|\Sigma\|_{\mathcal{L}_2}$  from disturbance  $w_k$  to output  $z_k$ . The following convex programming problem is given in Proposition 2, where the names and the dimensions of free variables are consistent with the ones of Proposition 1.

**Proposition 2.** *Given  $0 < \lambda \leq 1$  and  $\delta > 0$ , by considering symmetric positive definite matrices  $Q_i \in \mathbb{R}^{n \times n}$ , positive diagonal matrices  $\Delta_i \in \mathbb{R}^{p_y \times p_y}$  and  $T_i \in \mathbb{R}^{m \times m}$ , matrices  $U_i \in \mathbb{R}^{n \times n}$ ,  $Y_{1,i} \in \mathbb{R}^{m \times n}$ ,  $Z_{1,i} \in \mathbb{R}^{m \times n}$  and  $Y_{2,i} \in \mathbb{R}^{m \times p_y}$ , for  $i \in \mathcal{I}_N$  and scalar  $\gamma$ , the convex optimization problem*

$$\begin{aligned} & \min_{Q_i, U_i, \Delta_i, T_i, Z_{1,i}, Y_{1,i}, Y_{2,i}, \gamma} \gamma \\ & \text{subject to LMIs (30) and (41),} \end{aligned}$$

leads to a switching control law represented by (32), solution of Problem 2, related to the maximization of the disturbance rejection.

**Proof:** We proceed by recurrence to prove this proposition. Assume that  $x_k \in L_V(\delta)$ , that is  $V_{\sigma(k)}(x_k) < \frac{1}{\delta}$  for all  $k = 0, \dots, k_0$ . Let us prove firstly that  $x_{k_0+1}$  satisfies  $V_{\sigma(k_0+1)}(x_{k_0+1}) < \frac{1}{\delta}$ : since LMIs (41) are satisfied, Lemma 4 implies that the modified bounded sector condition for the dead-zone function (21) is verified at each instants  $k = 0, \dots, k_0$ . Being LMIs (30) satisfied, Lemma 2 allows to write Inequality (33), that is, we have

$$\Delta^\lambda V + \frac{1}{\gamma} z'_k z_k - w'_k w_k < 0. \quad (50)$$

By assuming that  $x_0 = 0$  ( $V_{\sigma(0)}(0) = 0$ ), Inequality (50) (or more particularly  $\Delta^\lambda V < w'_k w_k$ ) leads to  $\forall k_0 \in \mathbb{N}$ :

$$V_{\sigma(k_0+1)}(x_{k_0+1}) \leq \sum_{k=0}^{k=k_0} \Delta^\lambda V \leq \sum_{k=0}^{k=k_0} w'_k w_k \leq \frac{1}{\delta}. \quad (51)$$

The limit  $\lim_{k_0 \rightarrow +\infty} V_{\sigma(k_0)}(x_{k_0})$  exists and is positive and bounded by  $\frac{1}{\delta}$ . By summing Inequality (50), we have

$$\frac{1}{\gamma} \sum_{k \in \mathbb{N}} z'_k z_k \leq \sum_{k \in \mathbb{N}} w'_k w_k.$$

In other words, the system verifies  $\|\Sigma\|_{\mathcal{L}_2} \leq \sqrt{\gamma}$ . Since the optimization problem provides the minimal  $\gamma$ , this is the optimal  $\mathcal{L}_2$ -gain.  $\square$

**Remark 2.** In the case of non-trivial initial condition  $x_0$  belonging to  $\bigcap_{i \in \mathcal{I}_N} \mathcal{E}(P_i, \beta)$ , where  $\beta$  is a positive scalar, the definition of  $\mathcal{L}_2$ -gain should be extended to

$$\sum_{k \in \mathbb{N}} z'_k z_k \leq \gamma \sum_{k \in \mathbb{N}} w'_k w_k + \gamma x'_0 P_{\sigma(0)} x_0 \leq \gamma \left( \sum_{k \in \mathbb{N}} w'_k w_k + \beta \right). \quad (52)$$

**Remark 3.** The problem of maximizing the disturbance tolerance could be treated by an optimization problem similar to that one of Proposition 2. More precisely, the optimization problem aims at minimizing  $\delta$  instead of  $\gamma$ . The value of  $\gamma$  is thus not essential, but should remain finite to ensure that Inequality (30) is verified (see for more details [21]).

## 5. Illustrations

Consider system (1) switching between  $N = 2$  modes, with the following matrices ( $n = 2$ ;  $m = p_y = p_z = r = 1$ ):

$$\begin{aligned} A_1 &= \begin{bmatrix} -1.1 & 0.4 \\ -0.2 & 1.1 \end{bmatrix}; & A_2 &= \begin{bmatrix} -0.2 & 0.7 \\ 0.6 & 1.3 \end{bmatrix}; & B_1 &= \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}; & B_2 &= \begin{bmatrix} 0 \\ 1.3 \end{bmatrix}; \\ G_1 &= \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}; & G_2 &= \begin{bmatrix} 1.6 \\ 0 \end{bmatrix}; & E_1^x &= \begin{bmatrix} 0.1 \\ 0.12 \end{bmatrix}; & E_2^x &= \begin{bmatrix} 0.1 \\ 0.11 \end{bmatrix}; \\ C_1^z &= C_2^z = [1 \ 1]; & L_1^z &= 1; & L_2^z &= 2; & D_1^z &= D_2^z = 1; \\ E_1^z &= 0.3; & E_2^z &= 0.22; & C_1^y &= C_2^y = [1 \ 0]; & E_1^y &= 0.5; & E_2^y &= 0.1. \end{aligned}$$

The nonlinearities  $\varphi_1$  and  $\varphi_2$  verifying the sector condition with  $\Omega_1 = 0.6$ ,  $\Omega_2 = 0.5$  and  $\rho = 0.5$  are depicted in Figure 1 and are given by

$$\varphi_1(y) = 0.6 \frac{y(1 + \sin(y))}{2}; \quad \varphi_2(y) = 0.5 \frac{y(1 + \exp(-y^2/2))}{2}.$$

Applying the algorithm given in Proposition 1 with  $\lambda = 1$ , we obtain the following numerical results:  $\gamma = 1.4160 \times 10^5$  and  $\mu = 1.1687$ .



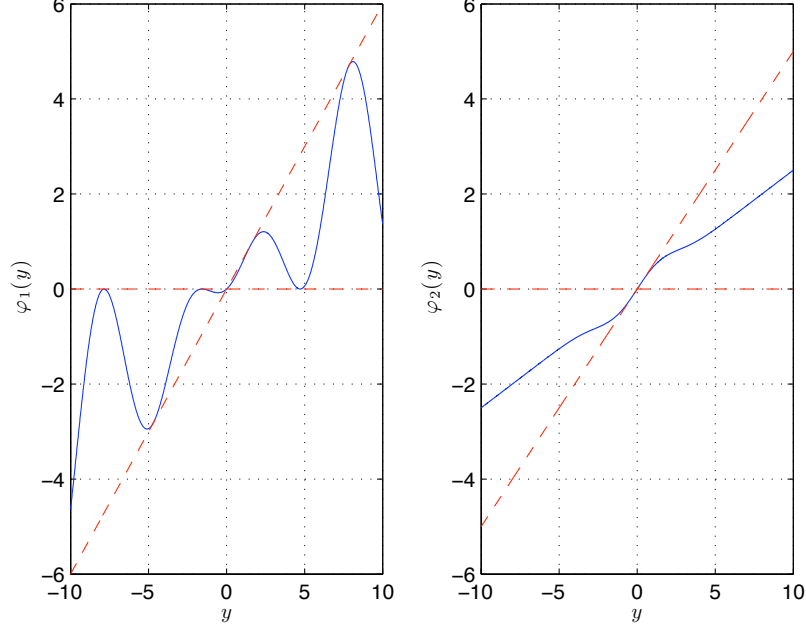


Figure 1: Nonlinearities  $\varphi_1(\cdot)$  and  $\varphi_2(\cdot)$  verifying the sector condition.

$$\begin{aligned}
Q_1 &= \begin{bmatrix} 1.5768 & 0.6927 \\ 0.6927 & 1.5211 \end{bmatrix}; & T_1 &= 0.3852; & \Delta_1 &= 1.3083; \\
Z_1 &= \begin{bmatrix} -0.2690 & -0.7397 \end{bmatrix}; & U_1 &= \begin{bmatrix} 1.5768 & 0.6927 \\ 0.6927 & 1.5211 \end{bmatrix}; \\
Y_{1,1} &= \begin{bmatrix} -0.7860 & -1.2797 \end{bmatrix}; & Y_{2,1} &= 0.2384; \\
Q_2 &= \begin{bmatrix} 4.8453 & -0.8933 \\ -0.8933 & 1.4350 \end{bmatrix}; & T_2 &= 1.1513; & \Delta_2 &= 0.2116; \\
Z_2 &= \begin{bmatrix} -2.4169 & -0.6905 \end{bmatrix}; & U_2 &= \begin{bmatrix} 4.8459 & -0.8935 \\ -0.8935 & 1.4351 \end{bmatrix}; \\
Y_{1,2} &= \begin{bmatrix} -2.7734 & -1.1579 \end{bmatrix}; & Y_{2,2} &= 0.1155.
\end{aligned}$$

In Figure 2, the sets  $\mathcal{E}(Q_1^{-1})$  and  $\mathcal{E}(Q_2^{-1})$  corresponding to the above synthesis result are plotted with symbol '+'. The solid line denotes the intersection of  $\mathcal{E}(Q_1^{-1})$  and  $\mathcal{E}(Q_2^{-1})$ , that is the set  $L_V(1)$ . Finally the solid

circle denotes the largest circle  $\mathcal{E}(\mu I)$ , included in  $L_V(1)$ . In the same figure, one illustrative example of trajectory is presented. The trajectory (circles linked by solid lines) starts from a point in  $L_V(1)$ . The initial point  $x_0 = (-0.3 \ -1)'$  is depicted with a square. One can note that  $L_V(1)$  is contractive.

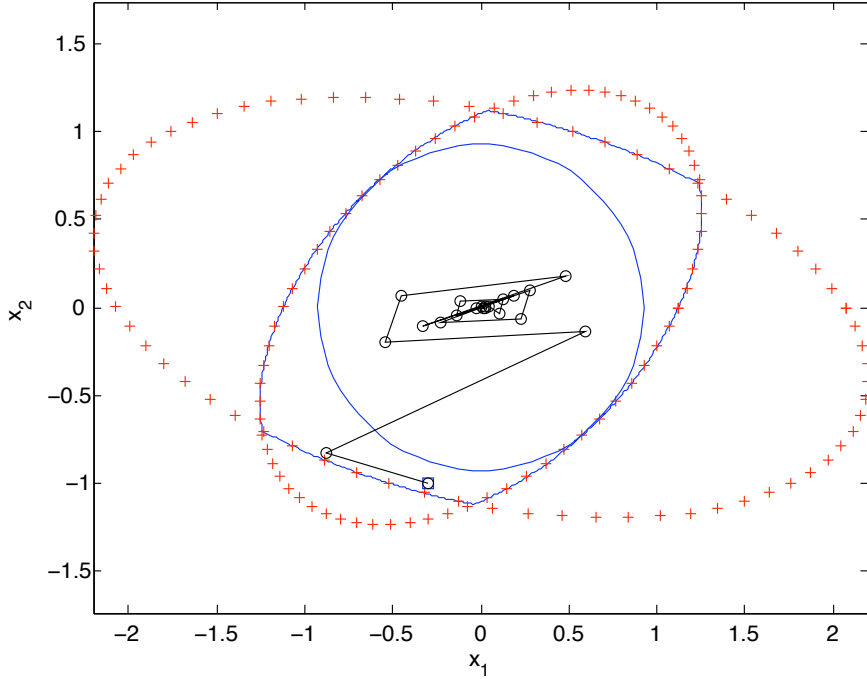


Figure 2: Sets  $\mathcal{E}(Q_1^{-1})$  and  $\mathcal{E}(Q_2^{-1})$  obtained from the optimization problem, the intersection  $L_V(\delta)$  and the largest circle included in  $L_V(1)$ .

The trajectory  $x_k$  is plotted with the control  $u_k$  and its saturation in Figure 3, in function of the time. The saturation  $\text{sat}(u_k)$  is emphasized at instants  $k = 0, 1, 3$ .

The convergence rate  $\lambda$  is conflicting with the size of  $L_V(\delta)$ . A compromise should be obtained by a correct choice of  $\lambda$ , as shown in Table 1. The associated sets  $L_V(1)$  are represented on Figure 4 for the different values of  $\lambda$  from Table 1.

If Assumption 2 does not hold, the active mode is not known. Furthermore if the Assumption 4 is not applicable, the value of the active nonlinearity is not available. By considering the additional constraints

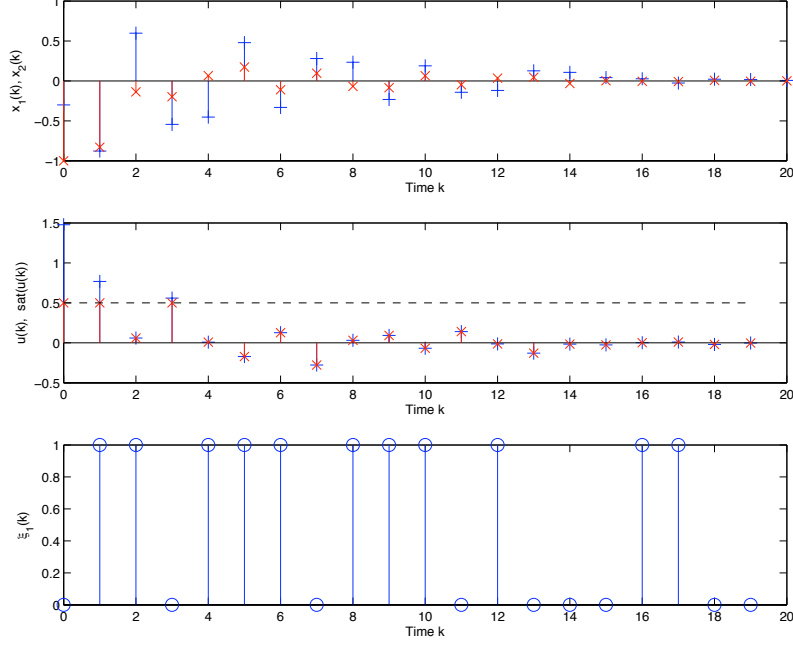


Figure 3: A particular trajectory: a) state components ( $x_1$  with '+';  $x_2$  with 'x'); b) control  $u_k$  with blue '+' and saturated control  $\text{sat}(u_k)$  with red 'x'; c) first component of the indicator function  $\xi_{(1)}(k)$ .

$\lambda$	$\mu$	$1/\sqrt{\mu}$
1	1.1687	0.9250
0.90	2.3543	0.6517
0.80	5.2330	0.4371
0.70	15.1100	0.2573

Table 1: Compromise between  $\lambda$  and  $\mu$ .

$(Y_{1,i}, Y_{2,i}, U_i, \Delta_i) = (Y_1, Y_2, U, \Delta)$  independent on the mode  $i$ ,  $\forall i \in \mathcal{I}_N$ , or  $Y_{2,i} = 0$ , it is possible to obtain particular control laws with constant gains, or a pure state feedback control law. For  $\lambda = 0.9$ , the conservatism is emphasized in Table 2.

Applying the algorithm provided in Proposition 2 leads, with  $\lambda = 1$  and

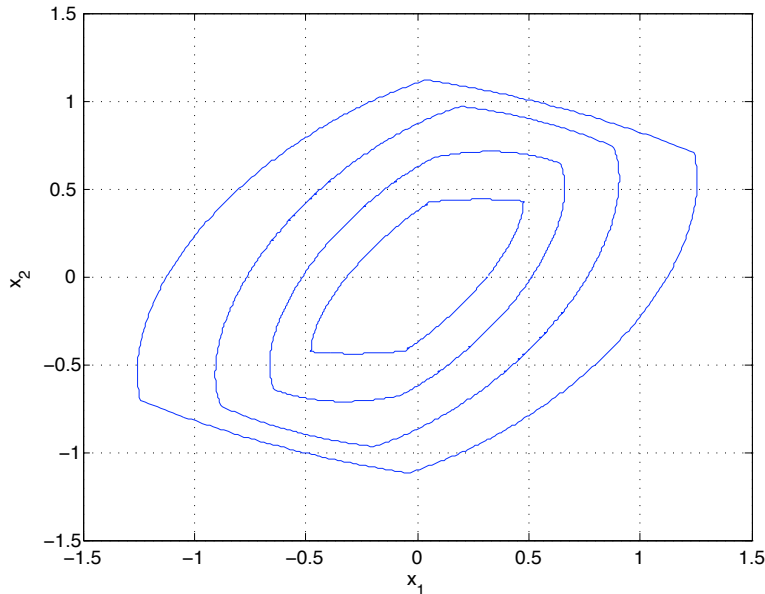


Figure 4: Sets  $L_V(1)$  for several  $\lambda$

$u_k$	$\mu$	$1/\sqrt{\mu}$
$K_{\sigma(k)}x_k + \Gamma_{\sigma(k)}\varphi_{\sigma(k)}(y_k)$	2.3542	0.4248
$Kx_k + \Gamma\varphi_{\sigma(k)}(y_k)$	3.7897	0.2639
$K_{\sigma(k)}x_k$	2.4804	0.4032
$Kx_k$	unfeasible	

Table 2: Comparison of results obtained by considering different classes of control laws.

$\delta = 0.5$ , to the following numerical results:  $\gamma = 1.1977$ . The considered disturbance which is applied is defined by:

$$w_k = \begin{cases} 0.6 \sin(17k), & 0 \leq k < 10, \\ 0, & k \geq 10. \end{cases}$$

The energy of this disturbance is equal to  $1.7616 \leq \frac{1}{\delta} = 2$ . A trajectory associated to this disturbance is depicted on Figure 5. We can note that the disturbance is rejected.

As in the previous problem, a trade-off should be found between the

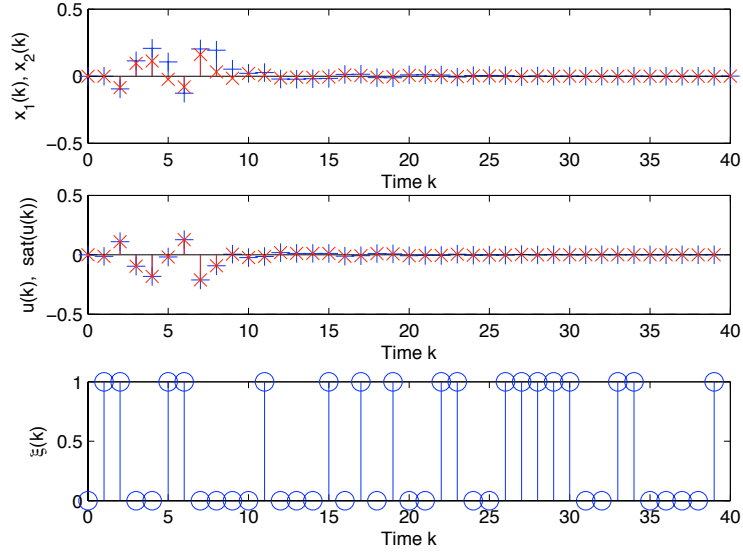


Figure 5: A particular trajectory: a) state components ( $x_1$  with '+';  $x_2$  with 'x'); b) control  $u_k$  with blue '+' and saturated control  $\text{sat}(u_k)$  with red 'x'; c) first component of the indicator function  $\xi_{(1)}(k)$ .

values  $\delta$  and  $\gamma$ , as presented in Table 3. For  $\delta \leq 0.092$ , the optimization problem has no solution.

$\delta$	$\gamma$
0.092	$2.2576 \times 10^7$
0.093	$6.0260 \times 10^4$
0.095	8.4925
0.097	4.2079
0.099	2.8066
0.1	2.4087
0.2	0.1976
1	0.1182
10	0.1182
100	0.1182

Table 3: Compromise between  $\delta$  and  $\gamma$ .

## 6. Conclusion

This paper has investigated the control problems of  $\lambda$ -contractivity and  $\mathcal{L}_2$  gain induced gain minimizing, related to discrete-time switching systems, which include modal nonlinearities, actuator saturations and additional energy-bounded disturbances. These modal nonlinearities are assumed to satisfy their own cone bounded sector condition. The class of switched control laws, composed of a state feedback and an active nonlinearity feedback is considered. Optimization problems under LMIs constraints are provided to solve the considered problems. A numerical example is studied to illustrate these results. Several problems are still open: for example designing controllers subject to decentralized control structure. Furthermore, the design problem of output dynamic controllers for both the cases where the cone bounded nonlinearity is available or not for feedback could be also investigated.

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