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► **To cite this version:**

| Wolfgang Steiner. On the structure of $(-\beta)$ -integers. 2010. hal-00533345v1

HAL Id: hal-00533345

<https://hal.science/hal-00533345v1>

Preprint submitted on 5 Nov 2010 (v1), last revised 26 Jun 2011 (v2)

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ON THE STRUCTURE OF $(-\beta)$ -INTEGERS

WOLFGANG STEINER

ABSTRACT. The $(-\beta)$ -integers are natural generalisations of the β -integers, and thus of the integers, for negative real bases. When β is a $(-\beta)$ -number, which is the analogue of a Parry number, we describe their structure by a fixed point of an anti-morphism.

1. INTRODUCTION

The aim of this paper is to study the structure of the set of real numbers having a digital expansion of the form

$$\sum_{k=0}^{n-1} a_k (-\beta)^k,$$

where $(-\beta)$ is a negative real base with $\beta > 1$, the digits $a_k \in \mathbb{Z}$ satisfy certain conditions specified below, and $n \geq 0$. These numbers are called $(-\beta)$ -integers, and have been recently studied by Ambrož, Dombek, Masáková and Pelantová [1].

Before dealing with these numbers, we recall some facts about β -integers, which are the real numbers of the form

$$\pm \sum_{k=0}^{n-1} a_k \beta^k \quad \text{such that} \quad 0 \leq \sum_{k=0}^{m-1} a_k \beta^k < \beta^m \quad \text{for all } 1 \leq m \leq n,$$

i.e., $\sum_{k=0}^{n-1} a_k \beta^k$ is a greedy β -expansion. Equivalently, we can define the set of β -integers as

$$\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \quad \text{with} \quad \mathbb{Z}_\beta^+ = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0),$$

where T_β is the β -transformation, defined by

$$T_\beta : [0, 1) \rightarrow [0, 1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor.$$

This map and the corresponding β -expansions were first studied by Rényi [17].

The notion of β -integers was introduced in the domain of quasicrystallography, see for instance [5], and the structure of the β -integers is very well understood now. We have $\mathbb{Z}_\beta \subseteq \beta \mathbb{Z}_\beta$, the set of distances between consecutive elements of \mathbb{Z}_β is

$$\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\},$$

where $T_\beta(x^-) = \lim_{y \rightarrow x, y < x} T_\beta(y)$, and the sequence of distances between consecutive elements of \mathbb{Z}_β^+ is coded by the fixed point of a substitution, see [8] for the case when Δ_β is a finite set, that is when β is a *Parry number*. We give short proofs of these facts in Section 2. More detailed properties of this sequence can be found e.g. in [2, 3, 4, 10, 14].

Closely related to \mathbb{Z}_β^+ are the sets

$$S_\beta(x) = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(x) \quad (x \in [0, 1)),$$

which were used by Thurston [18] to define (fractal) tilings of \mathbb{R}^{d-1} when β is a Pisot number of degree d , i.e., a root of a polynomial $x^d + p_1 x^{d-1} + \dots + p_d \in \mathbb{Z}[x]$ such that all other roots have modulus < 1 , and an algebraic unit, i.e., $p_d = \pm 1$. These tilings allow e.g. to determine the k -th digit a_k of a number without knowing the other digits, see [13].

It is widely agreed that the greedy β -expansions are the natural representations of real numbers in a real base $\beta > 1$. For the case of negative bases, the situation is not so clear. Ito and Sadahiro [12] proposed recently to use the $(-\beta)$ -transformation defined by

$$T_{-\beta} : \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1} \right), x \mapsto -\beta x - \left\lfloor -\beta x + \frac{\beta}{\beta+1} \right\rfloor.$$

see also [9]. This transformation has the important property that $T_{-\beta}(-x/\beta) = x$ for all $x \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1} \right)$. Some instances are depicted in Figures 1, 3 and 4.

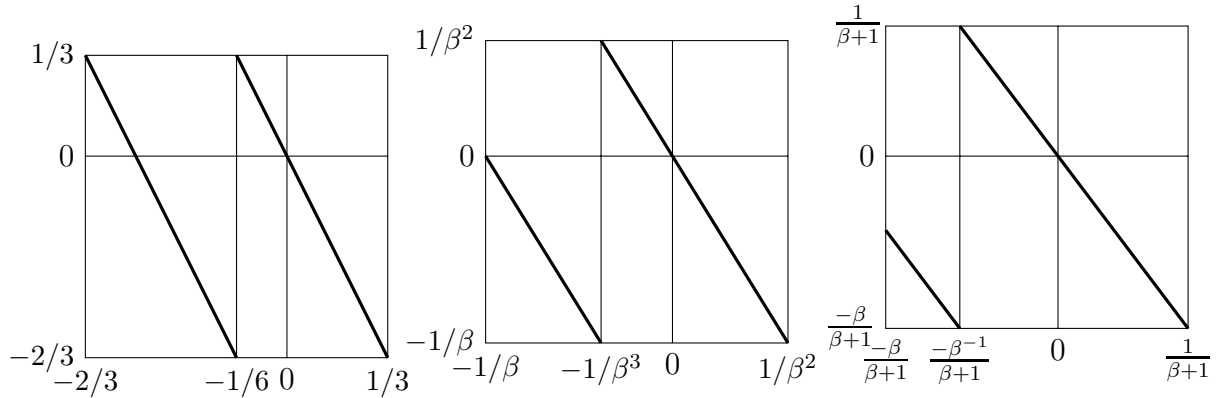


FIGURE 1. The $(-\beta)$ -transformation for $\beta = 2$ (left), $\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$ (middle), and $\beta = \frac{1}{\beta} + \frac{1}{\beta^2} \approx 1.325$ (right).

The set of $(-\beta)$ -integers is therefore defined by

$$\mathbb{Z}_{-\beta} = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0).$$

These are the numbers

$$\sum_{k=0}^{n-1} a_k (-\beta)^k \quad \text{such that} \quad \frac{-\beta}{\beta+1} \leq \sum_{k=0}^{m-1} a_k (-\beta)^{k-m} < \frac{1}{\beta+1} \quad \text{for all } 1 \leq m \leq n.$$

Note that, in the case of β -integers, we have to add $-\mathbb{Z}_\beta^+$ to \mathbb{Z}_β^+ in order to obtain a set resembling \mathbb{Z} . In the case of $(-\beta)$ -integers, this is not necessary because the $(-\beta)$ -transformation allows to represent positive and negative numbers.

It is not difficult to see that $\mathbb{Z}_{-\beta} = \mathbb{Z} = \mathbb{Z}_\beta$ when $\beta \in \mathbb{Z}$, $\beta \geq 2$. Some other properties of $\mathbb{Z}_{-\beta}$ can be found in [1], mainly for the case when $T_{-\beta}^n(\frac{-\beta}{\beta+1}) \in [\frac{1-|\beta|}{\beta}, 0]$ for all $n \geq 1$. The set

$$V'_\beta = \{T_{-\beta}^n(\frac{-\beta}{\beta+1}) \mid n \geq 0\}$$

plays a similar role for $(-\beta)$ -expansions as the set Δ_β for β -expansions. Consequently, we call $\beta > 1$ a $(-\beta)$ -number if V'_β is a finite set, recalling that Parry numbers were originally called β -numbers in [16]. In Theorem 3, we describe, for any $(-\beta)$ -number $\beta \geq (1 + \sqrt{5})/2$, the sequence of $(-\beta)$ -integers in terms of a two-sided infinite word on a finite alphabet which is a fixed point of an anti-morphism. Note that the orientation-reversing property of the map $x \mapsto -\beta x$ imposes the use of an anti-morphism instead of a morphism. Anti-morphisms were considered in a similar context by Enomoto [7].

For $1 < \beta < \frac{1+\sqrt{5}}{2}$, we have $\mathbb{Z}_{-\beta} = \{0\}$, as already proved in [1]. However, our study still makes sense for these bases $(-\beta)$ because we can also describe the sets

$$S_{-\beta}(x) = \lim_{n \rightarrow \infty} (-\beta)^n T_{-\beta}^{-n}(x) \quad (x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})),$$

where the limit set consists of the numbers lying in all but finitely many sets $(-\beta)^n T_{-\beta}^{-n}(x)$, $n \geq 0$. The reason for taking the limit instead of the union over all $n \geq 0$ is that $T_{-\beta}^2(\frac{-\beta^{-1}}{\beta+1}) \neq \frac{-\beta}{\beta+1}$ when $\beta \notin \mathbb{Z}$, see Section 3. In [15], the $(-\beta)$ -transformation is studied in detail for these values of β .

2. β -INTEGERS

In this section, we consider the structure of β -integers. The results are not new, but it is useful to state and prove them in order to understand the differences with $(-\beta)$ -integers.

Recall that $\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\}$, and let Δ_β^* be the free monoid generated by Δ_β . Elements of Δ_β^* will be considered as words on the alphabet Δ_β , and the operation is the concatenation of words. The β -substitution is the morphism $\varphi_\beta : \Delta_\beta^* \rightarrow \Delta_\beta^*$, defined by

$$\varphi_\beta(x) = \underbrace{11 \cdots 1}_{[\beta x] - 1 \text{ times}} T_\beta(x^-) \quad (x \in \Delta_\beta).$$

Here, 1 is an element of Δ_β and not the identity element of Δ_β^* (which is the empty word). Recall that, as φ_β is a morphism, we have $\varphi_\beta(uv) = \varphi_\beta(u)\varphi_\beta(v)$ for all $u, v \in \Delta_\beta^*$. Since $\varphi_\beta^{n+1}(1) = \varphi_\beta^n(\varphi_\beta(1))$ and $\varphi_\beta(1)$ starts with 1, $\varphi_\beta^n(1)$ is a prefix of $\varphi_\beta^{n+1}(1)$ for every $n \geq 0$.

Theorem 1. *For any $\beta > 1$, we have*

$$\mathbb{Z}_\beta^+ = \{z_k \mid k \geq 0\} \quad \text{with} \quad z_k = \sum_{j=1}^k u_j,$$

where $u_1 u_2 \cdots$ is the infinite word with letters in Δ_β which has $\varphi_\beta^n(1)$ as prefix for all $n \geq 0$.

The set of differences between consecutive β -integers is Δ_β .

The main observation for the proof of the theorem is the following. We use the notation $|v| = k$ and $L(v) = \sum_{j=1}^k v_j$ for any $v = v_1 \cdots v_k \in \Delta_\beta^k$, $k \geq 0$.

Lemma 1. *For any $n \geq 0$, $1 \leq k \leq |\varphi_\beta^n(1)|$, we have*

$$T_\beta^n\left(\left[\frac{z_{k-1}}{\beta^n}, \frac{z_k}{\beta^n}\right]\right) = [0, u_k],$$

and $z_{|\varphi_\beta^n(1)|} = L(\varphi_\beta^n(1)) = \beta^n$.

Proof. For $n = 0$, we have $|\varphi_\beta^0(1)| = 1$, $z_0 = 0$, $z_1 = 1$, $u_1 = 1$, thus the statements are true. Suppose that they hold for n , and consider

$$u_1 u_2 \cdots u_{|\varphi_\beta^{n+1}(1)|} = \varphi_\beta^{n+1}(1) = \varphi_\beta(\varphi_\beta^n(1)) = \varphi_\beta(u_1) \varphi_\beta(u_2) \cdots \varphi_\beta(u_{|\varphi_\beta^n(1)|}).$$

Let $1 \leq k \leq |\varphi_\beta^{n+1}(1)|$, and write $u_1 \cdots u_k = \varphi_\beta(u_1 \cdots u_{j-1}) v_1 \cdots v_i$ with $1 \leq j \leq |\varphi_\beta^n(1)|$, $1 \leq i \leq |\varphi_\beta(u_j)|$, i.e., $v_1 \cdots v_i$ is a non-empty prefix of $\varphi_\beta(u_j)$.

For any $x \in (0, 1]$, we have $T_\beta(x^-) = \beta x - [\beta x] + 1$, hence $L(\varphi_\beta(x)) = \beta x$ for $x \in \Delta_\beta$. This yields that

$$z_k = L(u_1 \cdots u_k) = \beta L(u_1 \cdots u_{j-1}) + L(v_1 \cdots v_i) = \beta z_{j-1} + i - 1 + v_i$$

and $z_{k-1} = \beta z_{j-1} + i - 1$, hence

$$\begin{aligned} \left[\frac{z_{k-1}}{\beta}, \frac{z_k}{\beta}\right] &= \left[z_{j-1} + \frac{i-1}{\beta}, z_{j-1} + \frac{i-1+v_i}{\beta}\right] \subseteq [z_{j-1}, z_{j-1} + u_j] = [z_{j-1}, z_j], \\ T_\beta^{n+1}\left(\left[\frac{z_{k-1}}{\beta^{n+1}}, \frac{z_k}{\beta^{n+1}}\right]\right) &= T_\beta\left(\left[\frac{i-1}{\beta}, \frac{i-1+v_i}{\beta}\right]\right) = [0, v_i] = [0, u_k]. \end{aligned}$$

Moreover, we have $L(\varphi_\beta^{n+1}(1)) = \beta L(\varphi_\beta^n(1)) = \beta^{n+1}$, thus the statements hold for $n+1$. \square

Proof of Theorem 1. By Lemma 1, we have $z_{|\varphi_\beta^n(1)|} = \beta^n$ for all $n \geq 0$, thus $[0, 1)$ is split into the intervals $[z_{k-1}/\beta^n, z_k/\beta^n)$, $1 \leq k \leq |\varphi_\beta^n(1)|$. Therefore, Lemma 1 yields that

$$T_\beta^{-n}(0) = \{z_{k-1}/\beta^n \mid 1 \leq k \leq |\varphi_\beta^n(1)|\},$$

hence

$$\bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0) = \{z_k \mid k \geq 0\}.$$

Since $u_k \in \Delta_\beta$ for all $k \geq 1$ and $u_{|\varphi_\beta^n(1)|} = T_\beta^n(1^-)$ for all $n \geq 0$, we have

$$\{z_k - z_{k-1} \mid k \geq 1\} = \{u_k \mid k \geq 1\} = \Delta_\beta. \quad \square$$

For the sets $S_\beta(x)$, Lemma 1 gives the following corollary.

Corollary 1. *For any $x \in [0, 1)$, we have*

$$S_\beta(x) = \{z_k + x \mid k \geq 0, u_{k+1} > x\} \subseteq x + S_\beta(0).$$

Note that $S_\beta(x)$ is always the union of a sequence of nested sets because $y \in [0, 1)$ implies $y/\beta \in [0, 1)$ and $T_\beta(y/\beta) = y$, thus $\beta^n T_\beta^{-n}(x) \subseteq \beta^{n+1} T_\beta^{-n-1}(x)$ for all $x \in [0, 1)$.

3. $(-\beta)$ -INTEGERS

We now turn to the discussion of $(-\beta)$ -integers and sets $S_{-\beta}(x)$, $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$. The first technical problem comes from the fact $(-\beta)^n T_{-\beta}^{-n}(x) \subseteq (-\beta)^{n+1} T_{-\beta}^{-n-1}(x)$ is not always true because $\frac{-y}{\beta} \notin [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ for $y = \frac{-\beta}{\beta+1}$. However, we have the following lemma.

Lemma 2. *For any $\beta > 1$, $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, $n \geq 0$, we have*

$$T_{-\beta}^{-n}(x) \setminus \left\{ \frac{-\beta}{\beta+1} \right\} \subseteq (-\beta) T_{-\beta}^{-n-1}(x).$$

If $T_{-\beta}(x) = x$, in particular if $x = 0$, then

$$T_{-\beta}^{-n}(x) \subseteq \beta^2 T_{-\beta}^{-n-2}(x).$$

Proof. For any $y \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, we have $T_{-\beta}^2(y/\beta^2) = T_{-\beta}(-y/\beta) = y$, which implies the first inclusion and $T_{-\beta}^{-n}(x) \setminus \left\{ \frac{-\beta}{\beta+1} \right\} \subseteq \beta^2 T_{-\beta}^{-n-2}(x)$. If $T_{-\beta}^n(\frac{-\beta}{\beta+1}) = x = T_{-\beta}(x)$, then

$$T_{-\beta}^{n+2}\left(\frac{-\beta-1}{\beta+1}\right) = T_{-\beta}^{n+1}\left(\frac{-\beta}{\beta+1}\right) = T_{-\beta}(x) = x,$$

thus $\frac{-\beta}{\beta+1} \in \beta^2 T_{-\beta}^{-n-2}(x)$ as well. \square

The first two statements of the following proposition can also be found in [1].

Proposition 1. *For any $\beta > 1$, we have $\mathbb{Z}_{-\beta} \subseteq (-\beta) \mathbb{Z}_{-\beta}$.*

If $\beta < (1 + \sqrt{5})/2$, then $\mathbb{Z}_{-\beta} = \{0\}$.

If $\beta \geq (1 + \sqrt{5})/2$, then

$$\mathbb{Z}_{-\beta} \cap (-\beta)^n [-\beta, 1] = \left\{ (-\beta)^n, (-\beta)^{n+1} \right\} \cup (-\beta)^{n+2} \left(T_{-\beta}^{-n-2}(0) \cap \left(\frac{-1}{\beta}, \frac{1}{\beta^2} \right) \right)$$

for all $n \geq 0$, in particular

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \begin{cases} \{-\beta, -\beta + 1, \dots, -\beta + \lfloor \beta \rfloor, 0, 1\} & \text{if } \beta^2 \geq \lfloor \beta \rfloor(\beta + 1), \\ \{-\beta, -\beta + 1, \dots, -\beta + \lfloor \beta \rfloor - 1, 0, 1\} & \text{if } \beta^2 < \lfloor \beta \rfloor(\beta + 1). \end{cases}$$

Proof. For any $\beta > 1$, we have $T_{-\beta}(0) = 0$, thus $T_{-\beta}^{-n-1}(0) \subseteq T_{-\beta}^{-n}(0)$. This means that $(-\beta)^{n+1} T_{-\beta}^{-n-1}(0) \subseteq -\beta((-\beta)^n T_{-\beta}^{-n}(0))$, hence $\mathbb{Z}_{-\beta} \subseteq (-\beta) \mathbb{Z}_{-\beta}$.

If $\beta < \frac{1+\sqrt{5}}{2}$, then $\frac{-1}{\beta} < \frac{-\beta}{\beta+1}$, hence $T_{-\beta}^{-1}(0) = \{0\}$ and $\mathbb{Z}_{-\beta} = \{0\}$, see Figure 1 (right).

If $\beta \geq \frac{1+\sqrt{5}}{2}$, then $\frac{-1}{\beta} \in T_{-\beta}^{-1}(0)$ implies $1 \in \mathbb{Z}_{-\beta}$, thus $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \geq 0$. Clearly,

$$(-\beta)^{n+2} \left(T_{-\beta}^{-n-2}(0) \cap \left(\frac{-1}{\beta}, \frac{1}{\beta^2} \right) \right) \subseteq \mathbb{Z}_{-\beta} \cap (-\beta)^n (-\beta, 1).$$

To show the other inclusion, let $z \in (-\beta)^m T_{-\beta}^{-m}(0) \cap (-\beta)^n (-\beta, 1)$ for some $m \geq 0$. If $m < n + 2$, then Lemma 2 yields that $z \in (-\beta)^{n+2} T_{-\beta}^{-n-2}(0)$, since $(-1)^n z < \beta^n \leq \frac{\beta^{n+2}}{\beta+1}$ implies $z \neq \frac{(-\beta)^{n+2}}{(-\beta)^m}$. If $m > n + 2$, then $(\frac{-1}{\beta}, \frac{1}{\beta^2}) \subseteq (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ implies that $T_{-\beta}^{n+2}\left(\frac{z}{(-\beta)^{n+2}}\right) = T_{-\beta}^m\left(\frac{z}{(-\beta)^m}\right) = 0$, thus we also have $z \in (-\beta)^{n+2} \left(T_{-\beta}^{-n-2}(0) \cap (-\beta)^n (-\beta, 1) \right)$.

Consider now $n = 0$, then

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \{-\beta, 1\} \cup \{z \in (-\beta, 1) \mid T_{-\beta}^2(z/\beta^2) = 0\}.$$

Since $\frac{-\lfloor \beta \rfloor}{\beta} \geq \frac{-\beta}{\beta+1}$ if and only if $\beta^2 \geq \lfloor \beta \rfloor(\beta+1)$, we obtain

$$(-\beta)T_{-\beta}^{-1}(0) = \begin{cases} \{0, 1, \dots, \lfloor \beta \rfloor\} & \text{if } \beta^2 \geq \lfloor \beta \rfloor(\beta+1), \\ \{0, 1, \dots, \lfloor \beta \rfloor - 1\} & \text{if } \beta^2 < \lfloor \beta \rfloor(\beta+1). \end{cases}$$

If $T_{-\beta}^2(z/\beta^2) = 0$, then $z = -a_1\beta + a_0$ with $a_0 \in (-\beta)T_{-\beta}^{-1}(0)$, $a_1 \in \{0, 1, \dots, \lfloor \beta \rfloor\}$, and $\mathbb{Z}_{-\beta} \cap [-\beta, 1]$ consists of those numbers $-a_1\beta + a_0$ lying in $[-\beta, 1]$. \square

This shows in particular that the maximal difference between consecutive $(-\beta)$ -integers exceeds 1 whenever $\beta^2 < \lfloor \beta \rfloor(\beta+1)$, i.e., $T_{-\beta}(\frac{-\beta}{\beta+1}) < 0$. This was already shown for an example in [1]. Example 3 shows that the distance between two consecutive $(-\beta)$ -integers can be even greater than 2, and the structure of $\mathbb{Z}_{-\beta}$ can be quite complicated. Therefore, we adapt a slightly different strategy as for \mathbb{Z}_β .

In the following, we always assume that the set

$$V_\beta = V'_\beta \cup \{0\} = \{T_{-\beta}^n(\frac{-\beta}{\beta+1}) \mid n \geq 0\} \cup \{0\}$$

is finite, i.e., β is a $(-\beta)$ -number, and let β be fixed. For $x \in V_\beta$, let

$$r_x = \min \{y \in V_\beta \cup \{\frac{1}{\beta+1}\} \mid y > x\}, \quad \hat{x} = \frac{x+r_x}{2}, \quad J_x = \{x\} \quad \text{and} \quad J_{\hat{x}} = (x, r_x).$$

Then $\{J_a \mid a \in A_\beta\}$ forms a partition of $[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, where

$$A_\beta = V_\beta \cup \hat{V}_\beta, \quad \text{with} \quad \hat{V}_\beta = \{\hat{x} \mid x \in V_\beta\}.$$

We have $T_{-\beta}(J_x) = J_{T_{-\beta}(x)}$ for every $x \in V_\beta$, and the following lemma shows that the image of every $J_{\hat{x}}$, $x \in V_\beta$, is a union of intervals J_a , $a \in A_\beta$.

Lemma 3. *Let $x \in V_\beta$ and*

$$J_{\hat{x}} \cap T_{-\beta}^{-1}(V_\beta) = \{y_1, \dots, y_m\}, \quad x = y_0 < y_1 < \dots < y_m < y_{m+1} = r_x.$$

For any $0 \leq i \leq m$, we have

$$T_{-\beta}((y_i, y_{i+1})) = J_{\hat{x}_i} \quad \text{with} \quad x_i = \lim_{y \rightarrow y_{i+1}, y < y_{i+1}} T_{-\beta}(y), \quad \text{i.e.,} \quad \hat{x}_i = T_{-\beta}(\frac{y_i + y_{i+1}}{2}),$$

and $\beta(y_{i+1} - y_i) = \lambda(J_{\hat{x}_i})$, where λ denotes the Lebesgue measure.

Proof. Since $T_{-\beta}$ maps no point in (y_i, y_{i+1}) to $\frac{-\beta}{\beta+1} \in V_\beta$, the map is continuous on this interval and $\lambda(T_{-\beta}((y_i, y_{i+1}))) = \beta(y_{i+1} - y_i)$. We have $x_i \in V_\beta$ since $x_i = T_{-\beta}(y_{i+1})$ in case $y_{i+1} < \frac{1}{\beta+1}$, and $x_i = \frac{-\beta}{\beta+1}$ in case $y_{i+1} = \frac{1}{\beta+1}$. Since $y_i = \max\{y \in T_{-\beta}^{-1}(V_\beta) \mid y < y_{i+1}\}$, we obtain that $r_{x_i} = \lim_{y \rightarrow y_i, y > y_i} T_{-\beta}(y)$, thus $T_{-\beta}((y_i, y_{i+1})) = (x_i, r_{x_i})$. \square

Therefore, we define an anti-morphism $\psi_\beta : A_\beta^* \rightarrow A_\beta^*$ by

$$\psi_\beta(x) = T_{-\beta}(x) \quad \text{and} \quad \psi_\beta(\hat{x}) = \hat{x}_m T_{-\beta}(y_m) \cdots \hat{x}_1 T_{-\beta}(y_1) \hat{x}_0 \quad (x \in V_\beta),$$

with m , x_i and y_i as in Lemma 3. Here, anti-morphism means that $\psi_\beta(uv) = \psi_\beta(v)\psi_\beta(u)$ for all $u, v \in A_\beta^*$. Now, the last letter of $\psi_\beta(\hat{0})$ is \hat{t} , $t = \max\{x \in V_\beta \mid x < 0\}$, and the first letter of $\psi_\beta(\hat{t})$ is $\hat{0}$. Therefore, $\psi_\beta^{2n}(\hat{0})$ is a prefix of $\psi_\beta^{2n+2}(\hat{0}) = \psi_\beta^{2n}(\psi_\beta^2(\hat{0}))$ and $\psi_\beta^{2n+1}(\hat{0})$ is a suffix of $\psi_\beta^{2n+3}(\hat{0})$ for every $n \geq 0$.

Theorem 2. For any $(-\beta)$ -number $\beta \geq (1 + \sqrt{5})/2$, we have

$$\mathbb{Z}_{-\beta} = \{z_k \mid k \in \mathbb{Z}, u_k = 0\} \quad \text{with} \quad z_k = \begin{cases} \sum_{j=1}^k \lambda(J_{u_j}) & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} \lambda(J_{u_{-j}}) & \text{if } k \leq 0, \end{cases}$$

where $\cdots u_{-1}u_0u_1\cdots$ is the two-sided infinite word on the finite alphabet A_β such that $u_0 = 0$, $\psi_\beta^{2n}(\widehat{0})$ is a prefix of $u_1u_2\cdots$ and $\psi_\beta^{2n+1}(\widehat{0})$ is a suffix of $\cdots u_{-2}u_{-1}$ for all $n \geq 0$.

Since $\psi_\beta(u_0) = u_0$, the word $\cdots u_{-1}u_0u_1\cdots$ can be seen as a fixed point of ψ_β .

The following lemma is the analogue of Lemma 1. Note that $u_{2k} \in V_\beta$ and $u_{2k+1} \in \widehat{V}_\beta$ for all $k \in \mathbb{Z}$, thus $z_{2k} = z_{2k-1}$ and $z_{-2k} = z_{1-2k}$ for all $k \geq 1$. We use the notation

$$L(v) = \sum_{j=1}^k \lambda(J_{v_j}) \quad \text{if } v = v_1 \cdots v_k \in A_\beta^k.$$

Lemma 4. For any $n \geq 0$, $0 \leq k < |\psi_\beta^n(\widehat{0})|/2$, we have

$$T_{-\beta}^n \left(\frac{z_{(-1)^n 2k}}{(-\beta)^n} \right) = u_{(-1)^n 2k}, \quad T_{-\beta}^n \left(\left(\frac{z_{(-1)^n 2k}}{(-\beta)^n}, \frac{z_{(-1)^n (2k+1)}}{(-\beta)^n} \right) \right) = J_{u_{(-1)^n (2k+1)}},$$

and $z_{(-1)^n |\psi_\beta^n(\widehat{0})|} = (-1)^n L(\psi_\beta^n(\widehat{0})) = r_0 (-\beta)^n$.

Proof. The statements are true for $n = 0$ since $|\psi_\beta^0(\widehat{0})| = 1$, $z_0 = 0$, $z_1 = \lambda(J_0) = r_0$.

Suppose that they hold for even n , and consider

$$u_{-|\psi_\beta^{n+1}(\widehat{0})|} \cdots u_{-2}u_{-1} = \psi_\beta^{n+1}(\widehat{0}) = \psi_\beta(\psi_\beta^n(\widehat{0})) = \psi_\beta(u_{|\psi_\beta^n(\widehat{0})|}) \cdots \psi_\beta(u_2)\psi_\beta(u_1).$$

Let $0 \leq k < |\psi_\beta^{n+1}(\widehat{0})|/2$, and write

$$u_{-2k-1} \cdots u_{-1} = v_{-2i-1} \cdots v_{-1} \psi_\beta(u_1 \cdots u_{2j})$$

with $0 \leq j < |\psi_\beta^n(\widehat{0})|/2$, $0 \leq i < |\psi_\beta(u_{2j+1})|/2$, i.e., $u_{-2i-1} \cdots u_{-1}$ is a suffix of $\psi_\beta(u_{2j+1})$.

By Lemma 3, we have $L(\psi_\beta(\widehat{x})) = \beta \lambda(J_{\widehat{x}})$ for any $x \in V_\beta$. This yields that

$$-z_{-2k-1} = \beta L(u_1 \cdots u_{2j}) + L(v_{-2i-1} \cdots v_{-1}) = \beta z_{2j} + L(v_{-2i-1} \cdots v_{-1})$$

and $-z_{-2k} = \beta z_{2j} + L(v_{-2i} \cdots v_{-1})$. With $\widehat{x} = u_{2j+1}$ and y_i as in Lemma 3, we obtain

$$\begin{aligned} T_{-\beta}^{n+1} \left(\frac{z_{-2k}}{(-\beta)^{n+1}} \right) &= T_{-\beta}^{n+1} \left(\frac{z_{2j}}{(-\beta)^n} - \frac{L(v_{-2i} \cdots v_{-1})}{(-\beta)^{n+1}} \right) \\ &= \begin{cases} T_{-\beta}(u_{2j}) = \psi_\beta(u_{2j}) = u_{-2k} & \text{if } i = 0, \\ T_{-\beta}(x + L(v_{-2i} \cdots v_{-1})/\beta) = T_{-\beta}(y_i) = v_{-2i} = u_{-2k} & \text{if } i \geq 1, \end{cases} \end{aligned}$$

and

$$T_{-\beta}^{n+1} \left(\left(\frac{z_{-2k}}{(-\beta)^{n+1}}, \frac{z_{-2k-1}}{(-\beta)^{n+1}} \right) \right) = T_{-\beta}((y_i, y_{i+1})) = J_{v_{-2i-1}} = J_{u_{-2k-1}}.$$

Moreover, we have $L(\psi_\beta^{n+1}(\widehat{0})) = \beta L(\psi_\beta^n(\widehat{0})) = r_0 \beta^{n+1}$, thus the statements hold for $n + 1$.

The proof for odd n runs along the same lines and is therefore omitted. \square

Proof of Theorem 2. By Lemma 4, we have $z_{(-1)^n |\psi_\beta^n(\widehat{0})|} = r_0 (-\beta)^n$ for all $n \geq 0$, thus $[0, r_0)$ splits into the intervals $(z_{(-1)^{2k}(-\beta)^{-n}}, z_{(-1)^{2k+1}(-\beta)^{-n}})$ and points $z_{(-1)^{2k}(-\beta)^{-n}}$, $0 \leq k < |\psi_\beta^n(\widehat{0})|/2$, hence

$$T_{-\beta}^{-n}(0) \cap [0, r_0) = \{z_{(-1)^{2k}(-\beta)^{-n}} \mid 0 \leq k < |\psi_\beta^n(\widehat{0})|/2, u_{(-1)^{2k}} = 0\}.$$

Let $m \geq 1$ be such that $\beta^{2m} r_0 > \frac{1}{\beta+1}$. Then we have $[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}) \subset (-\beta^{2m+1} r_0, \beta^{2m} r_0)$, and Lemma 2 yields that

$$T_{-\beta}^{-n}(0) \subseteq \beta^{2m} (T_{-\beta}^{-n-2m}(0) \cap [0, r_0)) \cup (-\beta)^{2m+1} (T_{-\beta}^{-n-2m-1}(0) \cap [0, r_0)),$$

thus

$$\bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0) = \bigcup_{n \geq 0} (-\beta)^n (T_{-\beta}^{-n}(0) \cap [0, r_0)) = \{z_{2k} \mid k \in \mathbb{Z}, u_{2k} = 0\}. \quad \square$$

As in the case of positive bases, the word $\cdots u_{-1} u_0 u_1 \cdots$ also describes the sets

$$S_{-\beta}(x) = \lim_{n \rightarrow \infty} (-\beta)^n T_{-\beta}^{-n}(x) = \bigcup_{n \geq 0} (-\beta)^n (T_{-\beta}^{-n}(x) \setminus \{\frac{-\beta}{\beta+1}\}) \quad (x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})),$$

where the second equality follows from Lemma 2. It is already indicated in the Introduction that $S_{-\beta}(x)$ can differ from $\bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(x)$. Indeed, if $T_{-\beta}^n(\frac{-\beta}{\beta+1}) = x \neq T_{-\beta}(x)$, then

$$T_{-\beta}^{n+2}(\frac{-\beta-1}{\beta+1}) = T_{-\beta}^{n+1}(\frac{-\beta}{\beta+1}) = T_{-\beta}(x) \neq x,$$

thus $\frac{(-\beta)^{n+1}}{\beta+1} \in (-\beta)^m T_{-\beta}^{-m}(x)$ if and only if $m = n$.

Theorem 2 and Lemma 4 give the following corollary.

Corollary 2. *For any $x \in V_\beta$, we have*

$$S_{-\beta}(x) = \{z_k \mid k \in \mathbb{Z}, u_k = x\}$$

and, for any $y \in J_{\widehat{x}}$,

$$S_{-\beta}(y) = \{z_k + y - x \mid k \geq 0, u_{k+1} = \widehat{x}\} \cup \{z_k + y - x \mid k < 0, u_k = \widehat{x}\}.$$

Recall that our main goal is to understand the structure of $\mathbb{Z}_{-\beta}$ (in case $\beta \geq (1 + \sqrt{5})/2$), i.e., to describe the occurrences of 0 in the word $\cdots u_{-1} u_0 u_1 \cdots$ defined in Theorem 2 and the words between two successive occurrences. Let

$$R_\beta = \{u_k u_{k+1} \cdots u_{s(k)-1} \mid k \in \mathbb{Z}, u_k = 0\} \quad \text{with} \quad s(k) = \min\{j \in \mathbb{Z} \mid u_j = 0, j > k\}$$

be the set of return words of 0 in $\cdots u_{-1} u_0 u_1 \cdots$. Note that $s(k)$ is defined for all $k \in \mathbb{Z}$ since $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \geq 0$ by Proposition 1.

For any $w \in R_\beta$, $\psi_\beta(w0)$ is a factor of $\cdots u_{-1} u_0 u_1 \cdots$ starting and ending with 0, thus we can write $\psi_\beta(w0) = w_1 \cdots w_m 0$ with $w_j \in R_\beta$, $1 \leq j \leq m$, and set

$$\varphi_{-\beta}(w) = w_1 \cdots w_m.$$

This defines an anti-morphism $\varphi_{-\beta} : R_\beta^* \rightarrow R_\beta^*$, which plays the role of the β -substitution.

Since $\cdots u_{-1}u_0u_1\cdots$ is generated by $u_1 = \widehat{0}$, we consider $w_\beta = u_0u_1\cdots u_{s(0)-1}$. We have

$$[0, 1] = [0, \frac{1}{\beta+1}) \cup [\frac{1}{\beta+1}, 1], \quad T_{-\beta}((-\beta)^{-1}[\frac{1}{\beta+1}, 1]) = [\frac{-\beta}{\beta+1}, 0],$$

thus $L(w_\beta) = 1$ and

$$w_\beta = 0\widehat{0}x_1\widehat{x}_1\cdots x_mx_m\widehat{x}_m x_{-\ell}\widehat{x}_{-\ell}\cdots x_{-1}\widehat{x}_{-1},$$

with $V_\beta = \{x_{-\ell}, \dots, x_{-1}, 0, x_1, \dots, x_m\}$, $x_{-\ell} < \cdots < x_{-1} < 0 < x_1 < \cdots < x_m$.

Theorem 3. For any $(-\beta)$ -number $\beta \geq (1 + \sqrt{5})/2$, we have

$$\mathbb{Z}_{-\beta} = \{z'_k \mid k \in \mathbb{Z}\} \quad \text{with} \quad z'_k = \begin{cases} \sum_{j=1}^k L(u'_j) & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} L(u'_{-j}) & \text{if } k \leq 0, \end{cases}$$

where $\cdots u'_{-2}u'_{-1}u'_1u'_2\cdots$ is the two-sided infinite word on the finite alphabet R_β such that $\varphi_{-\beta}^{2n}(w_\beta)$ is a prefix of $u'_1u'_2\cdots$ and $\varphi_{-\beta}^{2n+1}(w_\beta)$ is a suffix of $\cdots u'_{-2}u'_{-1}$ for all $n \geq 0$.

The set of distances between consecutive $(-\beta)$ -integers is

$$\Delta_{-\beta} = \{z'_{k+1} - z'_k \mid k \in \mathbb{Z}\} = \{L(w) \mid w \in R_\beta\}.$$

Note that the index 0 is omitted in $\cdots u'_{-2}u'_{-1}u'_1u'_2\cdots$ for reasons of symmetry.

Proof. The definitions of $\cdots u_{-1}u_0u_1\cdots$ in Theorem 2 and of $\varphi_{-\beta}$ imply that $\cdots u'_{-2}u'_{-1}u'_1u'_2\cdots$ is the derived word of $\cdots u_{-1}u_0u_1\cdots$ with respect to R_β , that is

$$u'_k = u_{|u'_1\cdots u'_{k-1}|} \cdots u_{|u'_1\cdots u'_k|-1}, \quad u'_{-k} = u_{-|u'_{-k}\cdots u'_{-1}|} \cdots u_{-|u'_{-k}\cdots u'_{-1}|-1} \quad (k \geq 1)$$

with

$$\{|u'_1\cdots u'_{k-1}| \mid k \geq 1\} \cup \{-|u'_{-k}\cdots u'_{-1}| \mid k \geq 1\} = \{k \in \mathbb{Z} \mid u_k = 0\}.$$

Here, for any $v \in R_\beta^*$, $|v|$ denotes the length of v as a word in A_β^* , not in R_β^* . Since

$$z'_k = \sum_{j=1}^k L(u'_j) = \sum_{j=0}^{|u'_1\cdots u'_k|-1} \lambda(J_{u_j}) = \sum_{j=1}^{|u'_1\cdots u'_k|} \lambda(J_{u_j}), \quad z'_{-k} = -\sum_{j=1}^k L(u'_{-j}) = -\sum_{j=1}^{|u'_{-k}\cdots u'_{-1}|} \lambda(J_{u_{-j}})$$

for all $k \geq 0$, Theorem 2 yields that $\{z'_k \mid k \in \mathbb{Z}\} = \mathbb{Z}_{-\beta}$.

It follows from the definition of R_β that $\Delta_{-\beta} = \{L(w) \mid w \in R_\beta\}$.

It remains to show that R_β is a finite set. We first show that the restriction of ψ_β to \widehat{V}_β is primitive, which means that there exists some $m \geq 1$ such that, for every $x \in V_\beta$, $\psi_\beta^m(\widehat{x})$ contains all elements of \widehat{V}_β . The proof is taken from [11, Proposition 8]. If $\beta > 2$, then the largest connected pieces of images of $J_{\widehat{x}}$ under $T_{-\beta}$ grow until they cover two consecutive discontinuity points $\frac{1}{\beta+1} - \frac{\alpha+1}{\beta}$, $\frac{1}{\beta+1} - \frac{\alpha}{\beta}$ of $T_{-\beta}$, and the next image covers all intervals $J_{\widehat{y}}$, $y \in V_\beta$. If $\frac{1+\sqrt{5}}{2} < \beta \leq 2$, then $\beta^2 > 2$ implies that the largest connected pieces of images of $J_{\widehat{x}}$ under $T_{-\beta}^2$ grow until they cover two consecutive discontinuity points of $T_{-\beta}^2$. Since

$$T_{-\beta}^2\left(\left(\frac{-\beta}{\beta+1}, \frac{\beta-2}{\beta+1} - \frac{1}{\beta}\right)\right) = \left(\frac{-\beta^3+\beta^2+\beta}{\beta+1}, \frac{1}{\beta+1}\right), \quad T_{-\beta}^2\left(\left(\frac{\beta-2}{\beta+1} - \frac{1}{\beta}, \frac{-\beta-1}{\beta+1}\right)\right) = \left(\frac{-\beta}{\beta+1}, \frac{\beta^2-\beta-1}{\beta+1}\right), \\ T_{-\beta}^2\left(\left(\frac{-\beta-1}{\beta+1}, \frac{\beta-2}{\beta+1}\right)\right) = \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right), \quad T_{-\beta}^2\left(\left(\frac{\beta-2}{\beta+1}, \frac{1}{\beta+1}\right)\right) = \left(\frac{-\beta}{\beta+1}, \frac{\beta^2-\beta-1}{\beta+1}\right),$$

the next image covers the fixed point 0, and further images grow until after a finite number of steps they cover all intervals $J_{\hat{y}}$, $y \in V_\beta$. The case $\beta = \frac{1+\sqrt{5}}{2}$ is treated in Example 1.

If $T_{-\beta}^n\left(\frac{-\beta}{\beta+1}\right) \neq 0$ for all $n \geq 0$, then $u_k = 0$ is equivalent with $u_{k+1} = \hat{0}$, see Proposition 2 below, thus we can consider the return words of $\hat{0}$ in $\cdots u_{-1}u_0u_1 \cdots$ instead of the return words of 0. Since $\psi_\beta^m(\hat{x}_0 x_1 \hat{x}_2)$ has at least two occurrences of $\hat{0}$ for all $x_0, x_1, x_2 \in V_\beta$, there are only finitely many such return words, cf. [6]. If $T_{-\beta}^n\left(\frac{-\beta}{\beta+1}\right) = 0$, then $\psi_\beta^n(x_0 \hat{x}_1 x_2)$ starts and ends with 0 for all $x_0, x_1, x_2 \in V_\beta$, hence R_β is finite as well. \square

We remark that, for practical reasons, the set R_β can be obtained from the set $R = \{w_\beta\}$ by adding to R iteratively all return words of 0 which appear in $\varphi_{-\beta}(w)$ for some $w \in R$ until R stabilises. The final set R is equal to R_β .

Now, we apply the theorems in the case of two quadratic examples.

Example 1. Let $\beta = \frac{1+\sqrt{5}}{2}$, i.e., $\beta^2 = \beta + 1$, and $t = \frac{-\beta}{\beta+1} = \frac{-1}{\beta}$. We have $V_\beta = \{t, 0\}$. Since

$$J_{\hat{t}} = (t, 0) = \left(t, \frac{-1}{\beta^3}\right) \cup \left\{\frac{-1}{\beta^3}\right\} \cup \left(\frac{-1}{\beta^3}, 0\right), \quad J_{\hat{t}} = \left(0, \frac{1}{\beta^2}\right),$$

see Figure 1 (middle), the anti-morphism ψ_β on A_β^* is defined by

$$\psi_\beta: \quad t \mapsto 0, \quad \hat{t} \mapsto \hat{0}t\hat{t}, \quad 0 \mapsto 0, \quad \hat{0} \mapsto \hat{t}.$$

Its two-sided fixed point $\cdots u_{-1}u_0u_1 \cdots$ is

$$\cdots \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{0}t\hat{t}}_{\psi_\beta(\hat{t})} \underbrace{0}_{\psi_\beta(t)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{0}t\hat{t}}_{\psi_\beta(\hat{t})} \underbrace{0}_{\psi_\beta(t)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{\hat{0}}_{\psi_\beta(\hat{0})} \underbrace{\hat{0}t\hat{t}}_{\psi_\beta(\hat{t})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{0}t\hat{t}}_{\psi_\beta(\hat{t})} \underbrace{0}_{\psi_\beta(t)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{0}_{\psi_\beta(0)} \cdots,$$

where $\hat{0}$ marks the central letter u_0 . The ψ_β -images of the complete return words of 0 are

$$\psi_\beta: \quad 0\hat{0}t\hat{t}0 \mapsto 0\hat{0}t\hat{t}0\hat{t}0, \quad 0\hat{t}0 \mapsto 0\hat{0}t\hat{t}0,$$

thus $R_\beta = \{A, B\}$ with $A = 0\hat{0}t\hat{t}$, $B = 0\hat{t}$. The anti-morphism

$$\varphi_{-\beta}: \quad A \mapsto AB, \quad B \mapsto A,$$

has the two-sided fixed point

$$\cdots u'_{-2}u'_{-1} u'_1u'_2 \cdots = \cdots AAB AAB AAB AB \cdot AAB AAB AB AAB AB AAB B \cdots.$$

We have $\lambda(J_{\hat{0}}) = \frac{1}{\beta^2}$, $\lambda(J_{\hat{t}}) = \frac{1}{\beta}$, thus $L(A) = 1$, $L(B) = \frac{1}{\beta} = \beta - 1$, and some $(-\beta)$ -integers are shown in Figure 2. Note that $(-\beta)^n$ can also be represented as $(-\beta)^{n+2} + (-\beta)^{n+1}$.

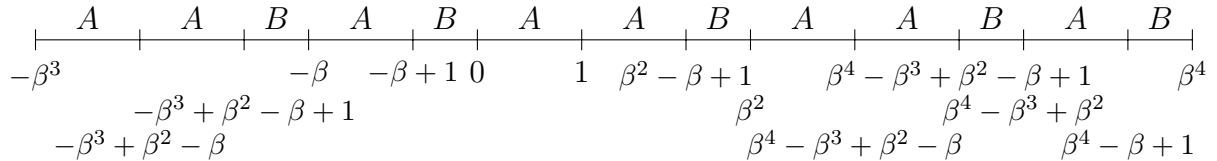


FIGURE 2. The $(-\beta)$ -integers in $[-\beta^3, \beta^4]$, $\beta = (1 + \sqrt{5})/2$.

Example 2. Let $\beta = \frac{3+\sqrt{5}}{2}$, i.e., $\beta^2 = 3\beta - 1$, then the $(-\beta)$ -transformation is depicted in Figure 3, where $t_0 = \frac{-\beta}{\beta+1}$, $t_1 = T_{-\beta}(t_0) = \frac{\beta^2}{\beta+1} - 2 = \frac{-\beta-1}{\beta+1}$, $T_{-\beta}(t_1) = \frac{1}{\beta+1} - 1 = t_0$. Therefore, $V_\beta = \{t_0, t_1, 0\}$ and the anti-morphism $\psi_\beta : A_\beta^* \rightarrow A_\beta^*$ is defined by

$$\psi_\beta : t_0 \mapsto t_1, \quad \widehat{t}_0 \mapsto \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0, \quad t_1 \mapsto t_0, \quad \widehat{t}_1 \mapsto \widehat{0}, \quad 0 \mapsto 0, \quad \widehat{0} \mapsto \widehat{t}_0 t_1 \widehat{t}_1,$$

which has the two-sided fixed point

$$\cdots \underbrace{0}_{\psi_\beta(0)} \underbrace{\widehat{0}}_{\psi_\beta(\widehat{t}_1)} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0}_{\psi_\beta(\widehat{t}_0)} \underbrace{t_1}_{\psi_\beta(t_0)} \underbrace{\widehat{t}_0 t_1 \widehat{t}_1}_{\psi_\beta(\widehat{0})} \underbrace{\widehat{0}}_{\psi_\beta(0)} \underbrace{\widehat{0}}_{\psi_\beta(\widehat{t}_1)} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0}_{\psi_\beta(\widehat{t}_0)} \cdots,$$

where $\widehat{0}$ marks the central letter u_0 . The ψ_β -images of the complete return words of 0 are

$$\begin{aligned} \psi_\beta : \quad & 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \mapsto 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0, \\ & 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \mapsto 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0, \\ & 0 \widehat{0} t_0 \widehat{t}_0 t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \mapsto 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_0 t_1 \widehat{t}_1 0. \end{aligned}$$

Note that $0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0$ and $0 \widehat{0} t_0 \widehat{t}_0 t_0 \widehat{t}_0 t_1 \widehat{t}_1 0$ differ only by a letter in V_β , and correspond therefore to intervals of same length. Since the letters t_0 and t_1 are never mapped to 0, we identify these two return words. Then we have $R_\beta = \{A, B\}$ with $A = 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1$, $B = 0 \widehat{0} t_0 \widehat{t}_0 \{t_0, t_1\} \widehat{t}_0 t_1 \widehat{t}_1$. The anti-morphism

$$\varphi_{-\beta} : A \mapsto AB, \quad B \mapsto ABB,$$

has the two-sided fixed point

$$\cdots ABB AB ABB ABB AB ABB AB ABB ABB AB \cdots.$$

We have $L(A) = 1$, $L(B) = \beta - 1 > 1$, and some $(-\beta)$ -integers are shown in Figure 3.

We remark that it is in general sufficient to consider the elements of \widehat{V}_β when one is only interested in $\mathbb{Z}_{-\beta}$. This is made precise in the following proposition.

Proposition 2. *Let β and $\cdots u_{-1} u_0 u_1 \cdots$ be as in Theorem 2, $t = \max\{x \in V_\beta \mid x < 0\}$. If $0 \notin V'_\beta$ or the size of V'_β is odd, then $u_k = 0$ is equivalent with $u_{k+1} = \widehat{0}$ for all $k \in \mathbb{Z}$. If $0 \notin V'_\beta$ or the size of V'_β is even, then $u_k = 0$ is equivalent with $u_{k-1} = \widehat{t}$ for all $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{Z}$ and $m \geq 0$ such that $z_{2k}/\beta^{2m} \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$. Then we have

- $u_{2k} = 0$ if and only if $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$,
- $u_{2k+1} = \widehat{0}$ if and only if $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$,
- $u_{2k-1} = \widehat{t}$ if and only if $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$.

Recall that $u_{2k} \in V_\beta$ and $u_{2k+1} \in \widehat{V}_\beta$ for all $k \in \mathbb{Z}$. If z_{2k}/β^{2m} is a point of discontinuity of $T_{-\beta}^{2m}$, then we must have $T_{-\beta}^\ell(z_{2k}/\beta^{2m}) = \frac{-\beta}{\beta+1}$ for some $1 \leq \ell \leq 2m$.

If $0 \notin V'_\beta = \{T_{-\beta}^n(\frac{-\beta}{\beta+1}) \mid n \geq 0\}$, then $T_{-\beta}^\ell(z_{2k}/\beta^{2m}) = \frac{-\beta}{\beta+1}$ is not possible when $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$, thus $u_{2k-1} = \widehat{t}$, $u_{2k} = 0$ and $u_{2k+1} = \widehat{0}$ are equivalent.

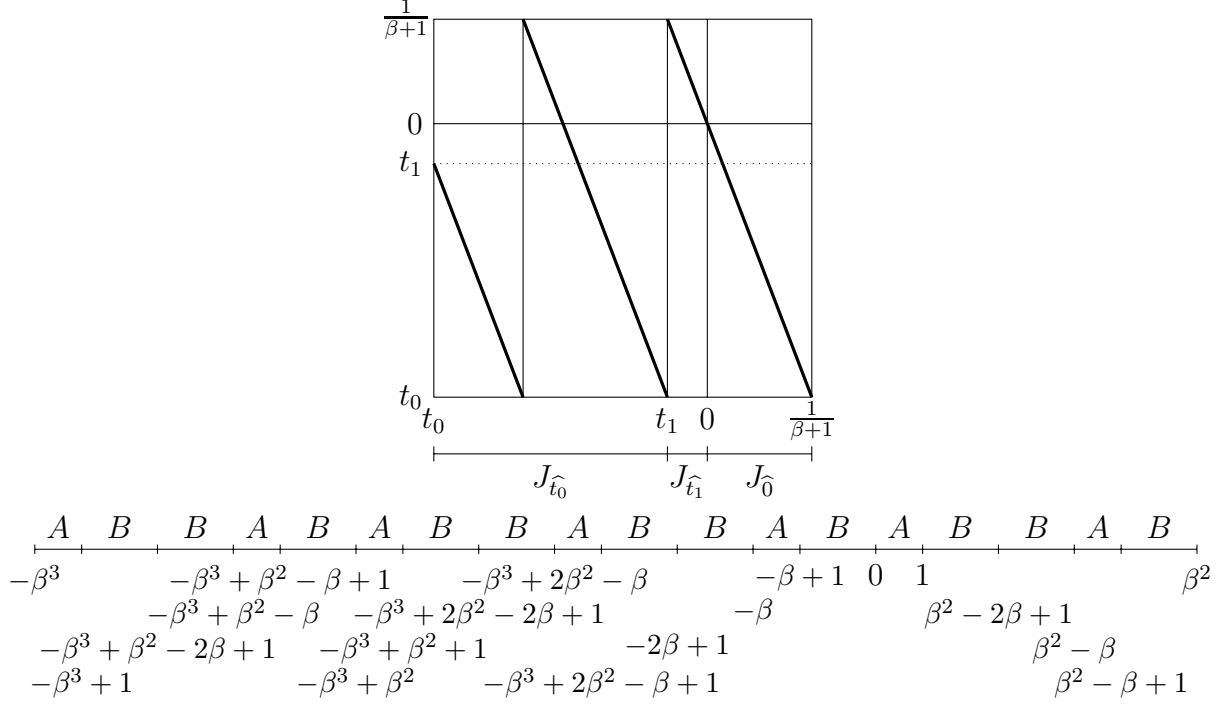


FIGURE 3. The $(-\beta)$ -transformation and $\mathbb{Z}_{-\beta} \cap [-\beta^3, \beta^2]$, $\beta = (3 + \sqrt{5})/2$.

Let now $T_{-\beta}^\ell(z_{2k}/\beta^{2m}) = \frac{-\beta}{\beta+1}$ and $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$, thus $0 \in V'_\beta$. Then the size of V'_β is the minimal $n \geq 2$ such that $T_{-\beta}^{n-1}(\frac{-\beta}{\beta+1}) = 0$. Moreover, $T_{-\beta}^j(z_{2k}/\beta^{2m}) \neq \frac{-\beta}{\beta+1}$ for all $j \neq \ell$.

If ℓ is even, then $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^\ell(y/\beta^{2m}) = \frac{-\beta}{\beta+1}$, thus $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$. From below, $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^\ell(y/\beta^{2m}) = \frac{1}{\beta+1}$ and $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{\ell+1}(y/\beta^{2m}) = \frac{-\beta}{\beta+1}$, thus $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = T_{-\beta}^{2m-\ell-1}(\frac{-\beta}{\beta+1})$. By the definition of n , we have $2m - \ell \geq n - 1$. If n is even, then we also have $2m - \ell - 1 \geq n - 1$, thus $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$.

If ℓ is odd, then the roles of $y > z_{2k}$ and $y < z_{2k}$ change, thus $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$, $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{\ell+1}(y/\beta^{2m}) = \frac{-\beta}{\beta+1}$. Now, $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$ if n is odd.

Therefore, $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$ is equivalent with $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$ if the size of V'_β is odd, and $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$ is equivalent with $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$ if the size of V'_β is even. \square

By Proposition 2, it suffices to consider the anti-morphism $\hat{\psi}_\beta : \hat{V}_\beta^* \rightarrow \hat{V}_\beta^*$ defined by

$$\hat{\psi}_\beta(\hat{x}) = \hat{x}_m \cdots \hat{x}_1 \hat{x}_0 \quad \text{when} \quad \hat{\psi}_\beta(\hat{x}) = \hat{x}_m T_{-\beta}(y_m) \cdots \hat{x}_1 T_{-\beta}(y_1) \hat{x}_0 \quad (x \in V_\beta).$$

Then, $\Delta_{-\beta}$ is given by the set \hat{R}_β which consists of the return words of $\hat{0}$ when $0 \notin V'_\beta$ or the size of V'_β is odd. When $0 \in V'_\beta$ and the size of V'_β is even, then \hat{R}_β consists of the words $w \hat{t}$ such that $\hat{t} w$ is a return word of \hat{t} .

Example 3. Let $\beta > 1$ with $\beta^6 = 3\beta^5 + 2\beta^4 + 2\beta^3 + \beta^2 - 2\beta - 1$, i.e., $\beta \approx 3.695$, then the $(-\beta)$ -transformation is depicted in Figure 4, where $t_n = T_{-\beta}^n\left(\frac{-\beta}{\beta+1}\right)$. We have $t_5 = \frac{-1}{\beta+1} = t_6$.

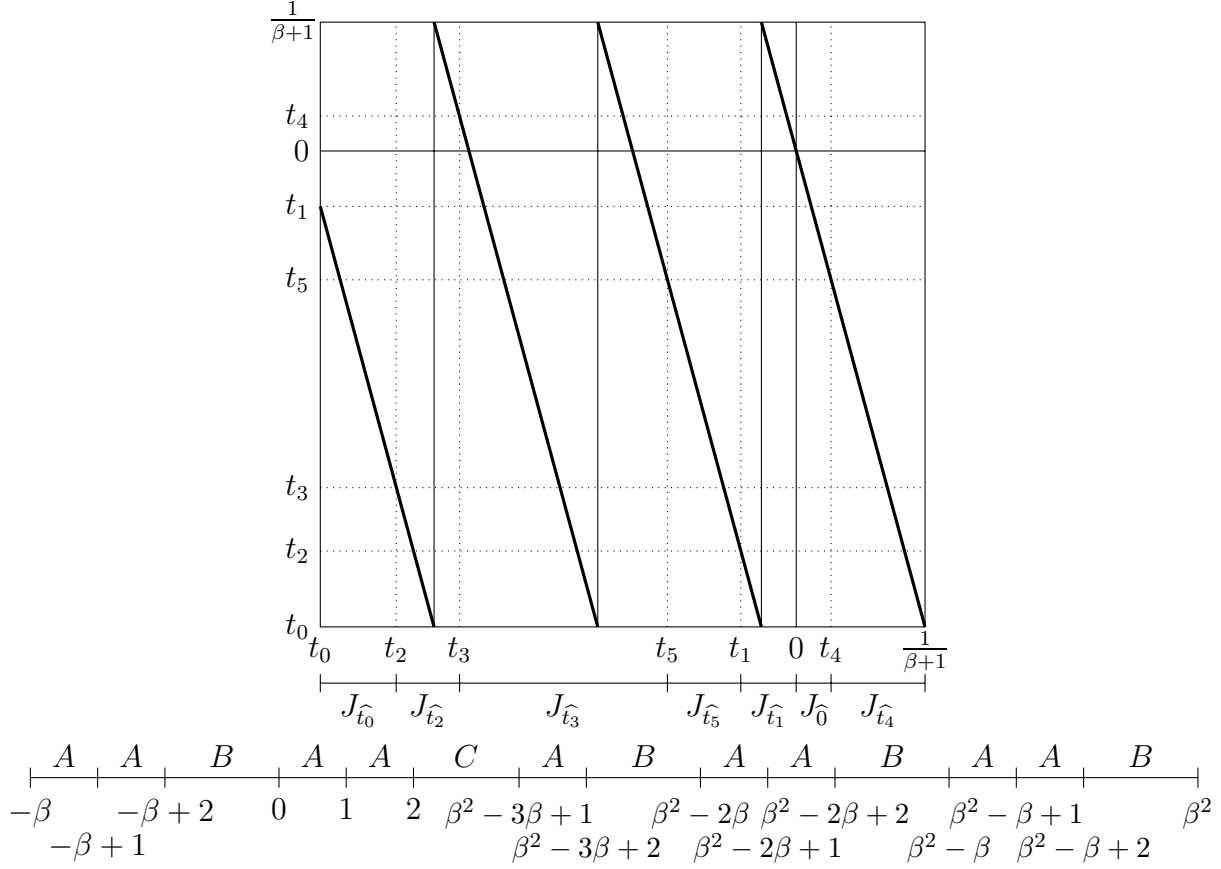


FIGURE 4. The $(-\beta)$ -transformation and $\mathbb{Z}_{-\beta} \cap [-\beta, \beta^2]$ from Example 3.

The anti-morphism $\widehat{\psi}_\beta : \widehat{V}_\beta^* \rightarrow \widehat{V}_\beta^*$ is therefore defined by

$$\widehat{\psi}_\beta : \begin{array}{lll} \widehat{t}_0 \mapsto \widehat{t}_3 \widehat{t}_5, & \widehat{t}_2 \mapsto \widehat{t}_4 \widehat{t}_0 \widehat{t}_2, & \widehat{t}_3 \mapsto \widehat{t}_5 \widehat{t}_1 \widehat{0} \widehat{t}_4 \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1 \widehat{0}, \\ \widehat{t}_5 \mapsto \widehat{t}_2 \widehat{t}_3, & \widehat{t}_1 \mapsto \widehat{0} \widehat{t}_4 \widehat{t}_0, & \widehat{0} \mapsto \widehat{t}_5 \widehat{t}_1, & \widehat{t}_4 \mapsto \widehat{t}_0 \widehat{t}_2 \widehat{t}_3. \end{array}$$

It is convenient to group to letters forming the words

$$\begin{array}{llll} a = \widehat{0} \widehat{t}_4, & b = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1, & c = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5, & d = \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1, \\ e = \widehat{t}_0 \widehat{t}_2, & f = \widehat{t}_4, & g = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3, & h = \widehat{t}_5 \widehat{t}_1, \end{array}$$

which correspond to the intervals

$$\begin{array}{llll} J_a = \left(0, \frac{1}{\beta+1}\right), & J_b = (t_0, 0), & J_c = (t_0, t_1), & J_d = (t_2, 0), \\ J_e = (t_0, t_3), & J_f = \left(t_4, \frac{1}{\beta+1}\right), & J_g = (t_0, t_5), & J_h = (t_5, 0). \end{array}$$

The anti-morphism $\widehat{\psi}_\beta$ acts on these words by

$$\begin{aligned} \widehat{\psi}_\beta : \quad a &\mapsto b, & b &\mapsto ababac, & c &\mapsto dabac, & d &\mapsto ababae, \\ e &\mapsto fc, & f &\mapsto g, & g &\mapsto habac, & h &\mapsto ag. \end{aligned}$$

Since $\widehat{0}$ only occurs at the beginning of a , the return words of $\widehat{0}$ with their $\widehat{\psi}_\beta$ -images are

$$\begin{aligned} ab &\mapsto ab ab acb, & aed &\mapsto ab ab aefcb, \\ acb &\mapsto ab ab acd ab acb, & aefcb &\mapsto ab ab acd ab acgfc b, \\ acd &\mapsto ab ab aed ab acb, & acgfc b &\mapsto ab ab acd ab \underbrace{acgh}_{=acb} ab acd ab acb. \end{aligned}$$

Then $\mathbb{Z}_{-\beta}$ is described by the anti-morphism $\widehat{\varphi}_{-\beta} : \widehat{R}_\beta^* \rightarrow \widehat{R}_\beta^*$ which is defined by

$$\begin{aligned} \widehat{\varphi}_{-\beta} : \quad A &\mapsto AAB, & L(A) &= 1, \\ B &\mapsto AACAB, & L(B) &= \beta - 2 \approx 1.695, \\ C &\mapsto AADAB, & L(C) &= \beta^2 - 3\beta - 1 \approx 1.569, \\ D &\mapsto AAE, & L(D) &= \beta^3 - 3\beta^2 - 2\beta - 1 \approx 1.104, \\ E &\mapsto AACAF, & L(E) &= \beta^4 - 3\beta^3 - 2\beta^2 - \beta - 2 \approx 2.081, \\ F &\mapsto AACABACAB, & L(F) &= \beta^5 - 3\beta^4 - 2\beta^3 - 2\beta^2 + \beta - 2 \approx 3.12. \end{aligned}$$

Some $(-\beta)$ -integers are represented in Figure 4, and the two-sided fixed point is

$$\dots AACAB AAB AADAB AAB AAB AACAB AAB AAB \dots$$

Note that grouping the letters as in Example 3 is always possible. It is usually a good idea to start directly with the corresponding intervals, and this is even possible when β is not a $(-\beta)$ -number. The drawback of this method is that the involved intervals can be a bit complicated to describe in the general case, e.g. $t_1 < \frac{1}{\beta+1} - \frac{|\beta|}{\beta}$ implies that (t_0, t_1) is mapped to (t_2, t_1) , an interval which does not occur in Example 3. Determining the return words is also a bit more complicated since $J_{\widehat{0}}$ can be contained in several intervals, and it should be taken care of the fact that the union of two intervals can be another interval (minus one point), as for $J_g \cup J_h = J_b \setminus \{t_5\}$ in Example 3. Therefore, we do not give a general account of this method here.

4. CONCLUSIONS AND OPEN QUESTIONS

With every $(-\beta)$ -number $\beta \geq (1 + \sqrt{5})/2$, we have associated an anti-morphism $\varphi_{-\beta}$ on a finite alphabet. The distances between consecutive $(-\beta)$ -integers are described by a fixed point of $\varphi_{-\beta}$, and $\varphi_{-\beta}$ is given by a simple algorithm. In a forthcoming version of [1], the anti-morphism will be described explicitly for any $\beta > 1$ such that $T_{-\beta}^n(\frac{-\beta}{\beta+1}) \in [\frac{1-|\beta|}{\beta}, 0]$ for all $n \geq 1$. Example 3 shows that the situation is more complicated when this condition is not fulfilled. It would be interesting to have a reasonably simple description of $\varphi_{-\beta}$ in the general case as well.

It is well known that the maximal distance between consecutive β -integers is bounded by 1. We have seen that this is not true for $(-\beta)$ -integers. Since the set $\Delta_{-\beta}$ is finite for any $(-\beta)$ -number $\beta \geq (1 + \sqrt{5})/2$, it is bounded. It is an open question whether there is a uniform bound on $\Delta_{-\beta}$. Another open question is whether $\Delta_{-\beta}$ is bounded when β is not a $(-\beta)$ -number. It is possible that these questions can be answered only when the structure of $\mathbb{Z}_{-\beta}$ is well understood in general.

Another topic which is probably worth investigating is the structure of the sets $S_{-\beta}(x)$ for $x \neq 0$, and the corresponding tilings when β is a Pisot unit.

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