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# ON THE STRUCTURE OF $(-\beta)$ -INTEGERS

WOLFGANG STEINER

ABSTRACT. The  $(-\beta)$ -integers are natural generalisations of the  $\beta$ -integers, and thus of the integers, for negative real bases. When  $\beta$  is a  $(-\beta)$ -number, which is the analogue of a Parry number, we describe their structure by a fixed point of an anti-morphism.

## 1. INTRODUCTION

The aim of this paper is to study the structure of the set of real numbers having a digital expansion of the form

$$\sum_{k=0}^{n-1} a_k (-\beta)^k,$$

where  $(-\beta)$  is a negative real base with  $\beta > 1$ , the digits  $a_k \in \mathbb{Z}$  satisfy certain conditions specified below, and  $n \geq 0$ . These numbers are called  $(-\beta)$ -integers, and have been recently studied by Ambrož, Dombek, Masáková and Pelantová [1].

Before dealing with these numbers, we recall some facts about  $\beta$ -integers, which are the real numbers of the form

$$\pm \sum_{k=0}^{n-1} a_k \beta^k \quad \text{such that} \quad 0 \leq \sum_{k=0}^{m-1} a_k \beta^k < \beta^m \quad \text{for all } 1 \leq m \leq n,$$

i.e.,  $\sum_{k=0}^{n-1} a_k \beta^k$  is a greedy  $\beta$ -expansion. Equivalently, we can define the set of  $\beta$ -integers as

$$\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \quad \text{with} \quad \mathbb{Z}_\beta^+ = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0),$$

where  $T_\beta$  is the  $\beta$ -transformation, defined by

$$T_\beta : [0, 1) \rightarrow [0, 1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor.$$

This map and the corresponding  $\beta$ -expansions were first studied by Rényi [17].

The notion of  $\beta$ -integers was introduced in the domain of quasicrystallography, see for instance [5], and the structure of the  $\beta$ -integers is very well understood now. We have  $\mathbb{Z}_\beta \subseteq \beta \mathbb{Z}_\beta$ , the set of distances between consecutive elements of  $\mathbb{Z}_\beta$  is

$$\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\},$$

where  $T_\beta(x^-) = \lim_{y \rightarrow x, y < x} T_\beta(y)$ , and the sequence of distances between consecutive elements of  $\mathbb{Z}_\beta^+$  is coded by the fixed point of a substitution, see [8] for the case when  $\Delta_\beta$  is a finite set, that is when  $\beta$  is a Parry number. We give short proofs of these facts in Section 2. More detailed properties of this sequence can be found e.g. in [2, 3, 4, 10, 14].

Closely related to  $\mathbb{Z}_\beta^+$  are the sets

$$S_\beta(x) = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(x) \quad (x \in [0, 1)),$$

which were used by Thurston [18] to define (fractal) tilings of  $\mathbb{R}^{d-1}$  when  $\beta$  is a Pisot number of degree  $d$ , i.e., a root of a polynomial  $x^d + p_1 x^{d-1} + \dots + p_d \in \mathbb{Z}[x]$  such that all other roots have modulus  $< 1$ , and an algebraic unit, i.e.,  $p_d = \pm 1$ . These tilings allow e.g. to determine the  $k$ -th digit  $a_k$  of a number without knowing the other digits, see [13].

It is widely agreed that the greedy  $\beta$ -expansions are the natural representations of real numbers in a real base  $\beta > 1$ . For the case of negative bases, the situation is not so clear. Ito and Sadahiro [12] proposed recently to use the  $(-\beta)$ -transformation defined by

$$T_{-\beta} : \left[ \frac{-\beta}{\beta+1}, \frac{1}{\beta+1} \right), x \mapsto -\beta x - \left\lfloor -\beta x + \frac{\beta}{\beta+1} \right\rfloor.$$

see also [9]. This transformation has the important property that  $T_{-\beta}(-x/\beta) = x$  for all  $x \in \left( \frac{-\beta}{\beta+1}, \frac{1}{\beta+1} \right)$ . Some instances are depicted in Figures 1, 3 and 4.

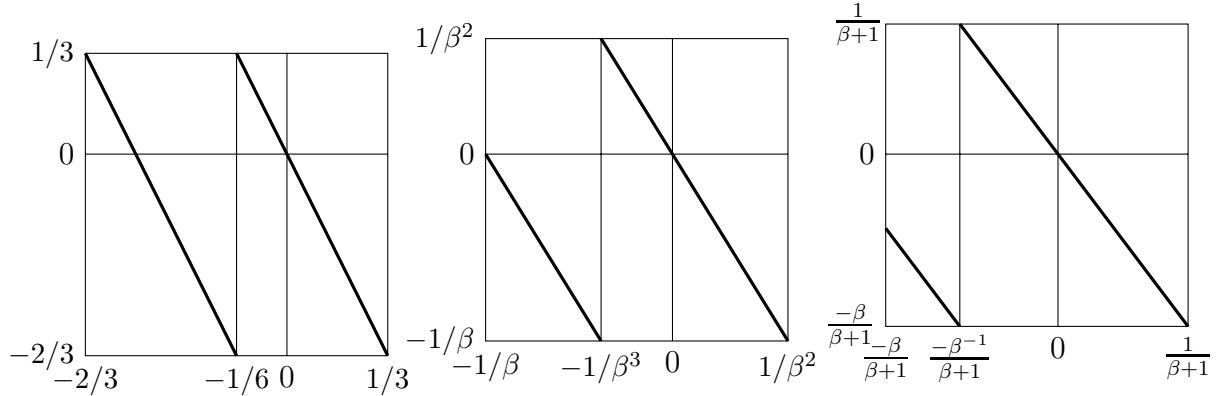


FIGURE 1. The  $(-\beta)$ -transformation for  $\beta = 2$  (left),  $\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$  (middle), and  $\beta = \frac{1}{\beta} + \frac{1}{\beta^2} \approx 1.325$  (right).

The set of  $(-\beta)$ -integers is therefore defined by

$$\mathbb{Z}_{-\beta} = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0).$$

These are the numbers

$$\sum_{k=0}^{n-1} a_k (-\beta)^k \quad \text{such that} \quad \frac{-\beta}{\beta+1} \leq \sum_{k=0}^{m-1} a_k (-\beta)^{k-m} < \frac{1}{\beta+1} \quad \text{for all } 1 \leq m \leq n.$$

Note that, in the case of  $\beta$ -integers, we have to add  $-\mathbb{Z}_\beta^+$  to  $\mathbb{Z}_\beta^+$  in order to obtain a set resembling  $\mathbb{Z}$ . In the case of  $(-\beta)$ -integers, this is not necessary because the  $(-\beta)$ -transformation allows to represent positive and negative numbers.

It is not difficult to see that  $\mathbb{Z}_{-\beta} = \mathbb{Z} = \mathbb{Z}_\beta$  when  $\beta \in \mathbb{Z}$ ,  $\beta \geq 2$ . Some other properties of  $\mathbb{Z}_{-\beta}$  can be found in [1], mainly for the case when  $T_{-\beta}^n(\frac{-\beta}{\beta+1}) \in [\frac{1-|\beta|}{\beta}, 0]$  for all  $n \geq 1$ . The set

$$V'_\beta = \{T_{-\beta}^n(\frac{-\beta}{\beta+1}) \mid n \geq 0\}$$

plays a similar role for  $(-\beta)$ -expansions as the set  $\Delta_\beta$  for  $\beta$ -expansions. Consequently, we call  $\beta > 1$  a  $(-\beta)$ -number if  $V'_\beta$  is a finite set, recalling that Parry numbers were originally called  $\beta$ -numbers in [16]. In Theorem 3, we describe, for any  $(-\beta)$ -number  $\beta \geq (1 + \sqrt{5})/2$ , the sequence of  $(-\beta)$ -integers in terms of a two-sided infinite word on a finite alphabet which is a fixed point of an anti-morphism. Note that the orientation-reversing property of the map  $x \mapsto -\beta x$  imposes the use of an anti-morphism instead of a morphism. Anti-morphisms were considered in a similar context by Enomoto [7].

For  $1 < \beta < \frac{1+\sqrt{5}}{2}$ , we have  $\mathbb{Z}_{-\beta} = \{0\}$ , as already proved in [1]. However, our study still makes sense for these bases  $(-\beta)$  because we can also describe the sets

$$S_{-\beta}(x) = \lim_{n \rightarrow \infty} (-\beta)^n T_{-\beta}^{-n}(x) \quad (x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})),$$

where the limit set consists of the numbers lying in all but finitely many sets  $(-\beta)^n T_{-\beta}^{-n}(x)$ ,  $n \geq 0$ . The reason for taking the limit instead of the union over all  $n \geq 0$  is that  $T_{-\beta}^2(\frac{-\beta^{-1}}{\beta+1}) \neq \frac{-\beta}{\beta+1}$  when  $\beta \notin \mathbb{Z}$ , see Section 3. In [15], the  $(-\beta)$ -transformation is studied in detail for these values of  $\beta$ .

## 2. $\beta$ -INTEGERS

In this section, we consider the structure of  $\beta$ -integers. The results are not new, but it is useful to state and prove them in order to understand the differences with  $(-\beta)$ -integers.

Recall that  $\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\}$ , and let  $\Delta_\beta^*$  be the free monoid generated by  $\Delta_\beta$ . Elements of  $\Delta_\beta^*$  will be considered as words on the alphabet  $\Delta_\beta$ , and the operation is the concatenation of words. The  $\beta$ -substitution is the morphism  $\varphi_\beta : \Delta_\beta^* \rightarrow \Delta_\beta^*$ , defined by

$$\varphi_\beta(x) = \underbrace{11 \cdots 1}_{[\beta x] - 1 \text{ times}} T_\beta(x^-) \quad (x \in \Delta_\beta).$$

Here, 1 is an element of  $\Delta_\beta$  and not the identity element of  $\Delta_\beta^*$  (which is the empty word). Recall that, as  $\varphi_\beta$  is a morphism, we have  $\varphi_\beta(uv) = \varphi_\beta(u)\varphi_\beta(v)$  for all  $u, v \in \Delta_\beta^*$ . Since  $\varphi_\beta^{n+1}(1) = \varphi_\beta^n(\varphi_\beta(1))$  and  $\varphi_\beta(1)$  starts with 1,  $\varphi_\beta^n(1)$  is a prefix of  $\varphi_\beta^{n+1}(1)$  for every  $n \geq 0$ .

**Theorem 1.** *For any  $\beta > 1$ , we have*

$$\mathbb{Z}_\beta^+ = \{z_k \mid k \geq 0\} \quad \text{with} \quad z_k = \sum_{j=1}^k u_j,$$

where  $u_1 u_2 \cdots$  is the infinite word with letters in  $\Delta_\beta$  which has  $\varphi_\beta^n(1)$  as prefix for all  $n \geq 0$ .

The set of differences between consecutive  $\beta$ -integers is  $\Delta_\beta$ .

The main observation for the proof of the theorem is the following. We use the notation  $|v| = k$  and  $L(v) = \sum_{j=1}^k v_j$  for any  $v = v_1 \cdots v_k \in \Delta_\beta^k$ ,  $k \geq 0$ .

**Lemma 1.** *For any  $n \geq 0$ ,  $1 \leq k \leq |\varphi_\beta^n(1)|$ , we have*

$$T_\beta^n\left(\left[\frac{z_{k-1}}{\beta^n}, \frac{z_k}{\beta^n}\right]\right) = [0, u_k],$$

and  $z_{|\varphi_\beta^n(1)|} = L(\varphi_\beta^n(1)) = \beta^n$ .

*Proof.* For  $n = 0$ , we have  $|\varphi_\beta^0(1)| = 1$ ,  $z_0 = 0$ ,  $z_1 = 1$ ,  $u_1 = 1$ , thus the statements are true. Suppose that they hold for  $n$ , and consider

$$u_1 u_2 \cdots u_{|\varphi_\beta^{n+1}(1)|} = \varphi_\beta^{n+1}(1) = \varphi_\beta(\varphi_\beta^n(1)) = \varphi_\beta(u_1) \varphi_\beta(u_2) \cdots \varphi_\beta(u_{|\varphi_\beta^n(1)|}).$$

Let  $1 \leq k \leq |\varphi_\beta^{n+1}(1)|$ , and write  $u_1 \cdots u_k = \varphi_\beta(u_1 \cdots u_{j-1}) v_1 \cdots v_i$  with  $1 \leq j \leq |\varphi_\beta^n(1)|$ ,  $1 \leq i \leq |\varphi_\beta(u_j)|$ , i.e.,  $v_1 \cdots v_i$  is a non-empty prefix of  $\varphi_\beta(u_j)$ .

For any  $x \in (0, 1]$ , we have  $T_\beta(x^-) = \beta x - [\beta x] + 1$ , hence  $L(\varphi_\beta(x)) = \beta x$  for  $x \in \Delta_\beta$ . This yields that

$$z_k = L(u_1 \cdots u_k) = \beta L(u_1 \cdots u_{j-1}) + L(v_1 \cdots v_i) = \beta z_{j-1} + i - 1 + v_i$$

and  $z_{k-1} = \beta z_{j-1} + i - 1$ , hence

$$\begin{aligned} \left[\frac{z_{k-1}}{\beta}, \frac{z_k}{\beta}\right] &= \left[z_{j-1} + \frac{i-1}{\beta}, z_{j-1} + \frac{i-1+v_i}{\beta}\right] \subseteq [z_{j-1}, z_{j-1} + u_j] = [z_{j-1}, z_j], \\ T_\beta^{n+1}\left(\left[\frac{z_{k-1}}{\beta^{n+1}}, \frac{z_k}{\beta^{n+1}}\right]\right) &= T_\beta\left(\left[\frac{i-1}{\beta}, \frac{i-1+v_i}{\beta}\right]\right) = [0, v_i] = [0, u_k]. \end{aligned}$$

Moreover, we have  $L(\varphi_\beta^{n+1}(1)) = \beta L(\varphi_\beta^n(1)) = \beta^{n+1}$ , thus the statements hold for  $n+1$ .  $\square$

*Proof of Theorem 1.* By Lemma 1, we have  $z_{|\varphi_\beta^n(1)|} = \beta^n$  for all  $n \geq 0$ , thus  $[0, 1)$  is split into the intervals  $[z_{k-1}/\beta^n, z_k/\beta^n)$ ,  $1 \leq k \leq |\varphi_\beta^n(1)|$ . Therefore, Lemma 1 yields that

$$T_\beta^{-n}(0) = \{z_{k-1}/\beta^n \mid 1 \leq k \leq |\varphi_\beta^n(1)|\},$$

hence

$$\bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0) = \{z_k \mid k \geq 0\}.$$

Since  $u_k \in \Delta_\beta$  for all  $k \geq 1$  and  $u_{|\varphi_\beta^n(1)|} = T_\beta^n(1^-)$  for all  $n \geq 0$ , we have

$$\{z_k - z_{k-1} \mid k \geq 1\} = \{u_k \mid k \geq 1\} = \Delta_\beta. \quad \square$$

For the sets  $S_\beta(x)$ , Lemma 1 gives the following corollary.

**Corollary 1.** *For any  $x \in [0, 1)$ , we have*

$$S_\beta(x) = \{z_k + x \mid k \geq 0, u_{k+1} > x\} \subseteq x + S_\beta(0).$$

Note that  $S_\beta(x)$  is always the union of a sequence of nested sets because  $y \in [0, 1)$  implies  $y/\beta \in [0, 1)$  and  $T_\beta(y/\beta) = y$ , thus  $\beta^n T_\beta^{-n}(x) \subseteq \beta^{n+1} T_\beta^{-n-1}(x)$  for all  $x \in [0, 1)$ .

3.  $(-\beta)$ -INTEGERS

We now turn to the discussion of  $(-\beta)$ -integers and sets  $S_{-\beta}(x)$ ,  $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ . The first technical problem comes from the fact  $(-\beta)^n T_{-\beta}^{-n}(x) \subseteq (-\beta)^{n+1} T_{-\beta}^{-n-1}(x)$  is not always true because  $\frac{-y}{\beta} \notin [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$  for  $y = \frac{-\beta}{\beta+1}$ . However, we have the following lemma.

**Lemma 2.** *For any  $\beta > 1$ ,  $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ ,  $n \geq 0$ , we have*

$$T_{-\beta}^{-n}(x) \setminus \left\{ \frac{-\beta}{\beta+1} \right\} \subseteq (-\beta) T_{-\beta}^{-n-1}(x).$$

If  $T_{-\beta}(x) = x$ , in particular if  $x = 0$ , then

$$T_{-\beta}^{-n}(x) \subseteq \beta^2 T_{-\beta}^{-n-2}(x).$$

*Proof.* For any  $y \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ , we have  $T_{-\beta}^2(y/\beta^2) = T_{-\beta}(-y/\beta) = y$ , which implies the first inclusion and  $T_{-\beta}^{-n}(x) \setminus \left\{ \frac{-\beta}{\beta+1} \right\} \subseteq \beta^2 T_{-\beta}^{-n-2}(x)$ . If  $T_{-\beta}^n(\frac{-\beta}{\beta+1}) = x = T_{-\beta}(x)$ , then

$$T_{-\beta}^{n+2}\left(\frac{-\beta-1}{\beta+1}\right) = T_{-\beta}^{n+1}\left(\frac{-\beta}{\beta+1}\right) = T_{-\beta}(x) = x,$$

thus  $\frac{-\beta}{\beta+1} \in \beta^2 T_{-\beta}^{-n-2}(x)$  as well.  $\square$

The first two statements of the following proposition can also be found in [1].

**Proposition 1.** *For any  $\beta > 1$ , we have  $\mathbb{Z}_{-\beta} \subseteq (-\beta) \mathbb{Z}_{-\beta}$ .*

*If  $\beta < (1 + \sqrt{5})/2$ , then  $\mathbb{Z}_{-\beta} = \{0\}$ .*

*If  $\beta \geq (1 + \sqrt{5})/2$ , then*

$$\mathbb{Z}_{-\beta} \cap (-\beta)^n [-\beta, 1] = \left\{ (-\beta)^n, (-\beta)^{n+1} \right\} \cup (-\beta)^{n+2} \left( T_{-\beta}^{-n-2}(0) \cap \left( \frac{-1}{\beta}, \frac{1}{\beta^2} \right) \right)$$

for all  $n \geq 0$ , in particular

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \begin{cases} \{-\beta, -\beta + 1, \dots, -\beta + \lfloor \beta \rfloor, 0, 1\} & \text{if } \beta^2 \geq \lfloor \beta \rfloor (\beta + 1), \\ \{-\beta, -\beta + 1, \dots, -\beta + \lfloor \beta \rfloor - 1, 0, 1\} & \text{if } \beta^2 < \lfloor \beta \rfloor (\beta + 1). \end{cases}$$

*Proof.* For any  $\beta > 1$ , we have  $T_{-\beta}(0) = 0$ , thus  $T_{-\beta}^{-n-1}(0) \subseteq T_{-\beta}^{-n}(0)$ . This means that  $(-\beta)^{n+1} T_{-\beta}^{-n-1}(0) \subseteq -\beta((-\beta)^n T_{-\beta}^{-n}(0))$ , hence  $\mathbb{Z}_{-\beta} \subseteq (-\beta) \mathbb{Z}_{-\beta}$ .

If  $\beta < \frac{1+\sqrt{5}}{2}$ , then  $\frac{-1}{\beta} < \frac{-\beta}{\beta+1}$ , hence  $T_{-\beta}^{-1}(0) = \{0\}$  and  $\mathbb{Z}_{-\beta} = \{0\}$ , see Figure 1 (right).

If  $\beta \geq \frac{1+\sqrt{5}}{2}$ , then  $\frac{-1}{\beta} \in T_{-\beta}^{-1}(0)$  implies  $1 \in \mathbb{Z}_{-\beta}$ , thus  $(-\beta)^n \in \mathbb{Z}_{-\beta}$  for all  $n \geq 0$ . Clearly,

$$(-\beta)^{n+2} \left( T_{-\beta}^{-n-2}(0) \cap \left( \frac{-1}{\beta}, \frac{1}{\beta^2} \right) \right) \subseteq \mathbb{Z}_{-\beta} \cap (-\beta)^n (-\beta, 1).$$

To show the other inclusion, let  $z \in (-\beta)^m T_{-\beta}^{-m}(0) \cap (-\beta)^n (-\beta, 1)$  for some  $m \geq 0$ . If  $m < n + 2$ , then Lemma 2 yields that  $z \in (-\beta)^{n+2} T_{-\beta}^{-n-2}(0)$ , since  $(-1)^n z < \beta^n \leq \frac{\beta^{n+2}}{\beta+1}$  implies  $z \neq \frac{(-\beta)^{n+2}}{\beta+1}$ . If  $m > n + 2$ , then  $(\frac{-1}{\beta}, \frac{1}{\beta^2}) \subseteq (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$  implies that  $T_{-\beta}^{n+2}\left(\frac{z}{(-\beta)^{n+2}}\right) = T_{-\beta}^m\left(\frac{z}{(-\beta)^m}\right) = 0$ , thus we also have  $z \in (-\beta)^{n+2} \left( T_{-\beta}^{-n-2}(0) \cap (-\beta)^n (-\beta, 1) \right)$ .

Consider now  $n = 0$ , then

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \{-\beta, 1\} \cup \{z \in (-\beta, 1) \mid T_{-\beta}^2(z/\beta^2) = 0\}.$$

Since  $\frac{-\lfloor \beta \rfloor}{\beta} \geq \frac{-\beta}{\beta+1}$  if and only if  $\beta^2 \geq \lfloor \beta \rfloor(\beta+1)$ , we obtain

$$(-\beta)T_{-\beta}^{-1}(0) = \begin{cases} \{0, 1, \dots, \lfloor \beta \rfloor\} & \text{if } \beta^2 \geq \lfloor \beta \rfloor(\beta+1), \\ \{0, 1, \dots, \lfloor \beta \rfloor - 1\} & \text{if } \beta^2 < \lfloor \beta \rfloor(\beta+1). \end{cases}$$

If  $T_{-\beta}^2(z/\beta^2) = 0$ , then  $z = -a_1\beta + a_0$  with  $a_0 \in (-\beta)T_{-\beta}^{-1}(0)$ ,  $a_1 \in \{0, 1, \dots, \lfloor \beta \rfloor\}$ , and  $\mathbb{Z}_{-\beta} \cap [-\beta, 1]$  consists of those numbers  $-a_1\beta + a_0$  lying in  $[-\beta, 1]$ .  $\square$

This shows in particular that the maximal difference between consecutive  $(-\beta)$ -integers exceeds 1 whenever  $\beta^2 < \lfloor \beta \rfloor(\beta+1)$ , i.e.,  $T_{-\beta}(\frac{-\beta}{\beta+1}) < 0$ . This was already shown for an example in [1]. Example 3 shows that the distance between two consecutive  $(-\beta)$ -integers can be even greater than 2, and the structure of  $\mathbb{Z}_{-\beta}$  can be quite complicated. Therefore, we adapt a slightly different strategy as for  $\mathbb{Z}_\beta$ .

In the following, we always assume that the set

$$V_\beta = V'_\beta \cup \{0\} = \{T_{-\beta}^n(\frac{-\beta}{\beta+1}) \mid n \geq 0\} \cup \{0\}$$

is finite, i.e.,  $\beta$  is a  $(-\beta)$ -number, and let  $\beta$  be fixed. For  $x \in V_\beta$ , let

$$r_x = \min \{y \in V_\beta \cup \{\frac{1}{\beta+1}\} \mid y > x\}, \quad \hat{x} = \frac{x+r_x}{2}, \quad J_x = \{x\} \quad \text{and} \quad J_{\hat{x}} = (x, r_x).$$

Then  $\{J_a \mid a \in A_\beta\}$  forms a partition of  $[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ , where

$$A_\beta = V_\beta \cup \hat{V}_\beta, \quad \text{with} \quad \hat{V}_\beta = \{\hat{x} \mid x \in V_\beta\}.$$

We have  $T_{-\beta}(J_x) = J_{T_{-\beta}(x)}$  for every  $x \in V_\beta$ , and the following lemma shows that the image of every  $J_{\hat{x}}$ ,  $x \in V_\beta$ , is a union of intervals  $J_a$ ,  $a \in A_\beta$ .

**Lemma 3.** *Let  $x \in V_\beta$  and*

$$J_{\hat{x}} \cap T_{-\beta}^{-1}(V_\beta) = \{y_1, \dots, y_m\}, \quad x = y_0 < y_1 < \dots < y_m < y_{m+1} = r_x.$$

*For any  $0 \leq i \leq m$ , we have*

$$T_{-\beta}((y_i, y_{i+1})) = J_{\hat{x}_i} \quad \text{with} \quad x_i = \lim_{y \rightarrow y_{i+1}, y < y_{i+1}} T_{-\beta}(y), \quad \text{i.e.,} \quad \hat{x}_i = T_{-\beta}(\frac{y_i + y_{i+1}}{2}),$$

*and  $\beta(y_{i+1} - y_i) = \lambda(J_{\hat{x}_i})$ , where  $\lambda$  denotes the Lebesgue measure.*

*Proof.* Since  $T_{-\beta}$  maps no point in  $(y_i, y_{i+1})$  to  $\frac{-\beta}{\beta+1} \in V_\beta$ , the map is continuous on this interval and  $\lambda(T_{-\beta}((y_i, y_{i+1}))) = \beta(y_{i+1} - y_i)$ . We have  $x_i \in V_\beta$  since  $x_i = T_{-\beta}(y_{i+1})$  in case  $y_{i+1} < \frac{1}{\beta+1}$ , and  $x_i = \frac{-\beta}{\beta+1}$  in case  $y_{i+1} = \frac{1}{\beta+1}$ . Since  $y_i = \max\{y \in T_{-\beta}^{-1}(V_\beta) \mid y < y_{i+1}\}$ , we obtain that  $r_{x_i} = \lim_{y \rightarrow y_i, y > y_i} T_{-\beta}(y)$ , thus  $T_{-\beta}((y_i, y_{i+1})) = (x_i, r_{x_i})$ .  $\square$

Therefore, we define an anti-morphism  $\psi_\beta : A_\beta^* \rightarrow A_\beta^*$  by

$$\psi_\beta(x) = T_{-\beta}(x) \quad \text{and} \quad \psi_\beta(\hat{x}) = \hat{x}_m T_{-\beta}(y_m) \cdots \hat{x}_1 T_{-\beta}(y_1) \hat{x}_0 \quad (x \in V_\beta),$$

with  $m$ ,  $x_i$  and  $y_i$  as in Lemma 3. Here, anti-morphism means that  $\psi_\beta(uv) = \psi_\beta(v)\psi_\beta(u)$  for all  $u, v \in A_\beta^*$ . Now, the last letter of  $\psi_\beta(\hat{0})$  is  $\hat{t}$ ,  $t = \max\{x \in V_\beta \mid x < 0\}$ , and the first letter of  $\psi_\beta(\hat{t})$  is  $\hat{0}$ . Therefore,  $\psi_\beta^{2n}(\hat{0})$  is a prefix of  $\psi_\beta^{2n+2}(\hat{0}) = \psi_\beta^{2n}(\psi_\beta^2(\hat{0}))$  and  $\psi_\beta^{2n+1}(\hat{0})$  is a suffix of  $\psi_\beta^{2n+3}(\hat{0})$  for every  $n \geq 0$ .

**Theorem 2.** For any  $(-\beta)$ -number  $\beta \geq (1 + \sqrt{5})/2$ , we have

$$\mathbb{Z}_{-\beta} = \{z_k \mid k \in \mathbb{Z}, u_k = 0\} \quad \text{with} \quad z_k = \begin{cases} \sum_{j=1}^k \lambda(J_{u_j}) & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} \lambda(J_{u_{-j}}) & \text{if } k \leq 0, \end{cases}$$

where  $\cdots u_{-1}u_0u_1\cdots$  is the two-sided infinite word on the finite alphabet  $A_\beta$  such that  $u_0 = 0$ ,  $\psi_\beta^{2n}(\widehat{0})$  is a prefix of  $u_1u_2\cdots$  and  $\psi_\beta^{2n+1}(\widehat{0})$  is a suffix of  $\cdots u_{-2}u_{-1}$  for all  $n \geq 0$ .

Since  $\psi_\beta(u_0) = u_0$ , the word  $\cdots u_{-1}u_0u_1\cdots$  can be seen as a fixed point of  $\psi_\beta$ .

The following lemma is the analogue of Lemma 1. Note that  $u_{2k} \in V_\beta$  and  $u_{2k+1} \in \widehat{V}_\beta$  for all  $k \in \mathbb{Z}$ , thus  $z_{2k} = z_{2k-1}$  and  $z_{-2k} = z_{1-2k}$  for all  $k \geq 1$ . We use the notation

$$L(v) = \sum_{j=1}^k \lambda(J_{v_j}) \quad \text{if } v = v_1 \cdots v_k \in A_\beta^k.$$

**Lemma 4.** For any  $n \geq 0$ ,  $0 \leq k < |\psi_\beta^n(\widehat{0})|/2$ , we have

$$T_{-\beta}^n \left( \frac{z_{(-1)^n 2k}}{(-\beta)^n} \right) = u_{(-1)^n 2k}, \quad T_{-\beta}^n \left( \left( \frac{z_{(-1)^n 2k}}{(-\beta)^n}, \frac{z_{(-1)^n (2k+1)}}{(-\beta)^n} \right) \right) = J_{u_{(-1)^n (2k+1)}},$$

and  $z_{(-1)^n |\psi_\beta^n(\widehat{0})|} = (-1)^n L(\psi_\beta^n(\widehat{0})) = r_0 (-\beta)^n$ .

*Proof.* The statements are true for  $n = 0$  since  $|\psi_\beta^0(\widehat{0})| = 1$ ,  $z_0 = 0$ ,  $z_1 = \lambda(J_0) = r_0$ .

Suppose that they hold for even  $n$ , and consider

$$u_{-|\psi_\beta^{n+1}(\widehat{0})|} \cdots u_{-2}u_{-1} = \psi_\beta^{n+1}(\widehat{0}) = \psi_\beta(\psi_\beta^n(\widehat{0})) = \psi_\beta(u_{|\psi_\beta^n(\widehat{0})|}) \cdots \psi_\beta(u_2)\psi_\beta(u_1).$$

Let  $0 \leq k < |\psi_\beta^{n+1}(\widehat{0})|/2$ , and write

$$u_{-2k-1} \cdots u_{-1} = v_{-2i-1} \cdots v_{-1} \psi_\beta(u_1 \cdots u_{2j})$$

with  $0 \leq j < |\psi_\beta^n(\widehat{0})|/2$ ,  $0 \leq i < |\psi_\beta(u_{2j+1})|/2$ , i.e.,  $u_{-2i-1} \cdots u_{-1}$  is a suffix of  $\psi_\beta(u_{2j+1})$ .

By Lemma 3, we have  $L(\psi_\beta(\widehat{x})) = \beta \lambda(J_{\widehat{x}})$  for any  $x \in V_\beta$ . This yields that

$$-z_{-2k-1} = \beta L(u_1 \cdots u_{2j}) + L(v_{-2i-1} \cdots v_{-1}) = \beta z_{2j} + L(v_{-2i-1} \cdots v_{-1})$$

and  $-z_{-2k} = \beta z_{2j} + L(v_{-2i} \cdots v_{-1})$ . With  $\widehat{x} = u_{2j+1}$  and  $y_i$  as in Lemma 3, we obtain

$$\begin{aligned} T_{-\beta}^{n+1} \left( \frac{z_{-2k}}{(-\beta)^{n+1}} \right) &= T_{-\beta}^{n+1} \left( \frac{z_{2j}}{(-\beta)^n} - \frac{L(v_{-2i} \cdots v_{-1})}{(-\beta)^{n+1}} \right) \\ &= \begin{cases} T_{-\beta}(u_{2j}) = \psi_\beta(u_{2j}) = u_{-2k} & \text{if } i = 0, \\ T_{-\beta}(x + L(v_{-2i} \cdots v_{-1})/\beta) = T_{-\beta}(y_i) = v_{-2i} = u_{-2k} & \text{if } i \geq 1, \end{cases} \end{aligned}$$

and

$$T_{-\beta}^{n+1} \left( \left( \frac{z_{-2k}}{(-\beta)^{n+1}}, \frac{z_{-2k-1}}{(-\beta)^{n+1}} \right) \right) = T_{-\beta}((y_i, y_{i+1})) = J_{v_{-2i-1}} = J_{u_{-2k-1}}.$$

Moreover, we have  $L(\psi_\beta^{n+1}(\widehat{0})) = \beta L(\psi_\beta^n(\widehat{0})) = r_0 \beta^{n+1}$ , thus the statements hold for  $n + 1$ .

The proof for odd  $n$  runs along the same lines and is therefore omitted.  $\square$



*Proof of Theorem 2.* By Lemma 4, we have  $z_{(-1)^n |\psi_\beta^n(\widehat{0})|} = r_0 (-\beta)^n$  for all  $n \geq 0$ , thus  $[0, r_0)$  splits into the intervals  $(z_{(-1)^{n2k}(-\beta)^{-n}}, z_{(-1)^{n(2k+1)}(-\beta)^{-n})$  and points  $z_{(-1)^{n2k}(-\beta)^{-n}$ ,  $0 \leq k < |\psi_\beta^n(\widehat{0})|/2$ , hence

$$T_{-\beta}^{-n}(0) \cap [0, r_0) = \{z_{(-1)^{n2k}(-\beta)^{-n} \mid 0 \leq k < |\psi_\beta^n(\widehat{0})|/2, u_{(-1)^{n2k}} = 0\}.$$

Let  $m \geq 1$  be such that  $\beta^{2m} r_0 > \frac{1}{\beta+1}$ . Then we have  $[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}) \subset (-\beta^{2m+1} r_0, \beta^{2m} r_0)$ , and Lemma 2 yields that

$$T_{-\beta}^{-n}(0) \subseteq \beta^{2m} (T_{-\beta}^{-n-2m}(0) \cap [0, r_0)) \cup (-\beta)^{2m+1} (T_{-\beta}^{-n-2m-1}(0) \cap [0, r_0)),$$

thus

$$\bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0) = \bigcup_{n \geq 0} (-\beta)^n (T_{-\beta}^{-n}(0) \cap [0, r_0)) = \{z_{2k} \mid k \in \mathbb{Z}, u_{2k} = 0\}. \quad \square$$

As in the case of positive bases, the word  $\cdots u_{-1} u_0 u_1 \cdots$  also describes the sets

$$S_{-\beta}(x) = \lim_{n \rightarrow \infty} (-\beta)^n T_{-\beta}^{-n}(x) = \bigcup_{n \geq 0} (-\beta)^n (T_{-\beta}^{-n}(x) \setminus \{\frac{-\beta}{\beta+1}\}) \quad (x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})),$$

where the second equality follows from Lemma 2. It is already indicated in the Introduction that  $S_{-\beta}(x)$  can differ from  $\bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(x)$ . Indeed, if  $T_{-\beta}^n(\frac{-\beta}{\beta+1}) = x \neq T_{-\beta}(x)$ , then

$$T_{-\beta}^{n+2}(\frac{-\beta-1}{\beta+1}) = T_{-\beta}^{n+1}(\frac{-\beta}{\beta+1}) = T_{-\beta}(x) \neq x,$$

thus  $\frac{(-\beta)^{n+1}}{\beta+1} \in (-\beta)^m T_{-\beta}^{-m}(x)$  if and only if  $m = n$ .

Theorem 2 and Lemma 4 give the following corollary.

**Corollary 2.** *For any  $x \in V_\beta$ , we have*

$$S_{-\beta}(x) = \{z_k \mid k \in \mathbb{Z}, u_k = x\}$$

and, for any  $y \in J_{\widehat{x}}$ ,

$$S_{-\beta}(y) = \{z_k + y - x \mid k \geq 0, u_{k+1} = \widehat{x}\} \cup \{z_k + y - x \mid k < 0, u_k = \widehat{x}\}.$$

Recall that our main goal is to understand the structure of  $\mathbb{Z}_{-\beta}$  (in case  $\beta \geq (1 + \sqrt{5})/2$ ), i.e., to describe the occurrences of 0 in the word  $\cdots u_{-1} u_0 u_1 \cdots$  defined in Theorem 2 and the words between two successive occurrences. Let

$$R_\beta = \{u_k u_{k+1} \cdots u_{s(k)-1} \mid k \in \mathbb{Z}, u_k = 0\} \quad \text{with} \quad s(k) = \min\{j \in \mathbb{Z} \mid u_j = 0, j > k\}$$

be the set of return words of 0 in  $\cdots u_{-1} u_0 u_1 \cdots$ . Note that  $s(k)$  is defined for all  $k \in \mathbb{Z}$  since  $(-\beta)^n \in \mathbb{Z}_{-\beta}$  for all  $n \geq 0$  by Proposition 1.

For any  $w \in R_\beta$ ,  $\psi_\beta(w0)$  is a factor of  $\cdots u_{-1} u_0 u_1 \cdots$  starting and ending with 0, thus we can write  $\psi_\beta(w0) = w_1 \cdots w_m 0$  with  $w_j \in R_\beta$ ,  $1 \leq j \leq m$ , and set

$$\varphi_{-\beta}(w) = w_1 \cdots w_m.$$

This defines an anti-morphism  $\varphi_{-\beta} : R_\beta^* \rightarrow R_\beta^*$ , which plays the role of the  $\beta$ -substitution.

Since  $\cdots u_{-1}u_0u_1\cdots$  is generated by  $u_1 = \widehat{0}$ , we consider  $w_\beta = u_0u_1\cdots u_{s(0)-1}$ . We have

$$[0, 1] = [0, \frac{1}{\beta+1}) \cup [\frac{1}{\beta+1}, 1], \quad T_{-\beta}((-\beta)^{-1}[\frac{1}{\beta+1}, 1]) = [\frac{-\beta}{\beta+1}, 0],$$

thus  $L(w_\beta) = 1$  and

$$w_\beta = 0\widehat{0}x_1\widehat{x}_1\cdots x_mx_m\widehat{x}_m x_{-\ell}\widehat{x}_{-\ell}\cdots x_{-1}\widehat{x}_{-1},$$

with  $V_\beta = \{x_{-\ell}, \dots, x_{-1}, 0, x_1, \dots, x_m\}$ ,  $x_{-\ell} < \cdots < x_{-1} < 0 < x_1 < \cdots < x_m$ .

**Theorem 3.** For any  $(-\beta)$ -number  $\beta \geq (1 + \sqrt{5})/2$ , we have

$$\mathbb{Z}_{-\beta} = \{z'_k \mid k \in \mathbb{Z}\} \quad \text{with} \quad z'_k = \begin{cases} \sum_{j=1}^k L(u'_j) & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} L(u'_{-j}) & \text{if } k \leq 0, \end{cases}$$

where  $\cdots u'_{-2}u'_{-1}u'_1u'_2\cdots$  is the two-sided infinite word on the finite alphabet  $R_\beta$  such that  $\varphi_{-\beta}^{2n}(w_\beta)$  is a prefix of  $u'_1u'_2\cdots$  and  $\varphi_{-\beta}^{2n+1}(w_\beta)$  is a suffix of  $\cdots u'_{-2}u'_{-1}$  for all  $n \geq 0$ .

The set of distances between consecutive  $(-\beta)$ -integers is

$$\Delta_{-\beta} = \{z'_{k+1} - z'_k \mid k \in \mathbb{Z}\} = \{L(w) \mid w \in R_\beta\}.$$

Note that the index 0 is omitted in  $\cdots u'_{-2}u'_{-1}u'_1u'_2\cdots$  for reasons of symmetry.

*Proof.* The definitions of  $\cdots u_{-1}u_0u_1\cdots$  in Theorem 2 and of  $\varphi_{-\beta}$  imply that  $\cdots u'_{-2}u'_{-1}u'_1u'_2\cdots$  is the derived word of  $\cdots u_{-1}u_0u_1\cdots$  with respect to  $R_\beta$ , that is

$$u'_k = u_{|u'_1\cdots u'_k|} \cdots u_{|u'_1\cdots u'_k|-1}, \quad u'_{-k} = u_{-|u'_{-k}\cdots u'_{-1}|} \cdots u_{-|u'_{-k}\cdots u'_{-1}|-1} \quad (k \geq 1)$$

with

$$\{|u'_1\cdots u'_{k-1}| \mid k \geq 1\} \cup \{-|u'_{-k}\cdots u'_{-1}| \mid k \geq 1\} = \{k \in \mathbb{Z} \mid u_k = 0\}.$$

Here, for any  $v \in R_\beta^*$ ,  $|v|$  denotes the length of  $v$  as a word in  $A_\beta^*$ , not in  $R_\beta^*$ . Since

$$z'_k = \sum_{j=1}^k L(u'_j) = \sum_{j=0}^{|u'_1\cdots u'_k|-1} \lambda(J_{u_j}) = \sum_{j=1}^{|u'_1\cdots u'_k|} \lambda(J_{u_j}), \quad z'_{-k} = -\sum_{j=1}^k L(u'_{-j}) = -\sum_{j=1}^{|u'_{-k}\cdots u'_{-1}|} \lambda(J_{u_{-j}})$$

for all  $k \geq 0$ , Theorem 2 yields that  $\{z'_k \mid k \in \mathbb{Z}\} = \mathbb{Z}_{-\beta}$ .

It follows from the definition of  $R_\beta$  that  $\Delta_{-\beta} = \{L(w) \mid w \in R_\beta\}$ .

It remains to show that  $R_\beta$  is a finite set. We first show that the restriction of  $\psi_\beta$  to  $\widehat{V}_\beta$  is primitive, which means that there exists some  $m \geq 1$  such that, for every  $x \in V_\beta$ ,  $\psi_\beta^m(\widehat{x})$  contains all elements of  $\widehat{V}_\beta$ . The proof is taken from [11, Proposition 8]. If  $\beta > 2$ , then the largest connected pieces of images of  $J_{\widehat{x}}$  under  $T_{-\beta}$  grow until they cover two consecutive discontinuity points  $\frac{1}{\beta+1} - \frac{\alpha+1}{\beta}$ ,  $\frac{1}{\beta+1} - \frac{\alpha}{\beta}$  of  $T_{-\beta}$ , and the next image covers all intervals  $J_{\widehat{y}}$ ,  $y \in V_\beta$ . If  $\frac{1+\sqrt{5}}{2} < \beta \leq 2$ , then  $\beta^2 > 2$  implies that the largest connected pieces of images of  $J_{\widehat{x}}$  under  $T_{-\beta}^2$  grow until they cover two consecutive discontinuity points of  $T_{-\beta}^2$ . Since

$$T_{-\beta}^2\left(\left(\frac{-\beta}{\beta+1}, \frac{\beta-2}{\beta+1} - \frac{1}{\beta}\right)\right) = \left(\frac{-\beta^3+\beta^2+\beta}{\beta+1}, \frac{1}{\beta+1}\right), \quad T_{-\beta}^2\left(\left(\frac{\beta-2}{\beta+1} - \frac{1}{\beta}, \frac{-\beta-1}{\beta+1}\right)\right) = \left(\frac{-\beta}{\beta+1}, \frac{\beta^2-\beta-1}{\beta+1}\right), \\ T_{-\beta}^2\left(\left(\frac{-\beta-1}{\beta+1}, \frac{\beta-2}{\beta+1}\right)\right) = \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right), \quad T_{-\beta}^2\left(\left(\frac{\beta-2}{\beta+1}, \frac{1}{\beta+1}\right)\right) = \left(\frac{-\beta}{\beta+1}, \frac{\beta^2-\beta-1}{\beta+1}\right),$$

the next image covers the fixed point 0, and further images grow until after a finite number of steps they cover all intervals  $J_{\hat{y}}$ ,  $y \in V_\beta$ . The case  $\beta = \frac{1+\sqrt{5}}{2}$  is treated in Example 1.

If  $T_{-\beta}^n\left(\frac{-\beta}{\beta+1}\right) \neq 0$  for all  $n \geq 0$ , then  $u_k = 0$  is equivalent with  $u_{k+1} = \hat{0}$ , see Proposition 2 below, thus we can consider the return words of  $\hat{0}$  in  $\cdots u_{-1}u_0u_1 \cdots$  instead of the return words of 0. Since  $\psi_\beta^m(\hat{x}_0 x_1 \hat{x}_2)$  has at least two occurrences of  $\hat{0}$  for all  $x_0, x_1, x_2 \in V_\beta$ , there are only finitely many such return words, cf. [6]. If  $T_{-\beta}^n\left(\frac{-\beta}{\beta+1}\right) = 0$ , then  $\psi_\beta^n(x_0 \hat{x}_1 x_2)$  starts and ends with 0 for all  $x_0, x_1, x_2 \in V_\beta$ , hence  $R_\beta$  is finite as well.  $\square$

We remark that, for practical reasons, the set  $R_\beta$  can be obtained from the set  $R = \{w_\beta\}$  by adding to  $R$  iteratively all return words of 0 which appear in  $\varphi_{-\beta}(w)$  for some  $w \in R$  until  $R$  stabilises. The final set  $R$  is equal to  $R_\beta$ .

Now, we apply the theorems in the case of two quadratic examples.

*Example 1.* Let  $\beta = \frac{1+\sqrt{5}}{2}$ , i.e.,  $\beta^2 = \beta + 1$ , and  $t = \frac{-\beta}{\beta+1} = \frac{-1}{\beta}$ . We have  $V_\beta = \{t, 0\}$ . Since

$$J_{\hat{t}} = (t, 0) = \left(t, \frac{-1}{\beta^3}\right) \cup \left\{\frac{-1}{\beta^3}\right\} \cup \left(\frac{-1}{\beta^3}, 0\right), \quad J_{\hat{t}} = \left(0, \frac{1}{\beta^2}\right),$$

see Figure 1 (middle), the anti-morphism  $\psi_\beta$  on  $A_\beta^*$  is defined by

$$\psi_\beta: \quad t \mapsto 0, \quad \hat{t} \mapsto \hat{0}t\hat{t}, \quad 0 \mapsto 0, \quad \hat{0} \mapsto \hat{t}.$$

Its two-sided fixed point  $\cdots u_{-1}u_0u_1 \cdots$  is

$$\cdots \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{0}t\hat{t}}_{\psi_\beta(\hat{t})} \underbrace{0}_{\psi_\beta(t)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{0}t\hat{t}}_{\psi_\beta(\hat{t})} \underbrace{0}_{\psi_\beta(t)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{\hat{0}}_{\psi_\beta(\hat{0})} \underbrace{\hat{0}t\hat{t}}_{\psi_\beta(\hat{t})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{0}t\hat{t}}_{\psi_\beta(\hat{t})} \underbrace{0}_{\psi_\beta(t)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{t}}_{\psi_\beta(\hat{0})} \underbrace{0}_{\psi_\beta(0)} \cdots,$$

where  $\hat{0}$  marks the central letter  $u_0$ . The  $\psi_\beta$ -images of the complete return words of 0 are

$$\psi_\beta: \quad 0\hat{0}t\hat{t}0 \mapsto 0\hat{0}t\hat{t}0\hat{t}0, \quad 0\hat{t}0 \mapsto 0\hat{0}t\hat{t}0,$$

thus  $R_\beta = \{A, B\}$  with  $A = 0\hat{0}t\hat{t}$ ,  $B = 0\hat{t}$ . The anti-morphism

$$\varphi_{-\beta}: \quad A \mapsto AB, \quad B \mapsto A,$$

has the two-sided fixed point

$$\cdots u'_{-2}u'_{-1} u'_1u'_2 \cdots = \cdots AAB AAB AAB AB AAB AAB AAB AB AAB AB AAB B \cdots.$$

We have  $\lambda(J_{\hat{0}}) = \frac{1}{\beta^2}$ ,  $\lambda(J_{\hat{t}}) = \frac{1}{\beta}$ , thus  $L(A) = 1$ ,  $L(B) = \frac{1}{\beta} = \beta - 1$ , and some  $(-\beta)$ -integers are shown in Figure 2. Note that  $(-\beta)^n$  can also be represented as  $(-\beta)^{n+2} + (-\beta)^{n+1}$ .

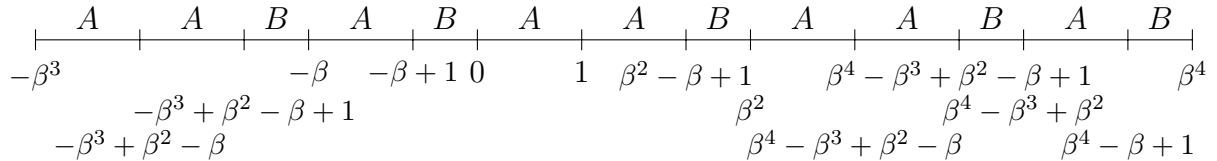


FIGURE 2. The  $(-\beta)$ -integers in  $[-\beta^3, \beta^4]$ ,  $\beta = (1 + \sqrt{5})/2$ .

*Example 2.* Let  $\beta = \frac{3+\sqrt{5}}{2}$ , i.e.,  $\beta^2 = 3\beta - 1$ , then the  $(-\beta)$ -transformation is depicted in Figure 3, where  $t_0 = \frac{-\beta}{\beta+1}$ ,  $t_1 = T_{-\beta}(t_0) = \frac{\beta^2}{\beta+1} - 2 = \frac{-\beta-1}{\beta+1}$ ,  $T_{-\beta}(t_1) = \frac{1}{\beta+1} - 1 = t_0$ . Therefore,  $V_\beta = \{t_0, t_1, 0\}$  and the anti-morphism  $\psi_\beta : A_\beta^* \rightarrow A_\beta^*$  is defined by

$$\psi_\beta : t_0 \mapsto t_1, \quad \widehat{t}_0 \mapsto \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0, \quad t_1 \mapsto t_0, \quad \widehat{t}_1 \mapsto \widehat{0}, \quad 0 \mapsto 0, \quad \widehat{0} \mapsto \widehat{t}_0 t_1 \widehat{t}_1,$$

which has the two-sided fixed point

$$\cdots \underbrace{0}_{\psi_\beta(0)} \underbrace{\widehat{0}}_{\psi_\beta(\widehat{t}_1)} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0}_{\psi_\beta(\widehat{t}_0)} \underbrace{t_1}_{\psi_\beta(t_0)} \underbrace{\widehat{t}_0 t_1 \widehat{t}_1}_{\psi_\beta(\widehat{0})} \underbrace{\widehat{0}}_{\psi_\beta(0)} \underbrace{\widehat{0}}_{\psi_\beta(\widehat{t}_1)} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0}_{\psi_\beta(\widehat{t}_0)} \cdots,$$

where  $\widehat{0}$  marks the central letter  $u_0$ . The  $\psi_\beta$ -images of the complete return words of 0 are

$$\begin{aligned} \psi_\beta : \quad & 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \mapsto 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0, \\ & 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \mapsto 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0, \\ & 0 \widehat{0} t_0 \widehat{t}_0 t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \mapsto 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_0 t_1 \widehat{t}_1 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_0 t_1 \widehat{t}_1 0. \end{aligned}$$

Note that  $0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1 0$  and  $0 \widehat{0} t_0 \widehat{t}_0 t_0 \widehat{t}_0 t_1 \widehat{t}_1 0$  differ only by a letter in  $V_\beta$ , and correspond therefore to intervals of same length. Since the letters  $t_0$  and  $t_1$  are never mapped to 0, we identify these two return words. Then we have  $R_\beta = \{A, B\}$  with  $A = 0 \widehat{0} t_0 \widehat{t}_0 t_1 \widehat{t}_1$ ,  $B = 0 \widehat{0} t_0 \widehat{t}_0 \{t_0, t_1\} \widehat{t}_0 t_1 \widehat{t}_1$ . The anti-morphism

$$\varphi_{-\beta} : A \mapsto AB, \quad B \mapsto ABB,$$

has the two-sided fixed point

$$\cdots ABB AB ABB ABB AB ABB AB ABB ABB AB \cdots.$$

We have  $L(A) = 1$ ,  $L(B) = \beta - 1 > 1$ , and some  $(-\beta)$ -integers are shown in Figure 3.

We remark that it is in general sufficient to consider the elements of  $\widehat{V}_\beta$  when one is only interested in  $\mathbb{Z}_{-\beta}$ . This is made precise in the following proposition.

**Proposition 2.** *Let  $\beta$  and  $\cdots u_{-1} u_0 u_1 \cdots$  be as in Theorem 2,  $t = \max\{x \in V_\beta \mid x < 0\}$ . If  $0 \notin V'_\beta$  or the size of  $V'_\beta$  is odd, then  $u_k = 0$  is equivalent with  $u_{k+1} = \widehat{0}$  for all  $k \in \mathbb{Z}$ . If  $0 \notin V'_\beta$  or the size of  $V'_\beta$  is even, then  $u_k = 0$  is equivalent with  $u_{k-1} = \widehat{t}$  for all  $k \in \mathbb{Z}$ .*

*Proof.* Let  $k \in \mathbb{Z}$  and  $m \geq 0$  such that  $z_{2k}/\beta^{2m} \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ . Then we have

- $u_{2k} = 0$  if and only if  $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$ ,
- $u_{2k+1} = \widehat{0}$  if and only if  $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$ ,
- $u_{2k-1} = \widehat{t}$  if and only if  $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$ .

Recall that  $u_{2k} \in V_\beta$  and  $u_{2k+1} \in \widehat{V}_\beta$  for all  $k \in \mathbb{Z}$ . If  $z_{2k}/\beta^{2m}$  is a point of discontinuity of  $T_{-\beta}^{2m}$ , then we must have  $T_{-\beta}^\ell(z_{2k}/\beta^{2m}) = \frac{-\beta}{\beta+1}$  for some  $1 \leq \ell \leq 2m$ .

If  $0 \notin V'_\beta = \{T_{-\beta}^n(\frac{-\beta}{\beta+1}) \mid n \geq 0\}$ , then  $T_{-\beta}^\ell(z_{2k}/\beta^{2m}) = \frac{-\beta}{\beta+1}$  is not possible when  $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$ , thus  $u_{2k-1} = \widehat{t}$ ,  $u_{2k} = 0$  and  $u_{2k+1} = \widehat{0}$  are equivalent.

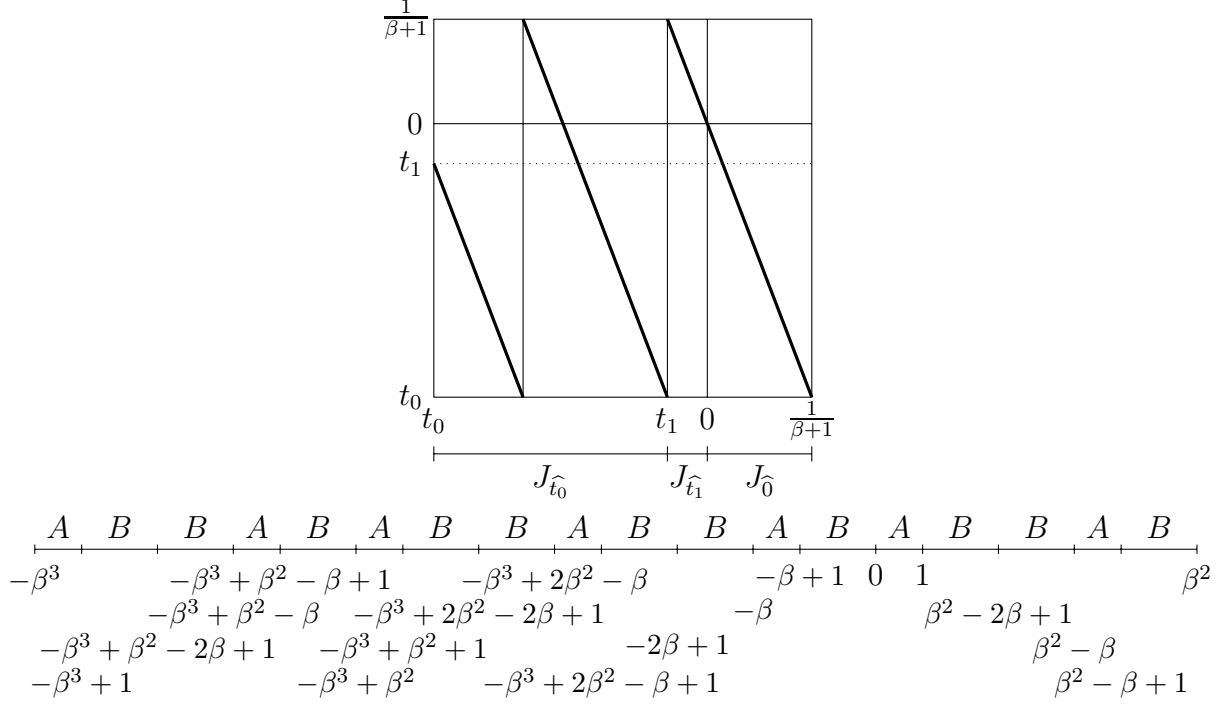


FIGURE 3. The  $(-\beta)$ -transformation and  $\mathbb{Z}_{-\beta} \cap [-\beta^3, \beta^2]$ ,  $\beta = (3 + \sqrt{5})/2$ .

Let now  $T_{-\beta}^\ell(z_{2k}/\beta^{2m}) = \frac{-\beta}{\beta+1}$  and  $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$ , thus  $0 \in V'_\beta$ . Then the size of  $V'_\beta$  is the minimal  $n \geq 2$  such that  $T_{-\beta}^{n-1}(\frac{-\beta}{\beta+1}) = 0$ . Moreover,  $T_{-\beta}^j(z_{2k}/\beta^{2m}) \neq \frac{-\beta}{\beta+1}$  for all  $j \neq \ell$ .

If  $\ell$  is even, then  $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^\ell(y/\beta^{2m}) = \frac{-\beta}{\beta+1}$ , thus  $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$ . From below,  $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^\ell(y/\beta^{2m}) = \frac{1}{\beta+1}$  and  $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{\ell+1}(y/\beta^{2m}) = \frac{-\beta}{\beta+1}$ , thus  $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = T_{-\beta}^{2m-\ell-1}(\frac{-\beta}{\beta+1})$ . By the definition of  $n$ , we have  $2m - \ell \geq n - 1$ . If  $n$  is even, then we also have  $2m - \ell - 1 \geq n - 1$ , thus  $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$ .

If  $\ell$  is odd, then the roles of  $y > z_{2k}$  and  $y < z_{2k}$  change, thus  $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$ ,  $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{\ell+1}(y/\beta^{2m}) = \frac{-\beta}{\beta+1}$ . Now,  $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$  if  $n$  is odd.

Therefore,  $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$  is equivalent with  $\lim_{y \rightarrow z_{2k}, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$  if the size of  $V'_\beta$  is odd, and  $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$  is equivalent with  $\lim_{y \rightarrow z_{2k}, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$  if the size of  $V'_\beta$  is even.  $\square$

By Proposition 2, it suffices to consider the anti-morphism  $\widehat{\psi}_\beta : \widehat{V}_\beta^* \rightarrow \widehat{V}_\beta^*$  defined by

$$\widehat{\psi}_\beta(\widehat{x}) = \widehat{x}_m \cdots \widehat{x}_1 \widehat{x}_0 \quad \text{when} \quad \widehat{\psi}_\beta(\widehat{x}) = \widehat{x}_m T_{-\beta}(y_m) \cdots \widehat{x}_1 T_{-\beta}(y_1) \widehat{x}_0 \quad (x \in V_\beta).$$

Then,  $\Delta_{-\beta}$  is given by the set  $\widehat{R}_\beta$  which consists of the return words of  $\widehat{0}$  when  $0 \notin V'_\beta$  or the size of  $V'_\beta$  is odd. When  $0 \in V'_\beta$  and the size of  $V'_\beta$  is even, then  $\widehat{R}_\beta$  consists of the words  $w \widehat{t}$  such that  $\widehat{t} w$  is a return word of  $\widehat{t}$ .

*Example 3.* Let  $\beta > 1$  with  $\beta^6 = 3\beta^5 + 2\beta^4 + 2\beta^3 + \beta^2 - 2\beta - 1$ , i.e.,  $\beta \approx 3.695$ , then the  $(-\beta)$ -transformation is depicted in Figure 4, where  $t_n = T_{-\beta}^n\left(\frac{-\beta}{\beta+1}\right)$ . We have  $t_5 = \frac{-1}{\beta+1} = t_6$ .

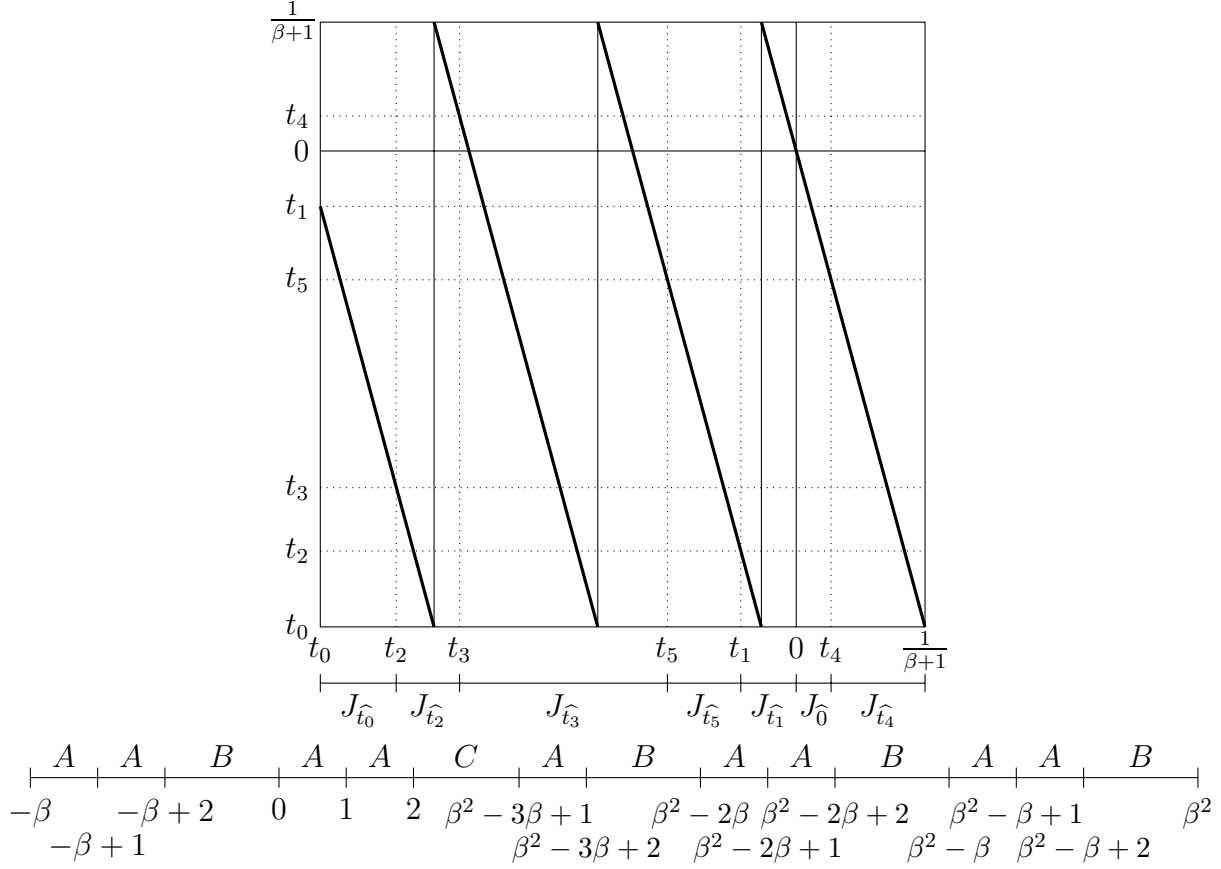


FIGURE 4. The  $(-\beta)$ -transformation and  $\mathbb{Z}_{-\beta} \cap [-\beta, \beta^2]$  from Example 3.

The anti-morphism  $\widehat{\psi}_\beta : \widehat{V}_\beta^* \rightarrow \widehat{V}_\beta^*$  is therefore defined by

$$\widehat{\psi}_\beta : \begin{array}{lll} \widehat{t}_0 \mapsto \widehat{t}_3 \widehat{t}_5, & \widehat{t}_2 \mapsto \widehat{t}_4 \widehat{t}_0 \widehat{t}_2, & \widehat{t}_3 \mapsto \widehat{t}_5 \widehat{t}_1 \widehat{0} \widehat{t}_4 \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1 \widehat{0}, \\ \widehat{t}_5 \mapsto \widehat{t}_2 \widehat{t}_3, & \widehat{t}_1 \mapsto \widehat{0} \widehat{t}_4 \widehat{t}_0, & \widehat{0} \mapsto \widehat{t}_5 \widehat{t}_1, & \widehat{t}_4 \mapsto \widehat{t}_0 \widehat{t}_2 \widehat{t}_3. \end{array}$$

It is convenient to group to letters forming the words

$$\begin{array}{llll} a = \widehat{0} \widehat{t}_4, & b = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1, & c = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5, & d = \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1, \\ e = \widehat{t}_0 \widehat{t}_2, & f = \widehat{t}_4, & g = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3, & h = \widehat{t}_5 \widehat{t}_1, \end{array}$$

which correspond to the intervals

$$\begin{array}{llll} J_a = \left(0, \frac{1}{\beta+1}\right), & J_b = (t_0, 0), & J_c = (t_0, t_1), & J_d = (t_2, 0), \\ J_e = (t_0, t_3), & J_f = \left(t_4, \frac{1}{\beta+1}\right), & J_g = (t_0, t_5), & J_h = (t_5, 0). \end{array}$$

The anti-morphism  $\widehat{\psi}_\beta$  acts on these words by

$$\begin{aligned} \widehat{\psi}_\beta : \quad a &\mapsto b, & b &\mapsto ababac, & c &\mapsto dabac, & d &\mapsto ababae, \\ e &\mapsto fc, & f &\mapsto g, & g &\mapsto habac, & h &\mapsto ag. \end{aligned}$$

Since  $\widehat{0}$  only occurs at the beginning of  $a$ , the return words of  $\widehat{0}$  with their  $\widehat{\psi}_\beta$ -images are

$$\begin{aligned} ab &\mapsto ab ab acb, & aed &\mapsto ab ab aefcb, \\ acb &\mapsto ab ab acd ab acb, & aefcb &\mapsto ab ab acd ab acgfc b, \\ acd &\mapsto ab ab aed ab acb, & acgfc b &\mapsto ab ab acd ab \underbrace{acgh}_{=acb} ab acd ab acb. \end{aligned}$$

Then  $\mathbb{Z}_{-\beta}$  is described by the anti-morphism  $\widehat{\varphi}_{-\beta} : \widehat{R}_\beta^* \rightarrow \widehat{R}_\beta^*$  which is defined by

$$\begin{aligned} \widehat{\varphi}_{-\beta} : \quad A &\mapsto AAB, & L(A) &= 1, \\ B &\mapsto AACAB, & L(B) &= \beta - 2 \approx 1.695, \\ C &\mapsto AADAB, & L(C) &= \beta^2 - 3\beta - 1 \approx 1.569, \\ D &\mapsto AAE, & L(D) &= \beta^3 - 3\beta^2 - 2\beta - 1 \approx 1.104, \\ E &\mapsto AACAF, & L(E) &= \beta^4 - 3\beta^3 - 2\beta^2 - \beta - 2 \approx 2.081, \\ F &\mapsto AACABACAB, & L(F) &= \beta^5 - 3\beta^4 - 2\beta^3 - 2\beta^2 + \beta - 2 \approx 3.12. \end{aligned}$$

Some  $(-\beta)$ -integers are represented in Figure 4, and the two-sided fixed point is

$$\dots AACAB AAB AADAB AAB AAB AACAB AAB AAB \dots$$

Note that grouping the letters as in Example 3 is always possible. It is usually a good idea to start directly with the corresponding intervals, and this is even possible when  $\beta$  is not a  $(-\beta)$ -number. The drawback of this method is that the involved intervals can be a bit complicated to describe in the general case, e.g.  $t_1 < \frac{1}{\beta+1} - \frac{|\beta|}{\beta}$  implies that  $(t_0, t_1)$  is mapped to  $(t_2, t_1)$ , an interval which does not occur in Example 3. Determining the return words is also a bit more complicated since  $J_{\widehat{0}}$  can be contained in several intervals, and it should be taken care of the fact that the union of two intervals can be another interval (minus one point), as for  $J_g \cup J_h = J_b \setminus \{t_5\}$  in Example 3. Therefore, we do not give a general account of this method here.

#### 4. CONCLUSIONS AND OPEN QUESTIONS

With every  $(-\beta)$ -number  $\beta \geq (1 + \sqrt{5})/2$ , we have associated an anti-morphism  $\varphi_{-\beta}$  on a finite alphabet. The distances between consecutive  $(-\beta)$ -integers are described by a fixed point of  $\varphi_{-\beta}$ , and  $\varphi_{-\beta}$  is given by a simple algorithm. In a forthcoming version of [1], the anti-morphism will be described explicitly for any  $\beta > 1$  such that  $T_{-\beta}^n(\frac{-\beta}{\beta+1}) \in [\frac{1-|\beta|}{\beta}, 0]$  for all  $n \geq 1$ . Example 3 shows that the situation is more complicated when this condition is not fulfilled. It would be interesting to have a reasonably simple description of  $\varphi_{-\beta}$  in the general case as well.

It is well known that the maximal distance between consecutive  $\beta$ -integers is bounded by 1. We have seen that this is not true for  $(-\beta)$ -integers. Since the set  $\Delta_{-\beta}$  is finite for any  $(-\beta)$ -number  $\beta \geq (1 + \sqrt{5})/2$ , it is bounded. It is an open question whether there is a uniform bound on  $\Delta_{-\beta}$ . Another open question is whether  $\Delta_{-\beta}$  is bounded when  $\beta$  is not a  $(-\beta)$ -number. It is possible that these questions can be answered only when the structure of  $\mathbb{Z}_{-\beta}$  is well understood in general.

Another topic which is probably worth investigating is the structure of the sets  $S_{-\beta}(x)$  for  $x \neq 0$ , and the corresponding tilings when  $\beta$  is a Pisot unit.

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