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To cite this version:
Wolfgang Steiner. On the structure of $(-\beta)$-integers. RAIRO - Theoretical Informatics and Applications (RAIRO: ITA), 2012, 46 (1), pp.181-200. 10.1051/ita/2011115. hal-00533345v2

HAL Id: hal-00533345
https://hal.science/hal-00533345v2
Submitted on 26 Jun 2011

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ON THE STRUCTURE OF \((-\beta)\)-INTEGERS

WOLFGANG STEINER

Abstract. The \((-\beta)\)-integers are natural generalisations of the \(\beta\)-integers, and thus of the integers, for negative real bases. When \(\beta\) is the analogue of a Parry number, we describe the structure of the set of \((-\beta)\)-integers by a fixed point of an anti-morphism.

1. Introduction

The aim of this paper is to study the structure of the set of real numbers having a digital expansion of the form
\[
\sum_{k=0}^{n-1} a_k (-\beta)^k,
\]
where \((-\beta)\) is a negative real base with \(\beta > 1\), the digits \(a_k \in \mathbb{Z}\) satisfy certain conditions specified below, and \(n \geq 0\). These numbers are called \((-\beta)\)-integers, and have been recently studied by Ambrož, Dombek, Masáková and Pelantová [1].

Before dealing with these numbers, we recall some facts about \(\beta\)-integers, which are the real numbers of the form
\[
\pm \sum_{k=0}^{m-1} a_k \beta^k \quad \text{such that} \quad 0 \leq \sum_{k=0}^{m-1} a_k \beta^k < \beta^m \quad \text{for all} \quad 1 \leq m \leq n,
\]
i.e., \(\sum_{k=0}^{n-1} a_k \beta^k\) is a greedy \(\beta\)-expansion. Equivalently, we can define the set of \(\beta\)-integers as
\[
\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \quad \text{with} \quad \mathbb{Z}_\beta^+ = \bigcup_{n \geq 0} \beta^n T^{-n}_\beta(0),
\]
where \(T_\beta\) is the \(\beta\)-transformation, defined by
\[
T_\beta : [0, 1) \to [0, 1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor.
\]
This map and the corresponding \(\beta\)-expansions were first studied by Rényi [20].

The notion of \(\beta\)-integers was introduced in the domain of quasicrystallography, see for instance [6], and the structure of the \(\beta\)-integers is very well understood now. We have \(\beta \mathbb{Z}_\beta \subseteq \mathbb{Z}_\beta\), the set of distances between consecutive elements of \(\mathbb{Z}_\beta\) is
\[
\Delta_\beta = \{T^n_\beta(1^-) \mid n \geq 0\},
\]
where \(T^n_\beta(x^-) = \lim_{y \to x^-} T^n_\beta(y)\), and the sequence of distances between consecutive elements of \(\mathbb{Z}_\beta^+\) is coded by the fixed point of a substitution, see [9] for the case when \(\Delta_\beta\) is a finite set, that is when \(\beta\) is a Parry number. We give short proofs of these facts in Section 2. More detailed properties of this sequence can be found e.g. in [2, 3, 4, 11, 16].
Closely related to $\mathbb{Z}_\beta^+$ are the sets

$$S_\beta(x) = \bigcup_{n \geq 0} \beta^n T_{-\beta}^{-n}(x) \quad (x \in [0,1]),$$

which were used by Thurston [21] to define (fractal) tilings of $\mathbb{R}^{d-1}$ when $\beta$ is a Pisot number of degree $d$, i.e., a root $> 1$ of a polynomial $x^d + p_1 x^{d-1} + \cdots + p_d \in \mathbb{Z}[x]$ such that all other roots have modulus $< 1$, and an algebraic unit, i.e., $p_d = \pm 1$. These tilings allow e.g. to determine the $k$-th digit $a_k$ of a number without knowing the other digits, see [15].

It is widely agreed that the greedy $\beta$-expansions are the natural representations of real numbers in a real base $\beta > 1$. For the case of negative bases, the situation is not so clear. Ito and Sadahiro [14] proposed recently to use the $(-\beta)$-transformation defined by

$$T_{-\beta} : \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right], \quad x \mapsto -\beta x - \left\lfloor \frac{\beta}{\beta+1} - \beta x \right\rfloor,$$

see also [10]. This transformation has the important property that $T_{-\beta}(-x/\beta) = x$ for all $x \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. Some instances are depicted in Figures 1, 3, 4 and 5.

The set of $(-\beta)$-integers is therefore defined by

$$\mathbb{Z}_{-\beta} = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0).$$

These are the numbers

$$\sum_{k=0}^{n-1} a_k (-\beta)^k \quad \text{such that} \quad \frac{-\beta}{\beta + 1} \leq \sum_{k=0}^{m-1} a_k (-\beta)^{k-m} < \frac{1}{\beta + 1} \quad \text{for all} \ 1 \leq m \leq n.$$

Note that, in the case of $\beta$-integers, we have to add $-\mathbb{Z}_\beta^+$ to $\mathbb{Z}_\beta^+$ in order to obtain a set resembling $\mathbb{Z}$. In the case of $(-\beta)$-integers, this is not necessary because the $(-\beta)$-transformation allows to represent positive and negative numbers.

**Figure 1.** The $(-\beta)$-transformation for $\beta = 2$ (left), $\beta = \frac{1 + \sqrt{5}}{2} \approx 1.618$ (middle), and $\beta = \frac{1}{\beta} + \frac{1}{\beta^2} \approx 1.325$ (right).
It is not difficult to see that $\mathbb{Z}_{-\beta} = \mathbb{Z} = \mathbb{Z}_{\beta}$ when $\beta \in \mathbb{Z}$, $\beta \geq 2$. Some other properties of $\mathbb{Z}_{-\beta}$ can be found in [1], mainly for the case when $T_{\beta}^{-n}(\beta + 1) \leq 0$ and $T_{\beta}^{2n-1}(\beta + 1) \geq \frac{1-|\beta|}{\beta}$ for all $n \geq 1$. (Note that $T_{\beta}^{-n}(\beta + 1) \in (\frac{1}{\beta + 1}, \frac{1-|\beta|}{\beta}) \cup (\frac{\beta - 1}{\beta + 1}, 0)$ implies $T_{\beta}^{2n+1}(\beta + 1) > 0$.)

The set

$$V_{\beta} = \{ T_{\beta}^{-n}(\beta + 1) \mid n \geq 0 \}$$

plays a similar role for $(-\beta)$-expansions as the set $\{ T_{\beta}^{n}(1^-) \mid n \geq 0 \}$ for $\beta$-expansions. If $V_{\beta}$ is a finite set, then we call $\beta > 1$ an Yrrap number. Note that the set of numbers are called Ito–Sadahiro numbers in [18], in reference to [14]. However, as the generalised $\beta$-transformations in [13] with $E = (1, \ldots, 1)$ are, up to conjugation by the map $x \mapsto \frac{1}{\beta+1} - x$, the same as our $(-\beta)$-transformations, these numbers were already considered by Góra and perhaps by other authors. Therefore, the neutral but intricate name $(-\beta)$-numbers was chosen in [17], referring to the original name $\beta$-numbers for Parry numbers [19]. The name Yrrap number, used in the present paper, refers to the connection with Parry numbers and to the fact that $T_{-\beta}$ is (locally) orientation-reversing.

For any Yrrap number $\beta \geq (1 + \sqrt{5})/2$, we describe the sequence of $(-\beta)$-integers in terms of a two-sided infinite word on a finite alphabet which is a fixed point of an anti-morphism (Theorem 3). Note that the orientation-reversing property of the map $x \mapsto -\beta x$ imposes the use of an anti-morphism instead of a morphism, and that anti-morphisms were considered in a similar context by Enomoto [8].

For $1 < \beta < \frac{1+\sqrt{5}}{2}$, we have $\mathbb{Z}_{-\beta} = \{0\}$, as already proved in [1]. However, our study still makes sense for these bases $(-\beta)$ because we can also describe the sets

$$S_{-\beta}(x) = \lim_{n \to \infty} (-\beta)^n T_{-\beta}^{-n}(x) \quad (x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right]),$$

where the limit set consists of the numbers lying in all but finitely many sets $(-\beta)^n T_{-\beta}^{-n}(x)$, $n \geq 0$. Taking the limit instead of the union over all $n \geq 0$ implies that every $y \in \mathbb{R}$ lies in exactly one set $S_{-\beta}(x)$, $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, see Lemma 2. Note that $T_{-\beta}^2(\frac{\beta - 1}{\beta + 1}) \neq \frac{\beta - 2}{\beta + 1}$ when $\beta \not\in \mathbb{Z}$. Other properties of the $(-\beta)$-transformation for $1 < \beta < \frac{1+\sqrt{5}}{2}$ are exhibited in [17].

2. \textbf{$\beta$-integers}

In this section, we consider the structure of $\beta$-integers. The results are not new, but it is useful to state and prove them in order to understand the differences with $(-\beta)$-integers.

Recall that $\Delta_{\beta} = \{ T_{\beta}^{n}(1^-) \mid n \geq 0 \}$, and let $\Delta_{\beta}^*$ be the free monoid generated by $\Delta_{\beta}$. Elements of $\Delta_{\beta}^*$ will be considered as words on the alphabet $\Delta_{\beta}$, and the operation is the concatenation of words. The $\beta$-substitution is the morphism $\varphi_{\beta} : \Delta_{\beta}^* \to \Delta_{\beta}^*$, defined by

$$\varphi_{\beta}(x) = \underbrace{1 \cdots 1}_{[\beta x] - 1 \text{ times}} T_{\beta}^{n}(x^-) \quad (x \in \Delta_{\beta}).$$

Here, 1 is an element of $\Delta_{\beta}$ and not the identity element of $\Delta_{\beta}^*$ (which is the empty word). Recall that, as $\varphi_{\beta}$ is a morphism, we have $\varphi_{\beta}(uv) = \varphi_{\beta}(u)\varphi_{\beta}(v)$ for all $u, v \in \Delta_{\beta}^*$. Since $\varphi_{\beta}^{n+1}(1) = \varphi_{\beta}(\varphi_{\beta}(1))$ and $\varphi_{\beta}(1)$ starts with 1, $\varphi_{\beta}^n(1)$ is a prefix of $\varphi_{\beta}^{n+1}(1)$ for every $n \geq 0$. 

ON THE STRUCTURE OF $(-\beta)$-INTEGERS
Lemma 1. For any $z$ and $u$, moreover, we have $\{ z_k | k \geq 0 \}$ with $z_k = \sum_{j=1}^{k} u_j$,

where $u_1 u_2 \cdots$ is the infinite word with letters in $\Delta_\beta$ which has $\varphi_\beta^n(1)$ as prefix for all $n \geq 0$.

The set of differences between consecutive $\beta$-integers is $\Delta_\beta$.

The main observation for the proof of the theorem is the following. We use the notation $\|v\| = k$ and $L(v) = \sum_{j=1}^{k} v_j$ for any $v = v_1 \cdots v_k \in \Delta_\beta$, $k \geq 0$.

Lemma 1. For any $n \geq 0$, $1 \leq k \leq |\varphi_\beta^n(1)|$, we have $T_\beta^n \left( \begin{bmatrix} z_{k-1} \beta \, \beta^n \, z_k \beta^n \end{bmatrix} \right) = [0, u_k]$, and $z_{|\varphi_\beta^n(1)|} = L(\varphi_\beta^n(1)) = \beta^n$.

Proof. For $n = 0$, we have $|\varphi_\beta^0(1)| = 1$, $z_0 = 0$, $z_1 = 1$, $u_1 = 1$, thus the statements are true. Suppose that they hold for $n$, and consider

$u_1 u_2 \cdots u_{|\varphi_\beta^{n+1}(1)|} = \varphi_\beta^{n+1}(1) = \varphi_\beta(\varphi_\beta^n(1)) = \varphi_\beta(u_1) \varphi_\beta(u_2) \cdots \varphi_\beta(u_{|\varphi_\beta^n(1)|})$.

Let $1 \leq k \leq |\varphi_\beta^{n+1}(1)|$, and write $u_1 \cdots u_k = \varphi_\beta(u_1 \cdots u_{j-1}) v_1 \cdots v_i$ with $1 \leq j \leq |\varphi_\beta^n(1)|$, $1 \leq i \leq |\varphi_\beta(u_j)|$, i.e., $v_1 \cdots v_i$ is a non-empty prefix of $\varphi_\beta(u_j)$.

For any $x \in (0, 1]$, we have $T_\beta(x^-) = \beta x - \lfloor \beta x \rfloor + 1$, hence $L(\varphi_\beta(x)) = \beta x$ for $x \in \Delta_\beta$. This yields that

$z_k = L(u_1 \cdots u_k) = \beta L(u_1 \cdots u_{j-1}) + L(v_1 \cdots v_i) = \beta z_{j-1} + i - 1 + v_i$

and $z_{k-1} = \beta z_{j-1} + i$, hence

$\left[ \frac{z_{k-1}}{\beta^n}, \frac{z_k}{\beta^n} \right] = \left[ z_{j-1} + \frac{i - 1}{\beta^n}, z_{j-1} + \frac{i - 1 + v_i}{\beta^n} \right] \subseteq [z_{j-1}, z_{j-1} + u_j] = [z_{j-1}, z_j]$

$T_\beta^{n+1} \left( \begin{bmatrix} z_{k-1} \beta^{n+1} \\
\frac{z_k}{\beta^{n+1}} \end{bmatrix} \right) = T_\beta \left( \begin{bmatrix} i - 1 \\
\frac{i - 1 + v_i}{\beta^n} \end{bmatrix} \right) = [0, v_i] = [0, u_k]$.

Moreover, we have $L(\varphi_\beta^{n+1}(1)) = \beta L(\varphi_\beta^n(1)) = \beta^{n+1}$, thus the statements hold for $n+1$. \( \square \)

Proof of Theorem. By Lemma, we have $z_{|\varphi_\beta^n(1)|} = \beta^n$ for all $n \geq 0$, thus $[0, 1)$ is split into the intervals $[z_{k-1}/\beta^n, z_k/\beta^n]$, $1 \leq k \leq |\varphi_\beta^n(1)|$. Therefore, Lemma yields that

$T_\beta^{n+1}(0) = \left\{ \frac{z_{k-1}}{\beta^n} | 1 \leq k \leq |\varphi_\beta^n(1)| \right\}$

hence

$\bigcup_{n \geq 0} \beta^n T_\beta^{n}(0) = \left\{ z_k | k \geq 0 \right\}$.

Since $u_k \in \Delta_\beta$ for all $k \geq 1$ and $u_{|\varphi_\beta^n(1)|} = T_\beta^n(1^-)$ for all $n \geq 0$, we have

$\left\{ z_k - z_{k-1} | k \geq 1 \right\} = \left\{ u_k | k \geq 1 \right\} = \Delta_\beta$.

For the sets $S_\beta(x)$, Lemma gives the following corollary.
Corollary 1. For any $x \in [0, 1)$, we have

$$S_\beta(x) = \{z_k + x \mid k \geq 0, u_{k+1} > x\} \subseteq x + S_\beta(0).$$

In particular, we have $S_\beta(x) - x = S_\beta(y) - y$ for all $x, y \in [0, 1)$ with $(x, y) \cap \Delta_\beta = \emptyset$. From the definition of $S_\beta(x)$ and since $x \in \beta T^{-1}_\beta(x)$, we also get that

$$S_\beta(x) = \bigcup_{y \in T^{-1}_\beta(x)} \beta S_\beta(y) \quad (x \in [0, 1)).$$

This shows that $S_\beta(x)$ is the solution of a graph-directed iterated function system (GIFS) when $\beta$ is a Parry number, cf. [15, Section 3.2].

3. ($-\beta$)-integers

We now turn to the discussion of ($-\beta$)-integers and sets $S_{-\beta}(x), x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$.

Lemma 2. For any $\beta > 1$, $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, we have

$$S_{-\beta}(x) = \bigcup_{n \geq 0} (-\beta)^n \left(T_{-\beta}^n(x) \setminus \left\{\frac{-\beta}{\beta+1}\right\}\right) = \bigcup_{y \in T^{-1}_{-\beta}(x)} (-\beta) S_{-\beta}(y).$$

For any $y \in \mathbb{R}$, there exists a unique $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$ such that $y \in S_{-\beta}(x)$. If $T_{-\beta}(x) = x$, then $S_{-\beta}(x) = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^n(x)$, in particular $S_{-\beta}(0) = \mathbb{Z}_{-\beta}$.

Proof. If $y \in S_{-\beta}(x)$, then we have $\frac{y}{(-\beta)^n} \in T_{-\beta}^n(x)$ for all sufficiently large $n$, thus $y \in (-\beta)^n \left(T_{-\beta}^n(x) \setminus \left\{\frac{-\beta}{\beta+1}\right\}\right)$ for some $n \geq 0$. On the other hand, $y \in (-\beta)^n \left(T_{-\beta}^n(x) \setminus \left\{\frac{-\beta}{\beta+1}\right\}\right)$ for some $n \geq 0$ implies that $T_{-\beta}^n(\frac{y}{(-\beta)^n}) = T_{-\beta}(\frac{y}{(-\beta)^n}) = x$ for all $m \geq n$, thus $y \in S_{-\beta}(x)$. This shows the first equation. Since $x \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$ implies that $x \in (-\beta) \left(T_{-\beta}^{-1}(x) \setminus \left\{\frac{-\beta}{\beta+1}\right\}\right)$, we obtain that $S_{-\beta}(x) = \bigcup_{y \in T_{-\beta}^{-1}(x)} (-\beta) S_{-\beta}(y)$ for all $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$.

For any $y \in \mathbb{R}$, we have $y \in S_{-\beta}(T_{-\beta}^n \left(\frac{y}{(-\beta)^n}\right))$ for all $n \geq 0$ such that $\frac{y}{(-\beta)^n} \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, thus $y \in S_{-\beta}(x)$ for some $x \in \left\{\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right\}$. To show that this $x$ is unique, let $y \in S_{-\beta}(x)$ and $y \in S_{-\beta}(x')$ for some $x, x' \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. Then we have $y \in (-\beta)^n \left(T_{-\beta}^n(x) \setminus \left\{\frac{-\beta}{\beta+1}\right\}\right)$ and $y \in (-\beta)^m \left(T_{-\beta}^m(x') \setminus \left\{\frac{-\beta}{\beta+1}\right\}\right)$ for some $m, n \geq 0$, thus $x = T_{-\beta}^n(\frac{y}{(-\beta)^n}) = T_{-\beta}^m(\frac{y}{(-\beta)^m}) = x'$.

If $T_{-\beta}(\frac{-\beta}{\beta+1}) = x = T_{-\beta}(x)$, then $T_{-\beta}^{n+2}(\frac{-\beta}{\beta+1}) = T_{-\beta}^{n+1}(\frac{-\beta}{\beta+1}) = T_{-\beta}(x) = x$ yields that $(-\beta)^n \frac{-\beta}{\beta+1} \in S_{-\beta}(x)$, which shows that $S_{-\beta}(x) = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^n(x)$ when $T_{-\beta}(x) = x$. \[\square\]

The first two statements of the following proposition can also be found in [1].

Proposition 1. For any $\beta > 1$, we have $(-\beta) \mathbb{Z}_{-\beta} \subseteq \mathbb{Z}_{-\beta}$.

If $\beta < (1 + \sqrt{5})/2$, then $\mathbb{Z}_{-\beta} = \{0\}$.
If $\beta \geq (1 + \sqrt{5})/2$, then

$$\mathbb{Z}_{-\beta} \cap (-\beta)^n [-\beta, 1] = \left\{(-\beta)^n, (-\beta)^{n+1}\right\} \cup (-\beta)^{n+2} (T_{-\beta}^{-n-2}(0) \cap \left(\frac{-1}{\beta}, \frac{1}{\beta^2}\right))$$
for all $n \geq 0$, in particular

$$\mathbb{Z}_-\beta \cap [-\beta, 1] = \begin{cases} 
\{ -\beta, -\beta + 1, \ldots, -\beta + \lfloor \beta \rfloor, 0, 1 \} & \text{if } \beta^2 \geq \lfloor \beta \rfloor (\beta + 1), \\
\{ -\beta, -\beta + 1, \ldots, -\beta + \lfloor \beta \rfloor - 1, 0, 1 \} & \text{if } \beta^2 < \lfloor \beta \rfloor (\beta + 1).
\end{cases}$$

**Proof.** The inclusion $(-\beta) \mathbb{Z}_-\beta \subseteq \mathbb{Z}_-\beta$ is a consequence of Lemma 2 and $0 \in T_{-\beta}^{-1}(0)$. If $\beta < \frac{1+\sqrt{5}}{2}$, then $\frac{1}{\beta} < -\frac{\beta}{\beta + 1}$, hence $T_{-\beta}^{-1}(0) = \{0\}$ and $\mathbb{Z}_-\beta = \{0\}$, see Figure 1(right).

If $\beta \geq \frac{1+\sqrt{5}}{2}$, then $\frac{1}{\beta} \in T_{-\beta}^{-1}(0)$ implies $1 \in \mathbb{Z}_-\beta$, thus $(-\beta)^n \in \mathbb{Z}_-\beta$ for all $n \geq 0$. Clearly,

$$(-\beta)^{n+2} \left( T_{-\beta}^{-n-2}(0) \cap \left( \frac{1}{\beta}, \frac{1}{\beta^2} \right) \right) \subseteq \mathbb{Z}_-\beta \cap (-\beta)^n (-\beta, 1).$$

To show the other inclusion, let $z \in (-\beta)^n T_{-\beta}^{-m}(0) \cap (-\beta)^n (-\beta, 1)$ for some $m \geq 0$. If $z \neq (-\beta)^{m-}\frac{\beta}{\beta + 1}$, then $\frac{z}{\beta^n+m} \in \left( -\frac{\beta}{\beta+1}, \frac{1}{\beta+1} \right)$ and $\frac{z}{\beta^n+m} \in \left( -\frac{\beta}{\beta+1}, \frac{1}{\beta+1} \right)$ imply that $T_{-\beta}^{-n} \left( \frac{z}{\beta^n+m} \right) = T_{-\beta}^{-m} \left( \frac{z}{\beta^n+m} \right) = 0$. If $z = (-\beta)^{m-}\frac{\beta}{\beta + 1}$, then

$$T_{-\beta}^{-n} \left( \frac{z}{\beta^n+m} \right) = T_{-\beta}^{-n} \left( \frac{\beta}{\beta+1} \right) = T_{-\beta}^{-m+2} \left( \frac{\beta}{\beta+1} \right) = T_{-\beta}^{-m+1} \left( \frac{\beta}{\beta+1} \right) = T_{-\beta}^{-m}(0) = 0 ,$$

where we have used that $\frac{z}{\beta^n+m} \in \left( -\frac{\beta}{\beta+1}, \frac{1}{\beta+1} \right)$ implies $m \leq n$. Therefore, we have $z \in (-\beta)^{n+2} T_{-\beta}^{-n-2}(0)$ for all $z \in \mathbb{Z}_-\beta \cap (-\beta)^n (-\beta, 1)$.

Consider now $n = 0$, then

$$\mathbb{Z}_-\beta \cap [-\beta, 1] = \{ -\beta, 1 \} \cup \{ z \in (-\beta, 1) \mid T_{-\beta}^2 (z/\beta^2) = 0 \} .$$

Since $\frac{-\beta}{\beta+1} \geq -\frac{\beta}{\beta+1}$ if and only if $\beta^2 \geq \lfloor \beta \rfloor (\beta + 1)$, we obtain that

$$(-\beta) T_{-\beta}^{-1}(0) = \begin{cases} 
\{ 0, 1, \ldots, \lfloor \beta \rfloor \} & \text{if } \beta^2 \geq \lfloor \beta \rfloor (\beta + 1), \\
\{ 0, 1, \ldots, \lfloor \beta \rfloor - 1 \} & \text{if } \beta^2 < \lfloor \beta \rfloor (\beta + 1).
\end{cases}$$

If $T_{-\beta}^2 (z/\beta^2) = 0$, then $z = -a_1 \beta + a_0$ with $a_0 \in (-\beta) T_{-\beta}^{-1}(0)$, $a_1 \in \{ 0, 1, \ldots, \lfloor \beta \rfloor \}$, and $\mathbb{Z}_-\beta \cap [-\beta, 1]$ consists of those numbers $-a_1 \beta + a_0$ lying in $[-\beta, 1]$. \hfill \Box

Proposition 1 shows that the maximal difference between consecutive $(-\beta)$-integers exceeds 1 whenever $\beta^2 < \lfloor \beta \rfloor (\beta + 1)$, i.e., $T_{-\beta}^{-1} \left( -\frac{\beta}{\beta+1} \right) < 0$. For an example, this was also proved in [1]. In Examples 3 and 4, we see that the distance between two consecutive $(-\beta)$-integers can be even greater than 2, and the structure of $\mathbb{Z}_-\beta$ can be quite complicated. Therefore, we adapt a slightly different strategy as for $\mathbb{Z}_\beta$.

In the following, we always assume that the set

$$V'_\beta = V_\beta \cup \{ 0 \} = \left\{ T_{-\beta}^{-n} \left( \frac{\beta}{\beta+1} \right) \mid n \geq 0 \right\} \cup \{ 0 \}$$

is finite, i.e., $\beta$ is an Yrrap number, and let $\beta$ be fixed. For $x \in V'_\beta$, let

$$r_x = \min \left\{ y \in V'_\beta \cup \left\{ \frac{1}{\beta+1} \right\} \mid y > x \right\} , \quad \hat{x} = \frac{x + r_x}{2} , \quad J_x = \{ x \} \quad \text{and} \quad J_{\hat{x}} = (x, r_x) .$$

Then $\{ J_a \mid a \in A_\beta \}$ forms a partition of $\left[ -\frac{\beta}{\beta+1}, \frac{1}{\beta+1} \right)$, where

$$A_\beta = V'_\beta \cup V_{-\beta}^\prime , \quad \text{with} \quad V_{-\beta}^\prime = \{ \hat{x} \mid x \in V'_\beta \}.$$
We have $T_{-\beta}(J_x) = J_{T_{-\beta}(x)}$ for every $x \in V'_\beta$, and the following lemma shows that the image of every $J_x$, $x \in V'_\beta$, is a union of intervals $J_a$, $a \in A_\beta$.

**Lemma 3.** Let $x \in V'_\beta$ and write

$$J_{\hat{x}} \cap T^{-1}_{-\beta}(V'_\beta) = \{y_1, \ldots, y_m\}, \text{ with } x = y_0 < y_1 < \cdots < y_m < y_{m+1} = r_x.$$  

For any $0 \leq i \leq m$, we have

$$T_{-\beta}((y_i, y_{i+1})) = J_{\hat{x}_i} \text{ with } x_i = \lim_{y \to y_i} T_{-\beta}(y), \text{ i.e., } \hat{x}_i = T_{-\beta}\bigg(\frac{y + y_{i+1}}{2}\bigg),$$

and $\beta(y_{i+1} - y_i) = \lambda(J_{\hat{x}_i})$, where $\lambda$ denotes the Lebesgue measure.

**Proof.** Since $T_{-\beta}$ maps no point in $(y_i, y_{i+1})$ to $\frac{\beta}{\beta + 1} \in V'_\beta$, the map is continuous on this interval and $\lambda(T_{-\beta}((y_i, y_{i+1}))) = \beta(y_{i+1} - y_i)$. We have $x_i \in V'_\beta$ since $x_i = T_{-\beta}(y_{i+1})$ in case $y_{i+1} < \frac{\beta}{\beta + 1}$, and $x_i = \frac{\beta}{\beta + 1}$ in case $y_{i+1} = \frac{1}{\beta + 1}$. Since $y_i = \max\{y \in T_{-\beta}^{-1}(V'_\beta) | y < y_{i+1}\}$, we obtain that $r_{x_i} = \lim_{y \to y_i} T_{-\beta}(y)$, thus $T_{-\beta}((y_i, y_{i+1})) = (x_i, r_{x_i})$. \hfill \Box

In view of Lemma 3, we define an anti-morphism $\psi_\beta : A_\beta^* \to A_\beta^*$ by

$$\psi_\beta(x) = T_{-\beta}(x) \quad \text{and} \quad \psi_\beta(\hat{x}) = \hat{x}_m T_{-\beta}(y_m) \cdots \hat{x}_1 T_{-\beta}(y_1) \hat{x}_0 \quad \text{(} x \in V'_\beta, \text{ with} \)$$

with $m$, $x_i$ and $y_i$ as in Lemma 3. Here, anti-morphism means that $\psi_\beta(uv) = \psi_\beta(v)\psi_\beta(u)$ for all $u, v \in A_\beta^*$. Now, the last letter of $\psi_\beta(\hat{t})$ is $\hat{t}$, with $t = \max\{x \in V_\beta | x < 0\}$, and the first letter of $\psi_\beta(\hat{t})$ is $\hat{0}$. Therefore, $\psi_\beta^{2n}(\hat{0})$ is a prefix of $\psi_\beta^{2n+2}(\hat{0}) = \psi_\beta^{2n}(\psi_\beta(\hat{0}))$ and $\psi_\beta^{2n+1}(\hat{0})$ is a suffix of $\psi_\beta^{2n+3}(\hat{0})$ for every $n \geq 0$.

**Theorem 2.** For any Yrrap number $\beta \geq (1 + \sqrt{5})/2$, we have

$$Z_{-\beta} = \{z_k | k \in \mathbb{Z}, u_{2k} = 0\} \quad \text{with} \quad z_k = \left\{ \begin{array}{ll} \sum_{j=1}^{k} \lambda(J_{u_{2j-1}}) & \text{if } k \geq 0, \\ -\sum_{j=1}^{k+1} \lambda(J_{u_{2j-1}}) & \text{if } k \leq 0, \end{array} \right.$$ 

where $\cdots u_{-1}u_0u_1 \cdots$ is the two-sided infinite word on the finite alphabet $A_\beta$ such that $u_0 = 0$, $\psi_\beta(\hat{0})$ is a prefix of $u_1 u_2 \cdots$ and $\psi_\beta^{2n+1}(\hat{0})$ is a suffix of $\cdots u_{-2}u_{-1}$ for all $n \geq 0$.

Note that $\cdots u_{-1}u_0u_1 \cdots$ is a fixed point of $\psi_\beta$, with $u_0$ being mapped to $u_0$.

The following lemma is the analogue of Lemma 1. We use the notation

$$L(v) = \sum_{j=1}^{k} \lambda(J_{v_j}) \quad \text{if } v = v_1 \cdots v_k \in A_\beta^k.$$ 

Note that $u_{2k} \in V'_\beta$ and $u_{2k+1} \in V'_\beta$ for all $k \in \mathbb{Z}$, thus $\lambda(J_{u_{2k}}) = 0$ for all $k \in \mathbb{Z}$.

**Lemma 4.** For any $n \geq 0$, $0 \leq k < |\psi_\beta^n(\hat{0})|/2$, we have

$$T_{-\beta}^n\left(\frac{z(-1)^n k}{(-\beta)^n}\right) = u(-1)^n 2k, \quad T_{-\beta}^n\left(\frac{z(-1)^n k}{(-\beta)^n}, \frac{z(-1)^n (k+1)}{(-\beta)^n}\right) = J_{u(-1)^n (2k+1)},$$

and $z(-1)^n|\psi_\beta^n(\hat{0})|/2 = (-1)^n L\left(\psi_\beta^n(\hat{0})\right) = \lambda(J_{\hat{0}})(-\beta)^n = r_0 (-\beta)^n$. 

Proof. The statements are true for \( n = 0 \) since \( |\psi^0_\beta(\overline{0})| = 1 \), \( z_0 = 0 \), \( z_1 = \lambda(J_0) = r_0 \).

Suppose that they hold for even \( n \), and consider

\[
u_{\psi^0_\beta(\overline{0})} \cdots u_{-2} u_{-1} = \psi^{n+1}_\beta(\overline{0}) = \psi_\beta^{u_{\psi^0_\beta(\overline{0})}}(\overline{0}) = \psi_\beta(u_{\psi^0_\beta(\overline{0})}) \cdots \psi_\beta(u_2) \psi_\beta(u_1).
\]

Let \( 0 \leq k < |\psi^{n+1}_\beta(\overline{0})|/2 \), and write

\[
u_{-2k-1} \cdots u_{-1} = \nu_{-2} \cdots \nu_{-1} = \psi_\beta(u_1) \cdots \psi_\beta(u_2)
\]

with \( 0 \leq j < |\psi^n_\beta(\overline{0})|/2 \), \( 0 \leq i < |\psi_\beta(u_{j+1})|/2 \), i.e., \( u_{-2i-1} \cdots u_{-1} \) is a suffix of \( \psi_\beta(u_{j+1}) \).

By Lemma \[3\] we have \( L(\psi_\beta(\overline{x})) = \beta \lambda(J_\overline{x}) \) for any \( x \in V'_\beta \). This yields that

\[-z_{-k-1} = \beta L(u_1 \cdots u_{2j}) + L(v_{-2i-1} \cdots v_{-1}) = \beta z_j + L(v_{-2i-1} \cdots v_{-1})
\]

and \( -z_{-k} = \beta z_j + L(v_{-2i-2} \cdots v_{-1}) \). By the induction hypothesis, we obtain that

\[
T^{n+1}_{-\beta}
\left(\frac{z_{-k}}{(-\beta)^{n+1}} \frac{z_{-k-1}}{(-\beta)^{n+1}} \right) = T^{n}_{-\beta}(v_{-2i-1} v_{-2i-2}) = T^{n}_{-\beta}(y_i) = v_{-2i} = u_{-2k}
\]

if \( i = 0 \),

\[
T^{n+1}_{-\beta}
\left(\frac{z_{-k}}{(-\beta)^{n+1}} \frac{z_{-k-1}}{(-\beta)^{n+1}} \right) = T^{n}_{-\beta}(y_i, y_{i+1}) = J_{v_{-2i-1}} = J_{u_{-2k-1}}.
\]

Moreover, we have \( L(\psi^{n+1}_\beta(\overline{0})) = \beta \lambda(\psi^{n+1}_\beta(\overline{0})) = r_0 \beta^{n+1} \), thus the statements hold for \( n + 1 \).

The proof for odd \( n \) runs along the same lines and is therefore omitted. \( \square \)

Proof of Theorem \[2\] By Lemma \[4\] we have \( z_{\nu_{\overline{0}} / (u_{\beta}(\overline{0}))} = r_0 \beta^n \) for all \( n \geq 0 \), thus \( [0, r_0) \) splits into the intervals \( (z_{\nu_{\overline{0}} / (u_{\beta}(\overline{0}))}, -\beta^n) \) and points \( z_{\nu_{\overline{0}} / (u_{\beta}(\overline{0}))}, 0 \leq k < |\psi_\beta(\overline{0})|/2 \), hence

\[
T^{n}_{-\beta}(0) \cap [0, r_0) = \{ z_{\nu_{\overline{0}} / (u_{\beta}(\overline{0}))} / 2 \}
\]

Let \( m \geq 1 \) be such that \( \beta^{2m} r_0 \geq \frac{1}{\beta^{m+1}} \). Then we have \( (-\beta^{m+1}, \frac{1}{\beta^{m+1}}) \subseteq (-\beta^{2m+1} r_0, \beta^{2m} r_0) \), and

\[
T^{n}_{-\beta}(0) \cap \{ -z_{\beta^{m+1}} \} \subseteq (-\beta)^{2m} (T^{-n}_{-\beta}(0) \cap [0, r_0)) \cup (-\beta)^{2m+1} (T^{-n}_{-\beta}(0) \cap [0, r_0)),
\]

thus

\[
\bigcup_{n \geq 0} (-\beta)^{n} (T^{n}_{-\beta}(0) \cap \{ -z_{\beta^{m+1}} \}) = \bigcup_{n \geq 0} (-\beta)^{n} (T^{-n}_{-\beta}(0) \cap [0, r_0)) = \{ z_k \mid k \in \mathbb{Z}, u_{2k} = 0 \}.
\]

Together with Lemma \[2\] this proves the theorem. \( \square \)

As in the case of positive bases, the word \( \cdots u_{-1} u_0 \cdots \) also describes the sets \( S_{-\beta}(x) \).

Theorem \[2\] and Lemma \[4\] give the following corollary.

Corollary 2. For any \( x \in V'_\beta \), \( y \in J_\beta \), we have

\[
S_{-\beta}(x) = \{ z_k \mid k \in \mathbb{Z}, u_{2k} = x \}
\]

and

\[
S_{-\beta}(y) = \{ z_k + y - x \mid k \in \mathbb{Z}, u_{2k+1} = x \}.
\]
Lemma 2 and Corollary 2 imply that $S_{-\beta}(x)$ is the solution of a GIFS for any Yrrap number $\beta \geq (1 + \sqrt{5})/2$, $x \in \left[\frac{-\beta}{\beta + 1}, \frac{1}{\beta + 1}\right]$, cf. the end of Section 2.

Recall that our main goal is to understand the structure of $\mathbb{Z}_{-\beta}$ (in case $\beta \geq (1 + \sqrt{5})/2$), i.e., to describe the occurrences of 0 in the word $\cdots u_{-1}u_0u_1 \cdots$ defined in Theorem 2 and the words between two successive occurrences. Let

$$R_{\beta} = \{u_{2k}u_{2k+1} \cdots u_{2s(k)-1} \mid k \in \mathbb{Z}, u_{2k} = 0\}$$

be the set of return words of 0 in $\cdots u_{-1}u_0u_1 \cdots$. Note that $s(k)$ is defined for all $k \in \mathbb{Z}$ since $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \geq 0$ by Proposition 1.

For any $w \in R_{\beta}$, the word $\psi_{\beta}(w)$ is a factor of $\cdots u_{-1}u_0u_1 \cdots$ starting and ending with 0, thus we can write $\psi_{\beta}(w) = w_1 \cdots w_m 0$ with $w_j \in R_{\beta}$, $1 \leq j \leq m$, and set

$$\varphi_{-\beta}(w) = w_1 \cdots w_m.$$

This defines an anti-morphism $\varphi_{-\beta} : R_{\beta}^* \to R_{\beta}^*$, which plays the role of the $\beta$-substitution.

Since $\cdots u_{-1}u_0u_1 \cdots$ is generated by $u_1 = \hat{0}$, as described in Theorem 2, we consider $w_\beta = u_0u_1 \cdots u_{2s(0)-1}$. We have

$$[0, 1] = [0, \frac{1}{\beta + 1}] \cup \left[\frac{1}{\beta + 1}, 1\right]$$

and $T_{-\beta}((-\beta)^{-1}[\frac{1}{\beta + 1}, 1]) = [\frac{-\beta}{\beta + 1}, 0]$;

thus $L(w_\beta) = 1$ and

$$w_\beta = \hat{0} x_1 \hat{x}_1 \cdots x_m \hat{x}_m x_{-\ell} \hat{x}_{-\ell} \cdots x_{-1} \hat{x}_{-1},$$

where the $x_j$ are defined by $V'_{\beta} = \{x_{-\ell}, \ldots, x_{-1}, 0, x_1, \ldots, x_m\}$, $x_{-\ell} < \cdots < x_{-1} < 0 < x_1 < \cdots < x_m$.

**Theorem 3.** For any Yrrap number $\beta \geq (1 + \sqrt{5})/2$, we have

$$\mathbb{Z}_{-\beta} = \{z_k' \mid k \in \mathbb{Z}\}$$

with $z_k' = \begin{cases} \sum_{j=1}^k L(u_j') & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} L(u_j') & \text{if } k \leq 0, \end{cases}$

where $\cdots u_{-1}u_0u_1u_2' \cdots$ is the two-sided infinite word on the finite alphabet $R_{\beta}$ such that $\varphi_{-\beta}^{2n}(w_{\beta})$ is a prefix of $u'_1u_2' \cdots$ and $\varphi_{-\beta}^{2n+1}(w_{\beta})$ is a suffix of $\cdots u_{-2}u_{-1}'$ for all $n \geq 0$.

The set of distances between consecutive $(-\beta)$-integers is

$$\Delta_{-\beta} = \{z_{k+1}' - z_k' \mid k \in \mathbb{Z}\} = \{L(w) \mid w \in R_{\beta}\}.$$

Note that the index 0 is omitted in $\cdots u_{-2}u_{-1}'u_1'u_2' \cdots$ for reasons of symmetry.

**Proof.** The definitions of $\cdots u_{-1}u_0u_1 \cdots$ in Theorem 2 and of $\varphi_{-\beta}$ imply that $\cdots u_{-2}u_{-1}'u_1'u_2' \cdots$ is the derived word of $\cdots u_{-1}u_0u_1 \cdots$ with respect to $R_{\beta}$, that is

$$u_k' = u_{|u_1 \cdots u_{k-1}'|} \cdots u_{|u_1 \cdots u_k'| - 1}, \quad u_{-k}' = u_{|u_{-k} \cdots u_{-1}'|} \cdots u_{|u_{-k} \cdots u_{-1}'| - 1} \quad (k \geq 1)$$

with

$$\{|u'_1 \cdots u'_{k-1}| \mid k \geq 1\} \cup \{-|u'_{-k} \cdots u'_{-1}| \mid k \geq 1\} = \{k \in \mathbb{Z} \mid u_k = 0\}. $$
Here, for any \( v \in R_\beta^* \), \( |v| \) denotes the length of \( v \) as a word in \( A_\beta^* \), not in \( R_\beta \). Since
\[
z'_k = \sum_{j=1}^{k} L(u'_j) = \sum_{j=0}^{\lfloor |u'_j|/2 \rfloor} \lambda(J_{u_j}) = \sum_{j=1}^{k} \lambda(J_{u_j}), \quad z'_{-k} = -\sum_{j=1}^{k} L(u'_{-j}) = -\sum_{j=1}^{k} \lambda(J_{u_{-j}})
\]
for all \( k \geq 0 \). Theorem 2 yields that \( \{z'_k \mid k \in \mathbb{Z}\} = \mathbb{Z}_-. \)

It follows from the definition of \( R_\beta \) that \( \Delta_- = \{L(w) \mid w \in R_\beta\} \).

It remains to show that \( R_\beta \) is a finite set. We first show that the restriction of \( \psi_\beta \) to \( \widehat{V}_\beta' \)
is primitive, which means that there exists some \( m \geq 1 \) such that, for every \( x \in V_\beta' \), \( \psi_\beta^m(\widehat{x}) \)
contains all elements of \( \widehat{V}_\beta' \). The proof is taken from [13, Proposition 8]. If \( \beta > 2 \), then the
largest connected pieces of images of \( J_\beta \) under \( T_\beta \) grow until they cover two consecutive
discontinuity points \( \frac{1}{\beta+1} - \frac{a+1}{\beta}, \frac{1}{\beta+1} - \frac{a}{\beta} \) of \( T_\beta \), and the next image covers all intervals \( J_\beta \), \( y \in V_\beta' \). If \( \frac{1+\sqrt{5}}{2} < \beta \leq 2 \), then \( \beta^2 > 2 \) implies that the largest connected pieces of images
of \( J_\beta \) under \( T_\beta \) grow until they cover two consecutive discontinuity points of \( T_\beta \). Since
\[
T_{\beta}^{-2}\left(\frac{\beta-1}{\beta+1}, \frac{\beta-2}{\beta+1} - 1 \right) = \left(\frac{\beta^2+\beta-1}{\beta+1}, \frac{1}{\beta+1} \right), \quad T_{\beta}^{-2}\left(\frac{\beta^2}{\beta+1} - 1, \frac{\beta-1}{\beta+1} \right) = \left(\frac{\beta^2-\beta}{\beta+1}, \frac{\beta^2-1}{\beta+1} \right),
\]
the next image covers the fixed point 0, and further images grow until after a finite number of steps they cover all intervals \( J_\beta \), \( y \in V_\beta' \). The case \( \beta = \frac{1+\sqrt{5}}{2} \) is treated in Example 1.

If \( T^n_\beta(\frac{\beta}{\beta+1}) \neq 0 \) for all \( n \geq 0 \), then \( T^n_\beta \) is continuous at all points \( x \in \left(\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right) \) with \( T_\beta(\widehat{x}) = 0 \), thus \( u_{2k} = 0 \) is equivalent to \( u_{2k+1} = \widehat{0} \) (see also Proposition 2 below). Hence
we can consider the return words of \( \widehat{0} \) in \( \cdots u_{-1} u_0 u_1 \cdots \) instead of the return words of 0.
Since \( \psi_\beta^m(\widehat{x}_0 x_1 \widehat{x}_2) \) has at least two occurrences of \( \widehat{0} \) for all \( x_0, x_1, x_2 \in V_\beta' \), there are only
finitely many such return words. If \( T^n_\beta(\frac{\beta}{\beta+1}) = 0 \), then \( \psi_\beta^m(x_0 \widehat{x}_1 x_2) \) starts and ends with 0
for all \( x_0, x_1, x_2 \in V_\beta' \), hence \( R_\beta \) is finite as well.

For details on derived words of primitive substitutive words, we refer to [7].

We remark that, for practical reasons, the set \( R_\beta \) can be obtained from the set \( R = \{w_\beta\} \)
by adding to \( R \) iteratively all return words of 0 which appear in \( \psi_\beta(w) \) for some \( w \in R \)
until \( R \) stabilises. The final set \( R \) is equal to \( R_\beta \).

Now, we apply the theorems in the case of two quadratic examples.

**Example 1.** Let \( \beta = \frac{1+\sqrt{5}}{2} \), i.e., \( \beta^2 = \beta + 1 \), and \( t = \frac{\beta}{\beta+1} = \frac{1}{2} \), then \( V_\beta = V_\beta' = \{t, 0\} \). Since
\[
J_t = (t, 0) = \left(\frac{1}{2}, \frac{1}{2}\right) \cup \{\frac{1}{2}\} \cup \{\frac{1}{2}\}, \quad J_0 = (0, \frac{1}{2}),
\]
see Figure II (middle), the anti-morphism \( \psi_\beta \) on \( A_\beta^* \) is defined by
\[
\psi_\beta: \quad t \mapsto 0, \quad \widehat{t} \mapsto \widehat{0} t \widehat{t}, \quad 0 \mapsto 0, \quad \widehat{0} \mapsto \widehat{t}.
\]

Its two-sided fixed point \( \cdots 0 \widehat{t} t \widehat{t} 0 \widehat{t} 0 \widehat{0} t \widehat{t} 0 \widehat{0} t \widehat{t} 0 \widehat{0} t \widehat{t} 0 \widehat{t} 0 \widehat{0} t \widehat{t} 0 \widehat{0} t \widehat{t} 0 \widehat{t} 0 \widehat{t} 0 \cdots \),
is
where \( \hat{0} \) marks the central letter \( u_0 \). The return word of \( 0 \) starting at \( u_0 \) is \( w_\beta = 0 \hat{0} t \hat{t} \). The image \( \psi_\beta(w_0) = 0 \hat{0} t \hat{t} 0 \hat{t} 0 \) contains the return words \( w_\beta \) and \( 0 \hat{t} \). Since \( \psi_\beta(0 \hat{t} 0) = 0 \hat{0} t \hat{t} \hat{0} \), there are no other return words of \( 0 \), i.e., \( R_\beta = \{ A, B \} \) with \( A = 0 \hat{0} t \hat{t} \), \( B = 0 \hat{t} \). Therefore, \( \cdots u'_2 u'_1 u'_1 u'_2 \cdots \) is a two-sided fixed point of the anti-morphism
\[
\varphi_{-\beta} : \ A \mapsto AB, \quad B \mapsto A,
\]
with
\[
u'_{-13} \cdots u'_1 \cdots u'_{21} = AABAABABAABAB AABAABABAABABABABABABAB.
\]
We have \( \lambda(J_0) = \frac{1}{\beta^2} \), \( \lambda(J_1) = \frac{1}{\beta} \), thus \( L(A) = 1 \), \( L(B) = \frac{1}{\beta} = \beta - 1 \), and some \((-\beta)\)-integers are shown in Figure 2. Note that \((-\beta)^n\) can also be represented as \((-\beta)^{n+2} + (-\beta)^{n+1}\).

\[
\begin{array}{cccccccccc}
-\beta^3 -\beta^2 +\beta^2 -\beta & -\beta & -\beta +1 & 0 & 1 & \beta^2 -\beta +1 & \beta^4 -\beta^3 +\beta^2 -\beta & \beta^4 -\beta^3 +\beta^2 & \beta^4 -\beta^3 +\beta^2 -\beta +1 & \beta^4 -\beta^3 +\beta^2
\end{array}
\]

**Figure 2.** The \((-\beta)\)-integers in \([-\beta^3, \beta^4]\), \( \beta = (1 + \sqrt{5})/2 \).

**Example 2.** Let \( \beta = \frac{3+\sqrt{5}}{2} \), i.e., \( \beta^2 = 3\beta - 1 \), then the \((-\beta)\)-transformation is depicted in Figure 3 where \( t_0 = \frac{-\beta}{\beta+1} \), \( t_1 = T_{-\beta}(t_0) = \frac{-\beta}{\beta+1} - 2 = \frac{-\beta^2}{\beta+1} \), \( T_{-\beta}(t_1) = \frac{1}{\beta+1} - 1 = t_0 \). Therefore, \( V'_\beta = \{ t_0, t_1, 0 \} \) and the anti-morphism \( \psi_\beta : A_\beta^* \rightarrow A_\beta^* \) is defined by

\[
\psi_\beta : \ t_0 \mapsto t_1 , \quad \hat{t}_0 \mapsto \hat{t}_0 t_1 \hat{t}_1 0 \hat{0} t_0 \hat{t}_0 , \quad t_1 \mapsto t_0 , \quad \hat{t}_1 \mapsto \hat{0} , \quad 0 \mapsto 0 , \quad \hat{0} \mapsto \hat{t}_0 t_1 \hat{t}_1 ,
\]

which has the two-sided fixed point
\[
\cdots 0 \overset{\psi_\beta(0)}{\longrightarrow} \overset{\psi_\beta(\hat{0})}{\longrightarrow} 0 \overset{\psi_\beta(0)}{\longrightarrow} \overset{\psi_\beta(\hat{0})}{\longrightarrow} t_0 \overset{\psi_\beta(0)}{\longrightarrow} \overset{\psi_\beta(\hat{0})}{\longrightarrow} \overset{\psi_\beta(0)}{\longrightarrow} t_0 \overset{\psi_\beta(0)}{\longrightarrow} \overset{\psi_\beta(\hat{0})}{\longrightarrow} t_0 \overset{\psi_\beta(0)}{\longrightarrow} \overset{\psi_\beta(\hat{0})}{\longrightarrow} \overset{\psi_\beta(0)}{\longrightarrow} t_0 \overset{\psi_\beta(0)}{\longrightarrow} \overset{\psi_\beta(\hat{0})}{\longrightarrow} \overset{\psi_\beta(0)}{\longrightarrow} \cdots ,
\]

where \( \hat{0} \) marks the central letter \( u_0 \). We have \( w_\beta = 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 \) and

\[
\psi_\beta : \quad 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 \mapsto 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 ,
\]

\[
0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_0 t_1 \hat{t}_1 0 \mapsto 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 ,
\]

\[
0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_0 t_1 \hat{t}_1 0 \mapsto 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 0 .
\]

Note that \( 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_0 t_1 \hat{t}_1 \) and \( 0 \hat{0} t_0 \hat{t}_0 t_0 \hat{t}_0 t_1 \hat{t}_1 \) differ only by a letter in \( V'_\beta \), and correspond therefore to intervals of the same length. Since the letters \( t_0 \) and \( t_1 \) are never mapped to \( 0 \), we identify these two return words. This means that \( R_\beta = \{ A, B \} \) with \( A = 0 \hat{0} t_0 \hat{t}_0 t_1 \hat{t}_1 \), \( B = 0 \hat{0} t_0 \hat{t}_0 \{ t_0, t_1 \} \hat{t}_0 t_1 \hat{t}_1 \), and

\[
\cdots u'_2 u'_1 u'_1 u'_2 \cdots = \cdots ABBABBABBAB ABBABABBAB ABBABABBAB \cdots
\]

is a two-sided fixed point of the anti-morphism
\[
\varphi_{-\beta} : \ A \mapsto AB, \quad B \mapsto ABB.
\]
We have $L(A) = 1$, $L(B) = \beta - 1 > 1$, and some $(-\beta)$-integers are shown in Figure 3.

We remark that it is sufficient to consider the elements of $V_\beta$ when one is only interested in $\mathbb{Z}_{-\beta}$. This is made precise in the following proposition.

**Proposition 2.** Let $\beta$ and $\cdots u_{-1}u_0u_1 \cdots$ be as in Theorem 2. $t = \max\{x \in V_\beta \mid x < 0\}.$

For any $k \in \mathbb{Z}$, $u_{2k} = 0$ is equivalent to $u_{2k-1} = \hat{t}$ or $u_{2k+1} = 0$.

If $0 \not\in V_\beta$ or the size of $V_\beta$ is even, then $u_{2k} = 0$ is equivalent to $u_{2k-1} = \hat{t}$.

If $0 \not\in V_\beta$ or the size of $V_\beta$ is odd, then $u_{2k} = 0$ is equivalent to $u_{2k+1} = \hat{t}$.

**Proof.** Let $k \in \mathbb{Z}$ and $m \geq 0$ such that $z_k/\beta^{2m} \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. Then we have

- $u_{2k} = 0$ if and only if $T_{-\beta}^{2m}(z_k/\beta^{2m}) = 0$,
- $u_{2k-1} = \hat{t}$ if and only if $\lim_{y \to z_k} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$,
- $u_{2k+1} = \hat{0}$ if and only if $\lim_{y \to z_k} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$.

Thus $u_{2k} = 0$, $u_{2k-1} = \hat{t}$ and $u_{2k+1} = \hat{0}$ are equivalent when $T_{-\beta}^{2m}$ is continuous at $z_k/\beta^{2m}$. Assume from now on that $z_k/\beta^{2m}$ is a discontinuity point of $T_{-\beta}^{2m}$. Then $T_{-\beta}^\ell(z_k/\beta^{2m}) = \frac{-\beta}{\beta+1}$ for some $1 \leq \ell \leq 2m$ and, if $\ell$ is minimal with this property,

$$\lim_{y \to z_k} T_{-\beta}^{2\ell/2}(y/\beta^{2m}) = \frac{-\beta}{\beta+1} \quad \text{and} \quad \lim_{y \to z_k} T_{-\beta}^{2\ell/2}(y/\beta^{2m}) = \frac{-\beta}{\beta+1}.$$

Hence, if $0 \not\in V_\beta$, we cannot have $u_{2k} = 0$, $u_{2k-1} = \hat{t}$ or $u_{2k+1} = \hat{0}$ at a discontinuity point, which proves the proposition in this case. If $0 \in V_\beta$, then $T_{-\beta}^{N V_\beta-1}(\frac{-\beta}{\beta+1}) = 0$, thus
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\[ T^i_\beta(z_k/\beta^{2m}) = 0 \text{ if and only if } j \geq \ell + \#V_\beta - 1, \]
\[ \lim_{y \to z_k} T^i_\beta(y/\beta^{2m}) = 0 \text{ if and only if } j \geq 2[\ell/2] + \#V_\beta, \]
\[ \lim_{y \to z_k} T^i_\beta(y/\beta^{2m}) = 0 \text{ if and only if } j \geq 2[\ell/2] + \#V_\beta - 1. \]

Since $2[\ell/2] \geq \ell - 1$ and $2[\ell/2] \geq \ell$, we obtain $u_{2k} = 0$ whenever $u_{2k-1} = \hat{t}$ or $u_{2k+1} = \hat{\theta}$. If $\#V_\beta$ is even, then $u_{2k} = 0$ implies that $u_{2k-1} = \hat{t}$ since $2m \geq \ell + \#V_\beta - 1$ implies that $2m \geq 2[\ell/2] + \#V_\beta$. If $\#V_\beta$ is odd, then $u_{2k} = 0$ implies that $u_{2k+1} = \hat{\theta}$ since $2m \geq \ell + \#V_\beta - 1$ implies that $2m \geq 2[\ell/2] + \#V_\beta - 1$. This proves the proposition. \( QED \)

By Proposition 2 it suffices to consider the anti-morphism $\hat{\phi}_\beta : \hat{V}_\beta^* \to \hat{V}_\beta^*$ defined by

\[ \hat{\phi}_\beta(x) = \hat{x}_m \cdots \hat{x}_0 \text{ when } \psi_\beta(x) = \hat{x}_m T_\beta(y_m) \cdots \hat{x}_1 T_\beta(y_1) \hat{x}_0 \quad (x \in V_\beta^*). \]

Then $\Delta_\beta$ is given by the set $\hat{R}_\beta$ which consists of the return words of $\hat{\theta}$ when $0 \notin V_\beta$ or the size of $V_\beta$ is odd. When $0 \in V_\beta$ and the size of $V_\beta$ is even, as in Example 1, the return word of $\hat{\theta}$ consists of the words $\hat{t} \hat{w}$ such that $\hat{t} \hat{w}$ is a return word of $\hat{t}$.

Example 3. Let $\beta > 1$ with $\beta^3 = 2\beta^2 + 1$, i.e., $\beta \approx 2.206$, and let $t_n = T_\beta^n(\frac{-\beta}{\beta+1})$ for $n \geq 0$. Then we have

\[ t_0 = \frac{-\beta}{\beta+1}, \quad t_1 = \frac{\beta^2 - 1}{\beta+1} - 2 = \frac{\beta^2 - 1}{\beta+1}, \quad t_2 = \frac{\beta^2 - 1}{\beta+1} - 1 = \frac{\beta^2}{\beta+1}, \quad t_3 = \frac{-\beta}{\beta+1}, \quad t_4 = \frac{1}{\beta+1} - 1 = t_0, \]

see Figure 4. The anti-morphism $\hat{\phi}_\beta : \hat{V}_\beta^* \to \hat{V}_\beta^*$ is therefore defined by

\[ \hat{\phi}_\beta : \quad \hat{t}_0 \mapsto \hat{t}_2 \hat{t}_0, \quad \hat{t}_1 \mapsto \hat{t}_0 \hat{t}_1 \hat{t}_3 \hat{\theta}, \quad \hat{t}_2 \mapsto \hat{t}_3, \quad \hat{\theta} \mapsto \hat{t}_3, \quad \hat{t}_2 \mapsto \hat{t}_0 \hat{t}_1 \hat{t}_3. \]

Since $0 \notin V_\beta$, we consider return words of $\hat{\theta}$ in the $\hat{\psi}_\beta$-images of $\hat{w}_\beta = \hat{\theta}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3$:

\[ \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3 \mapsto \hat{\theta}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3, \quad \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3 \mapsto \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3, \quad \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3 \mapsto \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3, \quad \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3 \mapsto \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3, \quad \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3 \mapsto \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3, \quad \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3 \mapsto \hat{t}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3. \]

Hence $\hat{R}_\beta = \{A, B, C, D, E\}$ with $A = \hat{\theta}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3$, $B = \hat{\theta}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3$, $C = \hat{\theta}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3$, $D = \hat{\theta}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3$, $E = \hat{\theta}_2 \hat{t}_0 \hat{t}_1 \hat{t}_3$, and $\mathbb{Z}_{\beta}$ is described by the anti-morphism $\hat{\varphi}_{\beta} : \hat{R}_\beta \to \hat{R}_\beta$ given by

\[ \hat{\varphi}_{\beta} : \quad A \mapsto AB, \quad B \mapsto AC, \quad C \mapsto AD, \quad D \mapsto AED, \quad E \mapsto ABD. \]

The $(-\beta)$-integers in $[-\beta^3, \beta^4]$ are represented in Figure 4 and we have

\[ L(A) = 1, \quad L(B) = \beta - 1, \quad L(C) = \beta^2 - \beta - 1, \quad L(D) = \beta^2 - \beta \approx 2.659, \quad L(E) = \beta. \]

Note that $L(D) > \beta > 2$. Moreover, the cardinality of $\Delta_{\beta}$ is larger than that of $V_\beta$, which in turn is larger than the algebraic degree $d$ of $\beta$ ($\#\Delta_{\beta} = 5$, $\#V_\beta = 4$, $d = 3$).
In order to deal with shorter words, we group the letters forming the words.

\[ \hat{\psi}_\beta : \hat{V}_\beta \to \hat{V}_\beta \] is therefore given by

\[
\begin{align*}
\hat{\psi}_\beta : & \quad \hat{t}_0 \mapsto \hat{t}_3 \hat{t}_5, & \hat{t}_2 \mapsto \hat{t}_4 \hat{t}_0 \hat{t}_2, & \hat{t}_3 \mapsto \hat{t}_5 \hat{t}_1 \hat{t}_4 \hat{t}_0 \hat{t}_2 \hat{t}_3 \hat{t}_5 \hat{t}_1 \hat{t}_0, \\
& \quad \hat{t}_5 \mapsto \hat{t}_2 \hat{t}_3, & \hat{t}_1 \mapsto \hat{t}_4 \hat{t}_0, & \hat{t}_0 \mapsto \hat{t}_5 \hat{t}_1, & \hat{t}_4 \mapsto \hat{t}_0 \hat{t}_2 \hat{t}_3.
\end{align*}
\]

In order to deal with shorter words, we group the letters forming the words

\[
a = \hat{0} \hat{t}_4, \quad b = \hat{t}_0 \hat{t}_2 \hat{t}_3 \hat{t}_5 \hat{t}_1, \quad c = \hat{t}_0 \hat{t}_2 \hat{t}_3 \hat{t}_5 \hat{t}_1, \quad d = \hat{t}_2 \hat{t}_3 \hat{t}_5 \hat{t}_1, \quad e = \hat{t}_0 \hat{t}_2, \quad f = \hat{t}_4, \quad g = \hat{t}_0 \hat{t}_2 \hat{t}_3, \quad h = \hat{t}_5 \hat{t}_1,
\]

which correspond to the intervals \(J_a = (0, \frac{1}{\beta+1})\), \(J_b = (t_0, 0)\), \(J_c = (t_0, t_1)\), \(J_d = (t_2, 0)\), \(J_e = (t_0, t_3)\), \(J_f = (t_4, \frac{1}{\beta+1})\), \(J_g = (t_0, t_5)\), \(J_h = (t_5, 0)\), occurring in iterated images of \(J_a\).

The anti-morphism \(\hat{\psi}_\beta\) acts on these words by

\[
\hat{\psi}_\beta : \quad a \mapsto b, \quad b \mapsto ababac, \quad c \mapsto dabac, \quad d \mapsto ababac, \\
e \mapsto fc, \quad f \mapsto g, \quad g \mapsto habac, \quad h \mapsto ag.
\]
Since \( \hat{0} \) only occurs at the beginning of \( a \), the return words of \( \hat{0} \) with their \( \hat{\psi}_\beta \)-images are

\[
\begin{align*}
ab &\mapsto ab ab acb, \
acd &\mapsto ab ab aed ab acb,
\end{align*}
\]

\[
\begin{align*}
acb &\mapsto ab ab acd ab acb, \
aefcb &\mapsto ab ab acd ab acgfcb, \\
acd &\mapsto ab ab aed ab acb, \
acgfcb &\mapsto ab ab acd ab acgh.
\end{align*}
\]

\( \Rightarrow \) \( = \) \( acb \)

Therefore, \( \mathbb{Z}_{-\beta} \) is described by the anti-morphism \( \hat{\varphi}_{-\beta} : \hat{R}_\beta^* \rightarrow \hat{R}_\beta^* \) which is defined by

\[
\begin{align*}
\hat{\varphi}_{-\beta} : & \quad A \mapsto AAB, \quad L(A) = 1, \\
& \quad B \mapsto AACAB, \quad L(B) = \beta - 2 \approx 1.695, \\
& \quad C \mapsto AADAB, \quad L(C) = \beta^2 - 3\beta - 1 \approx 1.569, \\
& \quad D \mapsto AAE, \quad L(D) = \beta^3 - 3\beta^2 - 2\beta - 1 \approx 1.104, \\
& \quad E \mapsto AACAF, \quad L(E) = \beta^4 - 3\beta^3 - 2\beta^2 - \beta - 2 \approx 2.081, \\
& \quad F \mapsto AACABACAB, \quad L(F) = \beta^5 - 3\beta^4 - 2\beta^3 - 2\beta^2 + \beta - 2 \approx 3.12.
\end{align*}
\]

\[ \begin{array}{cccccccccccc}
-\beta & -\beta + 1 & 0 & 1 & 2 & \beta^2 - 3\beta + 1 & \beta^2 - 2\beta & \beta^2 - 2\beta + 1 & \beta^2 - \beta + 1 & \beta^2 - \beta + 2 & \beta^2 \\
J_{\tilde{t}_0} & J_{\tilde{t}_2} & J_{\tilde{t}_4} & J_{\tilde{t}_5} & J_{\tilde{t}_0} & J_{\tilde{t}_2} & J_{\tilde{t}_4} & J_{\tilde{t}_5}
\end{array} \]

\textbf{Figure 5.} The \((-\beta)\)-transformation and \( \mathbb{Z}_{-\beta} \cap [-\beta, \beta^2] \) from Example [4]
4. Conclusions

With every Yrrap number $\beta \geq (1 + \sqrt{5})/2$, we have associated an anti-morphism $\varphi_{-\beta}$ on a finite alphabet. The distances between consecutive $(-\beta)$-integers are described by a fixed point of $\varphi_{-\beta}$. In [1], the anti-morphism is described explicitly for each $\beta > 1$ such that $T_n^\beta\left(\frac{-\beta}{\beta+1}\right) \leq 0$ and $T_{n-1}^\beta\left(\frac{-\beta}{\beta+1}\right) \geq \frac{1-\lfloor\beta\rfloor}{\beta}$ for all $n \geq 1$. Examples 3 and 4 show that the situation can be quite complicated when this condition is not fulfilled. Although $\varphi_{-\beta}$ can be obtained by a simple algorithm, it seems to be difficult to find a priori bounds for the number of different distances between consecutive $(-\beta)$-integers or for their maximal value.

Only the case of quadratic Pisot numbers $\beta$ is completely solved; here, we know from [14, 1] that $\# V_\beta = \# \Delta_{-\beta} = 2$.

Recall that the maximal distance between consecutive $\beta$-integers is 1, and the number of different distances is equal to the cardinality of the set $\{T_n^\beta(1^-) \mid n \geq 0\}$. Example 3 shows that the $(-\beta)$-integers do not satisfy similar properties. By generalising Example 4 to $\beta > 1$ with $\beta^6 = (m+1)\beta^5 + m\beta^4 + \beta^3 + \beta^2 - m\beta - 1$, $m \geq 2$, one sees that the maximal distance can be arbitrarily close to 4 for algebraic integers of degree 6 and $\# V_\beta = 6$.

In a forthcoming paper, we associate anti-morphisms $\varphi_{-\beta}$ on infinite alphabets with non-Yrrap numbers $\beta$, by considering the intervals occurring in the iterated $T_{-\beta}$-images of $((0, \frac{1}{\beta+1}))$, cf. Example 4, and we show that the distances between consecutive $(-\beta)$-integers can be unbounded, e.g. for $\beta > 1$ satisfying $\frac{-\beta}{\beta+1} = \sum_{k=1}^{\infty} a_k (-\beta)^{-k}$ where $a_1 a_2 \cdots = 312321231322 \cdots$ is a fixed point of the morphism $3 \mapsto 31232$, $2 \mapsto 2$, $1 \mapsto 1$. For Yrrap numbers $\beta$, this implies that there is no bound for the distance between consecutive $(-\beta)$-integers which is independent of $\beta$. However, large distances occur probably only far away from 0 and when $\# V_\beta$ is large, and it would be interesting to quantify these relations.

Another topic that is worth investigating is the structure of the sets $S_{-\beta}(x)$ for $x \neq 0$, and of the corresponding tilings when $\beta$ is a Pisot unit. A related question is whether $\mathbb{Z}_{-\beta}$ can be given by a cut and project scheme, cf. [5, 12].

References

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