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SPECTRAL STATISTICS FOR RANDOM SCHRÖDINGER OPERATORS IN THE LOCALIZED REGIME

FRANÇOIS GERMINET AND FRÉDÉRIC KLOPP

ABSTRACT. We study various statistics related to the eigenvalues and eigenfunctions of random Hamiltonians in the localized regime. Consider a random Hamiltonian at an energy E in the localized phase. Assume the density of states function is not too flat near E . Restrict it to some large cube Λ . Consider now I_Λ , a small energy interval centered at E that asymptotically contains infinitely many eigenvalues when the volume of the cube Λ grows to infinity. We prove that, with probability one in the large volume limit, the eigenvalues of the random Hamiltonian restricted to the cube inside the interval are given by independent identically distributed random variables, up to an error of size an arbitrary power of the volume of the cube.

As a consequence, we derive

- uniform Poisson behavior of the locally unfolded eigenvalues,
- a.s. Poisson behavior of the joint distributions of the unfolded energies and unfolded localization centers in a large range of scales.
- the distribution of the unfolded level spacings, locally and globally,
- the distribution of the unfolded localization centers, locally and globally.

RÉSUMÉ. Nous étudions différentes statistiques associées aux valeurs propres et vecteurs propres d'un opérateur aléatoire dans le régime localisé. Considérons un opérateur aléatoire au voisinage d'une énergie E supposée se trouver dans le régime localisé. On considère la restriction de l'opérateur à un grand cube Λ . Soit I_Λ , un petit intervalle d'énergie centré en E qui, asymptotiquement, contient une infinité de valeurs propres quand Λ grandit. Nous démontrons qu'asymptotiquement presque sûrement, les valeurs propres de l'opérateur aléatoire restreint au cube contenu dans l'intervalle sont très bien approchées par des variables aléatoires indépendantes identiquement distribuées.

De cette caractérisation, nous déduisons

- que localement, les valeurs propres renormalisées comportent uniformément poissonien,
- que les distributions jointes des valeurs propres et centres de localisation renormalisés sont asymptotiquement distribués selon une loi de Poisson,
- la distribution des espacements de niveaux renormalisés, localement et globalement,
- la distribution des espacements des centres de localisation renormalisés, localement et globalement.

0. INTRODUCTION

On $\ell^2(\mathbb{Z}^d)$, consider the random Anderson model

$$H_\omega = -\Delta + \lambda V_\omega$$

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where $-\Delta$ is the free discrete Laplace operator

$$(-\Delta u)_n = \sum_{|m-n|=1} u_m \quad \text{for } u = (u_n)_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$$

and V_ω is the random potential

$$(V_\omega u)_n = \omega_n u_n \quad \text{for } u = (u_n)_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d).$$

We assume that the random variables $(\omega_n)_{n \in \mathbb{Z}^d}$ are independent identically distributed and that their distribution admits a compactly supported smooth density, say g .

It is well known that, ω almost surely, the spectrum of H_ω is equal to a fixed closed set, say, Σ . Moreover, there exists a Lebesgue almost everywhere bounded density of states (see [Kir08]), say $\lambda \mapsto \nu(\lambda)$, such that, for any continuous function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, one has

$$\int_{\mathbb{R}} \varphi(\lambda) \nu(\lambda) d\lambda = \mathbb{E}(\langle \delta_0, \varphi(H_\omega) \delta_0 \rangle).$$

The function ν is the density of a probability measure on Σ .

For $L > 1$, consider $\Lambda = [-L, L]^d \cap \mathbb{Z}^d$, a cube on the lattice and let $H_\omega(\Lambda)$ be the random Hamiltonian H_ω restricted to Λ with periodic boundary conditions. It is a finite dimensional symmetric matrix; let us denote its eigenvalues ordered increasingly and repeated according to multiplicity by $E_1(\omega, \Lambda) \leq E_2(\omega, \Lambda) \leq \dots \leq E_N(\omega, \Lambda)$. For $x \geq 0$, define the empirical level spacings distribution of $H_\omega(\Lambda)$ as

$$DLS(x; \omega, \Lambda) = \frac{\#\{j; (E_{j+1}(\omega, \Lambda) - E_j(\omega, \Lambda))|\Lambda| \geq x\}}{|\Lambda|}.$$

A result that is typical of the results we prove in the present paper is

Theorem 0.1. *There exists $\lambda_0 > 0$ such that, for $|\lambda| > \lambda_0$, with probability 1, as $|\Lambda| \rightarrow +\infty$, $DLS(x; \omega, \Lambda)$ converges uniformly to the distribution $x \mapsto g(x)$ where*

$$g(x) = \int_{\Sigma} e^{-\nu(\lambda)x} \nu(\lambda) d\lambda \tag{0.1}$$

that is, ω almost surely,

$$\sup_{x \geq 0} |DLS(x; \omega, \Lambda) - g(x)| \xrightarrow{|\Lambda| \rightarrow +\infty} 0.$$

This result shows that, for the discrete Anderson Hamiltonian with smoothly distributed random potential at sufficiently large coupling, the limit of the level spacings distribution is that of i.i.d. random variables distributed according to the density of states of the random Hamiltonian.

To the best of our knowledge, this is the first rigorous study of the level spacings distribution of random Schrödinger operators.

The purpose of the paper is to study spectral statistics for random Hamiltonians in the localized regime. The large coupling Anderson Hamiltonian described above is the typical example. Spectral statistics have been studied in various works, mainly for discrete or continuous Anderson models (see e.g. [Mol82a, Min96, KN07, Wan01, CGK10]) but up to now, to the best of our knowledge, studies have only described the local spectral statistics. For a random Hamiltonian restricted to a cube Λ , the existence of the density of states, that is, of a limit for the number of eigenlevels per unit of volume (see (1.3)) implies that the average distance between levels is of order $|\Lambda|^{-1}$. Local spectral statistics are the statistics in energy intervals I of the size $|\Lambda|^{-1}$. Thus, such intervals contain typically a number of eigenvalues that is bounded uniformly in the volume of the cube. This, in particular, prevents the study of the empirical distribution of level spacings.

In the present paper, we go beyond this. Therefore, we introduce a new method of study of the

eigenlevels and localization centers that is quite close to the physical heuristics (see e.g. [Jan98, LR85, Mir00]). The method consists in approximating the eigenvalues of the true random operator (restricted to some finite cube) by a family of i.i.d. random variables that are constructed as eigenvalues of the random operator restricted to smaller cubes. That this is possible is a consequence of localization: due to their exponential falloff, eigenfunctions only see the random potential surrounding them. This construction is only feasible under some restrictions on the relative size of the region where one wants to study eigenvalues and the size of the cube on which one restricts the random operator. If one wants to control, with a good probability, all the eigenvalues in some interval I , then, one roughly needs I to be of size $|\Lambda|^{-\alpha}$, the inverse of the volume of the cube to some power α smaller than but close to 1 (see Theorem 1.15). If one wants to enlarge the interval I , one can go up to sizes that are of order a negative power of $\log |\Lambda|$ at the expense of being able to describe only most of the eigenvalues (see Theorem 1.16).

The basic tools that we use to control the eigenvalues are the so-called ‘‘Wegner’’ and ‘‘Minami’’ estimates (see (W) and (M)). In Theorem 1.1, we show an improved version of Wegner’s estimate where the control on the microscopic eigenvalues is obtained using the integrated density of states. For some systems, one can show an analogously improved Minami estimate (see [GKI]).

Using the approximation described above, we obtain uniform local statistics for the eigenvalues and localization centers when they are rescaled covariantly i.e. the scaling in energy is of order the scaling in space to the power $-d$ (see Theorem 1.3); we prove that the covariantly scaled local statistics are independent of the scale (if they are not too small) i.e. one always obtains Poisson statistics. In the non-covariant scaling case, we obtain almost sure results on the counting function (see Theorem 1.8). In the case of the standard scale i.e. energies are scaled by $|\Lambda|^{-1}$ on a cube of volume Λ , we also study the asymptotic independence of these local processes (see Theorems 1.4 and 1.5).

The next result is a large deviation estimate for the number of eigenvalues inside a possibly shrinking interval of a random operator restricted to some large cube (see Theorem 1.9). This bound shows that for intervals I that are not too small, the number of eigenvalues in I is given by the weight that the integrated density of states gives to I times the volume of the cube up to an error of smaller order.

Then, we derive the almost sure level spacings statistics near fixed energies as well as inside non trivial compact intervals (see Theorems 1.10, 1.12 and 1.13). We also compute the localization center spacings statistics (see Theorem 1.13).

We point out that our analysis goes beyond the previous results also in the sense that locally the images of the eigenvalues by the IDS are shown to exhibit a Poissonian behavior. When the DOS exists and is non zero, we recover the previous known statements, but a vanishing derivative of the IDS is allowed in this work.

We also consider the problem from a different point of view. The usual procedure consists in restricting the random Hamiltonian to some finite cube and study the statistics for this operator in the limit when the cube grows to be the whole space. One can also consider the Hamiltonian in the whole space. In a compact interval in the localized region, say I , the Hamiltonian admits countably many eigenvalues. We enumerate them using the localization center attached to an associated eigenfunction (see Proposition 1.1). I.e. we consider the eigenvalues in I having localization in some finite cube. We derive the almost sure statistics of the level spacings distributions (see Theorem 1.14); they are the same as the ones obtained for the Hamiltonian restricted to a cube.

1. THE MAIN RESULTS

We now turn to the main results of the present paper. They will be given for general random Schrödinger operators under a number of assumptions that are known to hold for, e.g. the Anderson

model and some continuous Anderson models.

We shall use the following standard notations: $a \lesssim b$ means there exists $c < \infty$ so that $a \leq cb$; $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. We write $a \asymp b$ when $a \lesssim b$ and $b \lesssim a$.

1.1. The random model. Consider $H_\omega = H_0 + V_\omega$, a \mathbb{Z}^d -ergodic random Schrödinger operator on $\mathcal{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$ (see e.g. [PF92, Sto01]). Typically, the background potential H_0 is the Laplacian $-\Delta$, possibly perturbed by a periodic potential. Magnetic fields can be considered as well; in particular, the Landau Hamiltonian is also admissible as a background Hamiltonian. For the sake of simplicity, we assume that V_ω is almost surely bounded; hence, almost surely, H_ω share the same domain $H^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$.

For Λ , a cube in either \mathbb{R}^d or \mathbb{Z}^d , we let $H_\omega(\Lambda)$ be the operator H_ω restricted to Λ with periodic boundary conditions. Our analysis stays valid for Dirichlet boundary conditions.

Furthermore, we shall denote by $\mathbf{1}_J(H)$ the spectral projector of the operator H on the energy interval J . $\mathbb{E}(\cdot)$ denotes the expectation with respect to ω ; Our first assumption will be an independence assumption on the local Hamiltonian that is

(IAD): Independence At a Distance: there exists $R_0 > 0$ such that for $\text{dist}(\Lambda, \Lambda') > R_0$, the random Hamiltonians $H_\omega(\Lambda)$ and $H_\omega(\Lambda')$ are independent.

Such an assumption is clearly satisfied by standard models like the Anderson model, the Poisson model or the random displacement model if the single site potential is compactly supported (see e.g. [PF92, Sto01]).

Remark 1.1. As will be seen in the course of the proofs, it can be weakened to assuming that the correlations between the local Hamiltonians decay polynomially at a rate that is faster than the $-d$ th power of the distance separating the cubes on which the local Hamiltonians are considered.

Let Σ be the almost sure spectrum of H_ω . Pick I a relatively compact open subset of Σ . Assume the following holds:

(W): a Wegner estimate holds in I , i.e. given I there exists $C > 0$ such that, for $J \subset I$, and Λ , a cube in \mathbb{R}^d or \mathbb{Z}^d , one has

$$\mathbb{E}[\text{tr}(\mathbf{1}_J(H_\omega(\Lambda)))] \leq C|J||\Lambda|. \quad (1.1)$$

(M): a Minami estimate holds in I , i.e. given I there exists $C > 0$ and $\rho > 0$ such that, for $J \subset I$, and Λ , a cube in \mathbb{R}^d or \mathbb{Z}^d , one has

$$\mathbb{E}[\text{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \cdot [\text{tr}(\mathbf{1}_J(H_\omega(\Lambda))) - 1]] \leq C(|J||\Lambda|)^{1+\rho}. \quad (1.2)$$

Remark 1.2. The Wegner estimate has been proved for many random Schrödinger models both discrete and continuous Anderson models under rather general conditions on the single site potential and on the randomness (see e.g. [His08, KM07, Ves08]). The right hand side in (1.1) can be lower bounded by the probability to have at least one eigenvalue in J (for J small).

As the proofs will show, one can weaken assumption (W) and replace the right hand side with $C|J|^\alpha |\Lambda|^\beta$ for arbitrary positive α and β . Such Wegner estimates are known to hold also for some non monotonous models (see e.g. [Kl95, HK02, GHK07]).

On the Minami estimate, much less is known: it holds for the discrete Anderson model with $I = \Sigma$ (see [Min96, GV07, BHS07, CGK09]). For the continuous Anderson model, in [CGK10], it is shown to hold at the bottom of the spectrum under more restrictive conditions on the single site potential than needed to prove the Wegner estimate. These proofs yield an optimal exponent $\rho = 1$. The right hand side in (1.2) can be lower bounded by the probability to have at least two eigenvalues in J . For $\rho = 1$, it behaves as the square of the probability to have one eigenvalue in J . So, close by eigenvalues behave as independent random variables.

The integrated density of states is defined as

$$N(E) := \lim_{|\Lambda| \rightarrow +\infty} \frac{\#\{\text{e.v. of } H_\omega(\Lambda) \text{ less than } E\}}{|\Lambda|} \quad (1.3)$$

By (W), $N(E)$ is the distribution function of a measure that is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} . Let ν be the density of state of H_ω i.e. the distributional derivative of N . In the sequel, for a set I , we will often write $N(I)$ for the mass the measure $\nu(E)dE$ puts on I i.e.

$$N(I) = \int_I \nu(E)dE. \quad (1.4)$$

1.2. The localized regime. For $L \geq 1$, Λ_L denotes the cube $[-L/2, L/2]^d$ in either \mathbb{R}^d or \mathbb{Z}^d . Let \mathcal{H}_Λ be $\ell^2(\Lambda \cap \mathbb{Z}^d)$ in the discrete case and $L^2(\Lambda)$ in the continuous one. For a vector $\varphi \in \mathcal{H}$, we define

$$\|\varphi\|_x = \begin{cases} \|\mathbf{1}_{\Lambda(x)}\varphi\|_2 & \text{where } \Lambda(x) = \{y; |y-x| \leq 1/2\} \\ |\varphi(x)| & \end{cases} \quad \begin{cases} \text{if } \mathcal{H} = L^2(\mathbb{R}^d), \\ \text{if } \mathcal{H} = \ell^2(\mathbb{Z}^d). \end{cases} \quad (1.5)$$

In the discrete case, the definition is that given in section 1.5.

Let I be a compact interval. We assume that I lies in the region of complete localization (see e.g. [GK04, GK06]) for which we use the following finite volume version:

(Loc): there exists $\xi \in]0, 1[$ such that

$$\sup_{L > 0} \sup_{\substack{\text{supp } f \subset I \\ |f| \leq 1}} \mathbb{E} \left(\sum_{\gamma \in \mathbb{Z}^d} e^{|\gamma|^\xi} \|\mathbf{1}_{\Lambda(0)} f(H_\omega(\Lambda_L)) \mathbf{1}_{\Lambda(\gamma)}\|_2 \right) < +\infty. \quad (1.6)$$

Whenever the fractional moment method is available, one may replace the factor $e^{|\gamma|^\xi}$ by an exponential one $e^{\eta|\gamma|}$, where $\eta > 0$. We note that the assumption (Loc) implies in particular that the spectrum of H_ω is pure point in I (see e.g. [GK06, Kir08]). We refer to Appendix 6.2 where we provide equivalent finite volume properties of the region of complete localization, and show it coincides with the infinite volume one. For the sake of the exposition, we extract from Theorem 6.1 the following lemma that we shall use intensively in this paper.

Lemma 1.1. *Under assumption (W) and (Loc), for any $p > 0$ and $\xi \in]0, 1[$, there exists $q > 0$ such that, for $L \geq 1$ large enough, with probability larger than $1 - L^{-p}$, if*

- (1) $\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda_L)$ associated to $E_{n,\omega} \in I$,
- (2) $x_n(\omega) \in \Lambda_L$ is a maximum of $x \mapsto \|\varphi_{n,\omega}\|_x$ in Λ_L ,

then, for $x \in \Lambda_L$, one has

$$\|\varphi_{n,\omega}\|_x \leq L^q e^{-|x-x_n(\omega)|^\xi}. \quad (1.7)$$

Such a result can essentially be found in [GK06] for the continuous case and in [K11] for the discrete case.

Clearly, the function $x \mapsto \|\varphi_{n,\omega}\|_x$ need not have a unique maximum in Λ_L . But, as, for any $x \in \Lambda_L$, one has

$$\sum_{\gamma \in \Lambda_L \cap \mathbb{Z}^d} \|\varphi_{n,\omega}\|_{x+\gamma}^2 = \|\varphi_{n,\omega}\|^2 = 1,$$

if $x_n(\omega)$ is a maximum, then $\|\varphi_{n,\omega}\|_{x_n(\omega)}^2 \geq (2L+1)^{-d}$. Hence, if $x_n(\omega)$ and $x'_n(\omega)$ are two maxima, then (Loc) implies that, for any p , there exists $C_p > 0$ such that, with a probability larger than $1 - L^{-p}$, we have

$$|x_n(\omega) - x'_n(\omega)| \leq C_p (\log L)^{1/\xi}.$$

For $\varphi \in \mathcal{H}_\Lambda$, define the set of localization centers for φ as

$$C(\varphi) = \{x \in \Lambda; \|\varphi\|_x = \max_{\gamma \in \Lambda} \|\varphi\|_\gamma\}. \quad (1.8)$$

As a consequence of Lemma 1.1, one has

Lemma 1.2. *Pick I in the localized regime for H_ω . For any $p > 0$, there exists $C_p > 0$, such that, with probability larger than $1 - L^{-p}$, if $E_j(\omega, \Lambda) \in I$ then the diameter of $C(\varphi_j(\omega, \Lambda))$ is less than $C_p \log^{1/\xi} |\Lambda|$.*

From now on, a localization center for a function φ will denote any point in the set of localization centers $C(\varphi)$ and let $x_j(\omega, \Lambda)$ be a localization center for $\varphi_j(\omega, \Lambda)$. One can e.g. order them lexicographically and pick the one with largest coefficients.

1.3. An improved Wegner estimate. In the localized regime, the Wegner estimate can be vastly improved so as to reflect the local behavior of the integrated density of states. Indeed, we prove

Theorem 1.1. *Assume (W) and (Loc) holds in I . Fix $\nu \in (0, 1)$. Then, for $L > 1$ and $J \subset I$ a compact interval such that $N(J) \geq e^{-L^\nu}$. Then,*

$$\mathbb{E}[\text{tr}(\mathbf{1}_J(H_\omega(\Lambda)))] = N(J)|\Lambda| (1 + O(|\Lambda|^{-1})). \quad (1.9)$$

The main gain over (W) is that (1.9) is an asymptotic; the loss is that it holds only for not too small intervals. The proof given in section 2.1 shows that, for larger intervals, the remainder term can be improved.

1.4. The local level statistics. For $L \in \mathbb{N}$, recall that $\Lambda = \Lambda_L$ and that $H_\omega(\Lambda)$ is the operator H_ω restricted to Λ with periodic boundary conditions. The notation $|\Lambda| \rightarrow +\infty$ is a shorthand for considering $\Lambda = \Lambda_L$ in the limit $L \rightarrow +\infty$.

Denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity by $E_1(\omega, \Lambda) \leq E_2(\omega, \Lambda) \leq \dots \leq E_N(\omega, \Lambda) \leq \dots$.

Let E_0 be an energy in I . The *unfolded local level statistics* near E_0 is the point process defined by

$$\Xi(\xi; E_0, \omega, \Lambda) = \sum_{j \geq 1} \delta_{\xi_j(E_0, \omega, \Lambda)}(\xi), \quad (1.10)$$

where

$$\xi_j(E_0, \omega, \Lambda) = |\Lambda|(N(E_j(\omega, \Lambda)) - N(E_0)). \quad (1.11)$$

The numbers $(|\Lambda|N(E_j(\omega, \Lambda)))_j$ are called the unfolded eigenvalues of $H_\omega(\Lambda)$ (see e.g. [Min07, Min08] for more details).

The unfolded local level statistics are described by

Theorem 1.2. *Assume (IAD), (W), (M) and (Loc) hold. Pick $\tilde{\rho} \in (0, \rho)$ where ρ is defined by (M). Pick E_0 be an energy in I such that the intergrated density of states satisfies*

$$\forall a > b, \exists C(a, b) > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0), \quad |N(E_0 + a\varepsilon) - N(E_0 + b\varepsilon)| \geq C(a, b)\varepsilon^{1+\tilde{\rho}}. \quad (1.12)$$

When $|\Lambda| \rightarrow +\infty$, the point process $\Xi(E_0, \omega, \Lambda)$ converges weakly to a Poisson process on \mathbb{R} with intensity the Lebesgue measure.

If one assume that N is differentiable at E_0 and that its derivative $\nu(E_0)$ is positive i.e.

$$0 < \nu(E_0) := \lim_{E \rightarrow E_0} \frac{N(E) - N(E_0)}{E - E_0}, \quad (1.13)$$

it is easy to check that (1.12) is satisfied and that, for $E_j - E_0$ small,

$$\xi_j(E_0, \omega, \Lambda) = |\Lambda| \nu(E_0) (E_j(\omega, \Lambda) - E_0) (1 + o(1)).$$

Thus, one recovers the convergence to a Poisson process when the point process (1.10) is replaced by the one defined by the points $(|\Lambda| \nu(E_0) (E_j(\omega, \Lambda) - E_0))_j$.

Theorem 1.2 under the additional assumption (1.13) was first obtained in [Mol82a] for a special one dimensional random Schrödinger model on the real line. For the discrete Anderson model, Theorem 1.14 was proved in [Min96] under the assumption (1.13). For continuous Anderson models, under the assumption (1.13), it was proved in [CGK10] following the strategy of [Min96].

Our method of proof is different from that of [Min96, CGK10] and, in spirit, closer to that of [Mol82a] and to the physical heuristics. Clearly, for (1.12) to be satisfied, we do not need N to be differentiable at E_0 nor its derivative to be positive. E.g. if N satisfies $N(E) = N(E_0) + c(E - E_0)^{1+\tilde{\rho}}(1 + o(1))$ near E_0 then (1.12) is satisfied. Condition (1.12) asks that at a given scale, N behaves roughly uniformly near E_0 . Note however that, if N is not differentiable at E_0 , then the local statistics of the eigenvalues themselves will not be Poissonian anymore.

Our method yields a uniform version of Theorem 1.2 to which we now turn.

1.4.1. *Uniform Poisson convergence over small intervals.* We now assume that (1.12) holds. Define

$$\alpha_{d,\rho,\tilde{\rho}} := \frac{d\rho + 1}{d\rho + \frac{1+\rho}{1+\tilde{\rho}}} < 1, \quad (1.14)$$

where ρ is determined in the Minami estimate (M).

Fix $\alpha \in (\alpha_{d,\rho,\tilde{\rho}}, 1)$. The uniform version of the Poisson process is a version that holds uniformly over an interval of energy, say, I centered at E_0 such that $N(I) \asymp |\Lambda|^{-\alpha}$. Such an interval is much larger than an interval $N(I) \asymp |\Lambda|^{-1}$. This is main improvement of Theorem 1.3 below over Theorem 1.2 or the statements found in [KN07, Min96, Mol82b]. It is natural to wonder what is the largest size of interval in which a result like Theorem 1.3. We do not know the answer to that question.

Let $I_\Lambda(E_0, \alpha)$ be the interval such that $N(I_\Lambda(E_0, \alpha))$ be centered at $N(E_0)$ of length $2|\Lambda|^{-\alpha}$. Denote by $N_\Lambda(\omega, E_0) := \text{tr } \mathbf{1}_{I_\Lambda(E_0, \alpha)}(H_\omega(\Lambda))$ the number of eigenvalues of $H_\omega(\Lambda)$ in $I_\Lambda(E_0, \alpha)$. For $1 \leq j \leq N_\Lambda(\omega, E_0)$, define the unfolded local eigenvalues $\xi_j(\omega, \Lambda)$ by (1.11). Hence, for all $1 \leq j \leq N_\Lambda(\omega, E_0)$, one has $\xi_j(\omega, \Lambda) \in |\Lambda|^{1-\alpha} \cdot [-1, 1]$.

We then prove

Theorem 1.3. *Assume (IAD), (W), (M) and (Loc) hold. Let E_0 be an energy in I such that (1.12) is satisfied. Pick $\alpha \in (\alpha_{d,\rho,\tilde{\rho}}, 1)$, and $p \geq 1$ an integer. Then, there exists $\delta > 0$ such that, for any sequences of intervals $I_1 = I_1^\Lambda, \dots, I_p = I_p^\Lambda$ in $|\Lambda|^{1-\alpha} \cdot [-1, 1]$ satisfying*

$$\inf_{j \neq k} \text{dist}(I_j, I_k) \geq e^{-|\Lambda|^\delta}, \quad (1.15)$$

we have, for any sequences of integers $k_1 = k_1^\Lambda, \dots, k_p = k_p^\Lambda \in \mathbb{N}^p$,

$$\lim_{|\Lambda| \rightarrow +\infty} \left| \mathbb{P} \left(\left\{ \begin{array}{l} \#\{j; \xi_j(\omega, \Lambda) \in I_1\} = k_1 \\ \omega; \quad \vdots \\ \#\{j; \xi_j(\omega, \Lambda) \in I_p\} = k_p \end{array} \right\} \right) - \frac{|I_1|^{k_1}}{k_1!} e^{-|I_1|} \dots \frac{|I_p|^{k_p}}{k_p!} e^{-|I_p|} \right| = 0.$$

In particular, $\Xi(\xi; E_0, \omega, \Lambda)$ defined in (1.10) converges weakly to a Poisson point process with intensity Lebesgue.

Note that, in Theorem 1.3, we do not require the limits

$$\begin{aligned} \lim_{|\Lambda| \rightarrow +\infty} \frac{|I_1|^{k_1}}{k_1!} e^{-|I_1|} &= \lim_{|\Lambda| \rightarrow +\infty} \frac{|I_1^\Lambda|^{k_1^\Lambda}}{k_1^\Lambda!} e^{-|I_1^\Lambda|}, \dots, \\ \lim_{|\Lambda| \rightarrow +\infty} \frac{|I_p|^{k_p}}{k_p!} e^{-|I_p|} &= \lim_{|\Lambda| \rightarrow +\infty} \frac{|I_p^\Lambda|^{k_p^\Lambda}}{k_p^\Lambda!} e^{-|I_p^\Lambda|} \end{aligned}$$

to exist.

1.4.2. *Asymptotic independence of the local processes.* Once Theorem 1.2 is known, it is natural to wonder how the point processes obtained at distinct energies relate to each other. To understand this, we assume

(GM): for $J \subset K \subset I$, one has

$$\mathbb{E} [\text{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \cdot \text{tr}(\mathbf{1}_K(H_\omega(\Lambda)) - 1)] \leq C|J||K||\Lambda|^2. \quad (1.16)$$

(D): for $\beta \in (0, 1)$ and $\{E_0, E'_0\} \subset I$ s.t. $E_0 \neq E'_0$, when $L \rightarrow +\infty$ and $\ell \asymp L^\beta$, one has

$$\mathbb{P} \left(\begin{array}{l} \sigma(H_\omega(\Lambda_\ell)) \cap (E_0 + L^{-d}[-1, 1]) \neq \emptyset, \\ \sigma(H_\omega(\Lambda_\ell)) \cap (E'_0 + L^{-d}[-1, 1]) \neq \emptyset \end{array} \right) = o((\ell/L)^d). \quad (1.17)$$

In their nature, assumptions (GM) and (D) are similar: they state that the probability to have two eigenvalues constrained to some intervals is much smaller than that of having a single eigenvalue in an interval. Note that $(\ell/L)^d$ is the order of magnitude of the right hand side in Wegner's estimate (W) for $H_\omega(\Lambda_\ell)$ and the interval $E_0 + L^{-d}[-1, 1]$.

Assumption (GM) was proved to hold for the discrete Anderson model in [CGK09] and for certain continuous Anderson model in [CGK]. In [K11], it is proved that assumption (D) holds for the discrete Anderson model in dimension 1 at any two distinct energies, and, in any dimension, for energies sufficiently far apart from each other.

Under these assumptions, we have

Theorem 1.4. *Assume (IAD), (W), (GM), (Loc), and (D) hold. Pick $E_0 \in I$ and $E'_0 \in I$ such that $E_0 \neq E'_0$ and (1.12) is satisfied at E_0 and E'_0 .*

When $|\Lambda| \rightarrow +\infty$, the point processes $\Xi(E_0, \omega, \Lambda)$ and $\Xi(E'_0, \omega, \Lambda)$, defined in (1.10), converge weakly respectively to two independent Poisson processes on \mathbb{R} with intensity the Lebesgue measure. That is, for $U_+ \subset \mathbb{R}$ and $U_- \subset \mathbb{R}$ compact intervals and $\{k_+, k_-\} \in \mathbb{N} \times \mathbb{N}$, one has

$$\mathbb{P} \left(\begin{array}{l} \#\{j; \xi_j(E_0, \omega, \Lambda) \in U_+\} = k_+ \\ \omega; \\ \#\{j; \xi_j(E'_0, \omega, \Lambda) \in U_-\} = k_- \end{array} \right) \xrightarrow{\Lambda \rightarrow \mathbb{Z}^d} \left(\frac{|U_+|^{k_+}}{k_+!} e^{-|U_+|} \right) \left(\frac{|U_-|^{k_-}}{k_-!} e^{-|U_-|} \right).$$

Theorem 1.4 naturally leads to wonder how far the energies E_0 and E'_0 need to be from each other with respect to the scaling used to renormalize the eigenvalues for such a result to still hold.

We prove

Theorem 1.5. *Assume (IAD), (W), (GM), (Loc). Pick $E_0 \in I$ such that (1.12) is satisfied. Assume moreover that the density of states ν is continuous at E_0 .*

Consider two sequences of energies, say $(E_\Lambda)_\Lambda$ and $(E'_\Lambda)_\Lambda$ such that

- (1) $E_\Lambda \xrightarrow{\Lambda \rightarrow \mathbb{Z}^d} E_0$ and $E'_\Lambda \xrightarrow{\Lambda \rightarrow \mathbb{Z}^d} E_0$,
- (2) $|\Lambda| \cdot |N(E_\Lambda) - N(E'_\Lambda)| \xrightarrow{\Lambda \rightarrow \mathbb{Z}^d} +\infty$.

Then, the point processes $\Xi(E_\Lambda, \omega, \Lambda)$ and $\Xi(E'_\Lambda, \omega, \Lambda)$, defined in (1.10), converge weakly respectively to two independent Poisson processes on \mathbb{R} with intensity the Lebesgue measure.

Theorem 1.5 shows that, in the localized regime, eigenvalues that are separated by a distance that is asymptotically infinite when compared to the mean spacing between the eigenlevels, behave as independent random variables. There are no interactions except at very short distances or possibly at distances of order 1 if (D) fails.

Assumption (2) can clearly not be omitted in Theorem 1.5; it suffices to consider e.g. E_Λ, E'_Λ s.t. $N(E_\Lambda) = N(E'_\Lambda) + a|\Lambda|^{-1}$ to see that the two limit random processes are obtained as a shift from one another.

1.5. The joint (energy - localization center) statistics. Recall that $E_1(\omega, \Lambda) \leq E_2(\omega, \Lambda) \leq \dots \leq E_N(\omega, \Lambda)$ denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity. Recall Lemma 1.1: it states that, to an eigenvector associated to $E_j(\omega, \Lambda)$, we can associate a center of localization that we denote by $x_j(\omega, \Lambda)$.

1.5.1. Uniform Poisson convergence for the joint (energy, center)-distribution. We now place ourselves in the same setting as in section 1.4.1. We prove

Theorem 1.6. *Assume (IAD), (W), (M) and (Loc) hold. Let E_0 be an energy in I such that (1.12) holds for some $\tilde{\rho} \in (0, \rho)$. Pick $\alpha \in (\alpha_{d, \rho, \tilde{\rho}}, 1)$, and $p \geq 1$ an integer. Then, there exists $\delta > 0$ such that,*

- for any sequences of intervals $I_1 = I_1^\Lambda, \dots, I_p = I_p^\Lambda$ in $|\Lambda|^{1-\alpha} \cdot [-1, 1]$ satisfying (1.15),
- for any sequences of cubes $C_1 = C_1^\Lambda, \dots, C_p = C_p^\Lambda$ in $[-1/2, 1/2]^d$

one has, for any sequences of integers $k_1 = k_1^\Lambda, \dots, k_p = k_p^\Lambda \in \mathbb{N}^p$,

$$\lim_{|\Lambda| \rightarrow +\infty} \mathbb{P} \left| \left(\begin{array}{c} \# \left\{ j; \begin{array}{l} \xi_j(\omega, \Lambda) \in I_1 \\ x_j(\omega, \Lambda)/L \in C_1 \end{array} \right\} = k_1 \\ \omega; \quad \vdots \\ \# \left\{ j; \begin{array}{l} \xi_j(\omega, \Lambda) \in I_p \\ x_j(\omega, \Lambda)/L \in C_p \end{array} \right\} = k_p \end{array} \right) - \prod_{j=1}^p e^{-|I_j||C_j|} \frac{(|I_j||C_j|)^{k_j}}{k_j!} \right| = 0$$

where the $\xi_j(\omega, \Lambda_L)$'s are defined in (1.11).

In particular the point process defined as

$$\Xi_\Lambda^2(\xi, x; E_0, \Lambda) = \sum_{j=1}^N \delta_{\xi_j(\omega, \Lambda_L)}(\xi) \otimes \delta_{x_j(\omega, \Lambda)/L}(x) \quad (1.18)$$

converges weakly to a Poisson point process on $\mathbb{R} \times \mathbb{R}^d$ with intensity the Lebesgue measure.

The joint (energy, center)-distribution given by $\Xi_\Lambda^2(\xi, x; E_0, \Lambda)$ in (1.18) have been studied in [KN07], where they prove it converges weakly to a Poisson process.

We point out that in Theorem 1.6 intervals I_j 's and cubes C_j 's may depend on Λ . But the limit only depends on the product $|I_j||C_j|$. We shall exploit this fact in the next result.

1.5.2. Covariant scaling joint (energy, center)-distribution. Fix an increasing sequence of scales $\ell = (\ell_\Lambda)_\Lambda$ such that

$$\frac{\ell_\Lambda}{\log |\Lambda|} \xrightarrow{|\Lambda| \rightarrow +\infty} +\infty \quad \text{and} \quad \ell_\Lambda \leq |\Lambda|^{1/d}. \quad (1.19)$$

Pick $E_0 \in I$ so that $\nu(E_0) > 0$. Consider the point process

$$\Xi_\Lambda^2(\xi, x; E_0, \ell) = \sum_{j=1}^N \delta_{\ell_\Lambda^d [N(E_j(\omega, \Lambda)) - N(E_0)]}(\xi) \otimes \delta_{x_j(\omega)/\ell_\Lambda}(x). \quad (1.20)$$

The process is valued in $\mathbb{R} \times \mathbb{R}^d$; actually, if $c\ell_\Lambda \geq |\Lambda|^{1/d}$, it is valued in $\mathbb{R} \times (-c, c)^d$. Define

$$c_\ell = \lim_{|\Lambda| \rightarrow +\infty} |\Lambda|^{1/d} \ell_\Lambda^{-1} \in [1, +\infty]. \quad (1.21)$$

Note that if $\ell_\Lambda = L$, we recover (1.18). We prove

Theorem 1.7. *Assume (IAD), (W), (M) and (Loc) hold. Let E_0 be an energy in I such that (1.12) holds for some $\tilde{\rho} \in (0, \rho)$. The point process $\Xi_\Lambda^2(\xi, x; E_0, \ell)$ converges weakly to a Poisson process on $\mathbb{R} \times (-c_\ell, c_\ell)^d$ with intensity measure the Lebesgue measure.*

As a result, we see that, once the energies and the localization centers are scaled covariantly, the convergence to a Poisson process is true at any scale that is essentially larger than the localization width. The scaling is very natural; it is the one prescribed by the Heisenberg uncertainty principle: the more precision we require in the energy variable, the less we can afford in the space variable. In this respect, the energies behave like a homogeneous symbol of degree d . This is quite different from what one obtains in the case of the Laplace operator.

1.5.3. Non-covariant scaling joint (energy, center)-distribution. One can also study what happens when the energies and localization centers are not scaled covariantly. Consider two increasing sequences of scales, say $\ell = (\ell_\Lambda)_\Lambda$ and $\tilde{\ell} = (\tilde{\ell}_\Lambda)_\Lambda$. Pick E_0 an energy in I such that (1.12) holds for some $\tilde{\rho} \in (0, \rho)$. Consider the point process

$$\Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) = \sum_{j=1}^N \delta_{\ell_\Lambda^d [N(E_j(\omega, \Lambda)) - N(E_0)]}(\xi) \otimes \delta_{x_j(\omega)/\tilde{\ell}_\Lambda}(x). \quad (1.22)$$

Then, we prove

Theorem 1.8. *Assume (IAD), (W), (M) and (Loc) hold. Assume the sequences of increasing scales $\ell = (\ell_\Lambda)_\Lambda$ and $\tilde{\ell} = (\tilde{\ell}_\Lambda)_\Lambda$ satisfy (1.19). Assume that*

$$\text{if } \ell' = o(L) \text{ then } \frac{\ell_{\Lambda L + \ell'}}{\ell_{\Lambda L}} \xrightarrow{|\Lambda| \rightarrow +\infty} 1 \text{ and } \frac{\tilde{\ell}_{\Lambda L + \ell'}}{\tilde{\ell}_{\Lambda L}} \xrightarrow{|\Lambda| \rightarrow +\infty} 1. \quad (1.23)$$

Let J and C be bounded measurable sets respectively in \mathbb{R} and $(-c_{\tilde{\ell}}, c_{\tilde{\ell}})^d \subset \mathbb{R}^d$. One has

(1) *if, for some $\delta > 0$, one has $\frac{\tilde{\ell}_\Lambda}{\ell_\Lambda} \leq |\Lambda|^{-\delta}$, then ω almost surely, for Λ sufficiently large,*

$$\int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi dx = 0.$$

(2) *if, for some $\delta > 0$, one has $\frac{\tilde{\ell}_\Lambda}{\ell_\Lambda} \geq |\Lambda|^\delta$, then ω almost surely,*

$$\left(\frac{\ell_\Lambda}{\tilde{\ell}_\Lambda}\right)^{-d} \int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi dx \xrightarrow{|\Lambda| \rightarrow +\infty} |J| \cdot |C|.$$

Theorem 1.8 proves that the local energy levels and the localization centers become uniformly distributed in large energy windows if one conditions the localization centers to live in a much smaller window. On the other hand, for a typical sample, if one looks for eigenvalues in an energy interval smaller than the correctly scaled one with localization center in a cube, then asymptotically there are none.

If one replaces the polynomial growth or decay conditions of the ratio of scales $\ell_\Lambda/\tilde{\ell}_\Lambda$ by the condition that they tend to 0 or ∞ , or if one omits condition (1.23), the results stays valid except for the fact that the convergence is not almost sure anymore but simply holds in probability; actually, one has convergence in some L^p norm (see Remark 4.1 in section 4.2.3).

1.6. The level spacings statistics. Our goal is now to understand the level spacings statistics for eigenvalues near $E_0 \in I$. Pick I_Λ a compact interval containing E_0 such that its density of states measure $N(I_\Lambda)$ stays bounded. We note that, by the existence of the density of states and also Theorem 1.2, the spacing between the image of the eigenvalues of $H_\omega(\Lambda)$ through N near E_0 is of size $|\Lambda|^{-1}$. Hence, to study the empirical statistics of level spacings in I_Λ , $N(I_\Lambda)$ should contain asymptotically infinitely many images of energy levels of $H_\omega(\Lambda)$. Let us first study this number.

1.6.1. *A large deviation principle for the eigenvalue counting function.* Define the random numbers

$$N(I_\Lambda, \Lambda, \omega) := \#\{j; E_j(\omega, \Lambda) \in I_\Lambda\}. \quad (1.24)$$

Write $I_\Lambda = [a_\Lambda, b_\Lambda]$ and recall that $N(I_\Lambda) = N(b_\Lambda) - N(a_\Lambda)$ where N is the integrated density of states. We show that $N(I_\Lambda, \Lambda, \omega)$ satisfies a large deviation principle, namely,

Theorem 1.9. *Assume (IAD), (W), (M) and (Loc) hold. Assume that $(I_\Lambda)_\Lambda$ is a sequence of compact intervals in the localization region I such that*

- for some $\nu \in (0, 1)$, one has $N(I_\Lambda)|\Lambda|^{1-\nu} \rightarrow +\infty$,
- $N(I_\Lambda)|I_\Lambda|^{-1-\rho} \rightarrow +\infty$ as $|\Lambda| \rightarrow +\infty$ where ρ is defined in Assumption (M).

Then, there exists $\delta > 0$ such that, for $|\Lambda|$ sufficiently large, one has

$$\mathbb{P}(N(I_\Lambda, \Lambda, \omega) - N(I_\Lambda)|\Lambda| = o(N(I_\Lambda)|\Lambda|)) \geq 1 - e^{-(N(I_\Lambda)|\Lambda|)^\delta}. \quad (1.25)$$

We note that we do not need that intervals $(I_\Lambda)_\Lambda$ lie near points E_0 where (1.12) is satisfied; the density of states may vanish near E_0 though not faster than the rate fixed by the condition $N(I_\Lambda)|I_\Lambda|^{-1-\rho} \rightarrow +\infty$. The large deviation principle (1.25) is meaningful only if $N(I_\Lambda)|\Lambda| \rightarrow +\infty$; as N is Lipschitz continuous as a consequence of (W), this implies that

$$|\Lambda| \cdot |I_\Lambda| \rightarrow +\infty \quad \text{when} \quad |\Lambda| \rightarrow +\infty. \quad (1.26)$$

By Theorem 1.1, (1.25) can also be written as

$$\mathbb{P}(N(I_\Lambda, \Lambda, \omega) = \mathbb{E}(N(I_\Lambda, \Lambda, \omega))(1 + o(1))) \geq 1 - e^{-(\mathbb{E}(N(I_\Lambda, \Lambda, \omega)))^\delta}$$

which is why we call (1.25) a large deviation estimate.

For same model, one can improve upon (1.25) by relaxing $N(I_\Lambda)|I_\Lambda|^{-1-\rho} \rightarrow +\infty$ into $N(I_\Lambda)|I_\Lambda|^{-\nu} \rightarrow +\infty$ for some arbitrarily large $\nu > 0$ and obtain precise estimates of the term $o(N(I_\Lambda)|\Lambda|)$ (see [GKI]).

1.6.2. *The level spacing statistics near a given energy.* Fix $E_0 \in I$. Pick $I_\Lambda = [a_\Lambda, b_\Lambda]$ so that $|a_\Lambda| + |b_\Lambda| \rightarrow 0$. Consider the unfolded eigenvalue spacings, for $1 \leq j \leq N$,

$$\delta N_j(\omega, \Lambda) = |\Lambda|(N(E_{j+1}(\omega, \Lambda)) - N(E_j(\omega, \Lambda))) \geq 0. \quad (1.27)$$

Define the empirical distribution of these spacings to be the random numbers, for $x \geq 0$

$$DLS(x; E_0 + I_\Lambda, \omega, \Lambda) = \frac{\#\{j; E_j(\omega, \Lambda) \in E_0 + I_\Lambda, \delta N_j(\omega, \Lambda) \geq x\}}{N(I_\Lambda, \Lambda, \omega)}. \quad (1.28)$$

We will now study the spacings distributions of energies inside intervals that shrink to a point but that asymptotically contain infinitely many eigenvalues.

We prove

Theorem 1.10. *Assume (IAD), (W), (M) and (Loc) hold. Fix $E_0 \in I$ such that, for some $\bar{\rho} \in (0, \rho)$, there exists a neighborhood of E_0 , say, U such that*

$$\forall(x, y) \in U^2, \quad |N(x) - N(y)| \geq |x - y|^{1+\bar{\rho}}. \quad (1.29)$$

Fix $(I_\Lambda)_\Lambda$ a sequence of intervals such that $\sup_{E \in I_\Lambda} |E| \xrightarrow{|\Lambda| \rightarrow +\infty} 0$.

Assume that, for some $\delta > 0$, one has

$$|\Lambda|^{1-\delta} \cdot N(E_0 + I_\Lambda) \xrightarrow{|\Lambda| \rightarrow +\infty} +\infty \text{ and if } \ell' = o(L) \text{ then } \frac{N(E_0 + I_{\Lambda_{L+\ell'}})}{N(E_0 + I_{\Lambda_L})} \xrightarrow{|\Lambda| \rightarrow +\infty} 1. \quad (1.30)$$

Then, with probability 1, as $|\Lambda| \rightarrow +\infty$, $DLS(x; E_0 + I_\Lambda, \omega, \Lambda)$ converges uniformly to the distribution $x \mapsto e^{-x}$, that is, with probability 1,

$$\sup_{x \geq 0} |DLS(x; E_0 + I_\Lambda, \omega, \Lambda) - e^{-x}| \xrightarrow{|\Lambda| \rightarrow +\infty} 0. \quad (1.31)$$

Hence, the unfolded level spacings behave as if the images of the eigenvalues by the IDS were i.i.d. uniformly distributed random variables (see [Wei55] or section 7 of [Pyk65]). The exponential distribution of the level spacings is the one predicted by physical heuristics in the localized regime ([Jan98, LR85, Mir00, Tho74]). It is also in accordance with Theorem 1.2. In [Mol82a, Min96, CGK10], the domains in energy where the statistics could be studied were much smaller than the ones considered in Theorem 1.10. Indeed, in these works, the energy interval is of order $|\Lambda|^{-1}$ whereas, here, it is assumed to tend to 0 but to be asymptotically infinite when compared to $|\Lambda|^{-1}$.

Remark 1.3. The first condition in (1.30) ensures that I_Λ contains sufficiently many eigenvalues of $H_\omega(\Lambda)$. The second condition in (1.30) is a regularity condition of the decay of $|I_\Lambda|$. If, in (1.30), one replaces the first condition by $|\Lambda|N(E_0 + I_\Lambda) \rightarrow +\infty$ or omits the second or does both, one still gets convergence in probability of $DLS(x; E_0 + I_\Lambda, \omega, \Lambda)$ to e^{-x} (see Remark 4.2) i.e.

$$\mathbb{P} \left(\sup_{x \geq 0} |DLS(x; E_0 + I_\Lambda, \omega, \Lambda) - e^{-x}| \geq \varepsilon \right) \xrightarrow{|\Lambda| \rightarrow +\infty} 0.$$

Condition (1.29) is slightly stronger than (1.12); it requires some uniformity in the lower bound. Theorem 1.10 can be applied to obtain the levelspacing distribution near regular points of the IDS. Define \mathcal{E} to be the set of energies E such that $\nu(E) = N'(E)$ exists and

$$\lim_{|x|+|y| \rightarrow 0} \frac{N(E+x) - N(E+y)}{x-y} = \nu(E). \quad (1.32)$$

Obviously the set \mathcal{E} contains the continuity points of $\nu(E)$. We prove

Lemma 1.3. *The set \mathcal{E} is of full Lebesgue measure.*

For $E_0 \in \mathcal{E}$ such that $\nu(E_0) > 0$, assumption (1.29) holds for any $\tilde{\rho} > 0$. Moreover, one has

$$\delta E_j(\omega, \Lambda) = \nu(E_0) |\Lambda| (E_{j+1}(\omega, \Lambda) - E_j(\omega, \Lambda)) (1 + o(1)).$$

Then as a corollary of Theorem 1.10, we immediately obtain

Theorem 1.11. *Assume (IAD), (W), (M) and (Loc) hold. Fix $E_0 \in I \cap \mathcal{E}$ such that $\nu(E_0) > 0$. Fix $(I_\Lambda)_\Lambda$ a sequence of intervals such that $\sup_{I_\Lambda} |x| \xrightarrow{|\Lambda| \rightarrow +\infty} 0$.*

Assume that, for some $\delta > 0$, one has

$$|\Lambda|^{1-\delta} \cdot |I_\Lambda| \xrightarrow{|\Lambda| \rightarrow +\infty} +\infty \text{ and if } \ell' = o(L) \text{ then } \frac{|I_{\Lambda_{L+\ell'}}|}{|I_{\Lambda_L}|} \xrightarrow{|\Lambda| \rightarrow +\infty} 1.$$

Then, with probability 1, as $|\Lambda| \rightarrow +\infty$, the empirical distribution function

$$\frac{\#\{j; E_j(\omega, \Lambda) \in I_\Lambda, \nu(E_0) |\Lambda| (E_{j+1}(\omega, \Lambda) - E_j(\omega, \Lambda)) \geq x\}}{N(I_\Lambda, \Lambda, \omega)}$$

converges uniformly to the distribution $x \mapsto e^{-x}$.

1.6.3. *The level spacings statistics on macroscopic energy intervals.* Theorem 1.10 seems optimal as the density of states at E_0 enters into the correct rescaling to obtain a universal result. Hence, the distribution of level spacings on larger intervals needs to take into account the variations of the density of states on these intervals. Indeed, on intervals of non vanishing size, under additional regularity assumption on ν , one can compute the asymptotic distribution of the level spacings when one omits the local density of states in the spacing and obtain

Theorem 1.12. *Assume (IAD), (W), (M) and (Loc) hold. Pick $J \subset I$ a compact interval such $\lambda \mapsto \nu(\lambda)$ be continuous on J and $N(J) := \int_J \nu(\lambda) d\lambda > 0$. Define the unfolded eigenvalue spacings, for $1 \leq j \leq N$,*

$$\delta_J E_j(\omega, \Lambda) = \frac{N(J)}{|J|} |\Lambda| (E_{j+1}(\omega, \Lambda) - E_j(\omega, \Lambda)) \geq 0, \quad (1.33)$$

and the empirical distribution of these spacings to be the random numbers, for $x \geq 0$

$$DLS'(x; J, \omega, \Lambda) = \frac{\#\{j; E_j(\omega, \Lambda) \in J, \delta_J E_j(\omega, \Lambda) \geq x\}}{N(J, \omega, \Lambda)}. \quad (1.34)$$

Then, with probability 1, as $|\Lambda| \rightarrow +\infty$, $DLS'(x; J, \omega, \Lambda)$ converges uniformly to the distribution $x \mapsto g_{\nu, J}(x)$ where

$$g_{\nu, J}(x) = \int_J e^{-\nu_J(\lambda)|J|x} \nu_J(\lambda) d\lambda \text{ where } \nu_J = \frac{1}{N(J)} \nu. \quad (1.35)$$

We see that, in the large volume limit, the unfolded levels spacings behave as if the eigenvalues were i.i.d. random variables distributed according to the density $\frac{1}{N(J)} \nu(\lambda)$ i.e. to the density of states renormalized to be a probability measure on J (see section 7 of [Pyk65]).

Theorem 0.1 is then an immediate consequence of Theorem 1.12 and the results on the regularity for the density of states of the discrete Anderson model at large disorder obtained in [BCKP88].

1.7. The localization center spacings statistics. Pick $E_0 \in I$. Inside a large cube Λ , the number of centers that corresponds to energies in I_Λ , $N(E_0 + I_\Lambda, \Lambda, \omega)$ (see (1.24)), is asymptotic to $N(E_0 + I_\Lambda)|\Lambda|$. Theorem 1.8 states that they are distributed uniformly. Thus, the reference mean spacing between localization centers is of size $(|\Lambda|/N(E_0 + I_\Lambda)|\Lambda|)^{1/d} = (N(E_0 + I_\Lambda))^{-1/d}$. This motivates the following definition. Define the empirical distribution of center spacing to be the random number

$$DCS(s; I_\Lambda, \Lambda, \omega) = \frac{\#\left\{j; \begin{array}{l} E_j(\omega, \Lambda) \in I_\Lambda, \\ \sqrt[d]{N(E_0 + I_\Lambda)} \min_{i \neq j} |x_i(\omega) - x_j(\omega)| \geq s \end{array} \right\}}{N(E_0 + I_\Lambda, \Lambda, \omega)}. \quad (1.36)$$

We prove an analogue of Theorem 1.10, namely

Theorem 1.13. *Assume (IAD), (W), (M) and (Loc) hold. Pick $E_0 \in I$ such that, for some $\bar{\rho} \in (0, \rho)$, in U some neighborhood of E_0 , one has (1.29). Assume that, for some $\nu \in (0, 1)$, one has*

$$N(E_0 + I_\Lambda) |\Lambda| \xrightarrow{|\Lambda| \rightarrow +\infty} +\infty \text{ and } N(E_0 + I_\Lambda) \log^{d/\nu} |\Lambda| \xrightarrow{|\Lambda| \rightarrow +\infty} 0. \quad (1.37)$$

Then, as $|\Lambda| \rightarrow +\infty$, in probability, $DCS(s; I_\Lambda, \Lambda, \omega)$ converges uniformly to the distribution $x \mapsto e^{-s^d}$, that is, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left\{\omega; \sup_{s \geq 0} \left|DCS(s; E_0 + I_\Lambda, \Lambda, \omega) - e^{-s^d}\right| \geq \varepsilon\right\}\right) \xrightarrow{\Lambda \nearrow \mathbb{R}^d} 0. \quad (1.38)$$

Of course, as Theorem 1.13 is the counterpart of Theorem 1.10, Theorem 1.12 also has its counterpart for localization centers.

1.8. Another point of view. In the present section, we want to adopt a different point of view on the spectral statistics. Instead of discussing the statistics of the eigenvalues of the random system restricted to some finite box in the large box limit, we will describe the spectral statistics of the infinite system in the localized phase. Therefore, we first need to explicit what we mean with the localized phase for the random Hamiltonian on the whole space i.e. state the appropriate replacement for assumption (Loc) in this setting.

Let $I \subset \mathbb{R}$ be an interval. We assume

(Loc'): there exists $\xi \in (0, 1]$, $q > 0$ and $\gamma > 0$ such that, with probability 1, if $E \in I \cap \sigma(H_\omega)$ and φ is a normalized eigenfunction associated to E then, for some $x(E, \omega) \in \mathbb{R}^d$ or \mathbb{Z}^d , a maximum of $x \mapsto \|\varphi\|_x$, for some $C_\omega > 0$, one has, for $x \in \mathbb{R}^d$,

$$\|\varphi\|_x \leq C_\omega (1 + |x(E, \omega)|^2)^{q/2} e^{-\gamma|x-x(E, \omega)|^\xi}; \quad (1.39)$$

moreover, one has $\mathbb{E}(C_\omega) < +\infty$.

As above $x(E)$ is called a *center of localization* for energy E or for the associated eigenfunction φ . It is well established that (Loc') holds in any interval contained in the region of complete localization. The first proof is due to [dRJLS96] for the discrete Anderson model where they show that (Loc') is a consequence of the fractional moment method [Aiz94, AM93]; there $\xi = 1$. The proof extends to continuous Hamiltonians thanks to [AEN⁺06]. In [GDB98], the multiscale analysis is shown to imply (1.39) with $\xi = 1$ but with no control on $\mathbb{E}(C_\omega)$. That the multiscale analysis yields (Loc') for any $\xi < 1$ follows from [GK06, Corollary 3] and [GK06, Eq. (4.17)] to see that $\mathbb{E}(C_\omega) < \infty$.

Pick I an interval where the Hamiltonian H_ω is localized i.e. satisfies (Loc'). Assume that, ω -almost surely, $\sigma(H_\omega) \cap I = I$. Hence, any subinterval of I contains infinitely many eigenvalues and to define statistics, we need to enumerate these eigenvalues in a way or another. To do this, we use the localization centers. First, we prove

Proposition 1.1. *Assume (Loc') for some $\xi \in (0, 1]$ and fix $q > 2d$. Then, there exists $\gamma > 0$ such that, ω -almost surely, there exists $C_\omega > 1$, $\mathbb{E}(C_\omega) < \infty$, such that*

- (1) *if $x(E, \omega)$ and $x'(E, \omega)$ are two centers of localization for $E \in I$ then, for some $C_d > 0$ (depending only on d),*

$$|x(E, \omega) - x'(E, \omega)| \leq \gamma^{-1/\xi} \log^{1/\xi} \left(C_d C_\omega \langle x(E, \omega) \rangle^q \langle x'(E, \omega) \rangle^q \frac{1}{\gamma^{d/\xi}} \right). \quad (1.40)$$

- (2) *for $L \geq 1$, pick $(I_L)_L$ a sequence of intervals, $I_L \subset I$, such that, for some $\varepsilon > 0$, one has $L^{d-\varepsilon} N(I_L) \rightarrow +\infty$ and $N(I_L) |I_L|^{-1-\rho} \rightarrow +\infty$ where ρ is given by (M) and $N(I_L)$ by (1.4); if $N(I_L, L)$ denotes the number of eigenvalues of H_ω having a center of localization in Λ_L , then*

$$N(I_L, L) = N(I_L) |\Lambda_L| (1 + o(1)). \quad (1.41)$$

For $L \geq 1$, pick $I_L \subset I$ such that $L^{d-\varepsilon} N(I_L) \rightarrow +\infty$ for some $\varepsilon > 0$. In view of Proposition 1.1, there are only finitely many eigenvalues of H_ω in I_L having a localization center in Λ_L . Thus, we can consider their levelspacings: let us enumerate these eigenvalues as $E_1(\omega, L) \leq E_2(\omega, L) \leq \dots \leq E_N(\omega, L)$ where we repeat them according to multiplicity. Define the empirical distributions DLS and DLS' as in (1.28) and (1.34) for the eigenvalues of H_ω in I_L having a localization center in Λ_L . We prove

Theorem 1.14. *Assume (IAD), (W), (M), (Loc') hold. One has*

- if $E_0 \in I_L$ s.t. (1.12) is satisfied for some $\tilde{\rho} \in (0, \rho)$ and $|I_L| \rightarrow 0$ and satisfies (1.30), then, ω -almost surely, for $x \geq 0$

$$\lim_{L \rightarrow +\infty} DLS(x; I_L, \omega, L) = e^{-x}; \quad (1.42)$$

- if, for all L large, $|I_L| = J$ such that $N(J) > 0$ and ν is continuous on J then, ω -almost surely, one has

$$\lim_{L \rightarrow +\infty} DLS'(x; I_L, \omega, L) = g_J(x), \quad (1.43)$$

where g_J is defined in (1.35).

We see that the level spacings distribution of the eigenvalues of H_ω having center of localization in I_L have the same limits as those of the eigenvalues of $H_\omega(\Lambda)$. This is a consequence of the localization assumption (Loc').

1.9. The asymptotic description of the eigenvalues. We now turn to the basic technical results that are at the heart of the proofs of the results stated until now. They consist in a precise approximation of the eigenvalues of $H_\omega(\Lambda)$ in I_Λ by independent random variables. For such a description to be possible, the interval I_Λ should not be too large. The results stated in section 1 are then consequences of this description and, in some cases, additional decorrelation estimates. We will give two different descriptions depending on the size of $N(I_\Lambda)$. When this quantity is sufficiently small with respect to $|\Lambda|^{-1}$, our procedure enables us to control all the eigenvalues. If it is not, we only control most of the eigenvalues.

1.9.1. *Controlling all the eigenvalues.* Assume E_0 is such that (1.12) holds. First, pick I_Λ centered at E_0 such that $N(I_\Lambda) \asymp |\Lambda|^{-\alpha}$ for $\alpha \in (\alpha_{d,\rho,\tilde{\rho}}, 1)$ where $\alpha_{d,\rho,\tilde{\rho}}$ is defined in (1.14). This will enable us to control all the eigenvalues of $H_\omega(\Lambda)$ in I_Λ . This result is at the heart of the proofs of Theorems 1.2, 1.3, 1.6, 1.7 and 1.8. Moreover, if the decorrelations estimates (GM) and (D) hold, this result will be sufficient to prove all the results in section 1.4.2.

Theorem 1.15. *Assume E_0 is such that (1.12) holds. Recall that $\alpha_{d,\rho,\tilde{\rho}}$ is defined in (1.14) and pick $\alpha \in (\alpha_{d,\rho,\tilde{\rho}}, 1)$. Pick I_Λ centered at E_0 such that $N(I_\Lambda) \asymp |\Lambda|^{-\alpha}$. There exists $\beta < \alpha$ and some $\ell = \mathcal{O}(L^\beta)$, so that for $\beta' = 0+$ small enough and for any L sufficiently large, we have a decomposition of Λ into disjoint cubes of the form $\Lambda_\ell(\gamma_j) := \gamma_j + [0, \ell]^d$:*

- $\cup_j \Lambda_\ell(\gamma_j) \subset \Lambda_L$,
- $\text{dist}(\Lambda_\ell(\gamma_j), \Lambda_\ell(\gamma_k)) \geq L^{\beta'}$ if $j \neq k$,
- $\text{dist}(\Lambda_\ell(\gamma_j), \partial\Lambda) \geq L^{\beta'}$
- $|\Lambda_L \setminus \cup_j \Lambda_\ell(\gamma_j)| \lesssim |\Lambda|^{1 - \frac{1}{d}(\beta - \beta')}$,

such that, there exists a set of configurations \mathcal{Z}_Λ s.t.:

- $\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - |\Lambda|^{-(\alpha - \alpha_{d,\rho,\tilde{\rho}})}$,
- for $\omega \in \mathcal{Z}_\Lambda$, each centers of localization associated to $H_\omega(\Lambda)$ belong to some $\Lambda_\ell(\gamma_j)$ and each box $\Lambda_\ell(\gamma_j)$ satisfies:
 - (1) the Hamiltonian $H_\omega(\Lambda_\ell(\gamma_j))$ has at most one eigenvalue in I_Λ , say, $E_j(\omega, \Lambda_\ell(\gamma_j))$;
 - (2) $\Lambda_\ell(\gamma_j)$ contains at most one center of localization, say $x_{k_j}(\omega, L)$, of an eigenvalue of $H_\omega(\Lambda)$ in I_Λ , say $E_{k_j}(\omega, L)$;
 - (3) $\Lambda_\ell(\gamma_j)$ contains a center $x_{k_j}(\omega, L)$ if and only if $\sigma(H_\omega(\Lambda_\ell(\gamma_j))) \cap I_\Lambda \neq \emptyset$; in which case, one has, with $\ell' = L^{\beta'}$,

$$|E_{k_j}(\omega, L) - E_j(\omega, \Lambda_\ell(\gamma_j))| \leq e^{-(\ell')^\xi} \text{ and } \text{dist}(x_{k_j}(\omega, L), \Lambda_L \setminus \Lambda_\ell(\gamma_j)) \geq \ell'. \quad (1.44)$$

In particular if $\omega \in \mathcal{Z}_\Lambda$, all the eigenvalues of $H_\omega(\Lambda)$ are described by (1.44).

1.9.2. *Controlling most eigenvalues.* Then, we now state a result that works on intervals I_Λ such that $N(I_\Lambda)$ be of size $(\log |\Lambda|)^{-d/\xi}$ but gives the control only on most of the eigenvalues. This is enough to control the level spacings on such sets. This is the main tool to obtain Theorems 1.10, 1.12, 1.13 and 1.14.

Theorem 1.16. *Set $\ell' = (R \log |\Lambda|)^{\frac{1}{\xi}}$ with R large and consider intervals $I_\Lambda = [E_0 - i_\Lambda/2, E_0 + i_\Lambda/2] \subset I$ with $i_\Lambda = |\Lambda|$ so that $|\Lambda|N(I_\Lambda) \rightarrow \infty$. Assume that $N(I_\Lambda)|I_\Lambda|^{-(1+\bar{\rho})} \rightarrow \infty$. Pick a sequence of scales ℓ_Λ so that $\ell' \ll \ell_\Lambda \ll L$ and $N(I_\Lambda)^{\frac{1}{1+\bar{\rho}}} \ell_\Lambda^d \rightarrow 0$ as $|\Lambda| \rightarrow \infty$.*

For $|\Lambda|$ sufficiently large, there exists a decomposition of Λ_L into $\mathcal{O}(|\Lambda|/\ell_\Lambda^d)$ disjoint cubes of the form $\Lambda_\ell(\gamma_j) := \gamma_j + [0, \ell]^d$, where $\ell = \ell_\Lambda(1 + \mathcal{O}(\ell_\Lambda/|\Lambda|)) = \ell_\Lambda(1 + o(1))$, so that:

- $\cup_j \Lambda_\ell(\gamma_j) \subset \Lambda_L$,
- $\text{dist}(\Lambda_\ell(\gamma_j), \Lambda_\ell(\gamma_k)) \geq \ell'$ if $j \neq k$,
- $\text{dist}(\Lambda_\ell(\gamma_j), \partial\Lambda) \geq \ell'$
- $|\Lambda_L \setminus \cup_j \Lambda_\ell(\gamma_j)| \lesssim |\Lambda| \ell' / \ell$,

such that, there exists a set of configurations \mathcal{Z}_Λ s.t.

- $\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - \exp\left(-cN(I_\Lambda)|\Lambda|(N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{1+\bar{\rho}}}\ell^{d\rho})\right) - \exp\left(-cN(I_\Lambda)|\Lambda|\ell'\ell^{-1}\right)$,
- for all $\omega \in \mathcal{Z}_\Lambda$, there exists at least $\frac{|\Lambda|}{\ell^d}(1 + \mathcal{O}(N(I_\Lambda)^{1/(1+\bar{\rho})}\ell^d))$ disjoint boxes $\Lambda_\ell(\gamma_j)$ satisfying the properties (1), (2) and (3) described in Theorem 1.15 with $\ell' = (R \log |\Lambda|)^{1/\xi}$ in (1.44); we note that $N(I_\Lambda)\ell^{d-1}\ell' = o(1)$ as $|\Lambda| \rightarrow +\infty$;
- the number of eigenvalues of $H_\omega(\Lambda)$ that are not described by the above picture is $\lesssim N(I_\Lambda)|\Lambda|((N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{1+\bar{\rho}}}\ell^{d(1+\rho)}) + N(I_\Lambda)|\Lambda|(\ell')^{d+1}\ell^{-1})$, which is $o(N(I_\Lambda)|\Lambda|)$ provided

$$(\ell')^{d+1} \ll \ell \ll N(I_\Lambda)^{-\frac{\rho-\bar{\rho}}{d(1+\rho)(1+\bar{\rho})}}. \quad (1.45)$$

1.9.3. *Comparing various cube sizes.* We prove

Proposition 1.2. *Assume (IAD) (Loc) and (W). Fix I a compact interval and C a compact cube in \mathbb{R}^d . Fix $p > 0$ and a sequence $(\ell_\Lambda)_\Lambda$ satisfying (1.19). Then, there exists a set of configurations \mathcal{Z}_Λ so that $\mathbb{P}(\mathcal{Z}_\Lambda) \rightarrow 1$ as $|\Lambda| \rightarrow +\infty$ and, for $\omega \in \mathcal{Z}_\Lambda$, to every eigenvalue of $H_\omega(\Lambda)$ in $E_0 + \ell_\Lambda^{-d}I$ associated to a localization center in $\ell_\Lambda C$, say $E_n(\omega, \Lambda)$, one can associate a unique eigenvalue of $H_\omega(\ell_\Lambda C)$, say $E_n(\omega, \ell_\Lambda C)$; moreover, these eigenvalues satisfy*

$$|E_n(\omega, \ell_\Lambda C) - E_n(\omega, \Lambda)| \leq |\Lambda|^{-p}. \quad (1.46)$$

Remark 1.4. As the proofs will show, the sizes of the intervals where the control of the eigenvalues is possible and the probability of the event where this control is possible both depend very much on the forms of the Wegner and Minami estimates, (W) and (M). In particular, if one replaces (W) by what is suggested in Remark 1.1, the constants appearing in Theorems 1.15 and 1.16 and Proposition 1.2 have to be modified.

1.10. **Outline of the article.** To complete this section, let us now briefly describe the architecture of the remaining parts of the paper that consist in the proofs of all the results stated in sections 0 and 1.

We start in section 2 with the computation of two important quantities related to our approximation scheme. Consider a cube Λ and an energy interval I such that $|I| \cdot |\Lambda|$ is small. In section 2, we compute

- the probability that $H_\omega(\Lambda)$ has exactly any eigenvalue in I ,
- the distribution of this eigenvalue conditioned on the fact that it is unique.

This distribution is used in the sequel to approximate the eigenvalue and localization center processes.

Section 3 is devoted to the proof of the approximation theorems, Theorems 1.15 and 1.16, in section 3. Section 4 is devoted to the proof of the results on the spectral statistics. In section 5, we derive the results for the full Hamiltonian i.e. we develop the other point of view presented in section 1.8. Finally, the appendix 6 is devoted to various technical results used in the course of the proofs, included a description of equivalent finite volume localization properties.

2. THE LOCAL DISTRIBUTION OF EIGENVALUES

In this section, we prove Theorem 1.1 and then use this result to compute the distribution of unfolded eigenvalues.

2.1. The proof of Theorem 1.1. To that end, note that by covariance

$$\mathbb{E}[\mathrm{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))] = |\Lambda| \mathbb{E}[\mathrm{tr}(\chi_0 \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) \chi_0)]. \quad (2.1)$$

Recall (1.4): the increase of the integrated density of states of H_ω on I_Λ is given by

$$N(I_\Lambda) = N(b_\Lambda) - N(a_\Lambda) = \mathbb{E}[\mathrm{tr}(\chi_0 \mathbf{1}_{I_\Lambda}(H_\omega) \chi_0)].$$

To control $|\mathbb{E}[\mathrm{tr}(\chi_0 \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) \chi_0)] - \mathbb{E}[\mathrm{tr}(\chi_0 \mathbf{1}_{I_\Lambda}(H_\omega) \chi_0)]|$, we use localization estimates after having smoothed out the characteristic function. Let f_δ be a compactly supported Gevrey class function such that $f_\delta = 1$ in I_Λ , and $f_\delta = 0$ outside a neighborhood of length δ of I_Λ . Note that, by Wegner's estimate (W) and the Lipschitz continuity of N ,

$$|\mathbb{E}[\mathrm{tr}(\chi_0 [\mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - f_\delta(H_\omega(\Lambda))] \chi_0)]| + |\mathbb{E}[\mathrm{tr}(\chi_0 (\mathbf{1}_{I_\Lambda}(H_\omega) - f_\delta(H_\omega)) \chi_0)]| \leq C\delta. \quad (2.2)$$

To estimate $|\mathbb{E}[\mathrm{tr}(\chi_0 (f_\delta(H_\omega(\Lambda)) - f_\delta(H_\omega)) \chi_0)]|$, we use a Helffer-Sjöstrand formula to represent $f_\delta(H_\omega(\Lambda))$ and $f_\delta(H_\omega)$. As the support of f_δ lies in the localization region and as f is of Gevrey regularity, the exponential decay estimate for the resolvents of $H_\omega(\Lambda)$ and H_ω imply that, for $\xi' \in (\xi, 1)$, there exists $C > 0$ such that, for Λ sufficiently large (see e.g. the computation of [CGK10, (6.23)]), one has,

$$|\mathbb{E}[\mathrm{tr}(\chi_0 (f_\delta(H_\omega(\Lambda)) - f_\delta(H_\omega)) \chi_0)]| \leq \delta^{-C} e^{-\ell^{\xi'}}.$$

Recall that $N(I_\Lambda) \geq e^{-\ell^\xi}$ and $\xi' > \xi$, and pick now $\delta \asymp N(I_\Lambda)$. Plugging this into (2.2) and (2.1) yields (1.9) and completes the proof of Theorem 1.1. \square

2.2. The distribution of unfolded eigenvalues. Consider a cube Λ of sidelength ℓ and an interval $I_\Lambda = [a_\Lambda, b_\Lambda] \subset I$ (i.e. I_Λ is contained in the localization region). Consider the following random variables:

- $X = X(\Lambda, I_\Lambda)$ is the Bernoulli random variable

$$X = \mathbf{1}_{H_\omega(\Lambda) \text{ has exactly one eigenvalue in } I_\Lambda};$$

- $\tilde{E} = \tilde{E}(\Lambda, I_\Lambda)$ is the eigenvalue of $H_\omega(\Lambda)$ in I_Λ conditioned on $X = 1$;
- $\tilde{\xi} = \tilde{\xi}(\Lambda, I_\Lambda) = (\tilde{E}(\Lambda, I_\Lambda) - a_\Lambda)/|I_\Lambda|$.

Clearly $\tilde{\xi}$ is valued in $[0, 1]$; let $\tilde{\Xi}$ be its distribution function.

In the present section, we will describe the distribution of these random variables as $|\Lambda| \rightarrow +\infty$ and $|I_\Lambda| \rightarrow 0$. We prove

Lemma 2.1. *Assume (W), (M) and (Loc) hold. For any $\nu \in (0, 1)$, there exists $C > 1$ such that, one has*

$$|\mathbb{P}(X = 1) - N(I_\Lambda)|\Lambda| \leq C \left(|\Lambda|^{1+\rho} |I_\Lambda|^{1+\rho} + |I_\Lambda|^{-C} e^{-\ell^\nu/C} \right) \quad (2.3)$$

where $N(E)$ denotes the integrated density of states of H_ω .
Moreover, one has

$$\begin{aligned} & \left| (\tilde{\Xi}(x) - \tilde{\Xi}(y)) P(X = 1) - (N(a_\Lambda + x|I_\Lambda|) - N(a_\Lambda + y|I_\Lambda|))|I_\Lambda| \right| \\ & \leq C \left(|x - y|^{1+\rho}|I_\Lambda|^{1+\rho} + (|I_\Lambda||x - y|)^{-C} e^{-\ell^\nu/C} \right). \end{aligned} \quad (2.4)$$

This lemma differs from the usual computation of the DOS in the sense that the size of the interval decays as the thermodynamic limit is taken. A joint limit in the volume and the size of the interval has to be taken here. The price we pay for this joint limit is that we shall restrict ourselves to the localization regime, while the IDS exists in a broader region.

Proof of Lemma 2.1. We first relate $\mathbb{P}(X = 1) = \mathbb{P}\{\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) = 1\}$ to the quantity that leads us to the DOS, namely $\mathbb{E}[\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))]$. To do so, we proceed as in [Kri08, CGK09] and notice that

$$\mathbb{E}[\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))] = \sum_{k=1}^{\infty} \mathbb{P}\{\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) \geq k\}$$

and

$$\begin{aligned} \sum_{k=2}^{\infty} \mathbb{P}\{\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) \geq k\} & \leq \sum_{k=2}^{\infty} k(k-1) \mathbb{P}\{\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) = k\} \\ & = \mathbb{E}\{\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) (\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - 1)\}. \end{aligned}$$

By our assumption (M), we know

$$0 \leq \mathbb{P}\{\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) \geq 1\} - \mathbb{P}(X = 1) = \mathbb{P}\{\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) \geq 2\} \leq C|\Lambda|^{1+\rho}|I_\Lambda|^{1+\rho}$$

So, it follows that

$$0 \leq \mathbb{E}(\text{tr}[\mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))]) - \mathbb{P}(X = 1) \leq C|\Lambda|^{1+\rho}|I_\Lambda|^{1+\rho} \quad (2.5)$$

We are left with the evaluation of $\mathbb{E}[\text{tr } \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))]$. We use (2.2) and pick $\delta = |\Lambda|^{1+\rho}|I_\Lambda|^{1+\rho}$ and plug this, (2.2) and (2.1) into (2.5). This yields (2.3).

To prove (2.4), we write

$$(\tilde{\Xi}(x) - \tilde{\Xi}(y))P(X = 1) = \mathbb{P}\{\text{tr}[\mathbf{1}_{I_{x,y,\Lambda}}(H_\omega(\Lambda))] = 1\} \quad (2.6)$$

where $I_{x,y,\Lambda} = [a_\Lambda + x|I_\Lambda|, a_\Lambda + y|I_\Lambda|]$.

Replacing I_Λ with the interval $I_{x,y,\Lambda}$ in the estimation of $P(X = 1)$ yields the proof of (2.4).

The proof of Lemma 2.1 is complete. \square

We now describe the distribution of the unfolded eigenvalues. Therefore we slightly change our notations to localize the quantities near some energy E_0 . Pick $E_0 \in I$ such that (1.12) be satisfied and $I_\Lambda = [a_\Lambda, b_\Lambda]$. Recall that

- $X = X(\Lambda, E_0 + I_\Lambda)$ is the Bernoulli random variable

$$X = \mathbf{1}_{H_\omega(\Lambda)} \text{ has exactly one eigenvalue in } E_0 + I_\Lambda$$

- $\tilde{E} = \tilde{E}(\Lambda, E_0 + I_\Lambda)$ is this eigenvalue conditioned on $X = 1$.

Define

$$\xi = \frac{N(\tilde{E}) - N(E_0 + a_\Lambda)}{N(E_0 + b_\Lambda) - N(E_0 + a_\Lambda)}. \quad (2.7)$$

The random variable ξ is valued in $[0, 1]$. Let Ξ be its distribution function. We prove

Lemma 2.2. *Assume (W), (M) and (Loc) hold. Pick E_0 such that (1.29) holds. Fix $\nu \in (0, 1)$. Assume, moreover, that*

$$N(E_0 + I_\Lambda)^{\frac{\rho-\tilde{\rho}}{1+\tilde{\rho}}} |\Lambda|^\rho \xrightarrow{|\Lambda| \rightarrow +\infty} 0. \quad (2.8)$$

Then, for $(x, y) \in [0, 1]^2$ such that $|x - y| \geq e^{-|\Lambda|^{-\nu/d}}$, one has

$$\Xi(x) - \Xi(y) = (x - y) \left(1 + o\left(N(E_0 + I_\Lambda)^{\frac{\rho-\tilde{\rho}}{1+\tilde{\rho}}} |\Lambda|^\rho \right) \right). \quad (2.9)$$

Proof. Pick $\nu < \tilde{\nu} < 1$. Recall that the IDS N is monotone by definition and continuous thanks to (W). Assumption (1.29) guarantees that, for Λ sufficiently large, for $[a, b] \subset E_0 + I_\Lambda$, one has

$$N^{-1}([a, b]) \leq (b - a)^{1/(1+\tilde{\rho})}. \quad (2.10)$$

By Lemma 2.1, for (x, y) as in (2.9), one has

$$\Xi(x) - \Xi(y) = (x - y) \frac{1 + A + B}{1 + C + D}$$

where, using (2.10) and the assumption on $|x - y|$ in (2.9), we compute

$$\begin{aligned} |A| &\leq C \frac{|N^{-1}(N(E_0 + a_\Lambda) + N(E_0 + I_\Lambda)[x, y])|^{1+\rho}}{|x - y|} \frac{|\Lambda|^\rho}{N(E_0 + I_\Lambda)} \leq |\Lambda|^\rho [|x - y| N(E_0 + I_\Lambda)]^{\frac{\rho-\tilde{\rho}}{1+\tilde{\rho}}} \\ &\leq N(E_0 + I_\Lambda)^{\frac{\rho-\tilde{\rho}}{1+\tilde{\rho}}} |\Lambda|^\rho, \\ |B| &\leq |N^{-1}(N(E_0 + a_\Lambda) + N(E_0 + I_\Lambda)[x, y])|^{-C} e^{-|\Lambda|^{\tilde{\nu}/d}/C} \leq e^{-|\Lambda|^{\tilde{\nu}/d}/C + C|\Lambda|^{\nu/d}} \\ &\leq N(E_0 + I_\Lambda)^{\frac{\rho-\tilde{\rho}}{1+\tilde{\rho}}} |\Lambda|^\rho \end{aligned}$$

and $|C|$ and $|D|$ are respectively bounded by the same bounds as $|A|$ and $|B|$ for $(x, y) = (0, 1)$. This completes the proof of Lemma 2.2. \square

2.3. The local counting function. From Theorem 1.16 and Lemma 2.1, we derive an asymptotic for the number of eigenvalue of $H_\omega(\Lambda)$ in I_Λ , which we denote by $N(\omega, \Lambda, I_\Lambda)$. Such a control is necessary in order obtain the level spacing statistics.

Corollary 2.1. *Assume we are in the case of Theorem 1.16. Assume moreover that $N(I_\Lambda)|\Lambda| \rightarrow +\infty$. Then, for any $\delta \in [\frac{1}{2}, 1[$,*

$$\begin{aligned} &\mathbb{P}\{|N(\omega, \Lambda, I_\Lambda) - N(I_\Lambda)|\Lambda|\} = o(N(I_\Lambda)|\Lambda|) \\ &\geq 1 - \exp(-c|\Lambda||I_\Lambda|(|I_\Lambda|\ell^d)^\rho) - \exp(-cN(I_\Lambda)|\Lambda|\ell^\ell \ell^{-1}) - \exp(-c(N(I_\Lambda)|\Lambda|)^\delta). \end{aligned}$$

Proof. We keep the notations of the proof of Theorem 1.16.

Consider the boxes $(\Lambda_\ell(\gamma_j))_{1 \leq j \leq \tilde{N}}$ given by Theorem 1.16. We know that their number \tilde{N} satisfies

$$\tilde{N} = (|\Lambda|/n)(1 + o(|\Lambda|/n)).$$

For $1 \leq j \leq \tilde{N}$, let $X_j = X(\Lambda_\ell(\gamma_j), I_\Lambda)$ i.e. X_j is the Bernoulli random variable equal to 1 if $H_\omega(\Lambda_\ell(\gamma_j))$ has exactly one eigenvalue in I_Λ and zero otherwise. Let $n = |\Lambda_\ell|$ thus $n \asymp \ell^d$. It follows from Lemma 2.1 that

$$\mathbb{P}(X_j = 1) = N(I_\Lambda) n + o(N(I_\Lambda) n).$$

We have

$$|N(\omega, \Lambda, I_\Lambda) - N(I_\Lambda)|\Lambda| \leq \left| N(\omega, \Lambda, I_\Lambda) - \sum_{j=1}^{\tilde{N}} X_j \right| + \left| \sum_{j=1}^{\tilde{N}} X_j - N(I_\Lambda)|\Lambda| \right|$$

By Theorem 1.16, provided (1.45) holds, with a probability larger than

$$1 - \exp(-c|\Lambda||I_\Lambda|(|I_\Lambda|\ell^d)^\rho) - \exp(-cN(I_\Lambda)|\Lambda|\ell'\ell^{-1}),$$

we have

$$\left| N(\omega, \Lambda, I_\Lambda) - \sum_j X_j \right| \lesssim |\Lambda|(|I_\Lambda|\ell^d)^{1+\rho} + |\Lambda|N(I_\Lambda)(\ell')^{d+1}\ell^{-1} = o(N(I_\Lambda)|\Lambda|).$$

Next, for $\delta \in (\frac{1}{2}, 1)$, the large deviation estimate for i.i.d. Bernoulli variables valued in $\{0, 1\}$ with expectation $p = p(\tilde{N})$ s.t. $p \in (0, 1/2]$ and $p\tilde{N} \rightarrow +\infty$ (see e.g. [Dur96]) yields, for $p\tilde{N}$ sufficiently large,

$$\mathbb{P} \left(\left| \sum_{j=1}^{\tilde{N}} X_j - p\tilde{N} \right| \geq (p\tilde{N})^\delta \right) \leq e^{-(p\tilde{N})^{2\delta-1}/4}. \quad (2.11)$$

The result of Corollary 2.1 follows since $p\tilde{N} = N(I_\Lambda)|\Lambda|(1+o(1))$. \square

Remark 2.1. We can get a more precise version of Corollary 2.1 by optimizing in the intermediate scale ℓ , the number of eigenvalues we miss in the picture of Theorem 1.16, namely by choosing ℓ so that $Kn = K'n'$. Estimates can even be improved by resorting to higher order Minami estimates in order to bound the missing eigenvalues (replacing the crude deterministic bound given by the Weyl formula).

Remark 2.2. If $N(I) \ll |I|^{1+\rho}$, Corollary 2.1 still holds if one can improve on the Minami estimate, replacing one power of the interval length $|I|$ by $N(I)$. This can be done in some cases [GKI]

Proof of Theorem 1.9. We now apply Corollary 2.1 to derive Theorem 1.9. It suffices to check that, under the assumptions of Theorem 1.9, for some $\delta > 0$, one has

$$\min(|\Lambda||I_\Lambda|(|I_\Lambda|\ell^d)^\rho, N(I_\Lambda)|\Lambda|\ell'\ell^{-1}) \geq (N(I_\Lambda)|\Lambda|)^\delta.$$

Pick the scale ℓ in Theorem 1.16 so that $\ell^d|I_\Lambda| \asymp (N(I_\Lambda)|\Lambda|)^{-\varepsilon}$ for some $\varepsilon > 0$ to be chosen. Thus, using (W), we compute $|\Lambda||I_\Lambda|(|I_\Lambda|\ell^d)^\rho \gtrsim (N(I_\Lambda)|\Lambda|)^{1-\varepsilon\rho}$. On the other hand,

$$N(I_\Lambda)|\Lambda|\ell'\ell^{-1} \gtrsim N(I_\Lambda)|\Lambda|N(I_\Lambda)^{-1/d}|\Lambda|^{-\varepsilon/d} = (N(I_\Lambda)|\Lambda|)^{1-\varepsilon}.$$

This completes the proof of Theorem 1.9. \square

3. THE PROOFS OF THEOREMS 1.16 AND 1.15

Recall Λ is a cube of side length L . Let I_Λ be an interval (included in I the region of localization). We first prove the useful

Lemma 3.1. *Assume (IAD), (W), (M) and (Loc). For any $q > 0$, the following holds:*

- (1) *Let x_0 be the (random) center of localization of an eigenfunction of $H_\omega(\Lambda)$ associated to an energy E in I_Λ . Pick $\ell' \leq \ell < L$, and consider a box $\Lambda_{\ell'}(\gamma) \subset \Lambda$. Assume that $x_0 \in \Lambda_{\ell'}(\gamma)$. Then, for $|\Lambda|$ large enough, with a probability at least $1 - |\Lambda|^{-q}$, there exists an eigenvalue of $H_\omega(\Lambda_{\ell'+\ell'}(\gamma))$ in a neighborhood of E of size $\mathcal{O}(e^{-c\ell'})$. As a consequence, if $e^{-c\ell'} \leq N(I_\Lambda)$, for $k = 1, 2$, the probability to find at least k such centers of localization of $H_\omega(\Lambda)$ in a given cube $\Lambda_{\ell'}(\gamma)$ is bounded respectively by $CN(I_\Lambda)\ell^d + |\Lambda|^{-q}$ for $k = 1$ and $C(|I_\Lambda|\ell^d)^{1+\rho} + |\Lambda|^{-q}$ for $k = 2$.*

- (2) Conversely, pick $\ell' \leq \ell$ so that $e^{-c\ell'} \leq N(I_\Lambda)$. For $|\Lambda|$ large enough, with probability at least $1 - |\Lambda|^{-q} - CN(I_\Lambda)\ell'\ell^{d-1}$, if $x_0 \in \Lambda_\ell \subset \Lambda$ is a center of localization of a given eigenfunction of $H_\omega(\Lambda_\ell)$ with energy E , then x_0 is close, up to a distance $(\log |\Lambda|)^{1/\xi}$, to the center of localization of some eigenfunction of $H_\omega(\Lambda)$ with energy in $[E - ce^{-c\ell'}, E + ce^{-c\ell'}]$.

Lemma 3.1 is similar to Lemma 3.1 in [K11]. It is central in our analysis of the eigenvalues in the localized regime ; it reflects the localized nature of the eigenstates.

Proof. We start with the point 1. Let φ be the eigenfunction associated to the center x_0 , and $E \in I_\Lambda$ the corresponding eigenvalue. Let Ψ_ℓ be a smooth characteristic function covering the cube $\Lambda_\ell(\gamma)$, i.e. Ψ_ℓ such that $\text{supp } \Psi_\ell \subset \Lambda_{\ell+\ell'}(\gamma)$, $\Psi_\ell = 1$ on $\Lambda_{\ell+\frac{1}{2}\ell'}(\gamma)$, $\text{supp } \nabla \Psi_\ell \subset \Lambda_{\ell+\ell'}(\gamma) \setminus \Lambda_{\ell+\frac{1}{2}\ell'}(\gamma)$.

By the localization property of the eigenfunction φ that holds true with probability at least $1 - |\Lambda|^{-q}$, we have $\|\Psi_\ell \varphi\| \geq \frac{1}{2}$ for ℓ' large enough. Set $\eta_\ell := \Psi_\ell \varphi / \|\Psi_\ell \varphi\|$. Then, η_ℓ is an approximate eigenvector of the Hamiltonian $H_\omega(\Lambda_{\ell+\ell'}(\gamma))$, in the sense that $\|\eta_\ell\| = 1$ and

$$\|(H_\omega(\Lambda_{\ell+\ell'}(\gamma)) - E)\eta_\ell\| \leq 2\|[H_\omega(\Lambda_{\ell+\ell'}(\gamma)), \Psi_\ell]\varphi\| \lesssim \sup_{\text{supp } \nabla \Psi_\ell} |\varphi| \lesssim e^{-c\ell'}.$$

It follows that $H_\omega(\Lambda_{\ell+\ell'}(\gamma))$ has an eigenvalue in the interval $[E - ce^{-c\ell'}, E + ce^{-c\ell'}]$.

Next, the probability estimate follows from Wegner if $k = 1$ and Minami if $k = 2$, provided we choose $e^{-c\ell'} \leq |I_\Lambda|$. This completes the proof of point 1.

Let us prove point 2. Assume first that $\text{dist}(x_0, \partial\Lambda_\ell) \geq \ell'$. Then, it is immediate to check that the eigenfunction $\varphi \in L^2(\Lambda_\ell)$ extended to $L^2(\Lambda)$ by setting $\varphi = 0$ on $\Lambda \setminus \Lambda_\ell$, is an approximate eigenfunction of $H_\omega(\Lambda)$ in the sense that

$$\|(H_\omega(\Lambda) - E)\varphi\| \lesssim \sup_{\Lambda_\ell \setminus \Lambda_{\ell-2}} |\varphi| \lesssim e^{-c\ell'}.$$

It remains to consider the situation where $\text{dist}(x_0, \partial\Lambda_\ell) \leq \ell'$. Cover the boundary $\partial\Lambda_\ell$ of the periodic box Λ_ℓ by $c_d(\ell/\ell')^{d-1}$ boxes of side length ℓ' . The center x_0 belongs to one of them. By the first part of the lemma, if a box $\Lambda_{\ell'}(\gamma) \subset \Lambda_\ell$ has a center of localization of $H_\omega(\Lambda_\ell)$ then $H_\omega(\Lambda_{2\ell'}(\gamma))$ has an eigenvalue as well in the interval I_Λ enlarged by a quantity $\mathcal{O}(e^{-c\ell'})$. The probability of such an event is $\lesssim |I_\Lambda|(\ell')^d$. This completes the proof of Lemma 3.1. \square

Proof of Proposition 1.2. It is an immediate consequence of (W) and Lemma 3.1. \square

Remark 3.1. If we replace assumption (IAD) with a weaker assumption e.g. so decaying correlation for remote cubes, then an analogue of Lemma 3.1 stays true up to the fact that the errors made in the eigenvalue approximations will now depend on these correlations.

Proof of Theorem 1.15. For $\beta' = 0+$ and $\beta > \beta'$ to be chosen later, set $\ell' = L^{\beta'}$ and ℓ so that $(\ell + \ell')k + \ell' = L$, where $k = \lceil L^{1-\beta} \rceil$. Note that $\ell = \mathcal{O}(L^\beta)$ in the large volume limit. With these definitions we can pick equally distributed boxes of size ℓ in Λ (with distance ℓ' between two neighbors) satisfying the conditions of Theorem 1.15.

Note that, for L large enough, one has $\ell' > R$, so that events based on distinct boxes $\Lambda_\ell(\gamma_j)$'s are independent.

Recall $N(I_\Lambda) \lesssim |\Lambda|^{-\alpha}$

Let $\mathcal{S}_{\ell,L}$ be the set of boxes $\Lambda_{\ell-\ell'}(\gamma_j) \subset \Lambda$ containing at least two centers of localization of $H_\omega(\Lambda)$. It follows from Lemma 3.1 and (1.12) that,

$$\mathbb{P}(\#\mathcal{S}_{\ell,L} \geq 1) \lesssim |\Lambda|^{1-\beta} (|\Lambda|^\beta |I_\Lambda|)^{1+\rho} \leq |\Lambda|^{1+\beta\rho} N(I_\Lambda)^{\frac{1+\rho}{1+\beta}} \lesssim |\Lambda|^{1+\beta\rho-\alpha\gamma^{-1}}, \quad (3.1)$$

where

$$\gamma := \frac{1+\tilde{\rho}}{1+\rho} \in]0, 1[. \quad (3.2)$$

To insure that all the centers of $H_\omega(\Lambda)$ fall inside one of the $\Lambda_\ell(\gamma_j)$'s and actually sufficiently well inside (by a distance ℓ'), we define $\Upsilon \subset \Lambda$ as the set $\Lambda \setminus \cup_j \Lambda_\ell(\gamma_j)$ enlarged by a length ℓ' . In other terms $\Upsilon = \Lambda \setminus \cup_j \Lambda_{\ell-\ell'}(\gamma_j)$. It follows from item (1) of Lemma 3.1 that,

$$\mathbb{P}(H_\omega(\Lambda) \text{ has a center of localization in } \Upsilon) \lesssim |\Upsilon|N(I_\Lambda) \lesssim |\Lambda|^{1-\frac{1}{d}(\beta-\beta')-\alpha} \quad (3.3)$$

We thus require that

$$\alpha > \max \left((1 + \beta\rho)\gamma, 1 - \frac{1}{d}(\beta - \beta') \right).$$

Optimization yields

$$\beta = \frac{1 - \gamma + \beta'/d}{\rho\gamma + d^{-1}}, \quad \alpha > 1 - \frac{1 - \gamma + \beta'/d}{\rho\gamma d + 1}. \quad (3.4)$$

Recall (1.14) that is

$$\alpha_{d,\rho,\bar{\rho}} = 1 - \frac{1 - \gamma}{\rho\gamma d + 1} = \frac{\rho d + 1}{\rho d + \gamma^{-1}}.$$

It follows from (3.1) and (3.3) that, with probability $\geq 1 - c|\Lambda|^{-(\alpha - \alpha_{d,\rho,\bar{\rho}})}$, item (2) of Theorem 1.15 holds, as well as the ‘‘only if’’ part of item (3).

Next, item (2) of Lemma 3.1 implies that item (1) as well as the ‘‘if’’ part of item (3) of Theorem 1.15 hold, with probability at least $1 - c|\Lambda|^{-\alpha + \beta - \frac{1}{d}(\beta - \beta')} \geq 1 - c|\Lambda|^{-(\alpha - \alpha_{d,\rho,\bar{\rho}})}$ for $\beta < 1$.

This completes the proof of Theorem 1.15. \square

Proof of Theorem 1.16. For a scale ℓ_Λ given, we set $q = [(L - \ell')/(\ell_\Lambda + \ell')]$. We may then adjust the scale ℓ_Λ by enlarging it to a new scale ℓ so that $(\ell + \ell')q + \ell' = L$ and $0 \leq \ell - \ell_\Lambda \lesssim \ell_\Lambda^2/|\Lambda| = o(\ell_\Lambda)$. As a consequence, we can consider a collection of boxes $\Lambda_\ell(\gamma_j)$ equally distant to their closest neighbors by a length ℓ' and satisfying the description of Theorem 1.16. In particular, events based on different boxes are independent.

Let $\mathcal{S}_{\ell,L}$ be the set of disjoint boxes $\Lambda_\ell(\gamma_j) \subset \Lambda$ containing at least 2 centers of localization of $H_\omega(\Lambda)$. We set $n = \ell^d$. It follows from Lemma 3.1 (taking into account $\ell' \ll \ell$) that, using independence and Stirling's formula,

$$\mathbb{P}(\#\mathcal{S}_{\ell,L} = k) \lesssim \binom{|\Lambda|/n}{k} (|I_\Lambda|n)^{(1+\rho)k} \lesssim (e|\Lambda|/(nk))^k (|I_\Lambda|n)^{(1+\rho)k} = \left(\frac{e|\Lambda|}{k} N(I_\Lambda)^{\frac{1+\rho}{1+\bar{\rho}}} n^\rho \right)^k \lesssim 2^{-k}$$

if we choose

$$k \geq K := \left[2eN(I_\Lambda)|\Lambda|(N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{1+\bar{\rho}}} n^\rho) \right] + 1 = \frac{|\Lambda|}{n} O\left([N(I_\Lambda)^{1/(1+\bar{\rho})}n]^{1+\rho}\right) = o\left(\frac{|\Lambda|}{n}\right), \quad (3.5)$$

by assumption. As a consequence, we get that

$$\mathbb{P}(\#\mathcal{S}_{\ell,L} \geq K) \lesssim 2^{-K}.$$

So that, with probability larger than $1 - 2^{-K}$, we can assume that the boxes $\Lambda_\ell(\gamma_j)$, except at most K of them, contain at most one center of localization.

We now control the number of centers of localization that may be contained in these K exceptional boxes. In a box of size ℓ , the deterministic a priori bound on the number of eigenvalues guarantees that this number is bounded by $\lesssim \ell^d = n$ (see e.g. [RS78]). Using this crude estimate the number of eigenvalues we miss with these K boxes is bounded by

$$Kn \lesssim N(I_\Lambda)|\Lambda|(N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{1+\bar{\rho}}} n^{1+\rho}) = o(N(I_\Lambda)|\Lambda|),$$

provided

$$N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{1+\bar{\rho}}} n^{1+\rho} = o(1). \quad (3.6)$$

We now turn to the complement of the $\Lambda_\ell(\gamma_j)$'s, and show that, with a good probability, it contains less than $|\Lambda||I_\Lambda|_{\alpha_\Lambda}$ centers of localization. We thus consider $\Upsilon = \Lambda \setminus \bigcup_{j=1}^{2^{d-1}} \Lambda_{\ell-\ell'}(\gamma_j)$. We partition Υ into 2^{d-1} sets of boxes of side length ℓ' . More precisely, we write $\Upsilon = \bigcup_{m=1}^{2^{d-1}} \Upsilon_m$, with $\Upsilon_m \cap \Upsilon_{m'} = \emptyset$ if $m \neq m'$, each Υ_m is a union of boxes of side length ℓ' which are two by two distant by at least ℓ' .

Clearly, one has $|\Upsilon| \lesssim |\Lambda|\ell'/\ell$.

Let $\mathcal{S}'_{\ell,L}$ be the set of boxes $\Lambda_{\ell'}(\gamma_j) \in \Upsilon$ containing at least one centers of localization of $H_\omega(\Lambda)$. Since we assumed $N(I_\Lambda) \gg |I_\Lambda|^{1+\rho}$, we shall replace the Wegner estimate by the more accurate approximation given by Lemma 2.1. Recall that the a priori bound $N(I_\Lambda) \leq C|I_\Lambda|$ holds by assumption (W). From considerations similar to those above, for each given m , taking into account that boxes in Υ_m may be enlarged by, say, $\frac{1}{10}\ell'$, for the distance between two of them is larger than ℓ' , it follows from Lemma 3.1 and Lemma 2.1 that, using independence and Stirling formula,

$$\mathbb{P}(\#\mathcal{S}'_{\ell,L} \cap \Upsilon_m \geq K') \lesssim \sum_{k \geq K'} \binom{|\Upsilon_m|/(\ell')^d}{k} (N(I_\Lambda)(\ell')^d)^k \lesssim \left(\frac{e|\Lambda|N(I_\Lambda)\ell'}{k\ell} \right)^k \lesssim 2^{-K'},$$

provided, for C , a large enough constant, one sets

$$K' := \left\lceil \frac{C|\Lambda|N(I_\Lambda)\ell'}{\ell} \right\rceil + 1 = \frac{|\Lambda|}{n} O(N(I_\Lambda)\ell^{d-1}\ell') = \frac{|\Lambda|}{n} O(N(I_\Lambda)^{1/(1+\bar{\rho})}n) = o\left(\frac{|\Lambda|}{n}\right). \quad (3.7)$$

As a consequence, one computes

$$\mathbb{P}(\#\mathcal{S}'_{\ell,L} \geq 2^{d-1}K') \leq \mathbb{P}(\exists m, \#\mathcal{S}'_{\ell,L} \cap \Upsilon_m \geq K') \leq \sum_{m=1}^{2^{d-1}} \mathbb{P}(\#\mathcal{S}'_{\ell,L} \cap \Upsilon_m \geq K') \lesssim 2^{-K'}.$$

Hence, up to $2^{d-1}K'$ boxes, we can assume that boxes of size ℓ' in Υ' contain at most one center of localization.

The maximal number of eigenvalues that can be contained in these $2^{d-1}K'$ bad boxes is $\lesssim K'n' = o(N(I_\Lambda)|\Lambda|)$ provided

$$\ell \gg (\ell')^{d+1}. \quad (3.8)$$

Combining (3.6) and (3.8), we see that the intermediate scale ℓ has to satisfy (1.45). To summarize, we proved that the picture described by Theorem 1.16 holds with a probability larger than

$$1 - e^{-cK} - e^{-cK'} \geq 1 - \exp\left(-cN(I_\Lambda)|\Lambda|(N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{1+\bar{\rho}}}\ell^{d\rho})\right) - \exp(-cN(I_\Lambda)|\Lambda|\ell'\ell^{-1}) \quad (3.9)$$

Moreover, the number of eigenvalues of $H_\omega(\Lambda)$ that are not described by this picture is bounded by

$$C(Kn + K'n') \lesssim N(I_\Lambda)|\Lambda| \left((N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{1+\bar{\rho}}}\ell^{d(1+\rho)} + \frac{(\ell')^{d+1}}{\ell}) \right) = o(N(I_\Lambda)|\Lambda|), \quad (3.10)$$

provided (1.45) holds.

This completes the proof of Theorem 1.16. \square

4. THE SPECTRAL STATISTICS

In this section, we prove most of the results on the local spectral statistics described in section 1. The whole of our analysis relies on Theorems 1.15 and 1.16.

4.1. Convergence of the local levels statistics. We first prove the uniform Poisson convergence, Theorem 1.3, of which Theorem 1.2 is an immediate consequence.

4.1.1. *Proof of Theorem 1.3.* We keep the notations of section 1.4.1. Recall that $\Lambda = \Lambda_L$. We know that Theorem 1.15 holds. Let \tilde{N} be the number of the cubes constructed in Theorem 1.15 and $(\Lambda_\ell(\gamma_j))_{1 \leq j \leq \tilde{N}}$ be those cubes. Then, $\tilde{N} = |\Lambda|^{1-\beta}(1 - O(|\Lambda|^{-(1-\beta)/d}))$. Recall that $\xi_j(E_0, \omega, \Lambda)$ is defined by (1.11) and consider the event

$$\Omega_{I_1, k_1; I_2, k_2; \dots; I_p, k_p}^\Lambda := \bigcap_{l=1}^p \{\omega; \#\{j; \xi_j(\omega, \Lambda) \in I_l\} = k_l\}.$$

Pick $\delta d < p := \beta' \xi$ where β' is given by Theorem 1.15 and δ by (1.15). Define $I_j^+ = I_j \cup [-e^{-L^p}, e^{-L^p}]$ and $I_j^- = I_j \cap ({}^c I_j + [-e^{-L^p}, e^{-L^p}])$. Clearly, $I_j^- \subset I_j \subset I_j^+$. Moreover, by (1.15), for L sufficiently large, we have that $I_j^+ \cap I_k^+ = I_j^- \cap I_k^- = \emptyset$ for $j < k$ and

- (1) either $|I_j^+| = |I_j|(1 + O(e^{-L^p/2}))$ and $|I_j^-| = |I_j|(1 + O(e^{-L^p/2}))$,
- (2) or $|I_j^-| \leq |I_j^+| \leq e^{-L^p/2}$.

If for some j such that $k_j > 0$, we are in alternative (2), the Wegner estimate tells us that, for L sufficiently large,

$$\mathbb{P}(\Omega_{I_1, k_1; I_2, k_2; \dots; I_p, k_p}^\Lambda) \leq e^{-L^p/4}.$$

Hence we get

$$\left| \mathbb{P}\left(\Omega_{I_1, k_1; I_2, k_2; \dots; I_p, k_p}^\Lambda\right) - e^{-|I_1| \frac{|I_1|^{k_1}}{k_1!}} \dots e^{-|I_p| \frac{|I_p|^{k_p}}{k_p!}} \right| \leq 2e^{-L^p/4}$$

as $e^{-|I| \frac{|I|^k}{k!}} \leq 1$.

Hence, we may assume that we are in alternative (1) for all j which we do from now on.

Recall that the function N is non decreasing and continuous. Thus, for I an interval, the set $\{E; N(E) \in I\}$ is an interval. For a cube Λ and an interval I , define the Bernoulli random variable

$$X_{\Lambda, I} = \mathbf{1}_{H_\omega(\Lambda) \text{ has an e.v. in } N^{-1}[N(E_0) + |\Lambda|^{-1}I]}$$

By Theorem 1.15, in particular (1.44), and the Wegner estimate (W), we know that

$$\begin{aligned} & \bigcap_{l=1}^p \left\{ \omega; \#\{j; X_{\Lambda_\ell(\gamma_j), I_l^-} = 1\} = k_l \right\} \cap \mathcal{Z}_\Lambda \subset \Omega_{I_1, k_1; I_2, k_2; \dots; I_p, k_p}^\Lambda \cap \mathcal{Z}_\Lambda, \\ & \Omega_{I_1, k_1; I_2, k_2; \dots; I_p, k_p}^\Lambda \cap \mathcal{Z}_\Lambda \subset \bigcap_{l=1}^p \left\{ \omega; \#\{j; X_{\Lambda_\ell(\gamma_j), I_l^-} = 1\} = k_l \right\} \cap \mathcal{Z}_\Lambda \end{aligned}$$

As the cubes $(\Lambda_\ell(\gamma_j))_j$ are at distance at least R from each other (see (IAD)) and the intervals $(I_l^+)_l$ (resp. $(I_l^-)_l$) are two by two disjoint, the events $\left(\left\{ \omega; \#\{j; X_{\Lambda_\ell(\gamma_j), I_l^-} = 1\} = k_l \right\} \right)_l$ (resp. $\left(\left\{ \omega; \#\{j; X_{\Lambda_\ell(\gamma_j), I_l^+} = 1\} = k_l \right\} \right)_l$) are two by two independent. Hence,

$$\begin{aligned} & \prod_{l=1}^p \mathbb{P}(\{\omega; \#\{j; X_{\Lambda_\ell(\gamma_j), I_l^-} = 1\} = k_l\}) - (1 - \mathbb{P}(\mathcal{Z}_\Lambda)) \\ & \leq \mathbb{P}(\Omega_{I_1, k_1; I_2, k_2; \dots; I_p, k_p}^\Lambda \cap \mathcal{Z}_\Lambda) \\ & \leq \prod_{l=1}^p \mathbb{P}(\{\omega; \#\{j; X_{\Lambda_\ell(\gamma_j), I_l^+} = 1\} = k_l\}) + (1 - \mathbb{P}(\mathcal{Z}_\Lambda)). \end{aligned}$$

As the random variables $(X_{\Lambda_\ell(\gamma_j), I_l^\pm})_j$ are independent and identically Bernoulli distributed, setting

$$p_l^\pm = \mathbb{P}\{H_\omega(\Lambda_\ell(0)) \text{ has an e.v. } E \text{ s.t. } N(E) \in N(E_0) + |\Lambda|^{-1}I_l^\pm\}$$

we get

$$\begin{aligned} \prod_{l=1}^p \binom{\tilde{N}}{k_l} (p_l^-)^{k_l} (1 - p_l^-)^{\tilde{N} - k_l} - (1 - \mathbb{P}(\mathcal{Z}_\Lambda)) &\leq \mathbb{P}(\Omega_{I_1, k_1; I_2, k_2; \dots; I_p, k_p}^\Lambda \cap \mathcal{Z}_\Lambda) \\ &\leq \prod_{l=1}^p \binom{\tilde{N}}{k_l} (p_l^+)^{k_l} (1 - p_l^+)^{\tilde{N} - k_l} + (1 - \mathbb{P}(\mathcal{Z}_\Lambda)) \end{aligned}$$

By (2.3) of Lemma 2.1, the assumption (1.12) and the definition of $(I_l^\pm)_l$, we have that

$$p_l^\pm = |I_l| |\Lambda|^{\beta-1} (1 + o(1)) = |I_l| \tilde{N}^{-1} (1 + o(1)).$$

Using Stirling's formula, as $\tilde{N} \geq \sup(k_l, |I_l|)$, one checks that

$$e^{-|I_l|} \frac{|I_l|^{k_l}}{k_l!} + \binom{\tilde{N}}{k_l} (p_l^\pm)^{k_l} (1 - p_l^\pm)^{\tilde{N} - k_l} \xrightarrow{|I_l| + k_l \rightarrow +\infty} 0.$$

Hence, we can assume that k_l and I_l are bounded and use the standard Poisson limit Theorem when \tilde{N} is large to get

$$\binom{\tilde{N}}{k_l} (p_l^\pm)^{k_l} (1 - p_l^\pm)^{\tilde{N} - k_l} = e^{-|I_l|} \frac{|I_l|^{k_l}}{k_l!} (1 + o(1)).$$

Hence we get

$$\lim_{|\Lambda| \rightarrow +\infty} \left| \mathbb{P} \left(\Omega_{I_1, k_1; I_2, k_2; \dots; I_p, k_p}^\Lambda \right) - e^{-|I_1|} \frac{|I_1|^{k_1}}{k_1!} \dots e^{-|I_p|} \frac{|I_p|^{k_p}}{k_p!} \right| = 0$$

and the proof of Theorem 1.3 is complete. \square

4.1.2. *The asymptotic independence.* We now turn to the proof of Theorem 1.4. Decompose Λ into the boxes constructed in Theorem 1.15 i.e. $\Lambda = \cup_{\gamma \in \Gamma_\Lambda} \Lambda_\ell(\gamma)$; the scale ℓ is determined by Theorem 1.15 and $N := \#\Gamma_\Lambda \sim |\Lambda| \ell^{-d}$. The set of boxes thus obtained are the same for both energies E and E' . For $I \subset \mathbb{R}$ a compact set, define the random variables

$$X_\gamma(E, I) = \begin{cases} 1 & \text{if } H_\omega(\Lambda_\ell(\gamma)) \text{ has an e.v. in } N^{-1}[N(E_0) + |\Lambda|^{-1}I], \\ 0 & \text{if not.} \end{cases} \quad (4.1)$$

Then, to prove Theorem 1.4, it suffices to prove that, for $(k, k') \in \mathbb{N}^2$ and $I \subset \mathbb{R}$ and $I' \subset \mathbb{R}$ two compact sets, one has

$$\begin{aligned} \lim_{|\Lambda| \rightarrow +\infty} \mathbb{E} \left[\left(\sum_{\gamma \in \Gamma_\Lambda} X_\gamma(E, I) \right)^k \left(\sum_{\gamma \in \Gamma_\Lambda} X_\gamma(E', I') \right)^{k'} \right] \\ = \lim_{|\Lambda| \rightarrow +\infty} \mathbb{E} \left[\left(\sum_{\gamma \in \Gamma_\Lambda} X_\gamma(E, I) \right)^k \right] \mathbb{E} \left[\left(\sum_{\gamma \in \Gamma_\Lambda} X_\gamma(E', I') \right)^{k'} \right]. \end{aligned} \quad (4.2)$$

Therefore, using the independence of the cubes, we expand the sums

$$\begin{aligned} S_N(k, k') &= \mathbb{E} \left[\left(\sum_{\gamma \in \Gamma_\Lambda} X_\gamma(E, I) \right)^k \left(\sum_{\gamma \in \Gamma_\Lambda} X_\gamma(E', I') \right)^{k'} \right] \\ &= \sum_{\gamma_1, \dots, \gamma_k} \sum_{\gamma'_1, \dots, \gamma'_{k'}} \mathbb{E} \left[X_{\gamma_1}(E, I) \cdots X_{\gamma_k}(E, I) \cdot X_{\gamma'_1}(E', I') \cdots X_{\gamma'_{k'}}(E', I') \right] \\ &= G_N(k, k') + R_N(k, k') \end{aligned}$$

where

$$G_N(k, k') = \sum_{\{\gamma_1, \dots, \gamma_k\} \cap \{\gamma'_1, \dots, \gamma'_{k'}\} = \emptyset} \mathbb{E} \left[X_{\gamma_1}(E, I) \cdots X_{\gamma_k}(E, I) \cdot X_{\gamma'_1}(E', I') \cdots X_{\gamma'_{k'}}(E', I') \right]$$

and

$$\begin{aligned} R_N(k, k') &= \sum_{\gamma} \sum_{\substack{\gamma_1, \dots, \gamma_l \\ \gamma'_1, \dots, \gamma'_{l'}}} \mathbb{E} \left[X_\gamma(E, I) X_\gamma(E', I') \cdot \prod_{j=1}^{k-1} X_{\gamma_j}(E, I) \cdot \prod_{j'=1}^{k'-1} X_{\gamma'_{j'}}(E', I') \right] \\ &= \sum_{\gamma} \sum_{l=1}^{k-1} \sum_{l'=1}^{k'-1} \binom{l}{k-1} \binom{l'}{k'-1} \\ &\quad \sum_{\substack{\gamma \notin \{\gamma_1, \dots, \gamma_l\} \\ \gamma \notin \{\gamma'_1, \dots, \gamma'_{l'}\}}} \mathbb{E} \left[X_\gamma(E, I) X_\gamma(E', I') \cdot \prod_{j=1}^l X_{\gamma_j}(E, I) \cdot \prod_{j'=1}^{l'} X_{\gamma'_{j'}}(E', I') \right] \\ &= \sum_{\gamma} \mathbb{E} [X_\gamma(E, I) X_\gamma(E', I')] \sum_{l=1}^{k-1} \sum_{l'=1}^{k'-1} \binom{l}{k-1} \binom{l'}{k'-1} \\ &\quad \sum_{\substack{\gamma \notin \{\gamma_1, \dots, \gamma_l\} \\ \gamma \notin \{\gamma'_1, \dots, \gamma'_{l'}\}}} \mathbb{E} \left[\prod_{j=1}^l X_{\gamma_j}(E, I) \cdot \prod_{j'=1}^{l'} X_{\gamma'_{j'}}(E', I') \right]. \end{aligned}$$

Hence,

$$R_N(k, k') = \sum_{\gamma} \mathbb{E} [X_\gamma(E, I) X_\gamma(E', I')] \sum_{l=1}^{k-1} \sum_{l'=1}^{k'-1} \binom{l}{k-1} \binom{l'}{k'-1} S_{N-1}(l, l'). \quad (4.3)$$

On the other hand, one computes

$$\begin{aligned} S_N(k) S_N(k') &= \mathbb{E} \left[\left(\sum_{\gamma \in \Gamma_\Lambda} X_\gamma(E, I) \right)^k \right] \mathbb{E} \left[\left(\sum_{\gamma \in \Gamma_\Lambda} X_\gamma(E', I') \right)^{k'} \right] \\ &= \sum_{\gamma_1, \dots, \gamma_k} \mathbb{E} [X_{\gamma_1}(E, I) \cdots X_{\gamma_k}(E, I)] \sum_{\gamma'_1, \dots, \gamma'_{k'}} \mathbb{E} [X_{\gamma'_1}(E', I') \cdots X_{\gamma'_{k'}}(E', I')] \\ &= G_N(k, k') + Q_N(k, k') \end{aligned}$$

where

$$\begin{aligned}
Q_N(k, k') &= \sum_{\gamma} \sum_{\gamma_1, \dots, \gamma_{k-1}} \mathbb{E} [X_{\gamma}(E, I) X_{\gamma_1}(E, I) \cdots X_{\gamma_{k-1}}(E, I)] \\
&\quad \cdot \sum_{\gamma'_1, \dots, \gamma'_{k'-1}} \mathbb{E} [X_{\gamma}(E', I') X_{\gamma'_1}(E', I') \cdots X_{\gamma'_{k'-1}}(E', I')] \\
&= \sum_{\gamma} \mathbb{E} [X_{\gamma}(E, I)] \mathbb{E} [X_{\gamma}(E', I')] \left[\sum_{l=1}^{k-1} \binom{l}{k-1} S_{N-1}(l) \right]^2.
\end{aligned} \tag{4.4}$$

Hence,

$$S_N(k, k') - S_N(k)S_N(k') = R_N(k, k') - Q_N(k, k'). \tag{4.5}$$

By Cauchy-Schwartz, one has

$$S_N(k, k') \leq \sqrt{S_N(2k)S_N(2k')}.$$

On the other hand, as $\mathbb{P}(X_{\gamma} = 1) \leq C/N$ by Lemma 2.1, one computes

$$\begin{aligned}
S_N(k) &\leq \sum_{j=0}^N \binom{j}{N} \left(\frac{C}{N}\right)^j j^k \leq C_k \sum_{j=0}^N \binom{j}{N} \left(\frac{Ck}{N}\right)^j \\
&\leq C_k e^{N \log(1+Ck/N)} \leq C_k e^{Ck} < +\infty
\end{aligned}$$

where $C_k = e^{k(\log k - 1 - \log \log k)}$ for $k \geq 2$.

So, for any k and k' , one has

$$\sup_{N \geq 1} |S_N(k)| + |S_N(k, k')| < +\infty,$$

Thus, using (4.3), (4.4) and (4.5), one obtains

$$\begin{aligned}
&|S_N(k, k') - S_N(k)S_N(k')| \\
&\leq C_{k, k'} \max \left(\sum_{\gamma} \mathbb{E} [X_{\gamma}(E, I)] \mathbb{E} [X_{\gamma}(E', I')], \sum_{\gamma} \mathbb{E} [X_{\gamma}(E, I) X_{\gamma}(E', I')] \right)
\end{aligned}$$

Hence, (4.2) and thus Theorem 1.4 and 1.5 follow from the following two properties

$$\begin{aligned}
&\sum_{\gamma} \mathbb{E} [X_{\gamma}(E, I)] \mathbb{E} [X_{\gamma}(E', I')] \xrightarrow{|\Lambda| \rightarrow +\infty} 0, \\
&\sum_{\gamma} \mathbb{E} [X_{\gamma}(E, I) X_{\gamma}(E', I')] \xrightarrow{|\Lambda| \rightarrow +\infty} 0.
\end{aligned}$$

As the operators $(H_{\omega}(\Lambda_{\ell}(\gamma)))_{\gamma}$ are i.i.d, this will be proved if we prove that

$$\left(\frac{L}{\ell}\right)^d \mathbb{E} [X_0(E, I)] \mathbb{E} [X_0(E', I')] \xrightarrow{|\Lambda| \rightarrow +\infty} 0, \tag{4.6}$$

$$\left(\frac{L}{\ell}\right)^d \mathbb{E} [X_0(E, I) X_0(E', I')] \xrightarrow{|\Lambda| \rightarrow +\infty} 0. \tag{4.7}$$

The limit (4.6) is an immediate consequence of the Wegner estimate as $\ell/L \rightarrow 0$. The limit (4.7) clearly is a consequence of (D) if $E \neq E'$ are fixed energies, and of (GM) if $E = E_{\Lambda}$ and $E' = E'_{\Lambda}$ satisfy the assumptions of Theorem 1.5.

This completes the proof of Theorem 1.4 and 1.5. \square

4.2. Study of the (levels, centers) statistics. In this section, we will prove Theorems 1.6, 1.7 and 1.8. To control the eigenvalues, as in the previous section, we use Theorem 1.15. So we keep the same notations here.

4.2.1. *The proof of Theorem 1.6.* The proof of Theorem 1.6 is very similar to that of Theorem 1.3. We only indicate the differences and keep the same notations. We need to estimate the probability of the event

$$\Omega_{(I_l, C_l, k_l)_{1 \leq l \leq p}}^\Lambda := \bigcap_{l=1}^p \left\{ \omega; \# \left\{ j; \begin{array}{l} \xi_j(\omega, \Lambda) \in I_l \\ x_j(\omega, \Lambda)/L \in C_l \end{array} \right\} = k_l \right\}$$

The same reasoning as in the proof of Theorem 1.3 yields

$$\begin{aligned} & \prod_{l=1}^p \mathbb{P}(\{\omega; \#\{j; \gamma_j/L \in C_l^- \text{ and } X_{\Lambda_\ell(\gamma_j), I_l^-} = 1\} = k_l\}) - (1 - \mathbb{P}(\mathcal{Z}_\Lambda)) \\ & \leq \mathbb{P}(\Omega_{(I_l, C_l, k_l)_{1 \leq l \leq p}}^\Lambda \cap \mathcal{Z}_\Lambda) \\ & \leq \prod_{l=1}^p \mathbb{P}(\{\omega; \#\{j; \gamma_j/L \in C_l^+ \text{ and } X_{\Lambda_\ell(\gamma_j), I_l^+} = 1\} = k_l\}) + (1 - \mathbb{P}(\mathcal{Z}_\Lambda)) \end{aligned}$$

where $C_l^+ = C_l + [-\ell/L, \ell/L]^d$ and $\mathbb{R}^d \setminus C_l^- = \mathbb{R}^d \setminus (C_l + (-\ell/L, \ell/L)^d)$. Hence,

$$\begin{aligned} & \prod_{l=1}^p \binom{\tilde{N}_l^-}{k_l} (p_l^-)^{k_l} (1 - p_l^-)^{\tilde{N}_l^- - k_l} - (1 - \mathbb{P}(\mathcal{Z}_\Lambda)) \leq \mathbb{P}(\Omega_{(I_l, C_l, k_l)_{1 \leq l \leq p}}^\Lambda \cap \mathcal{Z}_\Lambda) \\ & \leq \prod_{l=1}^p \binom{\tilde{N}_l^+}{k_l} (p_l^+)^{k_l} (1 - p_l^+)^{\tilde{N}_l^+ - k_l} + (1 - \mathbb{P}(\mathcal{Z}_\Lambda)) \end{aligned}$$

where

$$N_l^+ = |C_l| \tilde{N} (1 + O(L^{-1+\beta})), \quad N_l^- = |C_l| \tilde{N} (1 + O(L^{-1+\beta})).$$

One concludes in the same way as in the proof of Theorem 1.3. This completes the proof of Theorem 1.6. \square

4.2.2. *The proof of Theorem 1.7.* Let ℓ_Λ be the scale defined in section 1.5 satisfying (1.19). To prove Theorem 1.7, it is sufficient to prove that, for $(I_j)_{1 \leq j \leq l}$ disjoint segments of \mathbb{R} and $(C_j)_{1 \leq j \leq l}$ disjoint compact cubes in $[-c_\ell, c_\ell]$ (see (1.21)), one has

$$\begin{aligned} & \mathbb{P} \left(\forall 1 \leq j \leq l, \# \left\{ n; \begin{array}{l} N(E_n(\omega, \Lambda)) \in N(E_0) + \ell_\Lambda^{-d} I_j \\ x_n(\omega) \in \ell_\Lambda C_j \end{array} \right\} = k_j \right) \\ & \xrightarrow{|\Lambda| \rightarrow +\infty} \prod_{j=1}^l e^{-|I_j||C_j|} \frac{(|I_j||C_j|)^{k_j}}{k_j!} \quad (4.8) \end{aligned}$$

This is a consequence of Proposition 1.2 and Theorem 1.6. Indeed, using Proposition 1.2, as by assumption, the cubes $(C_l)_{1 \leq j \leq l}$ are at distance at least R from one another (see (IAD)) and, thus, the $(H_\omega(\ell_\Lambda C_j))_{1 \leq j \leq l}$ are two by two independent, for (C_j^\pm) cubes s.t. $\overline{C_j^-} \subset \overset{\circ}{C}_j \subset \overline{C_j^+} \subset \overset{\circ}{C}_j^+$,

we have that

$$\begin{aligned} & \prod_{j=1}^l \mathbb{P}(\#\{n; N(E_n(\omega, \ell C_j^-)) \in N(E_0) + \ell_\Lambda^{-d} I_j^-\} = k_j) - (1 - \mathbb{P}(\mathcal{Z}_\Lambda)) \\ & \leq \mathbb{P}\left(\forall 1 \leq j \leq l, \#\left\{n; \begin{array}{l} N(E_n(\omega, \Lambda)) \in N(E_0) + \ell_\Lambda^{-d} I_j \\ x_n(\omega) \in \ell_\Lambda C_j \end{array}\right\} = k_j\right) \\ & \leq \prod_{j=1}^l \mathbb{P}(\#\{n; N(E_n(\omega, \ell C_j^+)) \in N(E_0) + \ell_\Lambda^{-d} I_j^+\} = k_j) + (1 - \mathbb{P}(\mathcal{Z}_\Lambda)). \end{aligned}$$

Now, using Theorem 1.6, we compute the asymptotics of $\mathbb{P}(\#\{n; N(E_n(\omega, \ell_\Lambda C_j^\pm)) \in N(E_0) + \ell_\Lambda^{-d} I_j^\pm\})$; then, we let C_j^\pm tend to C_j and I_j^\pm tend to I_j to get the desired result. This completes the proof of Theorem 1.7. \square

4.2.3. *The proof of Theorem 1.8.* It is sufficient to consider the case of J a (non empty) segment and C a (non empty) cube. As the operators we considered are defined with periodic boundary conditions, we can restrict ourselves to cubes containing 0. We can also assume J is of the form $[0, a]$ or $[-a, 0]$ for some $a > 0$. Hence, the sets $\tilde{\ell}_\Lambda C$ are increasing and the sets $E_0 + \ell_\Lambda J$ are decreasing.

Pick $\ell^\pm = (\ell_\Lambda^\pm)_\Lambda$ and $\tilde{\ell}^\pm = (\tilde{\ell}_\Lambda^\pm)_\Lambda$ such that

$$\pm \frac{\ell_\Lambda^\pm - \ell_\Lambda}{\log^{1/\xi} |\Lambda|} \rightarrow +\infty, \quad \frac{\ell_\Lambda^\pm}{\ell_\Lambda} \rightarrow 1, \quad \pm \frac{\tilde{\ell}_\Lambda^\pm - \tilde{\ell}_\Lambda}{\log^{1/\xi} |\Lambda|} \rightarrow +\infty, \quad \frac{\tilde{\ell}_\Lambda^\pm}{\tilde{\ell}_\Lambda} \rightarrow 1. \quad (4.9)$$

Let χ_A be the characteristic function of A . Compute

$$\text{tr}(\chi_{\tilde{\ell}_\Lambda C} \mathbf{1}_{N^{-1}[N(E_0) + (\ell_\Lambda)^{-d} J]}(H_\omega(\Lambda))) = \sum_{\substack{E_n(\omega, \Lambda) \in \sigma(H_\omega(\Lambda)) \\ N(E_n(\omega, \Lambda)) \in N(E_0) + \ell_\Lambda^{-d} J}} \|\chi_{\tilde{\ell}_\Lambda C} \varphi_n(\omega, \Lambda)\|^2.$$

If $\varphi_n(\omega, \Lambda)$ has its localization center in $\tilde{\ell}_\Lambda C$, by (4.9) and (1.7), one has

$$\|\chi_{\tilde{\ell}_\Lambda^+ C} \varphi_n(\omega, \Lambda)\|^2 = 1 + O(|\Lambda|^{-\infty}),$$

and if it has its localization center outside $\tilde{\ell}_\Lambda C$, then

$$\|\chi_{\tilde{\ell}_\Lambda^- C} \varphi_n(\omega, \Lambda)\|^2 = O(|\Lambda|^{-\infty}).$$

Hence, as the number of eigenvalues of $H_\omega(\Lambda)$ in $N^{-1}[N(E_0) + (\ell_\Lambda)^{-d} J]$ is bounded by $C|\Lambda|$ for some $C > 0$, by Lemma 1.2, we get that, for any $p > 0$, for Λ sufficiently large, with a probability at least $1 - |\Lambda|^{-p}$, one has

$$\begin{aligned} \int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi dx - |\Lambda|^{-p} & \leq \text{tr}(\chi_{\tilde{\ell}_\Lambda^+ C} \mathbf{1}_{N^{-1}[N(E_0) + (\ell_\Lambda)^{-d} J]}(H_\omega(\Lambda))) \\ & \leq \int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi dx + |\Lambda|^{-p}. \end{aligned} \quad (4.10)$$

Partitioning $\tilde{\ell}_\Lambda^\pm C$ into cubes of side length 1 and using the covariance and (4.9), for χ_0 taken as in section 2, we get that

$$\begin{aligned} (\tilde{\ell}_\Lambda)^d |C| \mathbb{E}_{\ell_\Lambda}(1 + o(1)) & \leq \mathbb{E}(\text{tr}(\chi_{\tilde{\ell}_\Lambda^- C} \mathbf{1}_{N^{-1}[N(E_0) + (\ell_\Lambda)^{-d} J]}(H_\omega(\Lambda)))) \\ & \leq \mathbb{E}(\text{tr}(\chi_{\tilde{\ell}_\Lambda^+ C} \mathbf{1}_{N^{-1}[N(E_0) + (\ell_\Lambda)^{-d} J]}(H_\omega(\Lambda)))) \leq (\tilde{\ell}_\Lambda)^d |C| \mathbb{E}_{\ell_\Lambda}(1 + o(1)). \end{aligned}$$

where $\mathbb{E}_{\ell_\Lambda} = \mathbb{E}(\text{tr}(\chi_0 \mathbf{1}_{N^{-1}[N(E_0)+(\ell_\Lambda)^{-d}J]}(H_\omega(\Lambda))))$. The computations done in section 2 show that

$$\mathbb{E}_{\ell_\Lambda} = (\ell_\Lambda)^{-d} |J| (1 + o(1)).$$

Taking the expectation in (4.10), we immediately obtain

$$\mathbb{E} \left(\int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi \right) \leq C \left(\frac{\tilde{\ell}_\Lambda}{\ell_\Lambda} \right)^d \quad \text{if } \frac{\tilde{\ell}_\Lambda}{\ell_\Lambda} \xrightarrow{|\Lambda| \rightarrow +\infty} 0, \quad (4.11)$$

$$\left(\frac{\ell_\Lambda}{\tilde{\ell}_\Lambda} \right)^d \mathbb{E} \left(\int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi \right) \xrightarrow{|\Lambda| \rightarrow +\infty} |J| \cdot |C| \quad \text{if } \frac{\tilde{\ell}_\Lambda}{\ell_\Lambda} \xrightarrow{|\Lambda| \rightarrow +\infty} +\infty. \quad (4.12)$$

Assume now that $\frac{\tilde{\ell}_\Lambda}{\ell_\Lambda} \geq |\Lambda|^\rho$. Pick two scales $(\ell'_\Lambda)_\Lambda$ and $(\ell''_\Lambda)_\Lambda$ such that, for some $\rho' > 0$

$$\ell'_\Lambda \geq |\Lambda|^{\rho'}, \quad \frac{\tilde{\ell}_\Lambda}{\ell'_\Lambda} \geq |\Lambda|^{\rho'} \quad \text{and} \quad \frac{\ell'_\Lambda}{\ell_\Lambda} \geq |\Lambda|^{\rho'}. \quad (4.13)$$

Partition the cubes $\tilde{\ell}_\Lambda^\pm C$ into cubes of side length asymptotic to ℓ'_Λ : let $\Gamma_\Lambda^\pm = (\ell'_\Lambda \mathbb{Z}^d) \cap (\tilde{\ell}_\Lambda^\pm C)$ and

$$\tilde{\ell}_\Lambda^\pm C = \bigcup_{\gamma \in \Gamma_\Lambda} C_{\gamma, \ell'_\Lambda} \quad \text{where} \quad C_{\gamma, \ell'_\Lambda} = \gamma + \ell'_\Lambda [-1/2, 1/2]^d.$$

Then, we have

$$\text{tr}(\chi_{\tilde{\ell}_\Lambda^\pm C} \mathbf{1}_{N^{-1}[N(E_0)+(\ell_\Lambda)^{-d}J]}(H_\omega(\Lambda))) = \sum_{\gamma \in \Gamma_\Lambda^\pm} \text{tr}(\chi_{C_{\gamma, \ell'_\Lambda}} \mathbf{1}_{E_0+(\ell_\Lambda)^{-d}J}(H_\omega(\Lambda)))$$

Thus

$$\text{tr}^2(\chi_{\tilde{\ell}_\Lambda^\pm C} \mathbf{1}_{N^{-1}[N(E_0)+(\ell_\Lambda)^{-d}J]}(H_\omega(\Lambda))) = \sum_{\gamma \in \Gamma_\Lambda} \sum_{\gamma' \in \Gamma_\Lambda} T(\gamma, J, \ell'_\Lambda, \ell_\Lambda, \Lambda) T(\gamma', J, \ell'_\Lambda, \ell_\Lambda, \Lambda)$$

where

$$T(\gamma) = T(\gamma, J, \ell'_\Lambda, \ell_\Lambda, \Lambda) = \text{tr}(\chi_{C_{\gamma, \ell'_\Lambda}} \mathbf{1}_{N^{-1}[N(E_0)+(\ell_\Lambda)^{-d}J]}(H_\omega(\Lambda))).$$

We prove

Lemma 4.1. *If $|\gamma - \gamma'| \geq 2\ell'_\Lambda$ then*

$$|\mathbb{E}(T(\gamma) \cdot T(\gamma')) - \mathbb{E}(T(\gamma)) \cdot \mathbb{E}(T(\gamma'))| \leq C e^{-(\ell'_\Lambda)^{1/\varepsilon}/C}.$$

Hence, we have

$$\mathbb{E} \left| \text{tr}(\chi_{\tilde{\ell}_\Lambda^\pm C} \mathbf{1}_{N^{-1}[N(E_0)+(\ell_\Lambda)^{-d}J]}(H_\omega(\Lambda))) - \mathbb{E} \left(\text{tr}(\chi_{\tilde{\ell}_\Lambda^\pm C} \mathbf{1}_{N^{-1}[N(E_0)+(\ell_\Lambda)^{-d}J]}(H_\omega(\Lambda))) \right) \right|^2 \leq C (\ell'_\Lambda)^d$$

By (4.13) and (4.12), we get, for some $\rho > 0$,

$$\mathbb{E} \left| \left(\frac{\ell_\Lambda}{\tilde{\ell}_\Lambda} \right)^d \text{tr}(\chi_{\tilde{\ell}_\Lambda^\pm C} \mathbf{1}_{N^{-1}[N(E_0)+(\ell_\Lambda)^{-d}J]}(H_\omega(\Lambda))) - |J| \cdot |C| \right|^2 \leq C |\Lambda|^{-\rho} \quad (4.14)$$

If $\tilde{\ell}_\Lambda/\ell'_\Lambda \leq |\Lambda|^{-\rho}$, (4.11) becomes

$$\mathbb{E} \left(\int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi \right) \leq C |\Lambda|^{-d\rho}. \quad (4.15)$$

Now choose the scales $\tilde{\ell}_{\Lambda, L^p}^+ = \tilde{\ell}_{\Lambda, (L+1)^p}$ and $\ell_{\Lambda, L^p}^+ = \ell_{\Lambda, (L+1)^p}$. By (1.23), $\tilde{\ell}_{\Lambda, L^p}^+/\tilde{\ell}_{\Lambda, L^p} \rightarrow 1$ and $\ell_{\Lambda, L^p}^+/\ell_{\Lambda, L^p} \rightarrow 1$. Moreover, the estimates (4.15) and (4.14) hold for the pairs of scales $(\tilde{\ell}_{\Lambda, L^p}^+, \ell_{\Lambda, L^p}^+)$, and $(\tilde{\ell}_{\Lambda, L^p}, \ell_{\Lambda, L^p}^+)$. Moreover, for p large enough, they are summable. Thus, we have proved that

- in case (1) of Theorem 1.8, ω almost surely, for L sufficiently large,

$$\int_{J \times C} \Xi_{\Lambda_{L^p}}^2(\xi, x; E_0, \ell, \tilde{\ell}^+) d\xi dx = 0.$$

- in case (2) of Theorem 1.8, ω almost surely,

$$\begin{aligned} \left(\frac{\ell_{\Lambda_{L^p}}^+}{\tilde{\ell}_{\Lambda_{L^p}}^+} \right)^{-d} \int_{J \times C} \Xi_{\Lambda_{L^p}}^2(\xi, x; E_0, \ell, \tilde{\ell}^+) d\xi dx &\xrightarrow{|\Lambda| \rightarrow +\infty} |J| \cdot |C|, \\ \left(\frac{\ell_{\Lambda_{L^p}}}{\tilde{\ell}_{\Lambda_{L^p}}^+} \right)^{-d} \int_{J \times C} \Xi_{\Lambda_{L^p}}^2(\xi, x; E_0, \ell^+, \tilde{\ell}) d\xi dx &\xrightarrow{|\Lambda| \rightarrow +\infty} |J| \cdot |C| \end{aligned}$$

For $L^p \leq k \leq (L+1)^p$, as the sequences $(\ell_\Lambda)_\Lambda$ and $(\tilde{\ell}_\Lambda)_\Lambda$ are increasing, one has

$$\begin{aligned} \int_{J \times C} \Xi_{\Lambda_{L^p}}^2(\xi, x; E_0, \ell, \tilde{\ell}^+) d\xi dx &\geq \int_{J \times C} \Xi_{\Lambda_k}^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi dx \\ &\geq \int_{J \times C} \Xi_{\Lambda_{L^p}}^2(\xi, x; E_0, \ell^+, \tilde{\ell}) d\xi dx. \end{aligned}$$

By (1.23), for $L^p \leq k \leq (L+1)^p$, $\tilde{\ell}_{\Lambda_{L^p}}^+ \sim \tilde{\ell}_{\Lambda_k}^+ \sim \tilde{\ell}_{\Lambda_{L^p}}$ and $\ell_{\Lambda_{L^p}}^+ \sim \ell_{\Lambda_k}^+ \sim \ell_{\Lambda_{L^p}}$. Hence, we get

- in case (1) of Theorem 1.8, ω almost surely, for L sufficiently large,

$$\int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}^+) d\xi dx = 0.$$

As $\int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}^+) d\xi dx$ is an integer, this implies that this integer is 0.

- in case (2) of Theorem 1.8, ω almost surely,

$$\left(\frac{\ell_\Lambda}{\tilde{\ell}_\Lambda} \right)^{-d} \int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi dx \xrightarrow{|\Lambda| \rightarrow +\infty} |J| \cdot |C|.$$

This completes the proof of Theorem 1.8. \square

Remark 4.1. If we don't assume that either $\tilde{\ell}_\Lambda/\ell'_\Lambda \leq |\Lambda|^{-\rho}$ or $\tilde{\ell}_\Lambda/\ell'_\Lambda \leq |\Lambda|^{-\rho}$, but merely that either tends to 0, or we do not assume condition (1.23) then, (4.11), (4.10) and (4.14) show nevertheless that

$$\begin{aligned} \mathbb{E} \left(\int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi \right) &\rightarrow 0 \quad \text{if} \quad \frac{\tilde{\ell}_\Lambda}{\ell_\Lambda} \xrightarrow{|\Lambda| \rightarrow +\infty} 0, \\ \mathbb{E} \left| \left(\frac{\ell_\Lambda}{\tilde{\ell}_\Lambda} \right)^d \int_{J \times C} \Xi_\Lambda^2(\xi, x; E_0, \ell, \tilde{\ell}) d\xi - |J| \cdot |C| \right|^2 &\rightarrow 0 \quad \text{if} \quad \frac{\tilde{\ell}_\Lambda}{\ell_\Lambda} \xrightarrow{|\Lambda| \rightarrow +\infty} +\infty. \end{aligned}$$

This implies convergence in probability.

Proof of Lemma 4.1. Define $\tilde{C}_{\gamma, \ell'_\Lambda}^\pm = C_{\gamma, \ell'_\Lambda}^\pm + \ell'_\Lambda[-1/2, 1/2]^d$; hence, in view of the computations done in (2.2) and (2.1), there exists $C > 0$ such that, for $\delta > 0$, we have

$$\begin{aligned} \mathbb{E} \left| \text{tr}(\chi_{C_{\gamma, \ell'_\Lambda}} \mathbf{1}_{N^{-1}[N(E_0) + (\ell_\Lambda)^{-d}J]}(H_\omega(\Lambda))) - \text{tr}(\chi_{C_{\gamma, \ell'_\Lambda}^\pm} \mathbf{1}_{N^{-1}[N(E_0) + (\ell_\Lambda)^{-d}J]}(H_\omega(\tilde{C}_{\gamma, \ell'_\Lambda}^\pm))) \right| \\ \leq C \left(\delta (\ell_\Lambda)^d + \delta^{-C} (\ell_\Lambda)^{dC} e^{-(\ell_\Lambda)^{1/\xi}/C} \right) \quad (4.16) \end{aligned}$$

Pick $\delta = e^{-(\ell'_\Lambda)^{1/\xi}/C'}$ for some $C' > C$. As $|\gamma - \gamma'| \geq 2\ell'_\Lambda$, the operators $H_\omega(\tilde{C}_{\gamma, \ell'_\Lambda}^\pm)$ and $H_\omega(\tilde{C}_{\gamma', \ell'_\Lambda}^\pm)$ are stochastically independent of each other. Now, as by standard arguments of Schrödinger

operator theory (see e.g. [RS79]), $T(\gamma)$ and $T(\gamma')$ are bounded by $C(\ell'_\Lambda)^d$, (4.16) yields the result of Lemma 4.1 and completes its proof. \square

4.3. Study of the level spacings statistics. We will first prove Theorem 1.10. To do so, we first use a the reduction constructed in Theorem 1.16 and study the spacings for the approximated eigenvalues given by Theorem 1.16. Then, we derive the statistics described in Theorem 1.10 from those computations. Using the estimates obtained in the proof of Theorem 1.10, we will prove Theorem 1.12.

4.3.1. Some preliminary considerations. We now use Theorem 1.16. Let \mathcal{Z}_Λ be the set of realizations for which the conclusions of Theorem 1.16 hold. We know that $\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - O(|\Lambda|^{-\infty})$. Let \tilde{N} be the number of the good cubes (i.e. cubes that determine eigenvalues of $H_\omega(\Lambda)$) constructed in Theorem 1.16 and $(\Lambda_\ell(\gamma_j))_{1 \leq j \leq \tilde{N}}$ be those cubes. Then, $\tilde{N} = |\Lambda| \ell_\Lambda^{-d} (1 + o(1))$. Recall that ℓ_Λ is chosen so that, for some $\nu_1 < \nu_2 < 1$,

$$\ell_\Lambda \cdot \log^{-1/\nu_2} |\Lambda| \xrightarrow{|\Lambda| \rightarrow +\infty} +\infty \quad \text{and} \quad \ell_\Lambda \cdot \log^{-1/\nu_1} |\Lambda| \xrightarrow{|\Lambda| \rightarrow +\infty} 0. \quad (4.17)$$

As before, define the following random variables:

- $X_j = X_j(\ell, E_0 + I_\Lambda)$ is the Bernoulli random variable

$$X_j = \mathbf{1}_{H_\omega(\Lambda_\ell(\gamma_j)) \text{ has exactly one eigenvalue in } E_0 + I_\Lambda}$$

- $\tilde{E}_j = \tilde{E}_j(\ell, E_0 + I_\Lambda)$ is this eigenvalue conditioned on $X_j = 1$.

Assume $I_\Lambda = [a_\Lambda, b_\Lambda]$ and define

$$\xi_j = \frac{N(\tilde{E}_j) - N(E_0 + a_\Lambda)}{N(E_0 + b_\Lambda) - N(E_0 + a_\Lambda)}. \quad (4.18)$$

Note that ξ_j is valued in $[0, 1]$. Let Ξ denote the common distribution function of the $(\xi_j)_{1 \leq j \leq k}$. It was studied in lemma 2.2.

We first study the spacings for i.i.d. copies of the random variables $(\xi_j)_{1 \leq j \leq k}$. Let $(\bar{\xi}_j)_{1 \leq j \leq k}$ denote the $(\xi_j)_{1 \leq j \leq k}$ ordered increasingly and define

$$DLS_\xi(x, k; E_0 + I_\Lambda, \omega, \Lambda) = \frac{1}{k-1} \#\{1 \leq j \leq k; \bar{\xi}_{j+1} - \bar{\xi}_j > x/k\}. \quad (4.19)$$

Our main technical result is

Lemma 4.2. *Pick $E_0 \in I$ such that (1.29) be satisfied and $(I_\Lambda)_\Lambda$ intervals such that*

$$\sup_{E \in I_\Lambda} |E| \rightarrow 0, \quad N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{1+\bar{\rho}}} \ell_\Lambda^{d\rho} \rightarrow 0, \quad N(I_\Lambda) |\Lambda| \rightarrow +\infty \quad \text{as} \quad |\Lambda| \rightarrow +\infty. \quad (4.20)$$

Fix $m > 0$. Then, for $1 \leq k \leq |\Lambda|^m$, one

$$\sup_{m^{-1} \leq x \leq m} \mathbb{E} \left(|DLS_\xi(x, k; E_0 + I_\Lambda, \omega, \Lambda) - D(k, E_0 + I_\Lambda, \Lambda)|^2 \right) \leq \frac{C}{k}. \quad (4.21)$$

where

$$D(k, E_0 + I_\Lambda, \Lambda) := \int_0^1 (1 - \Xi(y + x/k) + \Xi(y))^{k-1} d\Xi(y). \quad (4.22)$$

Let $(J_\Lambda)_\Lambda$ be a sequence of intervals such that $\sup_{E \in J_\Lambda} |E| \rightarrow 0$ as $|\Lambda| \rightarrow +\infty$. Then

$$\sup_{\substack{I_\Lambda \subset J_\Lambda \\ I_\Lambda \text{ satisfying (4.20)}}} |D(k, E_0 + I_\Lambda, \Lambda) - e^{-x}| \xrightarrow[k \leq |\Lambda|^m]{k \rightarrow +\infty} 0. \quad (4.23)$$

Proof. To analyze the spacings of the $(\xi_j)_{1 \leq j \leq k}$, we use the computations of section 7 in [Pyk65] (see in particular (7.3)) that yield

$$\int_0^1 (1 - \Xi(y + x/k) + \Xi(y))^{k-1} d\Xi(y) = \mathbb{E}(DLS_\xi(x, k; E_0 + I_\Lambda, \omega, \Lambda)) = D(k, E_0 + I_\Lambda, \Lambda)$$

and

$$\begin{aligned} \mathbb{E}([DLS_\xi(x, k; E_0 + I_\Lambda, \omega, \Lambda)]^2) &= O\left(\frac{1}{k}\right) + \\ &+ 2 \int_{\mathbb{R}} \int_{y+x/k}^{+\infty} (1 - \Xi(y + x/k) + \Xi(y) - \Xi(z + x/k) + \Xi(z))^{k-2} d\Xi(z) d\Xi(y). \end{aligned}$$

Fix $\nu \in (\nu_2, 1)$. By (4.17), for $|\Lambda|$ large, one has $|\Lambda|^{-m-1} \geq e^{-\ell^\nu}$. Thus Lemma 2.2 yields that, for $y - x \geq |\Lambda|^{-m-1}$, one has has,

$$\Xi(y) - \Xi(x) = (y - x) \left(1 + o\left(N(I_\Lambda)^{\frac{\rho-\bar{\rho}}{4+\rho}} \ell_\Lambda^{d\rho}\right)\right) \quad (4.24)$$

Hence, for $m^{-1} \leq x \leq m$ and $1 \leq k \leq |\Lambda|^m$, one has

$$\begin{aligned} &\mathbb{E}(DLS_\xi^2(x, k; I_\Lambda, \omega, \Lambda)) \\ &= \int_0^1 \int_0^1 (1 - \Xi(y + x/k) + \Xi(y) - \Xi(z + x/k) + \Xi(z))^{k-2} d\Xi(z) d\Xi(y) \\ &- 2 \int_{\mathbb{R}} \int_y^{y+x/k} (1 - \Xi(y + x/k) + \Xi(y) - \Xi(z + x/k) + \Xi(z))^{k-2} d\Xi(z) d\Xi(y) + O\left(\frac{1}{k}\right) \\ &= \int_0^1 \int_0^1 (1 - \Xi(y + x/k) + \Xi(y) - \Xi(z + x/k) + \Xi(z))^{k-2} d\Xi(z) d\Xi(y) + O\left(\frac{1}{k}\right). \end{aligned}$$

Compute

$$\begin{aligned} &(1 - \Xi(y + x/k) + \Xi(y) - \Xi(z + x/k) + \Xi(z)) \\ &= (1 - \Xi(y + x/k) + \Xi(y))(1 - \Xi(z + x/k) + \Xi(z)) \\ &\quad - (\Xi(z + x/k) - \Xi(z))(\Xi(y + x/k) - \Xi(y)) \\ &= (1 - \Xi(y + x/k) + \Xi(y))(1 - \Xi(z + x/k) + \Xi(z)) + O(k^{-2}). \end{aligned}$$

Hence, plugging this into the previous formula, we get

$$\begin{aligned} \mathbb{E}(DLS_\xi^2(x, k; I_\Lambda, \omega, \Lambda)) &= \left[\int_0^1 (1 - \Xi(y + x/k) + \Xi(y))^{k-2} d\Xi(y) \right]^2 + O\left(\frac{1}{k}\right) \\ &= \left[\int_0^1 (1 - \Xi(y + x/k) + \Xi(y))^{k-1} \left(1 + O\left(\frac{1}{k}\right)\right) d\Xi(y) \right]^2 + O\left(\frac{1}{k}\right) \\ &= D(k, E_0 + I_\Lambda, \Lambda)^2 + O\left(\frac{1}{k}\right) \end{aligned}$$

This completes the proof of (4.21).

Fix E_0 such that (1.29) be satisfied and let us prove (4.23). For $I_\Lambda \subset J_\Lambda$, by Lemma 2.2, for $x \in [0, 1]$, we have,

$$\sup_{y \in [0, 1-x/k]} \left| \Xi(y + x/k) - \Xi(y) - \frac{x}{k} \right| \leq \frac{x}{k} \alpha_k \quad \text{and} \quad \sup_{y \in [1-x/k, 1]} |\Xi(y + x/k) - \Xi(y)| \leq \frac{x(1 + \alpha_k)}{k}$$

where $\sup_{\substack{I_\Lambda \subset J_\Lambda \\ I_\Lambda \text{ satisfying (4.20)}}} |\alpha_k| \xrightarrow[k \rightarrow +\infty]{k \leq |\Lambda|^m} 0$.

Hence, for k large,

$$|D(k, E_0 + I_\Lambda, \Lambda) - e^{-x}| \leq \int_0^{1-x/k} |e^{C\alpha_k} - 1| d\Xi(y) + \int_{1-x/k}^1 e^{Cx(1+\alpha_k)/k} d\Xi(y) \leq C \left(\alpha_k + \frac{1}{k} \right).$$

This completes the proof of (4.23) and, thus, of Lemma 4.2. \square

4.3.2. *The proof of Theorem 1.10.* By a classical result (see e.g. [PS98]), as the functions we are considering are monotonous and as $x \mapsto e^{-x}$ is continuous on $[0, +\infty)$, it suffices to prove the almost sure pointwise convergence of $DLS(x; E_0 + I_\Lambda, \omega, \Lambda)$ to e^{-x} (for $x > 0$).

Fix $\xi' < \xi$. Pick I_Λ as in Theorem 1.10. We distinguish two cases:

- (1) if $(N(I_\Lambda)(\log |\Lambda|)^{-d/\xi'})_\Lambda$ is unbounded: we can then cover it with disjoint intervals $(I_{k,\Lambda})_{1 \leq k \leq K_\Lambda}$ of length of order $(\log |\Lambda|)^{-d/\xi'}$ (so that $K_\Lambda \asymp N(I_\Lambda)(\log |\Lambda|)^{d/\xi'}$) and such that, for each $I_{k,\Lambda}$, we can apply Theorem 1.16 and Lemma 4.2.
- (2) if $(N(I_\Lambda)(\log |\Lambda|)^{-d/\xi'})_\Lambda$ stays bounded: we can then apply Theorem 1.16 and Lemma 4.2 to I_Λ itself. We then set $K_\Lambda = 1$ and $I_{1,\Lambda} = I_\Lambda$.

This yields

$$\left| DLS(x; E_0 + I_\Lambda, \omega, \Lambda) - \sum_{k=1}^{K_\Lambda} DLS(x; E_0 + I_{k,\Lambda}, \omega, \Lambda) \frac{N_\Lambda(E_0 + I_{k,\Lambda})}{N_\Lambda(I_\Lambda)} \right| \leq \frac{K_\Lambda}{N_\Lambda(I_\Lambda)} \quad (4.25)$$

where we recall that $N_\Lambda(I)$ is the (random) number of eigenvalues of $H_\omega(\Lambda)$ in I .

We first study $DLS(x; E_0 + I_{k,\Lambda}, \omega, \Lambda)$.

By (1.28), (1.24) and the approximation of the eigenvalues given by Theorem 1.16, for $\omega \in \mathcal{Z}_\Lambda$, if $J = \#\{1 \leq j \leq \tilde{N}; X_j = 1\}$ then we have

$$D^-(\omega, E_0 + I_{k,\Lambda}, \Lambda) + \alpha_\Lambda \geq DLS(x; E_0 + I_{k,\Lambda}, \omega, \Lambda) \geq D^+(\omega, E_0 + I_{k,\Lambda}, \Lambda) - \alpha_\Lambda \quad (4.26)$$

where

- $D^\pm(\omega, E_0 + I_{k,\Lambda}, \Lambda; J) := \frac{\#\{1 \leq j \leq J; \bar{\xi}_{j+1} - \bar{\xi}_j \geq x/N_{k,\Lambda} \pm |\Lambda|^{-2}\}}{N_{k,\Lambda}}$;
- $(\xi_j)_j$ are the renormalized eigenvalues defined in (4.18) at the beginning of section 4.3 for the energy interval $E_0 + I_{k,\Lambda}$;
- $N_{k,\Lambda} = N(E_0 + I_{k,\Lambda})|\Lambda|$ and $\alpha_\Lambda \rightarrow 0$ as $|\Lambda| \rightarrow +\infty$.

Define the random variables

$$D^\pm(\omega, E_0 + I_{k,\Lambda}, \Lambda) = D^\pm(\omega, E_0 + I_{k,\Lambda}, \Lambda; \#\{1 \leq j \leq \tilde{N}; X_j = 1\}) \mathbf{1}_{\mathcal{Z}_\Lambda}.$$

We prove

Lemma 4.3. *One has*

$$\sup_{1 \leq k \leq K_\Lambda} \left[(N_\Lambda^k)^{1/4} \cdot \mathbb{E} \left(|D^\pm(\omega, E_0 + I_{k,\Lambda}, \Lambda) - D(N_{k,\Lambda}, E_0 + I_{k,\Lambda}, \Lambda)|^2 \right) \right] \leq C. \quad (4.27)$$

Before proving Lemma 4.3, let us use it to complete the proof of Theorem 1.10. We first prove the almost sure convergence for a subsequence.

For $\Lambda = \Lambda_{L^{5/\rho}}$, by the first condition in (1.30), summing the estimate of Lemma 4.3 for $1 \leq k \leq K_{\Lambda_{L^{5/\rho}}}$, we get that

$$\sum_{k=1}^{K_{\Lambda_{L^{5/\rho}}}} \sum_{L \geq 1} \mathbb{E} \left(\left| D^\pm(\omega, E_0 + I_{k,\Lambda_{L^{5/\rho}}}, \Lambda_{L^{5/\rho}}) - D(N_{\Lambda_{L^{5/\rho}}}, E_0 + I_{k,\Lambda_{L^{5/\rho}}}, \Lambda_{L^{5/\rho}}) \right|^2 \right) < +\infty.$$

By (4.23) of Lemma 4.2, as $E_0 \in I$ satisfying (1.29), we know that

$$\sup_{1 \leq k \leq K_\Lambda} |D(N_\Lambda, E_0 + I_{k,\Lambda}, \Lambda) - e^{-x}| \xrightarrow{|\Lambda| \rightarrow +\infty} 0. \quad (4.28)$$

Hence, ω -almost surely,

$$\sup_{1 \leq k \leq K_{\Lambda_{L^{5/\rho}}}} |D^\pm(\omega, E_0 + I_{k,\Lambda_{L^{5/\rho}}}, \Lambda_{L^{5/\rho}}) - e^{-x}| \xrightarrow{L \rightarrow +\infty} 0.$$

Thus, by (4.26), ω -almost surely

$$\sup_{1 \leq k \leq K_{\Lambda_{L^{5/\rho}}}} |DLS(x; E_0 + I_{k,\Lambda}, \omega, \Lambda) - e^{-x}| \xrightarrow{L \rightarrow +\infty} 0. \quad (4.29)$$

Plugging this into (4.25), we obtain that, ω -almost surely

$$DLS(x; E_0 + I_{\Lambda_{L^{5/\rho}}}, \omega, \Lambda_{L^{5/\rho}}) \xrightarrow{L \rightarrow +\infty} e^{-x}.$$

To derive the almost sure convergence of $(DLS(x; E_0 + I_\Lambda, \omega, \Lambda))_\Lambda$, we use

Lemma 4.4. *Fix $p > 0$ and $r > 0$. Let $(I_\Lambda)_\Lambda$ be as Theorem 1.10 or $I_\Lambda = J$, J as in Theorem 1.12. Recall that $N_{\Lambda_L} = N_{\Lambda_L}(I_L)$.*

Then, ω -almost surely, for L sufficiently large, if $L' \in [L^p, (L+1)^p]$, except for at most $o(N_{\Lambda_{L^p}})$ of them, all the eigenvalues of $H_\omega(\Lambda_{L'})$ in $E_0 + I_{\Lambda_{L'}}$ are at a distance less than L^{-r} to an eigenvalue of $H_\omega(\Lambda_{L^p})$ in $E_0 + I_{\Lambda_{L^p}}$.

Pick $p = 5/\rho$ and $r > d$. For $L^p \leq L' \leq (L+1)^p$, by assumption (1.30) on I_Λ , one has $N(E_0 + I_{\Lambda_{L'}}) = N(E_0 + I_{\Lambda_{L^p}})(1 + o(1))$; so $N_{\Lambda_{L'}} = N_{\Lambda_{L^p}}(1 + o(1))$. Lemma 4.4 then implies that, ω -almost surely, for $L^p \leq L' \leq (L+1)^p$ and L sufficiently large, one has

$$DLS(x; I_{\Lambda_{L'}}, \omega, \Lambda_{L'}) = DLS(x; I_{\Lambda_{L^p}}, \omega, \Lambda_{L^p}) + o(1).$$

Hence, ω -almost surely, $DLS(x; I_\Lambda, \omega, \Lambda)$ converges to e^{-x} when $|\Lambda| \rightarrow +\infty$. This completes the proof of Theorem 1.10. \square

Remark 4.2. If one does not assume one of the conditions in (1.30), the estimate (4.27) and Lemma 4.4 are not sufficient anymore to obtain the almost sure convergence for $DLS(x; I_\Lambda, \omega, \Lambda)$. Nevertheless, plugging the estimates (4.28), (4.27) and (4.26) into (4.25), we see that

$$\mathbb{E}(|DLS(x; I_\Lambda, \omega, \Lambda) - e^{-x}|) \xrightarrow{|\Lambda| \rightarrow +\infty} 0.$$

This guarantees the convergence in probability.

Proof of Lemma 4.3. We now fix $1 \leq k \leq K_\Lambda$ and, to alleviate the notations, we will write $I_\Lambda := I_{k,\Lambda}$ and $N_\Lambda = N_{k,\Lambda} = N(I_\Lambda)|\Lambda|$. The interval I_Λ satisfies the requirements to apply Theorem 1.16 and Lemma 4.2.

As D^\pm is bounded by $C|\Lambda|$, as the $(X_j)_j$ are i.i.d. and as $\mathbb{P}(\mathcal{Z}_\Lambda) \leq |\Lambda|^{-p}$ for any $p > 0$, we compute

$$\begin{aligned} & \mathbb{E}(|D^\pm(\omega, I_\Lambda, \Lambda) - D(N_\Lambda, E_0 + I_\Lambda, \Lambda)|^2) \\ & \leq \frac{1}{|\Lambda|^p} + \sum_{k=0}^{\tilde{N}} \binom{\tilde{N}}{k} \mathbb{P}(X_1 = 1)^k (1 - \mathbb{P}(X_1 = 1))^{\tilde{N}-k} \\ & \quad \cdot \mathbb{E}(|DLS_\xi(x \pm |\Lambda|^{-2}, k; E_0 + I_\Lambda, \omega, \Lambda) - D(N_\Lambda, E_0 + I_\Lambda, \Lambda)|^2). \end{aligned}$$

Recall that $N_\Lambda = N(I_\Lambda)|\Lambda|$. As $\mathbb{P}(X_1 = 1) = N(I_\Lambda)\ell_\Lambda^d(1 + o(1))$ and $\tilde{N} = |\Lambda|\ell_\Lambda^{-d}(1 + o(1))$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{|k - N_\Lambda| \geq N_\Lambda^{1/2+\varepsilon}} \binom{\tilde{N}}{k} \mathbb{P}(X_1 = 1)^k (1 - \mathbb{P}(X_1 = 1))^{\tilde{N}-k} \leq e^{-N_\Lambda^\delta}. \quad (4.30)$$

Hence, by the first condition in (1.30), one gets

$$\begin{aligned} & \mathbb{E} (|D^\pm(\omega, I_\Lambda, \Lambda) - D(N_\Lambda, E_0 + I_\Lambda, \Lambda)|^2) \\ & \leq \frac{1}{|\Lambda|^p} + e^{-N_\Lambda^\delta/2} + 2 \sum_{|k - N_\Lambda| < N_\Lambda^{1/2+\varepsilon}} |D(k, E_0, \Lambda) - D(N_\Lambda, E_0 + I_\Lambda, \Lambda)|^2 \\ & \quad + 2 \sum_{|k - N_\Lambda| < N_\Lambda^{1/2+\varepsilon}} \mathbb{E} (|DLS_\xi(x \pm |\Lambda|^{-2}, k; E_0 + I_\Lambda, \omega, \Lambda) - D(k, E_0, \Lambda)|^2). \end{aligned} \quad (4.31)$$

For $|k - N_\Lambda| < N_\Lambda^{1/2+\varepsilon}$, let us estimate $D(k, E_0, \Lambda) - D(N_\Lambda, E_0 + I_\Lambda, \Lambda)$ that is

$$\begin{aligned} & D(k, E_0, \Lambda) - D(N_\Lambda, E_0 + I_\Lambda, \Lambda) \\ & = \int_0^1 ((1 - \Xi(y + x/k) + \Xi(y))^{k-1} - (1 - \Xi(y + x/N_\Lambda) + \Xi(y))^{N_\Lambda-1}) d\Xi(y). \end{aligned} \quad (4.32)$$

Using (4.24), for $|k - N_\Lambda| < N_\Lambda^{1/2+\varepsilon}$, compute

$$\begin{aligned} & |(1 - \Xi(y + x/k) + \Xi(y))^{k-1} - (1 - \Xi(y + x/N_\Lambda) + \Xi(y))^{N_\Lambda-1}| \\ & = (1 - \Xi(y + x/k) + \Xi(y))^{k-1} \\ & \quad \cdot \left| 1 - \left(\frac{1 - \Xi(y + x/N_\Lambda) + \Xi(y)}{1 - \Xi(y + x/k) + \Xi(y)} \right)^{N_\Lambda} (1 - \Xi(y + x/N_\Lambda) + \Xi(y))^{N_\Lambda-k} \right| \\ & \leq C \left(1 - e^{-C(N_\Lambda-k)/N_\Lambda} e^{-C(N_\Lambda-k)/k} \right) \leq CN_\Lambda^{-1/2+\varepsilon}. \end{aligned}$$

Hence, plugging this into (4.32), one obtains

$$|D(k, E_0, \Lambda) - D(N_\Lambda, E_0 + I_\Lambda, \Lambda)| \leq CN_\Lambda^{-1/2+\varepsilon}.$$

Plugging this and the result of Lemma 4.2 into (4.31), we obtain

$$\mathbb{E} (|D^\pm(\omega, I_\Lambda, \Lambda) - D(N_\Lambda, E_0 + I_\Lambda, \Lambda)|^2) \leq CN_\Lambda^{-1/2+3\varepsilon}.$$

We pick $\varepsilon = 1/12$ to completes the proof of Lemma 4.3. \square

Proof of Lemma 4.4. Pick $\alpha \in (0, 1)$ and $q > 1$. By Lemma 3.1, with a probability at least $1 - L^{-q}$, the eigenvalues of $H_\omega(\Lambda_{L'})$ in $E_0 + I_{\Lambda_{L'}}$ that are at a distance more than L^{-r} to an eigenvalue of $H_\omega(\Lambda_{L^p})$ in $E_0 + I_{\Lambda_{L^p}}$ fall into two categories:

- (1) either they belong to $E_0 + (I_{\Lambda_{L^p}} \setminus I_{\Lambda_{L'}})$ which may be empty,
- (2) or they have a localization center that belongs e.g. to the annulus $\Lambda_{L'} \setminus \Lambda_{L^p-L^\alpha}$.

The numbers of eigenvalues in the first category is bounded by $o(N_{\Lambda_{L^p}})$ as a consequence of the second part of assumption (1.30). The numbers of eigenvalues in the second category is bounded by $o(N_{\Lambda_{L^p}})$ as a consequence of the improved Wegner estimate, Theorem 1.1. The Borel-Cantelli lemma then implies Lemma 4.4. \square

4.3.3. *The proof of Theorem 1.12.* The proof is analogous to that of Theorem 1.10. As in that proof, it suffices to prove the almost sure pointwise convergence of $DLS'(x; J, \omega, \Lambda)$ to $g_{\nu, J}(x)$ for $x > 0$.

Fix $\xi' < \xi$. Pick $J = [a, b]$ as in Theorem 1.12. We can then cover it with disjoint intervals $(I_{j, \Lambda})_{1 \leq j \leq J_\Lambda}$ of length $|J|(\log |\Lambda|)^{-d/\xi'}$ and containing the point $e_{j, \Lambda} := a + (j-1/2)|J|(\log |\Lambda|)^{-d/\xi'}$ (so that $J_\Lambda \sim (\log |\Lambda|)^{d/\xi'}$) and such that, for each $I_{j, \Lambda}$, we can apply Lemma 4.2. This yields

$$\left| DLS'(x; J, \omega, \Lambda) - \sum_{j=1}^{J_\Lambda} DLS(\nu(e_{j, \Lambda})x/N(J); I_{j, \Lambda}, \omega, \Lambda) \frac{N_\Lambda(I_{j, \Lambda})}{N_\Lambda(J)} \right| \leq \frac{CJ_\Lambda}{|\Lambda|} \quad (4.33)$$

where DLS' is defined in (1.34).

Using the uniform continuity of ν , the same computations as in the proof of Theorem 1.10 yield the following analogue of (4.29): for all $\varepsilon > 0$,

$$\sup_{\substack{1 \leq j \leq J_\Lambda \\ \nu(e_{j, \Lambda}) \geq \varepsilon \\ \Lambda = \Lambda_{L^{5/\rho}}} \left| DLS(\nu(e_{j, \Lambda})x/N(J); I_{j, \Lambda}, \omega, \Lambda) - e^{-\nu(e_{j, \Lambda})x/N(J)} \right| \xrightarrow{L \rightarrow +\infty} 0, \quad \omega - \text{a.s.}$$

The large deviation principle for the eigenvalue counting function, Theorem 1.9, ensures that, almost surely, for $|\Lambda|$ large, $N_\Lambda(J) \geq |\Lambda||J|/2$ and

$$\lim_{L \rightarrow +\infty} \sup_{\substack{1 \leq j \leq J_\Lambda \\ \nu(e_{j, \Lambda}) \geq \varepsilon}} \left| \frac{N_\Lambda(I_{j, \Lambda})}{N_\Lambda(J)} \cdot \frac{N(J)}{\nu(e_{j, \Lambda})|I_{j, \Lambda}|} - 1 \right| = 0.$$

Moreover, using the uniform continuity of ν on J , we have that for any $\delta > 0$, there exists $\varepsilon > 0$ such that, for L sufficiently large, one has

$$\sup_{\substack{1 \leq j \leq J_\Lambda \\ \nu(e_{j, \Lambda}) \leq \varepsilon}} \left| \frac{N_\Lambda(I_{j, \Lambda})}{|I_{j, \Lambda}| N_\Lambda(J)} \right| \leq \delta.$$

Hence, by (4.33), ω -almost surely, there exists $C > 0$ such that $\varepsilon > 0$, one has

$$\sum_{\substack{1 \leq j \leq J_\Lambda \\ \nu(e_{j, \Lambda}) \leq \varepsilon \\ \Lambda = \Lambda_{L^{5/\rho}}} DLS(\nu(e_{j, \Lambda})x/N(J); I_{j, \Lambda}, \omega, \Lambda) \frac{N_\Lambda(I_{j, \Lambda})}{N_\Lambda(J)} \leq \delta \sum_{\substack{1 \leq j \leq J_\Lambda \\ \nu(e_{j, \Lambda}) \leq \varepsilon \\ \Lambda = \Lambda_{L^{5/\rho}}} |I_{j, \Lambda}| \leq C\delta,$$

thus

$$\limsup_{\substack{L \rightarrow +\infty \\ \Lambda = \Lambda_{L^{5/\rho}}} \left| DLS'(x; J, \omega, \Lambda) - \sum_{\substack{1 \leq j \leq J_\Lambda \\ \nu(e_{j, \Lambda}) \geq \varepsilon \\ \Lambda = \Lambda_{L^{5/\rho}}} \frac{\nu(e_{j, \Lambda})|I_{j, \Lambda}|}{N(J)} e^{-\nu(e_{j, \Lambda})x} \right| \leq C\delta.$$

On the other hand, as ν is continuous, non negative on J and $x > 0$, one has, for any $\delta > 0$, taking $\varepsilon > 0$ sufficiently small, one has

$$\left| \sum_{\substack{1 \leq j \leq J_\Lambda \\ \nu(e_{j, \Lambda}) < \varepsilon}} \frac{\nu(e_{j, \Lambda})|I_{j, \Lambda}|}{N(J)} e^{-\nu(e_{j, \Lambda})x/N(J)} \right| \leq \delta$$

and

$$\begin{aligned} \lim_{L \rightarrow +\infty} \sum_{\substack{1 \leq j \leq J_\Lambda \\ \Lambda = \Lambda_{L^{5/\rho}}} \frac{\nu(e_{j,\Lambda}) |I_{j,\Lambda}|}{N(J)} e^{-\nu(e_{j,\Lambda}) x/N(J)} &= \frac{|J|}{N(J)} \int_0^1 e^{-\nu(a+|J|y) x/N(J)} \nu(a+|J|y) dy \\ &= \frac{1}{N(J)} \int_J e^{-\nu(\lambda) x/N(J)} \nu(\lambda) d\lambda. \end{aligned}$$

Thus, for $\delta > 0$, we get

$$\limsup_{\substack{L \rightarrow +\infty \\ \Lambda = \Lambda_{L^{5/\rho}}} \left| DLS'(x; J, \omega, \Lambda) - \int_J e^{-\nu_J(\lambda)x} \nu_J(\lambda) d\lambda \right| \leq \delta.$$

Hence, letting δ tend to 0, we get that, ω almost surely,

$$\lim_{\substack{L \rightarrow +\infty \\ \Lambda = \Lambda_{L^{5/\rho}}} DLS'(x; J, \omega, \Lambda) = \int_J e^{-\nu_J(\lambda)x} \nu_J(\lambda) d\lambda.$$

To complete the proof of Theorem 1.12, we use Lemma 4.4 as in the proof of Theorem 1.10. \square

4.4. Study of the localization centers statistics.

We prove Theorems 1.13. As in the proofs of Theorems 1.10 and 1.12, it suffices to prove the simple convergence in (1.38) to obtain the uniform convergence as we are dealing with non increasing functions and the limit is continuous. Thus, pick $s > 0$ and to prove (1.38), it suffices to prove

$$\mathbb{E} \left(\left| DCS(s; I_\Lambda, \Lambda, \omega) - e^{-s^d} \right|^2 \right)_{\Lambda \nearrow \mathbb{R}^d} \rightarrow 0. \quad (4.34)$$

We apply Theorem 1.16 to the cube Λ and the interval I_Λ satisfying (1.37). We keep the same notations as in Theorem 1.16. Let Γ denote the set of centers γ_i constructed in Theorem 1.16. Let Γ_b denote the set of $\gamma \in \Gamma$ that do not satisfy (1), (2) and (3) of Theorem 1.15. Theorem 1.16 states that, for $\omega \in \mathcal{Z}_\Lambda$, one has

$$\#\Gamma_b = |\Lambda| \ell^{-d} O \left[(|I_\Lambda| \ell^d)^{1+\rho} + N(E_0 + I_\Lambda) \ell^{d-1} \ell' \right]$$

where ℓ and ℓ' are defined in Theorem 1.16.

By (1.37), one has

$$N(E_0 + I_\Lambda)^{-1/d} \gg \ell. \quad (4.35)$$

Thus, if we define

$$\Gamma' = \{ \gamma \in \Gamma; \text{dist}(\gamma, \Gamma_b) \geq 10s(N(E_0 + I_\Lambda))^{-1/d} \}$$

then

$$\#\Gamma' = O(\#\Gamma_b \cdot N(E_0 + I_\Lambda)^{-1} \ell^{-d}) = o(|\Lambda| \ell^{-d}). \quad (4.36)$$

For $\gamma \in \Gamma$, define the random variable X_γ to be equal

- to 1 if $H_\omega(\Lambda_\ell(\gamma))$ has an eigenvalue in I_Λ , and, for all $\gamma' \in \Gamma$ such that $|\gamma' - \gamma| \leq (N(E_0 + I_\Lambda))^{-1/d} s$, $H_\omega(\Lambda_\ell(\gamma'))$ has no eigenvalue in I_Λ ,
- to 0 if not.

Then, by (4.35), (4.36) and the estimate given in Theorem 1.16 on the number of eigenvalues of $H_\omega(\Lambda)$ not associated to an eigenvalue for a cube $(\Lambda_\ell(\gamma))_{\gamma \in \Gamma}$, we get that, for any $\varepsilon > 0$, for $|\Lambda|$ sufficiently large, one has

$$\begin{aligned} \sup_{\omega \in \mathcal{Z}_\Lambda} (DCS'(s - \varepsilon; I_\Lambda, \Lambda, \omega) - DCS(s; I_\Lambda, \Lambda, \omega)) \\ + \sup_{\omega \in \mathcal{Z}_\Lambda} (DCS(s; I_\Lambda, \Lambda, \omega) - DCS'(s + \varepsilon; I_\Lambda, \Lambda, \omega)) \leq \varepsilon \end{aligned} \quad (4.37)$$

where $DCS(s; I_\Lambda, \Lambda, \omega)$ is defined in (1.36) and

$$DCS'(s; I_\Lambda, \Lambda, \omega) = \frac{1}{N(E_0 + I_\Lambda, \Lambda, \omega)} \sum_{\gamma \in \Gamma} X_\gamma.$$

As $0 \leq DCS(s; I_\Lambda, \Lambda, \omega) \leq 1$ and $0 \leq DCS'(s; I_\Lambda, \Lambda, \omega) \leq 2$ (for Λ large), in view of (4.37), to prove (4.34), it suffices to prove

$$\mathbb{E} \left(\left| DCS'(s; I_\Lambda, \Lambda, \omega) - e^{-s^d} \right|^2 \right) \xrightarrow{\Lambda \nearrow \mathbb{R}^d} 0. \quad (4.38)$$

As the Hamiltonians $H_\omega(\Lambda_\ell(\gamma))$ and $H_\omega(\Lambda_\ell(\gamma'))$ are independent when $\gamma \neq \gamma'$, $(\gamma, \gamma') \in \Gamma^2$, using Lemma 2.1, we compute

$$\begin{aligned} \mathbb{E}(X_\gamma) &= (1 - N(E_0 + I_\Lambda)\ell^d)^{(N(E_0 + I_\Lambda)\ell^d)^{-1}s^d - 1} N(E_0 + I_\Lambda)\ell^d + o(|I_\Lambda|\ell^d) \\ &= e^{-s^d} N(E_0 + I_\Lambda)\ell^d + o(|I_\Lambda|\ell^d). \end{aligned}$$

Thus, using Theorem 1.9, one computes

$$\begin{aligned} \mathbb{E}(DCS'(s; I_\Lambda, \Lambda, \omega)) &= \mathbb{E} \left(\frac{1}{N(E_0 + I_\Lambda, \Lambda, \omega)} \sum_{\gamma \in \Gamma} X_\gamma \right) = \frac{1}{N(E_0 + I_\Lambda)|\Lambda|} \frac{|\Lambda|}{\ell^d} e^{-s^d} N(E_0 + I_\Lambda)\ell^d + o(1) \\ &= e^{-s^d} + o(1). \end{aligned} \quad (4.39)$$

On the other hand, by (4.35) and their definition, X_γ and $X_{\gamma'}$ are independent when $|\gamma' - \gamma| \leq 2(N(E_0 + I_\Lambda))^{-1}s + 1$. Hence, using Theorem 1.9, one computes

$$\begin{aligned} \Delta V &:= \mathbb{E}(DCS'(s; I_\Lambda, \Lambda, \omega)^2) - \mathbb{E}(DCS'(s; I_\Lambda, \Lambda, \omega))^2 \\ &= \frac{1 + o(1)}{(N(E_0 + I_\Lambda)|\Lambda|)^2} \sum_{\gamma \in \Gamma} \sum_{\substack{\gamma' \in \Gamma \\ |\gamma' - \gamma| \leq 2(N(E_0 + I_\Lambda))^{-1}s + 1}} (\mathbb{E}(X_\gamma X_{\gamma'}) - \mathbb{E}(X_\gamma)\mathbb{E}(X_{\gamma'})). \end{aligned}$$

Thus, as $\#\Gamma \leq C|\Lambda|/\ell^d$, by Lemma 2.1, one has

$$\Delta V \leq \frac{2}{(N(E_0 + I_\Lambda)|\Lambda|)^2} \frac{|\Lambda|}{\ell^d} \frac{1}{\ell^d N(E_0 + I_\Lambda)} N(E_0 + I_\Lambda)\ell^d = \frac{2}{N^2(E_0 + I_\Lambda)|\Lambda|\ell^d}. \quad (4.40)$$

Condition (1.37) then ensures that the choice of the length ℓ in Theorem 1.16 can be made such that $N^2(E_0 + I_\Lambda)|\Lambda|\ell^d \rightarrow +\infty$ as $|\Lambda| \rightarrow +\infty$.

Thus, (4.39) and (4.40) imply (4.38). This completes the proof of Theorem 1.13. \square

5. PROOF OF PROPOSITION 1.1 AND THEOREM 1.14

We start with the proof of Proposition 1.1 then we prove Theorem 1.14.

5.1. The proof of Proposition 1.1. Let us prove point (1). Consider $x(E)$ and $x'(E)$ two centers of localization for some energy $E \in I$. Let φ be a normalized eigenstate associated to E . Then, by the assumption (Loc'), we know that, for all x ,

$$\|\varphi\|_x^2 \leq C_\omega \langle x(E) \rangle^q e^{-\gamma|x-x(E)|^\xi} \langle x'(E) \rangle^q e^{-\gamma|x-x'(E)|^\xi}.$$

Hence, summing over x , we get that

$$1 \leq C_d C_\omega \langle x(E) \rangle^q \langle x'(E) \rangle^q \frac{1}{\gamma^{d/\xi}} e^{-\gamma|x(E)-x'(E)|^\xi}.$$

Taking the logarithm, we immediately get (1.40).

Let us now prove point (2). First, note that, by the Wegner estimate (W), the condition $N(I_L)L^d \rightarrow +\infty$ implies that $|I_L| \gtrsim L^{-d}$ for L large. Hence, if J is an interval centered at 0 such that $|J| = o(L^{-d})$, then $N(I_L + J) = N(I_L)(1 + o(1))$.

By definition, one has

$$N(I_L, L) = \sum_{\substack{n; E_n \in I_L \\ \gamma_n \in \Lambda_L}} \|\varphi_n\|^2.$$

where, for n , E_n is an eigenvalues of H_ω , φ_n an associated eigenfunction and γ_n a localization center of φ_n in Λ_L .

Fix $\alpha \in (0, 1)$. Using the estimate (1.39) in the same way as the estimates (Loc) was used in the proof of Lemma 3.1, we get that

$$\left| N(I_L, L) - \sum_{\substack{n; E_n \in I_L \\ \gamma_n \in \Lambda_L}} \|\varphi_n\|_{\Lambda_{L+L^\alpha}}^2 \right| \leq C_\omega L^{q+d} e^{-2\gamma L^{\alpha\varepsilon}}$$

where $\|\cdot\|_{\Lambda_{L+L^\alpha}}$ denotes the L^2 -norm on the cube Λ_{L+L^α} .

Hence, we have

$$\begin{aligned} N(I_L, L) &\leq \sum_{\substack{n; E_n \in I_L \\ \gamma_n \in \Lambda_L}} \|\varphi_n\|_{\Lambda_{L+L^\alpha}}^2 + C_\omega L^{q+d} e^{-2\gamma L^{\alpha\varepsilon}} \\ &\leq \text{tr}(\mathbf{1}_{I_L}(H_\omega)\mathbf{1}_{\Lambda_{L+L^\alpha}}) + C_\omega L^{q+d} e^{-2\gamma L^{\alpha\varepsilon}}. \end{aligned}$$

By standard estimates on Schrödinger operators (see e.g. [Sim05]), we know that

$$0 \leq \text{tr}(\mathbf{1}_{I_L}(H_\omega)\mathbf{1}_{\Lambda_{L+L^\alpha}}) \leq CL^d. \quad (5.1)$$

Pick χ a smooth cut-off function with gradient vanishing outside $\Lambda_{L+L^\alpha} \setminus \Lambda_{L+L^\alpha/2}$. Then, using the localization estimate (Loc'), one easily checks that, for L sufficiently large

- for n such that $E_n \in I_L$ and $\gamma_n \in \Lambda_L$, $\chi\varphi_n$ satisfies

$$\|(H_\omega(\Lambda_{L+L^\alpha}) - E_n)(\chi\varphi_n)\| \leq e^{-L^\alpha/C};$$

- the Gram matrix for the family $(\chi\varphi_n)_{E_n \in I_L, \gamma_n \in \Lambda_L}$ satisfies

$$((\langle \chi\varphi_n, \chi\varphi_m \rangle))_{\substack{E_n \in I_L, \gamma_n \in \Lambda_L \\ E_m \in I_L, \gamma_m \in \Lambda_L}} = Id + O\left(e^{-L^\alpha/C}\right). \quad (5.2)$$

As (5.1) implies that the number of n such that $E_n \in I_L$ and $\gamma_n \in \Lambda_L$ is bounded by CL^d , we get that

$$N(I_L, L) \leq \text{tr}(\mathbf{1}_{\tilde{I}_L}(H_\omega(\Lambda_{L+2L^\alpha}))) + C_\omega L^{q+d} e^{-2\gamma L^{\alpha\varepsilon}}$$

where $\tilde{I}_L = I_L + [-e^{L^\alpha/C}, e^{L^\alpha/C}]$.

Hence, if, as in section 3, $N(\omega, \Lambda_{L+L^\alpha}, I_L)$ denotes the number of eigenvalues of $H_\omega(\Lambda_{L+L^\alpha})$ in I_L , we have

$$N(I_L, L) \leq N(\omega, \Lambda_{L+L^\alpha}, I_L) + C_\omega L^{q+d} e^{-2\gamma L^{\alpha\varepsilon}}. \quad (5.3)$$

Recall that we assumed $N(I_L)|I_L|^{-1-\rho} \rightarrow +\infty$ where ρ is given by (M). The proof of Corollary 2.1 then also shows that, if $N(\omega, \Lambda_{L+L^\alpha}, \Lambda_L, I_L)$ denotes the number of eigenvalues of $H_\omega(\Lambda_{L+L^\alpha})$ in I_L having a localization center in Λ_L , then, for any q ,

$$\begin{aligned} \mathbb{P}\{|N(\omega, \Lambda_{L+L^\alpha}, \Lambda_L, I_L) - N(I_\Lambda)|\Lambda|\} = o(N(I_\Lambda)|\Lambda|)\} &\geq 1 - C_q(N(I_\Lambda)|\Lambda|)^{-q} \\ &\geq 1 - C_q L^{-2} \end{aligned}$$

as $N(I_L)L^{d-\varepsilon} \rightarrow +\infty$ for some $\varepsilon > 0$.

So, by the Borel-Cantelli Lemma and Corollary 2.1, we know that, ω -almost surely,

$$N(\omega, \Lambda_{L+L^\alpha}, \Lambda_L, I_L) = N(\omega, \Lambda_{L+L^\alpha}, I_L)(1 + o(1)) = N(I_L)L^d(1 + o(1)). \quad (5.4)$$

One has

$$N(I_L, L) \geq \sum_{\substack{n; E_n \in I_L \\ \gamma_n \in \Lambda_L}} \|\varphi_n\|_{\Lambda_{L+L^\alpha}}^2.$$

Using the same cut-off of the eigenfunctions as above, we then see that

$$N(I_L, L) \geq N(\omega, \Lambda_{L+L^\alpha}, \Lambda_L, I_L).$$

Using this estimate, (5.4), (5.3) and Theorem 1.9, we get (1.41) and complete the proof of Proposition 1.1. \square

5.2. The proof of Theorem 1.14. Let us now consider ω in the full measure set where the statements of Theorems 1.10, 1.12 and Proposition 1.1 hold. As in section 5.1, ω -almost surely, for L large, if E is an eigenvalue of H_ω with localization center in Λ_L , then there exists E' an eigenvalue of $H_\omega(\Lambda_{L+L^\alpha})$ such that $|E - E'| \leq e^{-L^\alpha/C}$. To avoid confusion, rename $DLS(x; I_L, \omega, L)$ defined by (1.28) to $DLS^f(x; I_L, \omega, L)$. Hence, for $\varepsilon > 0$ fixed, we have, for L large enough,

- when $|I_L| \rightarrow 0$ satisfying the assumptions of Theorem 1.14, we have

$$\begin{aligned} DLS(\nu(E_0)x - \varepsilon; I_L, \omega, L) &\geq \frac{N(I_L, L)}{N(I_L, \omega, \Lambda_L)} \cdot DLS^f(x; I_L, \omega, L) \\ &\geq DLS(\nu(E_0)x + \varepsilon; I_L, \omega, L); \end{aligned}$$

- when $I_L = J$ a fixed interval satisfying the assumptions of Theorem 1.14, we have

$$DLS'(x - \varepsilon; J, \omega, L) \geq \frac{N(I_L, L)}{N(J, \omega, \Lambda_L)} DLS^f(x; J, \omega, L) \geq DLS'(x + \varepsilon; J, \omega, L).$$

Thus, Theorem 1.14 is an immediate corollary of Theorems 1.10 and 1.12, and the second point of Proposition 1.1. \square

6. APPENDIX

6.1. The proof of Lemma 1.3. Compute

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{N(E+x) - N(E+y)}{x-y} - \nu(E) \right| &= \int_{\mathbb{R}} \left| \frac{1}{x-y} \int_{E+y}^{E+x} (\nu(e) - \nu(E)) de \right| \\ &\leq \int_0^1 \left(\int_{\mathbb{R}} |\nu(E+y+(x-y)e) - \nu(E)| dE \right) de. \end{aligned}$$

As $\nu \in L^1(\mathbb{R})$, Lebesgue's dominated convergence theorem ensures that the last integral converges to 0 when x and y tend to 0. Hence, the quotient in the first integral converges to 0 for almost every E . This completes the proof of Lemma 1.3. \square

6.2. Local control on the exponential decay of eigenfunctions. In this section, we establish SUDEC and SULE estimates for the eigenfunctions associated to an eigenvalue in the localized regime of a random operator restricted to a finite volume. These are the analogues of the infinite volume estimates introduced in [dRJLS96] and proved in [dRJLS96, GK06]. We denote by Σ_{CL} the region of complete localization for the random operator H_ω . It is defined as the set of energies $E \in \mathcal{I}$ where we have all the conclusions of the bootstrap multiscale analysis of [GK01] or the fractional moment method of [AM93, Aiz94]. These conclusions turn out to be equivalent properties describing the localized regime [GK04, GK06]. In Theorem 6.1, we provided new equivalent

characterizations of the region of complete localization, involving this time finite volume operators, rather than the infinite volume one. We prove

Theorem 6.1. *Let $I \subset \Sigma$ be a compact interval and assume that Wegner's estimate **(W)** holds in I . For L given, consider $\Lambda = \Lambda_L(0)$ a cube of side length L centered at 0, and denote by $\varphi_{\omega, \Lambda, j}$, $j = 1, \dots, \text{tr} \mathbf{1}_I(H_\omega(\Lambda))$, the normalized eigenvectors of $H_\omega(\Lambda)$ with corresponding eigenvalue in I . The following are equivalent*

- (1) $I \subset \Sigma_{\text{CL}}$
(2) For all $E \in I$, there exists $\theta > 3d - 1$,

$$\limsup_{L \rightarrow \infty} \mathbb{P} \left\{ \forall x, y \in \Lambda, |x - y| \geq \frac{L}{2}, \|\chi_x(H_\omega(\Lambda) - E)^{-1}\chi_y\| \leq L^{-\theta} \right\} = 1. \quad (6.1)$$

- (3) For all $\xi \in (0, 1)$,

$$\sup_{y \in \Lambda} \mathbb{E} \left\{ \sum_{x \in \Lambda} e^{|x-y|^\xi} \sup_j \|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \right\} < \infty. \quad (6.2)$$

- (4) There exists $\xi \in (0, 1)$,

$$\sup_{y \in \Lambda} \mathbb{E} \left\{ \sum_{x \in \Lambda} e^{|x-y|^\xi} \sup_j \|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \right\} < \infty. \quad (6.3)$$

- (5) For all $\xi \in (0, 1)$,

$$\sup_{y \in \Lambda} \mathbb{E} \left\{ \sum_{x \in \Lambda} e^{|x-y|^\xi} \sup_{\substack{\text{supp } f \subset I \\ |f| \leq 1}} \|\chi_x f(H_\omega(\Lambda))\chi_y\|_2 \right\} < \infty. \quad (6.4)$$

- (6) There exists $\xi \in (0, 1)$,

$$\sup_{y \in \Lambda} \mathbb{E} \left\{ \sum_{x \in \Lambda} e^{|x-y|^\xi} \sup_{\substack{\text{supp } f \subset I \\ |f| \leq 1}} \|\chi_x f(H_\omega(\Lambda))\chi_y\|_2 \right\} < \infty. \quad (6.5)$$

- (7) There exists $\xi \in (0, 1)$,

$$\sup_{y \in \Lambda} \sup_{\substack{\text{supp } f \subset I \\ |f| \leq 1}} \mathbb{E} \left\{ \sum_{x \in \Lambda} e^{|x-y|^\xi} \|\chi_x f(H_\omega(\Lambda))\chi_y\|_2 \right\} < \infty. \quad (6.6)$$

- (8) *(SUDEC for finite volume)* For all $p > 0$ and $\xi \in (0, 1)$, there exists $q > 0$ such that for any L large enough, the following holds with probability at least $1 - L^{-p}$: for any eigenvector $\varphi_{\omega, \Lambda, j}$ of $H_{\omega, \Lambda}$, with energy in I , for any $(x, y) \in \Lambda^2$,

$$\|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \leq L^q e^{-|x-y|^\xi}. \quad (6.7)$$

- (9) *(SULE for finite volume)* For all $p > d$ and $\xi \in (0, 1)$, there exists $q > 0$ such that for any L large enough, the following holds with probability at least $1 - L^{-p}$: for any eigenvector $\varphi_{\omega, \Lambda, j}$ of $H_{\omega, \Lambda}$, with energy in I , there is a center of localization $x_{\omega, \Lambda, j} \in \Lambda$, so that for any $x \in \Lambda$,

$$\|\varphi_{\omega, \Lambda, j}\|_x \leq L^q e^{-|x-x_{\omega, \Lambda, j}|^\xi}. \quad (6.8)$$

Remark 6.1. Theorem 6.1 provides a finite volume analog of [GK06, Theorem 1 and Corollary 3]. If generalized eigenfunctions are needed in the infinite volume case, the normalized eigenfunctions are good enough in the finite volume one for the spectrum intersected with the interval I is discrete.

For discrete models, using e.g. the finite volume fractional moment criteria of [ASFH01], one can derive bounds of the same type as (6.8) where $\xi = 1$. This has been done in [K111].

Proof of Theorem 6.1. We start by describing precisely Σ_{CL} , which is the set of energies where the conclusion of the bootstrap multiscale analysis of [GK01] is valid: Σ_{CL} for the random operator H_ω is defined as the set of $E \in I$ for which there exists some open interval $I(E) \subset I$, with $E \in I(E)$, such that, given any $\zeta \in (0, 1)$ and $\alpha \in (1, \zeta^{-1})$, there exists a length scale $L_0 \in 6\mathbb{N}$ and a mass $m > 0$, so that if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$ for $k = 0, 1, \dots$, we have

$$\mathbb{P} \{R(m, L_k, I(E), x, y)\} \geq 1 - e^{-L_k^\zeta} \quad (6.9)$$

for all $k = 0, 1, \dots$, and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + R_0$, where

$$R(m, L, I, x, y) = \left\{ \begin{array}{l} \omega; \\ \text{for every } E' \in I, \text{ either } \Lambda_L(x) \\ \text{or } \Lambda_L(y) \text{ is } (\omega, m, E')\text{-regular} \end{array} \right\}. \quad (6.10)$$

Here $[K]_{6\mathbb{N}} = \max\{L \in 6\mathbb{N}; L \leq K\}$; we work with scales in $6\mathbb{N}$ for convenience; $R_0 > 0$ is given in Assumption (IAD).

Given $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$ and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (ω, m, E) -regular for a given $m > 0$ if $E \notin \sigma(H_\omega(\Lambda_L(x)))$ and

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x, \frac{L}{3}}\| \leq e^{-mL}, \quad (6.11)$$

where $R_{\omega,x,L}(E) = (H_\omega(\Lambda_L(x)) - E)^{-1}$ and $\Gamma_{x,L}$ denotes the characteristic function of the ‘‘belt’’ $\overline{\Lambda}_{L-1}(x) \setminus \Lambda_{L-3}(x)$ when $\mathcal{H} = L^2(\mathbb{R}^d, dx)$ (the arguments can be easily modified for $\mathcal{H} = \ell^2(\mathbb{Z}^d)$). The interval I in Theorem 6.1 being compact, we can extract from $\cup_{E \in I} I(E) \supset I$ a finite number of intervals $I(E)$ that cover I . Thus, with no loss, we may assume that (6.9) is valid for the interval I itself.

We turn to the proof of Theorem 6.1 per se. That (1) \iff (2) is due to [GK04]. Note that if (6.1) holds, then

$$\limsup_{L \rightarrow \infty} \mathbb{P} \left\{ \|\Gamma_{0,L} R_{\omega,0,L}(E) \chi_{0, \frac{L}{3}}\| \leq L^{-\theta-2d+1} \right\} = 1.$$

Assume (2), that is (6.9) holds for the interval I . We show that (3) holds. A standard computation (see [GK04, (EDI)], [Kir08, Eq.(8.10)]) shows that if $\Lambda_L(x)$ is (ω, m, E) -regular, then, if $H_\omega(\Lambda_L(x))\varphi_{\omega,\Lambda,j} = \tilde{E}\varphi_{\omega,\Lambda,j}$ and $\|\varphi_{\omega,\Lambda,j}\| = 1$, then

$$\|\varphi_{\omega,\Lambda,j}\|_x \lesssim L^{d-1} \|\Gamma_{x,L} R_{\omega,x,L}(\tilde{E}) \chi_{x, \frac{L}{3}}\| \leq e^{-\frac{m}{2}L}, \quad (6.12)$$

whenever L is large enough, depending only on d, m .

It follows that for any $(x, y) \in \Lambda^2$, if k is so that $L_k \leq |x - y| < L_{k+1}$, then for any $\omega \in R(m, L, I, x, y)$, for any normalized $\varphi_{\omega,\Lambda,j}$,

$$\|\varphi_{\omega,\Lambda,j}\|_x \|\varphi_{\omega,\Lambda,j}\|_y \leq e^{-\frac{m}{2}L_k} \leq e^{-\frac{m}{2}|x-y|^{\frac{1}{\alpha}}}. \quad (6.13)$$

As a consequence,

$$\mathbb{E} \left\{ \sup_j \|\varphi_{\omega,\Lambda,j}\|_x \|\varphi_{\omega,\Lambda,j}\|_y \right\} \leq e^{-\frac{m}{2}|x-y|^{\frac{1}{\alpha}}} + e^{-|x-y|^{\frac{\zeta}{\alpha}}} \leq 2e^{-|x-y|^{\frac{\zeta}{\alpha}}}, \quad (6.14)$$

for $|x - y|$ large enough. Thus, (6.2) follows for any $\xi < \zeta\alpha^{-1}$. Since $\zeta < 1 < \alpha$ can be chosen arbitrarily close to 1, (3) follows.

(3) clearly implies (4).

To see that (3) implies (5), it is enough to decompose the operator $f(H_\omega(\Lambda))$ over the eigenvectors $(\varphi_{\omega,\Lambda,j})_j$ and note that $\|\chi_x \Pi_{\varphi_{\omega,\Lambda,j}} \chi_y\|_2 = \|\varphi_{\omega,\Lambda,j}\|_x \|\varphi_{\omega,\Lambda,j}\|_y$, when $\Pi_{\varphi_{\omega,\Lambda,j}}$ is the orthogonal projection on $\varphi_{\omega,\Lambda,j}$.

That (5) \implies (6) \implies (7) is immediate.

Assume (3). We prove (8). We have

$$\mathbb{P} \left(\exists (x, y) \in \Lambda^2, e^{|x-y|^\xi} \sup_j \|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \geq L^{p+d} \right) \quad (6.15)$$

$$\leq L^d \sup_{y \in \Lambda} \sum_{x \in \Lambda} \mathbb{P} \left(e^{|x-y|^\xi} \sup_j \|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \geq L^{p+d} \right) \quad (6.16)$$

$$\leq L^{-p} \sup_{y \in \Lambda} \sum_{x \in \Lambda} \mathbb{E} \left(e^{|x-y|^\xi} \sup_j \|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \right) \quad (6.17)$$

$$\lesssim L^{-p}. \quad (6.18)$$

In other terms, with probability at least $1 - L^{-p}$, we have $\|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \leq L^{p+d} e^{-|x-y|^\xi}$, for any j and $(x, y) \in \Lambda^2$, and (8) holds.

We show that (8) and (9) are equivalent. Assume (8) and let $x_{\omega, \Lambda, j}$ be a center of localization for $\varphi_{\omega, \Lambda, j}$. Since $\|\varphi_{\omega, \Lambda, j}\| = 1$, note that $\|\varphi_{\omega, \Lambda, j}\|_x \gtrsim L^{-d/2}$. We write (6.7) with $y = x_{\omega, \Lambda, j}$ and (6.8) follows from the last observation. Conversely, if (9) holds, then it follows from (6.8) that

$$\|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \leq L^{2q} e^{-|x-x_{\omega, \Lambda, j}|^\xi} e^{-|y-x_{\omega, \Lambda, j}|^\xi} \leq L^{2q} e^{-|x-y|^\xi}. \quad (6.19)$$

It remains to show (7) \implies (2) and (8) \implies (2).

Assume (8). For any given $E \in I$, thanks to (W), we may assume that $d(E, \sigma(H_\omega(\Lambda))) \geq L^{-p}$, with probability at least $1 - L^{-p}$. As a consequence, for any L large enough, for any $(x, y) \in \Lambda^2$, $|x - y| \geq L/2$,

$$\|\chi_x(H_\omega(\Lambda) - E)^{-1} \chi_y\| \leq L^p \sum_j \|\varphi_{\omega, \Lambda, j}\|_x \|\varphi_{\omega, \Lambda, j}\|_y \quad (6.20)$$

$$\leq L^{p+q} e^{-L^\xi}. \quad (6.21)$$

(2) follows.

Assume (7) and pick $E \in I$. Set $\delta = L^{-p}$. By (W), with probability at least $1 - L^{-p}$, we have

$$(H_\omega(\Lambda) - E)^{-1} = (H_\omega(\Lambda) - E)^{-1} \mathbf{1}_{\Sigma \setminus [E-\delta, E+\delta]}(H_\omega(\Lambda)). \quad (6.22)$$

Define the function $f_{E, \delta}(\lambda) = \delta(\lambda - E)^{-1} \mathbf{1}_{\Sigma \setminus [E-\delta, E+\delta]}(\lambda)$. Note that $|f_{E, \delta}| \leq 1$. It follows that

$$\mathbb{P} \left(\exists (x, y) \in \Lambda^2, |x - y| \geq \frac{L}{2}, \|\chi_x(H_\omega(\Lambda) - E)^{-1} \chi_y\| \geq L^{-\theta} \right) \quad (6.23)$$

$$\leq L^{-p} + \sum_{\substack{(x, y) \in \Lambda^2 \\ |x - y| \geq L/2}} \mathbb{P} (\|\chi_x f_{E, \delta}(H_\omega(\Lambda)) \chi_y\| \geq L^{-\theta-p}) \quad (6.24)$$

$$\leq L^{-p} + L^{-p} \sup_{y \in \Lambda} \sum_{\substack{x \in \Lambda \\ |x - y| \geq L/2}} L^{\theta+2p+d} \mathbb{E} (\|\chi_x f_{E, \delta}(H_\omega(\Lambda)) \chi_y\|) \quad (6.25)$$

$$\leq L^{-p} + L^{-p} \sup_{y \in \Lambda} \sum_{x \in \Lambda} (2|x - y|)^{\theta+2p+d} \mathbb{E} (\|\chi_x f_{E, \delta}(H_\omega(\Lambda)) \chi_y\|_2) \lesssim L^{-p}. \quad (6.26)$$

□

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