



HAL
open science

Finite volume discretization for dynamic viscosities on Voronoi grids

Christian Hüttig, Kai Stemmer

► **To cite this version:**

Christian Hüttig, Kai Stemmer. Finite volume discretization for dynamic viscosities on Voronoi grids. *Physics of the Earth and Planetary Interiors*, 2008, 171 (1-4), pp.137. <10.1016/j.pepi.2008.07.007>. <hal-00532161>

HAL Id: hal-00532161

<https://hal.science/hal-00532161v1>

Submitted on 4 Nov 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Accepted Manuscript

Title: Finite volume discretization for dynamic viscosities on Voronoi grids

Authors: Christian Hüttig, Kai Stemmer

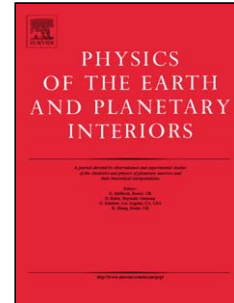
PII: S0031-9201(08)00165-9
DOI: doi:10.1016/j.pepi.2008.07.007
Reference: PEPI 4996

To appear in: *Physics of the Earth and Planetary Interiors*

Received date: 15-11-2007
Revised date: 15-5-2008
Accepted date: 7-7-2008

Please cite this article as: Hüttig, C., Stemmer, K., Finite volume discretization for dynamic viscosities on Voronoi grids, *Physics of the Earth and Planetary Interiors* (2007), doi:10.1016/j.pepi.2008.07.007

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



1 **Finite volume discretization for dynamic viscosities on Voronoi** 2 **grids**

3 *Christian Hüttig¹ and Kai Stemmer, Institute of Planetary Research, Department of Planetary*
4 *Physics, German Aerospace Center (DLR), Berlin, Germany*

5 **Abstract**

6 We present a new formulation to discretize the viscous term in the momentum equa-
7 tion of the Navier-Stokes set. A technique based on the finite volume method enables ther-
8 mal convection models to utilize spatially varying viscosity on a collocated variable arrange-
9 ment. This technique can be applied to various grids in two or three dimensions with Voronoi
10 properties, either irregular as the spiral grid or regular like the cubed sphere grid, icosahedral
11 or simple boxes. A model for mantle convection implements this discretization and is com-
12 pared to other published models. Further computational aspects are illuminated to efficiently
13 reduce required resources.

14 **1. Introduction**

15 Over the last three decades, local discretization methods like finite-element (FE), fi-
16 nite-volume and finite-difference (FD) methods have firmly established themselves as the
17 approach for computational fluid-flow problems in geophysics, especially with regard to
18 thermal convection problems involving spatially varying viscosities in two (Parmentier, 1978;
19 Christensen, 1984; Hansen and Yuen 1993; Solomatov and Moresi 2000) and three dimen-
20 sions (Ogawa et al., 1991; Christensen and Harder 1991; Trompert and Hansen, 1998; Tackley
21 1998; Stein et al. 2004) as well as in spherical shells (Tabata and Suzuki, 2000; Zhong et al
22 2000; Yoshida and Kageyama 2004; McNamara and Zhong 2005; Choblet 2005; Stemmer et
23 al. 2006). The advantages of irregular grids like arbitrary refinements and resolution choices
24 were left outside because of more complicated discretization techniques and computational

¹ Corresponding Author

25 requirements. Nevertheless, these advantages become especially interesting in cases with
26 complex geometries like spheres in three dimensions (3D) or disks in two dimensions (2D),
27 which occur regularly in geophysics. This work demonstrates a technique to utilize the FV
28 method on irregular grids in 3D with a focus on the spatial derivative of the stress tensor, re-
29 quired for the implementation of dynamic viscosities, which is of primary interest in the field
30 of mantle convection modeling, e.g. Stemmer et al., 2006.

31 Most of the published models for mantle convection that include locally varying vis-
32 cosities implement the necessary spatial derivative of the stress tensor with finite-elements
33 (Zhong et al. 2000) or utilize grid-regularities (Yoshida and Kageyama 2004, Choblet 2005,
34 Stemmer et al. 2006). Staggered approaches where velocity and scalar fields do not coincide
35 at the same locations were preferred because of their straightforward implementation and
36 elimination of numerical oscillations (Peric et al., 1988). Advantages of the FVM are the direct
37 connection to the underlying physical problem and the conservative nature of its scheme
38 that enables a direct implementation of boundary conditions similar to the finite-difference
39 method. A disadvantage arises in irregular grids. The effort to discretize the desired equa-
40 tions with the FV method increases once the walls between the computational cells become
41 tilted and spatial derivatives besides the normal direction are required. Ferziger and Péric,
42 2001 describe an effective approach to acquire a correct solution for tilted walls. The only
43 remaining problem within an irregular grid was the implementation of a spatial derivative of
44 the relatively complex stress tensor that is required for locally varying viscosities in geophysi-
45 cal flow problems.

46 The approach presented in this paper bases on the proposed dual-grid approach in
47 Ferziger and Péric, 2001. It utilizes a Voronoi grid as cellular discrete basis for the domain and
48 its dual, the Delaunay triangulation to setup shape functions for an arbitrary linear interpola-
49 tion mechanism (Baranger et al., 1996), as shown in figure 1. The triangulation in combina-
50 tion with a Voronoi discretization is able to fulfill all requirements for irregular grids in a FV

51 scheme. In recent years, computational and storage improvements have opened the door to
52 embrace the use of complex grids with their advantages and increased computational re-
53 quirements.

54 **2. Numerical Method**

55 **2.1. Preface**

56 The domain of interest in FV and FE methods is defined as control volumes, inte-
57 grated over the governing differential equations of interest (Ferziger and Péric, 2001). Mov-
58 ing further to fluid mechanics, the basis of the FVM is therefore the integral form of the con-
59 servation equations. The solution domain is divided into a finite number of control volumes
60 (CVs), and the conservation equations are applied to each CV. At the centroid of each CV lies
61 a computational node at which the variable values are to be calculated. Interpolation is used
62 to express variable values at the CV surface in terms of the nodal (CV-center) values. Surface
63 and volume integrals are approximated using suitable quadrature formulae. As a result, one
64 obtains an algebraic equation for each CV, in which a possible varying number of neighbor
65 nodal values appear (Fletcher, 2001).

66 The FVM can accommodate any type of grid, so it is suitable for regular and irregular
67 grids. Most regular grids are Voronoi grids by nature, which means that within a CV every
68 point is closer to its associated nodal location than to any other. This nodal location serves as
69 generator point for a Voronoi region and results in a CV. The resulting cell structure offers
70 interesting properties: the face dividing two neighboring CVs lies always midway and per-
71 pendicular between them, which is one requirement for the FV method. The amount of faces
72 is the direct count of neighbors for a specific CV and is always minimal (Okabe et al., 2000).
73 This neighbor count is directly associated with the matrix column count per row.

74 Exploiting the detail that a face lies midway between two neighboring nodes reduces
75 the difficulties of interpolating scalar values to the face centers, which is essential for the FV

76 method. To determine the face-value ϕ_f between the nodes $N1$ and $N2$ of a scalar ϕ , a central
 77 difference scheme (CDS) results in

$$\phi_f = \frac{1}{2}(\phi_{N1} + \phi_{N2}).$$

78 (1)

79 The CDS results in a second order accuracy as long as the interpolated point coincides
 80 with the center of the face. On condition that the mid-point stays within the face, the FVM
 81 has at least a first order accuracy (Ferziger and Péric, 2001). A complete FVM discretization
 82 with CDS interpolation for the Nabla operator, where P specifies a discrete node index, N the
 83 neighbor index and PN face quantities for the face between P and N like area A and normal
 84 vector \vec{n} results in

$$\begin{aligned} \nabla \phi_P &= \int_V \nabla \phi_P dV = \frac{1}{V_P} \int_S \phi_f \cdot \vec{n} dS = \frac{1}{V_P} \sum_N \phi_f A_{PN} \cdot \vec{n}_{PN} \\ &= \frac{1}{V_P} \sum_N \frac{1}{2} (\phi_P + \phi_N) A_{PN} \cdot \vec{n}_{PN} \end{aligned}$$

86 . (2)
 87

88 If ϕ is a vector, equation (2) reduces to the divergence operator $div(\phi)$, for a scalar field the

89 result is the gradient operator $grad(\phi)$.

90 2.2. Barycentric Interpolation

91 A more complex interpolation scenario arises for irregular grids. Faces often undergo
 92 a perpendicular shift that moves the face center away from the interpolated midpoint. To
 93 hold on to the second order accuracy, the necessity for a more complex interpolation arises.

94 An obvious choice is linear interpolation from a triangulated domain. Since the Delaunay
 95 triangulation is the dual of the Voronoi diagram (Okabe et al., 2000), utilizing it is straight-
 96 forward. The barycentric coordinates (Shepard, 1968) offer a simple way to interpolate at
 97 any point within a triangulated domain. As they are coordinates, they provide position infor-
 98 mation relative to a simplex (D - dimensional tetrahedral region). As a side effect, these co-
 99 ordinates in a normalized homogeneous form (their sum equals one) provide weight infor-
 100 mation for the specified location, as Figure 2 illustrates. In contrast to the natural neighbor
 101 interpolation introduced by Sambridge et al., 1995, this method requires always a minimal
 102 amount ($D+1$) of nodal values and does not require recursive formulas to obtain the weights.

103 Correct interpolation to the center of the face PN is now possible and by the use of
 104 barycentric interpolation remains of second order accuracy (Ferziger and Péric, 2001, chapter
 105 8.6). Equation (1) changes with the help of barycentric weights λ to

$$\phi^{fc} = \sum_{d=1}^{D+1} \lambda_d^w \phi_{\lambda_d^i}$$

106 (3)

107 where λ denotes the barycentric coordinates for a D -dimensional space, including the
 108 nodal indices λ^i of the according Delaunay simplex edge and the weight (or normalized coor-
 109 dinate) λ^w . Figure 3 illustrates all variables for the two-dimensional case within a single cell.

110 2.3. Spatial derivative of the deviatoric stress tensor

111 Many geophysical fluids require a spatial derivative of the stress tensor as part of the
 112 Navier-Stokes momentum equation. This enables the simulation to handle spatially varying
 113 viscosities. The definition of the deviatoric stress tensor of a velocity u and dynamic viscosity
 114 η is

$$\tau_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \eta (\nabla \vec{u} + (\nabla \vec{u})^T)$$

115 (4)

116 and results therefore in a rank 2 symmetric tensor of $D \times D$ dimensions. It can be expressed as

117 components of gradients of velocity components as shown in equation (5) for the two-
 118 dimensional case, where the diagonal parts represent the normal stresses and the off-
 119 diagonal parts shear stresses:

$$\tau_{ij} = \eta \begin{bmatrix} 2(\nabla \vec{u}_x)_x & (\nabla \vec{u}_y)_x + (\nabla \vec{u}_x)_y \\ (\nabla \vec{u}_x)_y + (\nabla \vec{u}_y)_x & 2(\nabla \vec{u}_y)_y \end{bmatrix} \quad (5)$$

121 The spatial derivative of this tensor results in a vector and according to the FV
 122 scheme is

$$\nabla \cdot \tau = \int_V \nabla \cdot \tau dV = \frac{1}{V} \int_S \tau \cdot \vec{n} dS \quad (6)$$

124 For constant η , $\nabla \cdot \tau$ reduces to $\eta \nabla^2 \vec{u}$. The problem for the FV method arises now that the
 125 gradients of the velocity components are required at the faces. The literature describes sev-
 126 eral ways to approximate the normal component to implement the Laplacian operator ∇^2
 127 (Ferziger and Péric 2001; Fletcher 1991). Usually finite difference schemes express deriva-
 128 tives at the faces. The problem with varying viscosities is that also at least one other non-
 129 normal spatial derivate is required. Stemmer et al., 2006 recently formulated an elegant solu-
 130 tion for the regular cubed sphere grid for collocated variable arrangement, utilizing midpoint
 131 interpolation of neighboring nodes in several directions. This technique is useful for cubical
 132 grid-setups but fails on irregular grids.

133 Including the neighbors of a nodes neighbor would resolve this issue and guarantees
 134 second order accuracy, but also increases the solution matrix size dramatically and would
 135 therefore make this approach impractical. Another choice is a change to the finite-difference
 136 approach on the faces. With the help of shape functions from the barycentric coordinates
 137 (2.2), a Cartesian cross can be constructed from the center of a face. This enables an FD for-
 138 mulation at the faces as presented in figure 4. The FV integral for an arbitrary u_i would

139 evolve with the interpolation sources $f_{i+/-}$ as in equation (2) for the Cartesian interpolation
 140 cross and f_c for the center as

$$\int_S \tau \cdot \vec{n} \, dS = \int_S \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \cdot \vec{n} \, dS = \sum_N \sum_{j=1}^D \frac{u_i^{f_{j+}} - u_i^{f_{j-}} + u_j^{f_{i+}} - u_j^{f_{i-}}}{d_s} \eta^{f_c} A_{PN} \cdot \vec{n}_{PN}$$

141 ,
 142 (7)

143 where d_s specifies the size of the interpolation cross. This distance can be either fixed for the
 144 whole grid or varying for every face, depending on resolution differences. It is untested what
 145 the optimal value for d_s is, but tests have shown that for a certain face area A_{PN} in a D -
 146 dimensional space $d_s = \sqrt[D]{A_{PN}}$ leads to a minimum of foreign neighbors (not depending on
 147 neither one of the original nodes that the face divides) on strongly irregular grids.

148 A different problem arises with the required viscosity at the cell face (η^{f_c}). Assuming
 149 the viscosity is correct at the face center, the method guarantees through pure use of second
 150 order interpolants a second order truncation error. This might not hold true if the viscosity
 151 needs to be interpolated. For example, in regular grids it can be shown that the harmonic
 152 interpolant of viscosity between adjacent cells preserves continuity of the normal stress
 153 component resolved onto the face from cell-to-cell, which is closer to the physical truth of
 154 the situation. However, this does not necessarily hold true for continuity of shear stress, and
 155 suggests another viscosity interpolant might be used. An implementation of a dual stencil
 156 composed of shear and normal components of stress and their associated continuity could
 157 ensure a proper viscosity interpolant at the faces.

158 2.4. Laplacian

159 The energy conservation as well as the momentum conservation equation for con-
 160 stant viscosities consists of the Laplacian operator ∇^2 . The FV discretization results in

$$\nabla^2 \phi = \int_V \nabla^2 \phi \, dV = \frac{1}{V} \int_S \nabla \phi \cdot \vec{n} \, dS \quad (8)$$

161

162 which requires the gradient in normal direction at the cell face. The fact that only the normal
163 direction of the gradient is required can be exploited and replaced by a CDS (Fletcher 1991):

$$\int_S \nabla \phi \cdot \vec{n} \, dS = \sum_N \frac{\phi_N - \phi_P}{d_{PN}} A_{PN} \quad (9)$$

164

165 This approximation for the face stays second order accurate even in irregular grids
166 since the derivative between P and N does not change within all simplexes that have both, P
167 and N , as connectors. However, a face center can move out of those simplexes on strongly
168 deformed cells. In this case, it is useful to introduce barycentric weights of the normal direc-
169 tion from the face center $f_{n+/-}$, as illustrated in Figure 3 as well. Equation (9) changes to

$$\int_S \nabla \phi \cdot \vec{n} \, dS = \sum_N \frac{\phi^{f_{n+}} - \phi^{f_{n-}}}{d_S} A_{PN} \quad (10)$$

170

171 and guarantees second order accuracy for all Voronoi cells.

171

172 3. Computational Aspects and Tests

173 3.1. Setup phase

173

174 In order to utilize the in chapter two presented scheme, a pre-calculated Voronoi di-
175 agram and Delaunay triangulation for the same discrete set of points should exist with the
176 following information:

176

- 177 • Nodal positions
- 178 • Cell volumes
- 179 • List of faces with their two neighbor indices (the node indices they divide), area and
180 center

180

- 181 • List of simplex indices

182

183 The interpolation information for each face must exist and should be pre-calculated
 184 to run a simulation efficiently. They stay in a fixed position during the simulation and can
 185 therefore be pre-calculated. One set of interpolation information consists of a simplex index
 186 that in turn holds the $D+1$ nodal indices λ_d^i and weight information λ_w^i . Each face usually has
 187 $2D+3$ or $2D+1$ interpolation points consisting of the center, the Cartesian cross and two more
 188 depending on the use of the $fn+/-$ weights as described in chapter 2.4.

189 To find a certain point within a Delaunay triangulation, it is possible to calculate the
 190 barycentric coordinates of a desired point relative to every simplex with equation (3) until
 191 each single coordinate lies between zero and one, as described in chapter 2.2. The time com-
 192 plexity of this procedure falls into $O(n^2)$. To reduce this to $O(n \log n)$, it is possible to index
 193 the simplex indices within a search tree. These indices correspond to node indices, which can
 194 be found in the face neighbor information as well. Since only locations close to a certain face
 195 are looked up, the simplexes containing the neighbor nodes of a face can be checked first.
 196 For extremely deformed cells, the fallback to a check-every-simplex should exist as well.

197 **3.2. Storage requirements**

198 Including the $fn+/-$ weights, interpolation information for one face requires

$$199 \quad (1 + D)(3 + 2D)(W + I) \tag{11}$$

200 bytes if a weight is stored with W bytes and an index with I bytes. On an average ir-
 201 regular grid in three dimensions, one node contains of approx. 14 neighbors (Huettig and
 202 Stemmer 2006), resulting in $7R$ faces, where R denotes the total node count. Assuming fur-
 203 ther double floating-point precision for the weights ($W=8$) and standard 32-bit integer for
 204 indices ($I=4$), the total memory requirement for the interpolation information is $3024R$ byte.

205 To reduce this enormous extra memory requirement one can exploit the fact that a weight is
206 always only in the range between zero and one. Utilizing fast integer arithmetic can turn a
207 16-bit integer into a floating point consisting of 2^{16} steps between zero and one, introducing
208 a global error of $\frac{D+1}{2^{16}}$ and a reduction to $W=2$, resulting in 1512R byte. A reduction to $l=3$ is
209 only theoretically possible because it leads, if even possible, practically to misaligned mem-
210 ory that dramatically reduces performance.

211 **3.3. Validation**

212 In order to test the presented numerical technique several results from other pub-
213 lished mantle convection models were reproduced to verify its accuracy. According to linear
214 stability analysis (Busse, 1975), there are two stable solutions for thermal convection in the
215 three-dimensional spherical shell with an inner to outer radius ratio of 0.55 that is compara-
216 ble to the Earth's mantle and is purely bottom heated. The complete set of equations is

217

$$\nabla \cdot \mathbf{u} = 0$$

218

$$\frac{\partial T}{\partial t} + \mathbf{u} \nabla T - \nabla^2 T = 0$$

219

$$\nabla \cdot [\eta (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] + Ra T \mathbf{e}_r - \nabla p = 0$$

$$\eta(T) = \exp(\gamma(T_{ref} - T)) \quad Ra = \frac{\rho g \alpha \Delta T d^3}{\kappa \eta_{ref}} \quad \eta_{ref} = \eta(T_{ref})$$

221

, where T is the Temperature, \mathbf{u} the velocity, p the hydrostatic pressure, t the time and \mathbf{e}_r the

222

unity vector towards the gravitational center. The viscosity follows an Arrhenius law and is

223

purely temperature dependent, as in Stemmer et al., 2006.

224

The stationary stable flow patterns in tetrahedral and cubic symmetry are suitable to

225

compare output values like the heat flow at the core-mantle boundary and the volume-

226

averaged rms-velocity against other published models that use different discretization

227

schemes and numerical techniques. To initiate these steady state patterns, the initial conduc-

228

tive temperature field is perturbed with a normalized spherical harmonic mode Y_l^m with

229

small amplitude. The mode Y_3^2 forces a tetrahedral flow pattern and $Y_4^0 + Y_4^4$ results in a

230

cubical symmetric flow. A convergence test for an isoviscous tetrahedral flow with a Rayleigh

231

number of 7000 similar to the published test by Stemmer et al., 2006, results in the bottom

232

Nusselt numbers (3.3105, 3.4514, 3.4848, 3.4897) for a radial/lateral grid resolution of

233

(8/642, 16/2562, 32/10242, 48/10242) nodes. Table 1 shows a detailed comparison to other

234

published models of isoviscous convection $\Delta\eta T = 1$ and convection with temperature de-

235

pendent viscosity of $\Delta\eta T = 10$, $\Delta\eta T = 20$ and $\Delta\eta T = 30$ respectively as an extension to the

236

published results by Stemmer et al., 2006.

237

For the advection-diffusion equation of temperature a similar discretization approach

238

as described in 2.4 with a three-level implicit time stepping and a BiCGStab solver turned out

239

to be an efficient combination. A major advantage of the method discussed here lies in its

240

robustness for a fully implicit treatment: instead of a classical use of the Courant-criteria

241 which restricts the time step length on the maximal velocity within the system, an approach
242 utilizing the maximal difference in a velocity compared to the previous inner iteration lead to
243 equally good results. An effect of this restriction is that less chaotic convection models re-
244 quire less time steps and therefore computational time. Of course, on very turbulent models,
245 the difference is less pronounced and the computing time required for a single time step
246 becomes relatively large.

247 Table 2 shows, besides a more detailed comparison of local values such as radial min
248 / max velocities and temperatures at mid-depth, the computational effort for these kind of
249 problems. This table shows also the volatile nature of these control values as two different
250 initial conditions are compared to each other.

251 Figure 5a and b shows detailed flow patterns and temperature distributions of the te-
252 trahedral and cubical steady state pattern with low temperature dependent viscosity
253 ($\Delta\eta T=20$). Once $\Delta\eta T$ reaches a certain limit, the convective regime changes to stagnant-lid
254 convection. This transition is smooth and between contrasts of $\Delta\eta T=1e4$ and $5e5$, as further
255 examined in three-dimensional spherical geometry by Stemmer et al., 2006. The here intro-
256 duced method shows the same behavior around this contrast and develops a stagnant lid, as
257 illustrated in figure 5c. Figure 6 displays the typical plume-thinning phenomena caused by
258 temperature dependent viscosity (Ratcliff et al. 1996; Hansen and Yuen, 1993). If the viscos-
259 ity contrast based on temperature is increased for a tetrahedral or cubical mode with mod-
260 erate Rayleigh numbers, the plume-tail gets thinner and faster.

261 **4. Summary**

262 The discretization method presented in chapter 2 is a proof of concept for mantle
263 convection simulations in irregular n-dimensional grids. The spatial derivative of the stress
264 tensor offers further thermal convection models the possibility to utilize varying viscosities.
265 The discretization bases on the finite volume method and the dual-grid approach (Ferziger

266 and Péric, 2001). A Voronoi diagram is constructed around the locations of the nodes that act
267 as cells for the control volumes. Furthermore, the dual of the Voronoi diagram, the Delaunay
268 triangulation, serves as interpolation basis for values between the cells.

269 Chapter 3 illuminates computational aspects dealing with an effective way to store
270 certain information within a setup phase and demonstrates how to reduce the extra resource
271 requirements for this kind of discretization. There is also a hint on how to speed up the setup
272 phase, reducing the time complexity back to $O(n \log n)$.

273 A comparison with other published models for mantle convection validates the
274 scheme and concludes this document. The verification bases on isoviscous and weakly tem-
275 perature dependent steady state flow patterns in a bottom heated three-dimensional spheri-
276 cal shell with an aspect ratio of 0.55. Bottom heat flow (Nusselt number) and RMS velocity
277 are compared to a collection of other published models of the same kind and shows promis-
278 ing results. The accuracy stays below two percent to other published values. The volatile
279 nature of local control values to initial conditions is also demonstrated in table 2.

280 Another general advantage of this technique is the physical view on the problem. As
281 Fletcher, 1991 noted, an advantage of the Finite Volume method is the direct view on the
282 underlying physical problem without the need for higher order abstractions as required for a
283 finite element approach, while maintaining its robustness and versatility.

284 **Acknowledgements**

285 The authors thank John Hernlund, David Yuen, and Paul Meakin for thoughtful and
286 constructive reviews. The authors would also like to thank D. Breuer and T. Spohn of the
287 German Aerospace Center (DLR) for providing the facilities utilized during this study. This
288 work was carried out under the HPC-EUROPA project (RII3-CT-2003-506079), with the sup-
289 port of the European Community - Research Infrastructure Action of the FP6.

290

- 291 Baranger, J., Matre, J.F., Oudin, F., 1996. Connection between finite volume and mixed finite
292 element methods, *Math. Mod. Numer. Anal.* 30 (4) 445–465.
- 293 Bercovici, D., Schubert, G., Glatzmaier, G.A., 1989. Threedimensional spherical models of
294 convection in the Earth's mantle. *Science* 244, 950–955.
- 295 Blankenbach, B., Busse, F., Christensen, U., Cserepes, L., Gunkel, D., Hansen, U., Harder, H.,
296 Jarvis, G., Koch, M., Marquart, G., Moore, D., Olson, P., Schmeling, H., Schnaubelt, T., 1989. A
297 benchmark comparison for mantle convection codes. *Geophys. J. Int.* 98, 23–38.
- 298 Busse, F.H., 1975. Patterns of convection in spherical shells. *J. Fluid Mech.* 72, 67–85.
- 299 Busse, F.H., Christensen, U., Clever, R., Cserepes, L., Gable, C., Giannandrea, E., Guillou, L.,
300 Houseman, G., Nataf, H.-C., Ogawa, M., Parmentier, M., Sotin, C., Travis, B., 1993. 3D convec-
301 tion at infinite Prandtl number in Cartesian geometry—a benchmark comparison. *Geophys.*
302 *Astro. Fluid* 75, 39–59.
- 303 Choblet, G., 2005. Modelling thermal convection with large viscosity gradients in one block of
304 the 'cubed sphere'. *J. Comput. Phys.* 205, 269–291.
- 305 Christensen, U., 1984. Convection with pressure- and temperature-dependent non-
306 Newtonian rheology. *Geophys. J. R. Astro. Soc.* 77, 343–384.
- 307 Christensen, U., Harder, H., 1991. 3D convection with variable viscosity. *Geophys. J. Int.* 104,
308 213–226.
- 309 Ferziger, J.H., Peric, M., 2001. *Computational Methods for Fluid Dynamics*, 3rd rev. ed. Edi-
310 tion, Springer, Berlin.
- 311 Fletcher, C.A.J., 1991. *Computational Techniques for Fluid Dynamics*, vol. 2., second ed.
312 Springer, Berlin.
- 313 Hansen, U., Yuen, D.A., 1993. High Rayleigh number regime of temperature-dependent vis-
314 cosity convection and the Earth's early thermal history. *Geophys. Res. Lett.* 20, 2191.
- 315 Harder, H., 1998. Phase transitions and the three-dimensional planform of thermal convec-
316 tion in the Martian mantle. *J. Geophys. Res.* 103, 16775–16797.
- 317 Harder, H., Hansen, U., 2005. A finite-volume solution method for thermal convection and
318 dynamo problems in spherical shells. *Geophys. J. Int.* 161, 522–532.
- 319 Huettig, C., Stemmer, K., 2007. The spiral grid: A new approach to discretize the sphere and
320 its application to mantle convection. *Geochem. Geophys. Geosyst.*, in press
- 321 Iwase, Y., 1996. Three-dimensional infinite Prandtl-number convection in a spherical shell
322 with temperature-dependent viscosity. *J. Geomag. Geoelectr.* 48, 1499–1514.
- 323 McNamara, A.K., Zhong, S., 2005. Degree-one mantle convection: dependence on internal
324 heating and temperature-dependent rheology. *Geophys. Res. Lett.* 32, L01301.

- 325 Ogawa, M., Schubert, G., Zebib, A., 1991. Numerical simulations of three-dimensional ther-
326 mal convection in a fluid with strongly temperature-dependent viscosity. *J. Fluid Mech.* 233,
327 299–328.
- 328 Okabe, A., Boots, B., Sugihara, K., Chiu, S.K., 2000. *Spatial Tessellations: Concepts and Appli-*
329 *cations of Voronoi Diagrams*, Wiley, ISBN 0471986356
- 330 Parmentier, E.M., 1978. A study of convection in non-Newtonian fluids. *J. Fluid Mech.* 84, 1–
331 11.
- 332 Peric, M., Kessler, R., Scheurer, G., 1988. Comparison of finite-volume numerical methods
333 with staggered and colocated grids. *Comp. Fluids* 16, 389–403.
- 334 Ratcliff, J.T., Schubert, G., Zebib, A., 1996. Effects of temperature-dependent viscosity on
335 thermal convection in a spherical shell. *Physica D* 97, 242–252.
- 336 Sambridge, M., Braun, J., McQueen, H., 1995. Geophysical parametrization and interpolation
337 of irregular data using natural neighbours. *Geophysical Journal International*, 122 (3), pp.
338 837–857.
- 339 Shepard, D., 1968. Two-dimensional interpolation function for irregularly-spaced data, *Proc*
340 *23rd Nat Conf*, pp. 517–524
- 341 Stein, C., Schmalz, J., Hansen, U., 2004. The effect of rheological parameters on plate behav-
342 iour in a self-consistent model of mantle convection. *Phys. Earth Planet. Int.* 142, 225–255.
- 343 Tabata, M., Suzuki, A., 2000. A stabilized finite element method for the Rayleigh-Benard equa-
344 tions with infinite Prandtl number in a spherical shell. *Comput. Methods Appl. Mech. Eng.*
345 190, 387–402.
- 346 Tackley, P.J., 1998. Self-consistent generation of tectonic plates in three-dimensional mantle
347 convection. *Earth Planet. Sci. Lett.* 157, 9–22.
- 348 Trompert, R.A., Hansen, U., 1996. The application of a finite volume multi-grid method to
349 three-dimensional flow problems in highly viscous fluids with a variable viscosity. *Geophys.*
350 *Astro. Fluid* 83, 261–291.
- 351 Trompert, R.A., Hansen, U., 1998. Mantle convection simulations with rheologies that gener-
352 ates plate-like behaviour. *Nature* 395, 686–689.
- 353 Yoshida, M., Kageyama, A., 2004. Application of the Ying–Yang grid to a thermal convection
354 of a Boussinesq fluid with infinite Prandtl number in a three-dimensional spherical shell.
355 *Geophys. Res. Lett.* 31, L12609.
- 356 Zhong, S., McNamara, A., Tan, E., Moresi, L., Gurnis, M., 2008. A Benchmark Study on Mantle
357 Convection in a 3-D Spherical Shell Using CitcomS. Submitted to *G³* on April 4, 2008.

358 Zhong, S., Zuber, M.T., Moresi, L., Gurnis, M., 2000. Role of temperature-dependent viscosity
359 and surface plates in spherical shell models of mantle convection. *J. Geophys. Res.* 105,
360 11063–11082.

Accepted Manuscript

361

362 Figure 1 shows the Voronoi diagram in red and the according Delaunay triangulation in blue
363 for a random set of generator points

364

365 Figure 2 illustrates the barycentric coordinates for a point P within the triangle ABC.

366

367 Figure 3: A node P surrounded by its neighbors N, with the associated Voronoi diagram in red
368 and the Delaunay triangulation in blue. The difference between the true face center with its
369 barycentric coordinates λ and the central difference scheme (mid-point) is highlighted as well
370 as important components for the FVM such as area A , distance d and normal vector \vec{n} .

371

372 Figure 4 shows as an extension to figure 3 the barycentric coordinates of the constructed
373 Cartesian cross around the face center and the two interpolation points along the normal
374 path of a pre-defined size d_s .

375

376 Figure 5 shows different convection patterns for a selected set of cases. All views display
377 temperature, while the upper left part shows the residual iso-surface of temperature with
378 the respective iso-values printed below, with a cutout of the positive domain for a better
379 insight. The upper right part displays a slice with streamlines and the lower part a projection
380 of the annotated shell. All cases are purely bottom heated with free-slip boundary conditions
381 and purely temperature dependent viscosity variations. $Ra_{0.5}=7000$ in all cases. The different
382 cases illustrate: a) $\Delta\eta T=20$, tetrahedral initial condition, b) $\Delta\eta T=20$, cubical initial condition, c)
383 $\Delta\eta T=1e6$, random initial condition. Case c shows a similar pattern on all other initial condi-
384 tions.

385

386 Figure 6 illustrates the difference of a weakly temperature dependent case ($\Delta\eta T=20$) with
387 $Ra_{0.5}=7000$ to the isoviscous case. While a, b, c and e display the difference in velocity, d
388 shows the difference in temperature. Because the slice views d and e are not centered
389 around zero, they contain an extra red contour line.

390

391 Table 1: Comparison of the bottom Nusselt number and the rms-velocity for the tetrahedral
392 and cubic steady-state convection with a viscosity contrast of $\Delta\eta T=1$, $\Delta\eta T=20$ and $\Delta\eta T=30$.
393 The Rayleigh number is 7000 for $T=0.5$. The abbreviation 'Be89' stands for the results from
394 Bercovici et al. (1989), 'Zh00' from Zhong et al. (2000), 'lw96' from lwase (1996), 'TS00' from
395 Tabata and Suzuki (2000), 'Ha98' from Harder (1998), 'Ra96' from Ratcliff (1996) and 'YK04'
396 from Yoshida and Kageyama (2004). The respective discretization method is listed as well,
397 where 'SP' denotes spectral, 'FE' finite elements, 'FD' finite differences and 'FV' finite vol-
398 umes.

399

400 Table 2 shows a comparison of global and local quantities as well as compute time measure-
401 ments for some selected cases. The sole influence on the initial condition (I.C.) in the first
402 two cases demonstrates the volatile nature of these values. All computations took place on
403 an 8 CPU shared memory machine (Opteron 875 w/ 2.2GHz), while t_{CPU} shows the compute
404 time in hours until an adequate steady state was reached. This final time is shown as non-
405 dimensional diffusion time t , combined with n_{iter} time steps. The grid consisted in all compu-
406 tations of a projected icosahedron with 32 radial levels and 10.242 lateral nodes. Other val-
407 ues consist of volume averaged temperature $\langle T \rangle$ and velocity $\langle V \rangle$, as well as their interior
408 (mid-shell) minima and maxima. The velocity minima and maxima are taken only from the
409 radial component. Values in square brackets are from Zhong et al., 2008 and round brackets
410 from Stemmer et al., 2006.

Tetrahedral symmetry			$\Delta\eta T = 1$		$\Delta\eta T = 10$		$\Delta\eta T = 20$	
Model	Method	Nodes	NU_{T0}	V_{rms}	NU_{T0}	V_{rms}	NU_{T0}	V_{rms}
Be89	SP	2400	3.4657	-	-	-	-	-
Zh00	FE	165888	3.519	-	-	-	-	-
lw96	FV	532480	3.45	32.4173	-	-	-	-
TS00	FE	324532	3.6565	32.936	-	-	-	-
Ha98	SP	552960	3.4955	32.6375	-	-	-	-
Ra96	FV	200000	3.4423	32.19	3.2337	26.80	3.1615	25.69
YK04	FD	2122416	3.4430	32.0481	-	-	3.133	26.1064
St06	FV	663552	3.4864	32.5894	3.2398	27.2591	3.1447	25.7300
This	FV	327744	3.4848	32.6535	3.2346	27.2513	3.1444	25.7139
Cubic symmetry			$\Delta\eta T = 1$		$\Delta\eta T = 20$		$\Delta\eta T = 30$	
Ha98	SP	552960	3.6086	31.0765	-	-	-	-
Ra96	FV	200000	3.5806	30.87	3.3663	25.17	3.3285	24.57
YK04	FD	2122416	3.5554	30.5197	3.3280	25.3856	-	-
St06	FV	663552	3.5982	31.0226	3.3423	24.9819	3.2864	24.1959
This	FV	327744	3.5953	31.0704	3.3315	24.9496	3.2747	24.1568

$Ra_{0.5}$	I.C.	$\Delta\eta T$	t	n_{iter}	t_{CPU}	Nu_t	Nu_b	$\langle T \rangle$	$T_{i,min}$	$T_{i,max}$	$\langle V \rangle$	$V_{ri,min}$	$V_{ri,max}$
1e5	Cubic	1	0.688 [0.315]	2110 [35000]	35.85	7.873 [7.850]	7.890 [7.770]	0.1826 [0.1728]	0.0304 [0.0228]	0.8804 [0.9454]	154.64 [154.8]	-265.58 [-261.5]	979.28 [982.6]
1e5	Random	1	0.748	1660	36.35	7.524 (7.371)	7.514 (7.372)	0.2493 (0.1941)	0.0306 (0.0271)	0.7970 (0.8973)	163.64 (153.13)	-396.70 (-275.65)	764.33 (949.34)
7000	Tetra	100	2.046 [2.0]	113 [30000]	3.33	2.909 [2.935]	2.920 [2.929]	0.2595 [0.2653]	0.0318 [0.0332]	0.9105 [0.9255]	23.13 [23.11]	-10.45 [-10.74]	166.59 [171.3]
7000	Tetra	1e3	1.001 [1.5]	201 [31000]	3.65	2.526 [2.546]	2.532 [2.535]	0.3027 [0.3124]	0.0662 [0.0695]	0.9292 [0.9452]	22.36 [22.90]	-6.40 [-6.97]	215.03 [226.7]

Accepted Manuscript

Manuscript

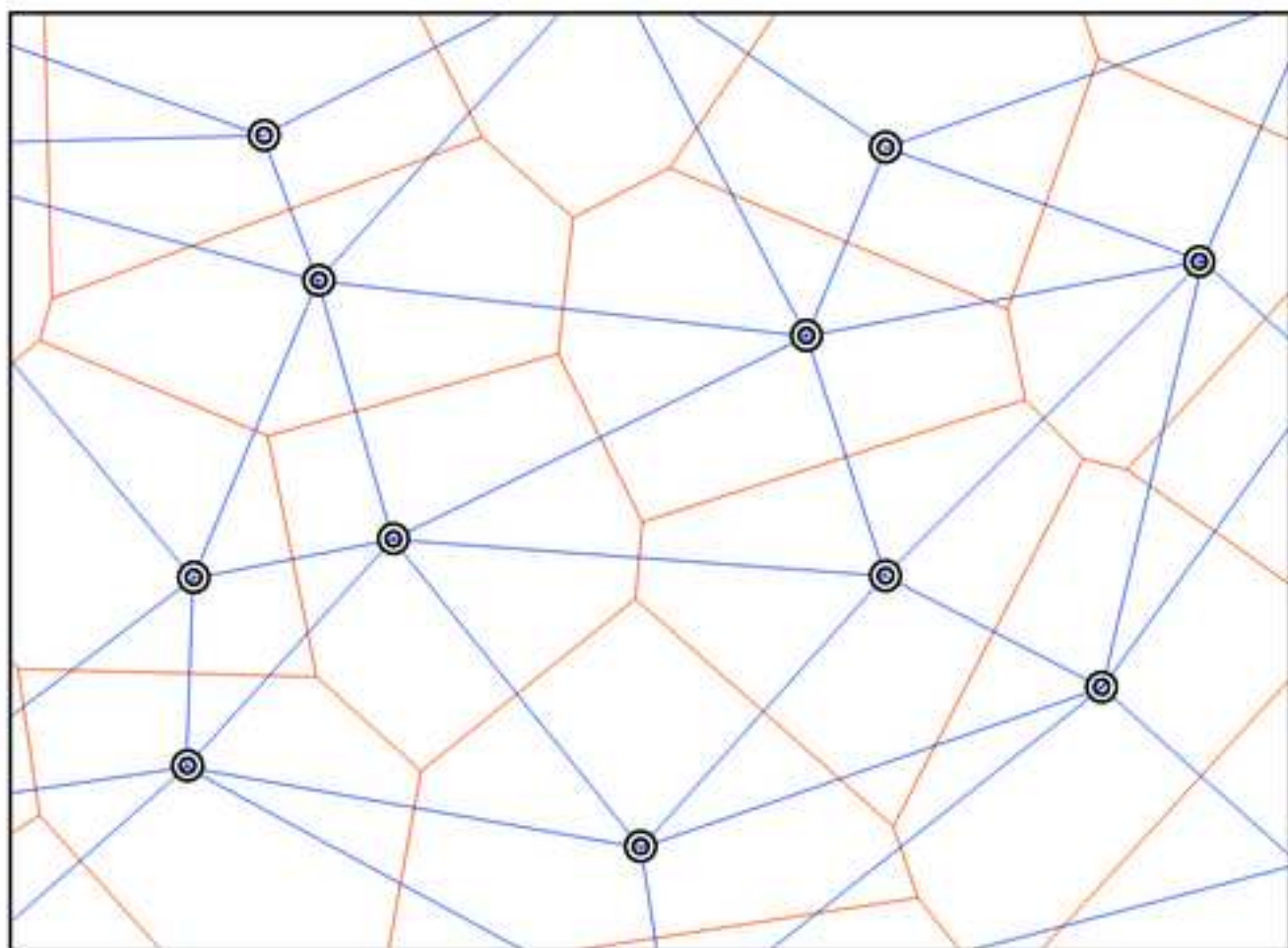
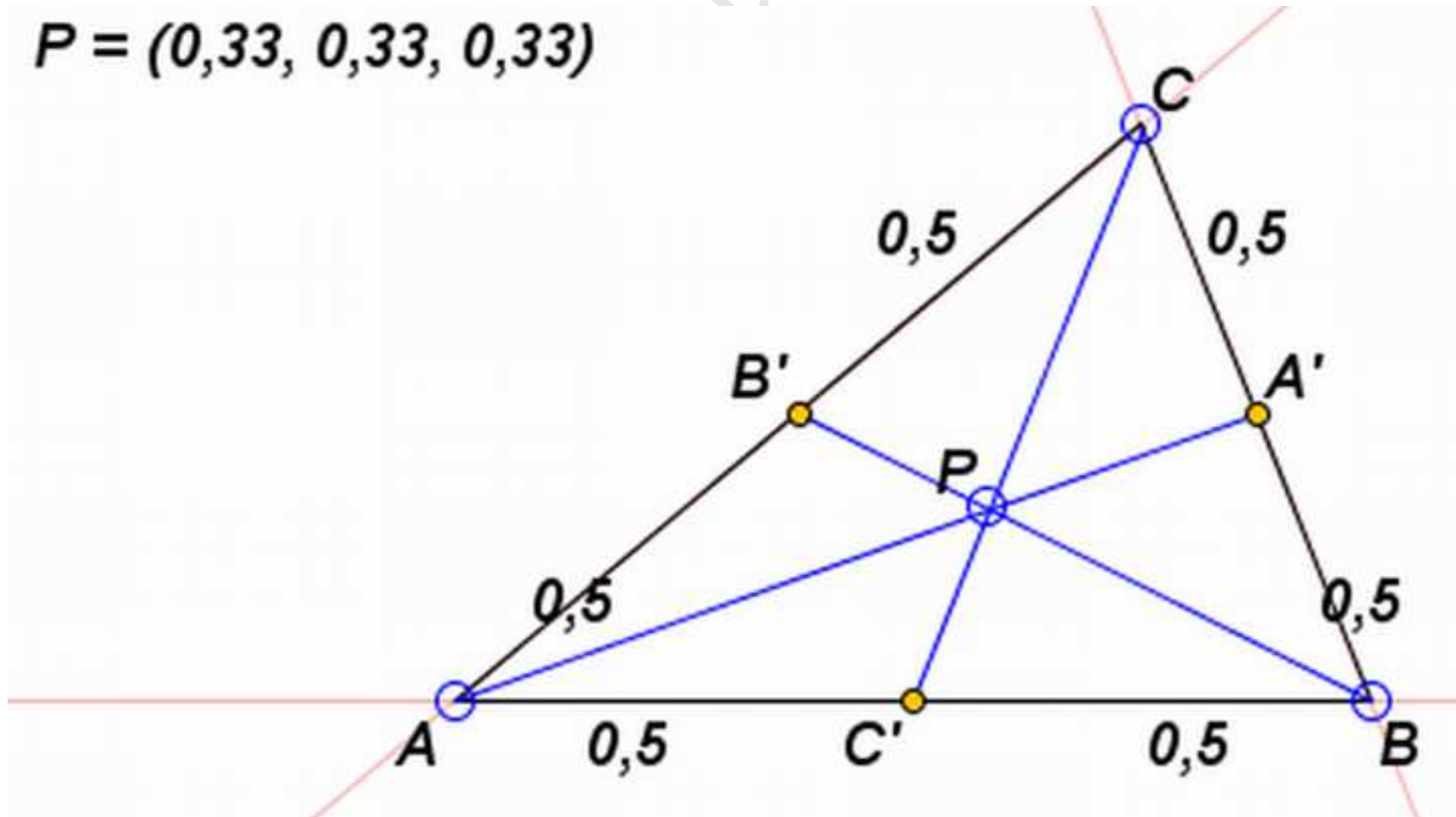
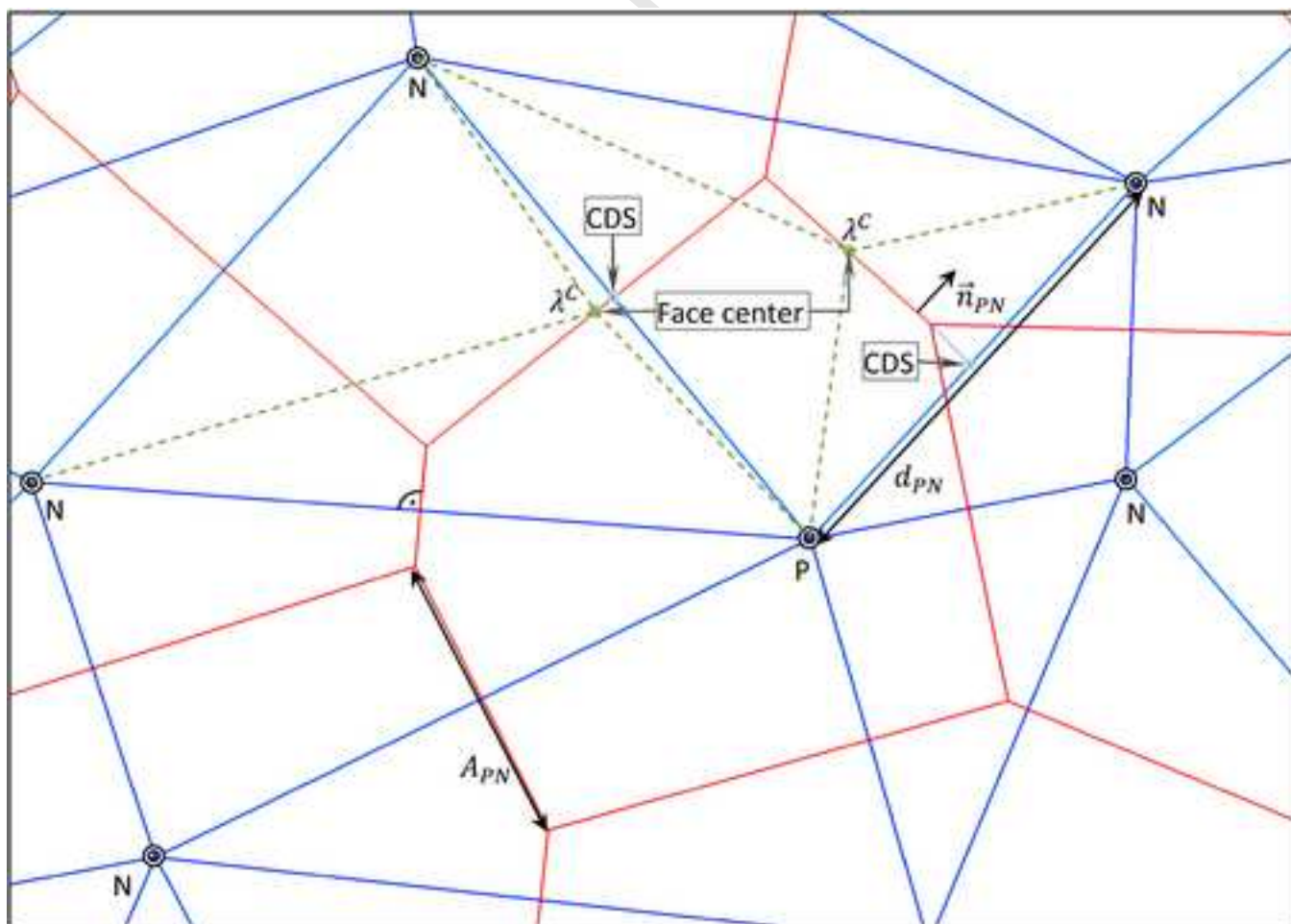
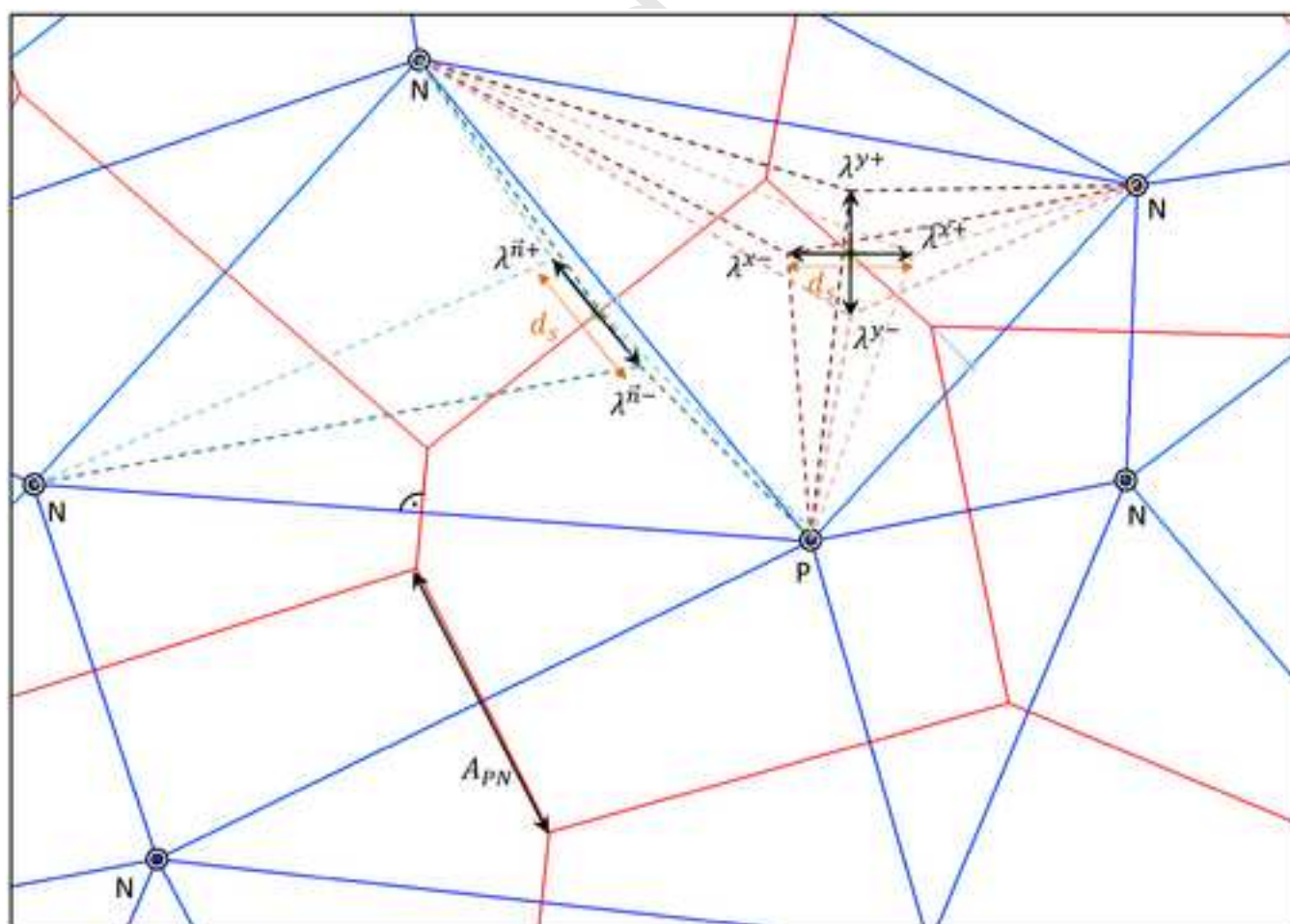


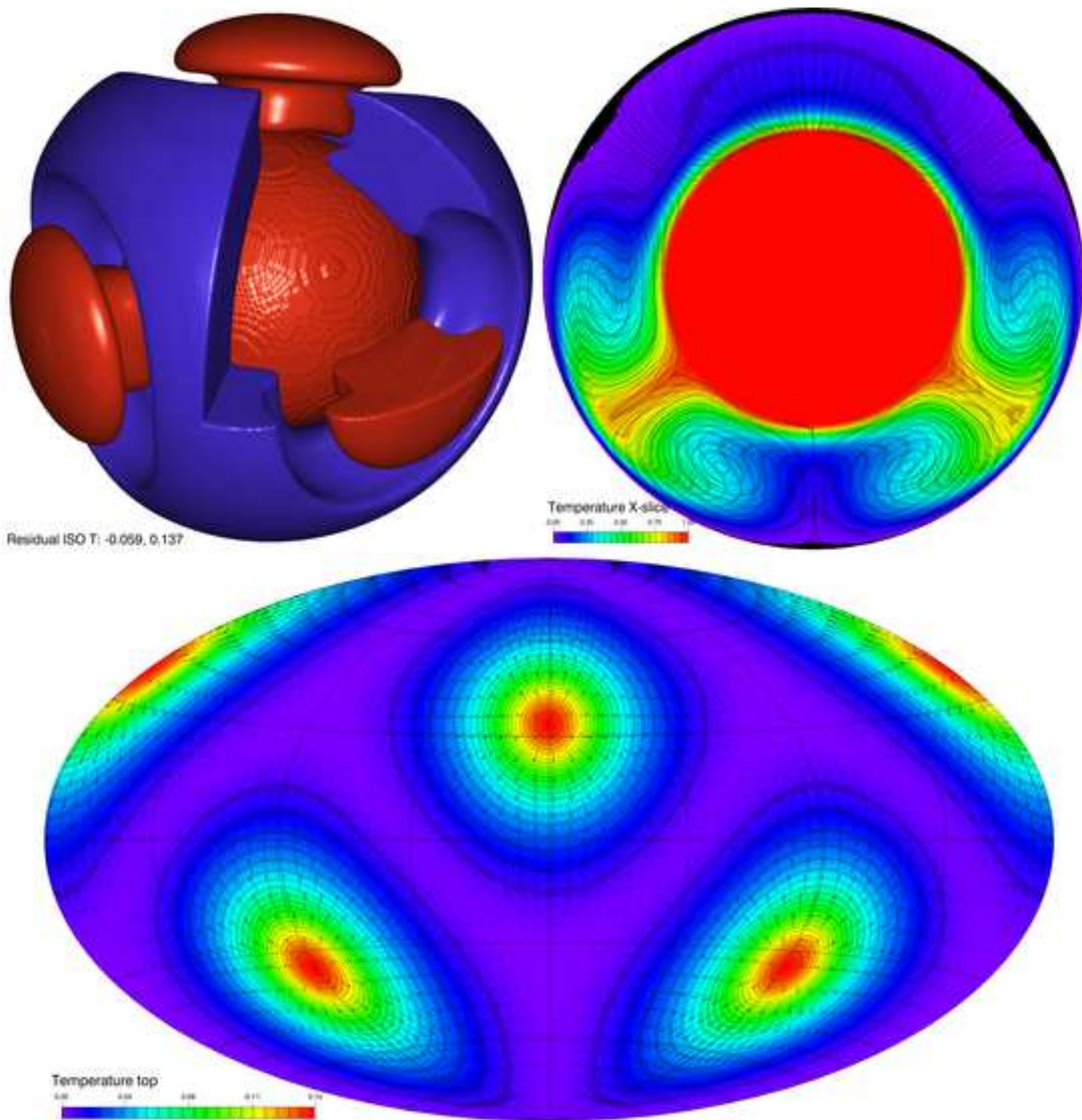
Figure 2

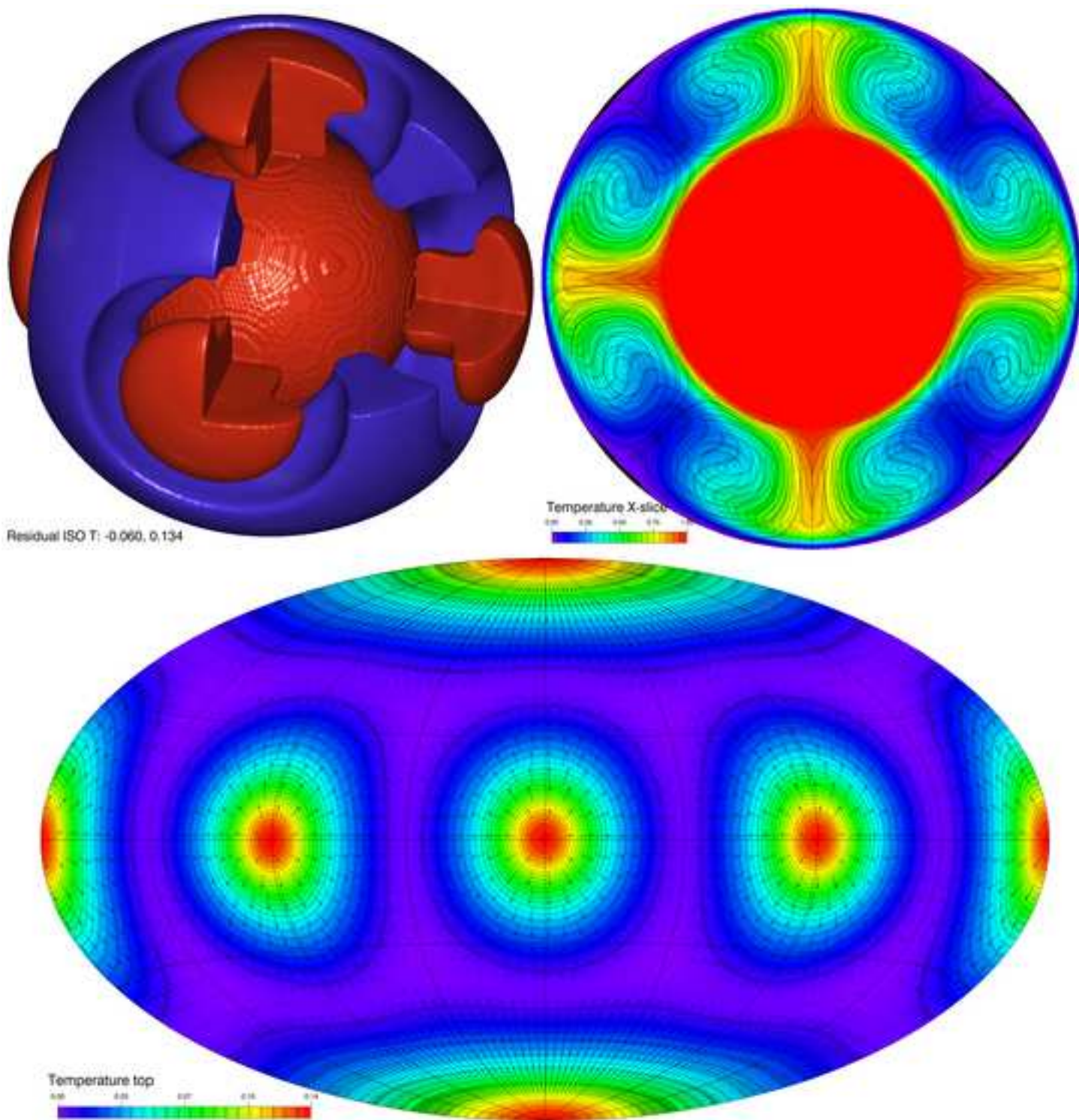
$$P = (0,33, 0,33, 0,33)$$











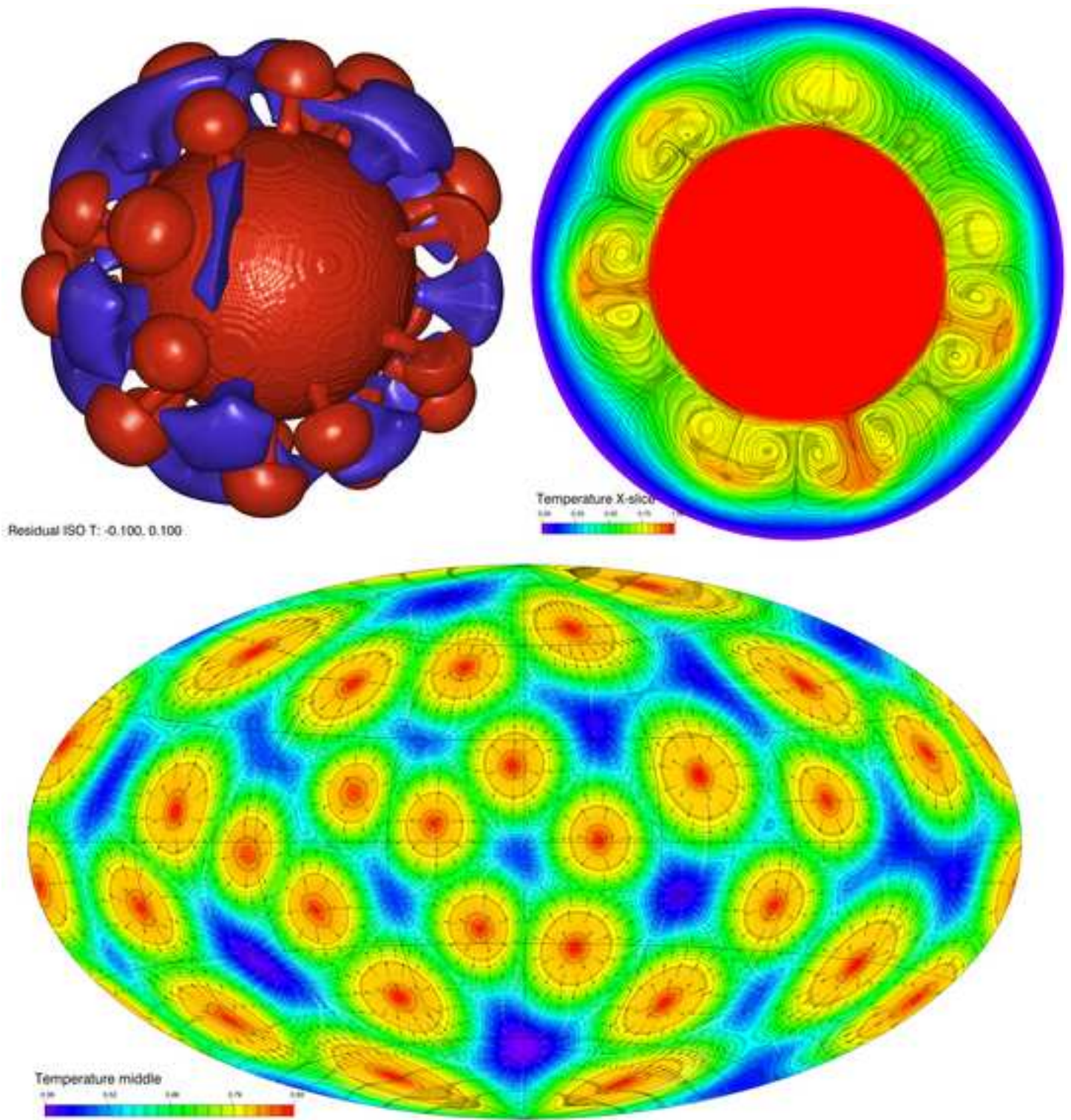


Figure 6a

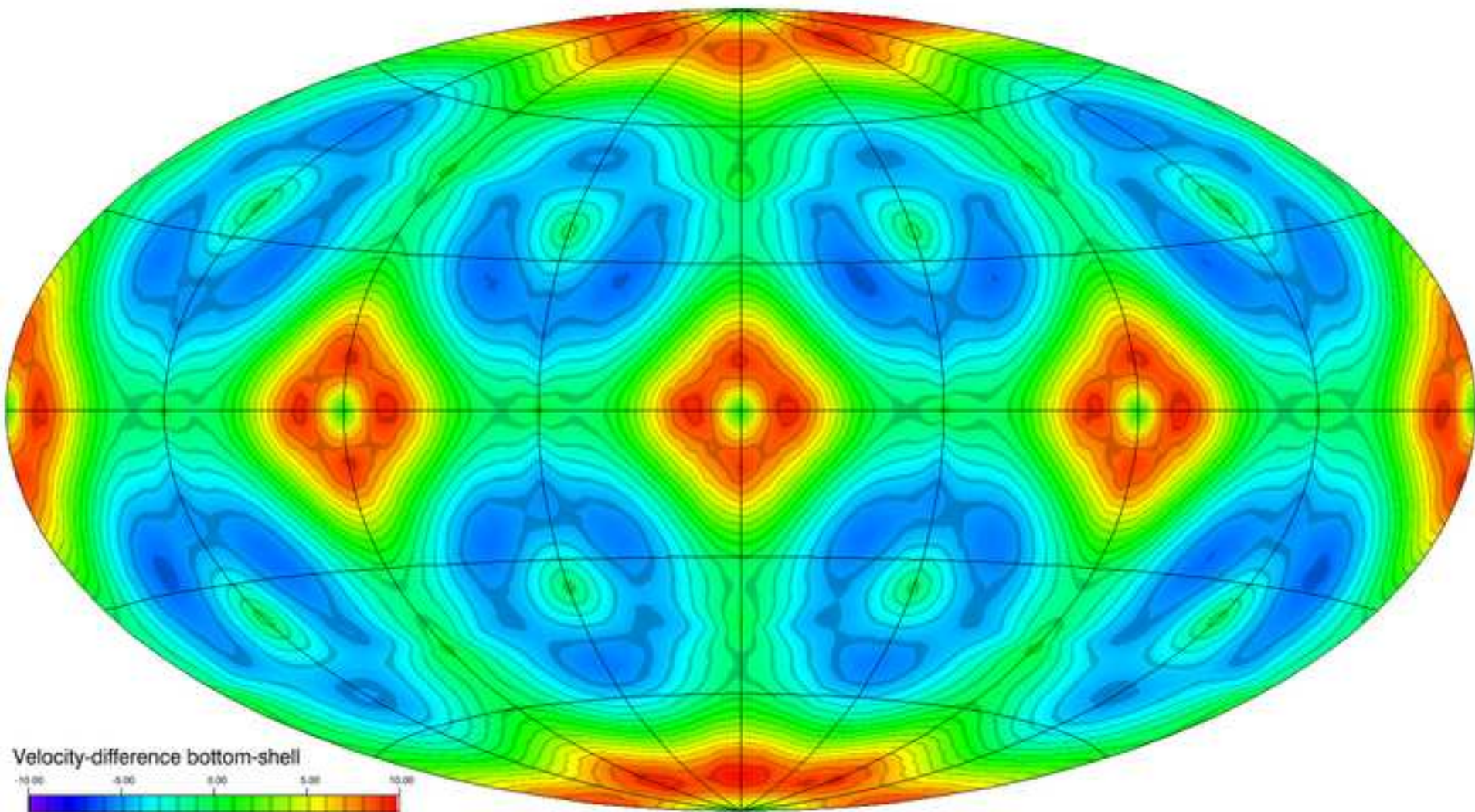


Figure 6b

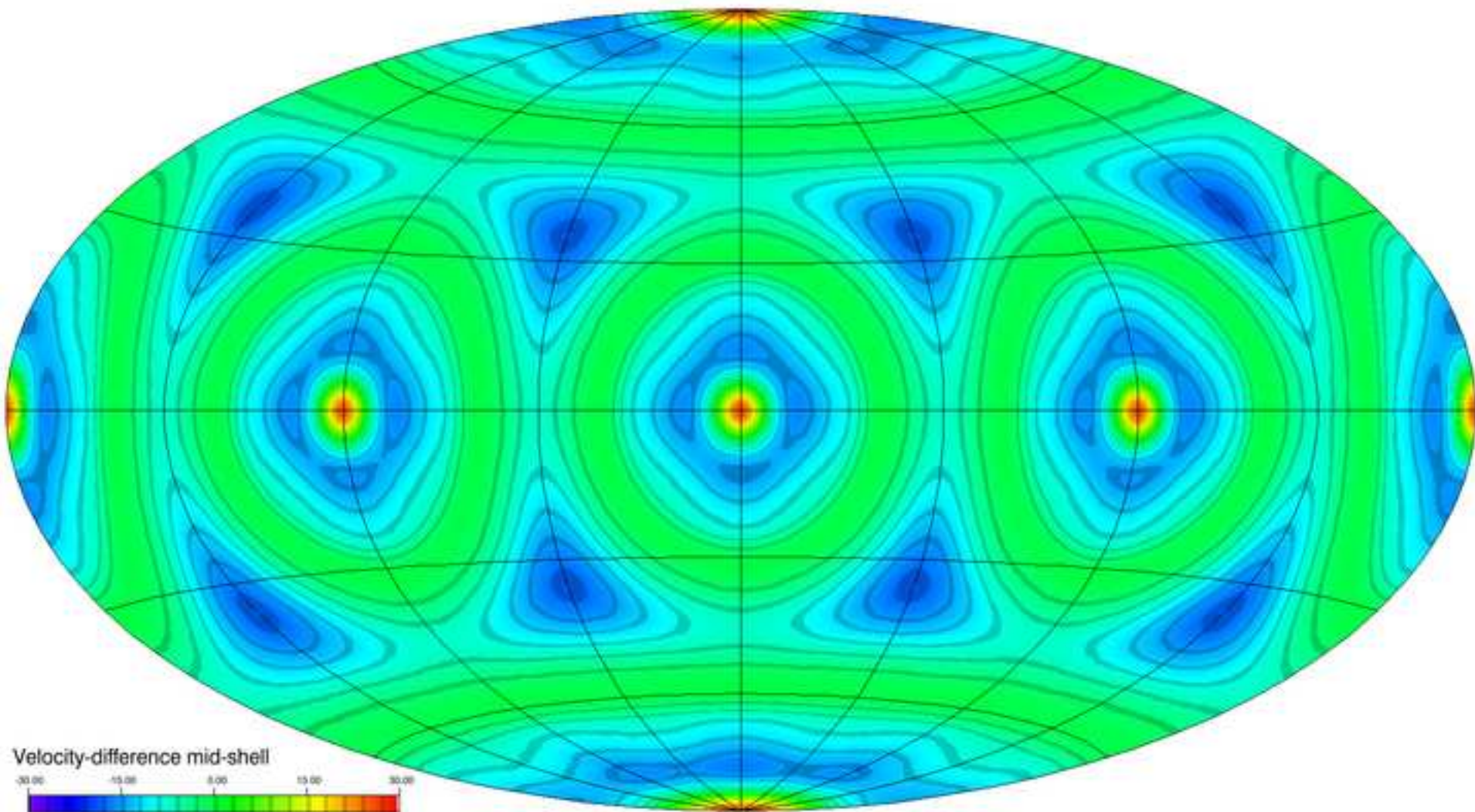


Figure 6c

