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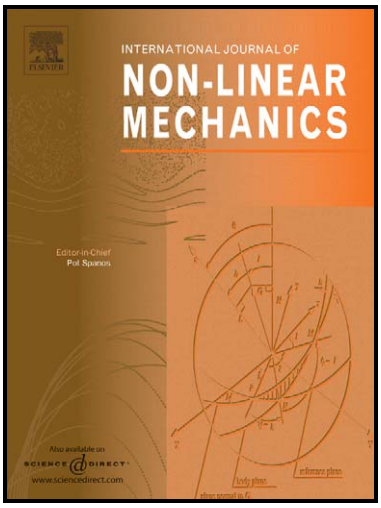

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Superposition of finite-amplitude transverse waves in deformed Mooney-Rivlin and neo-Hookean materials

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Dedicated to Philippe Boulanger on the occasion of his 60th birthday with esteem and gratitude.

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Abstract

In this paper, waves propagating in Mooney-Rivlin and neo-Hookean nonlinear elastic materials subjected to a homogeneous pre-strain are considered. In a previous paper, Boulanger and Hayes (Q. Jl Mech. Appl. Math. 45 (1992) 575–593) showed, for deformed Mooney-Rivlin materials, that the superposition of two finite-amplitude shear waves polarized in different directions (orthogonal to each other) and propagating along the same direction is an exact solution of the equations of motion. The two waves do not interact. Here, we are interested in superpositions of waves propagating in different directions. Two types of superpositions are considered: superpositions of waves polarized in the same direction, and also superposition of waves polarized in different directions. It is shown that such superpositions are exact solutions of the equations of motion with appropriate choices of the propagation and polarization directions.

Key words: finite elasticity, pre-strained materials, exact wave solutions, superpositions of waves.

1. Introduction

In this paper we consider, in the context of finite elasticity theory, two models of incompressible materials, the Mooney-Rivlin model and the neo-

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Hookean model. The Mooney-Rivlin model is characterized by two constitutive constants. The neo-Hookean model is characterized by one constitutive constant, the shear modulus.

Here, we are interested in the superposition of finite-amplitude linearly polarized transverse homogeneous plane waves propagating in different directions in Mooney-Rivlin and neo-Hookean materials which are first subjected to an arbitrary static homogeneous deformation.

In [1], Currie and Hayes showed that two linearly polarized finite-amplitude transverse waves, polarized in directions orthogonal to each other and to the propagation direction, may propagate along any direction in a deformed Mooney-Rivlin material. Later, Boulanger and Hayes [2] [3] [4] made a detailed study of homogeneous plane waves propagating in deformed Mooney-Rivlin materials. They also showed that the superposition of the two waves propagating along the same direction, and polarized along directions orthogonal to each other, is again an exact solution of the equations of motion.

Here, on the contrary, we are interested in superpositions of waves propagating in *different* directions. We first consider the possibility of superposition of finite-amplitude transverse plane waves propagating in different directions and polarized in the same direction in Mooney-Rivlin and neo-Hookean materials. Although the theory is nonlinear, it is shown that appropriate superpositions of this type are exact solutions of the equations of motion. Next, for neo-Hookean materials, it is shown that two appropriate finite-amplitude transverse homogeneous plane waves propagating and polarized in different directions may also be superposed. Although the theory is nonlinear such a superposition is also an exact solution of the equations of motion.

The plan of the paper is as follows. In Section 2, we present the Mooney-Rivlin and neo-Hookean models for nonlinear incompressible elastic materials.

Then (Section 3), the equations governing a motion superimposed on a state of arbitrary static homogeneous deformation are recalled. Note that the displacements due to this motion are not assumed to be small.

In Section 4, we consider the superposition of transverse homogeneous plane waves propagating in different directions and polarized in the same direction in deformed Mooney-Rivlin materials. It is shown that such a superposition is an exact solution of the equations of motion provided that the polarization direction of the two waves is along a principal direction of the static strain. The propagation directions are then arbitrary in the principal plane orthogonal to the polarization direction. Moreover, the waves must be

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of sinusoidal type.

In Section 5, we consider the same superposition as in Section 4, but in neo-Hookean materials. For neo-Hookean materials, such a superposition becomes possible for two arbitrary directions of propagation, both waves being polarized in the same direction orthogonal to the plane of the propagation directions.

In Section 6, we consider the superposition of two transverse homogeneous plane waves propagating and polarized in different directions in deformed neo-Hookean materials. It is shown that such a superposition is an exact solution of the equations of motion in the case when the direction of propagation of one of the transverse wave is orthogonal to the polarization direction of the other.

2. Mooney-Rivlin and Neo-Hookean materials

Incompressible isotropic elastic materials of the Mooney-Rivlin type are characterized by a strain-energy density W per unit volume given by [5]

$$2W = C(\mathrm{I}_1 - 3) + D(\mathrm{I}_2 - 3) \ , \tag{1}$$

where C and D are material constants, and $\mathrm{I}_1, \mathrm{I}_2$ are the first two principal invariants of the left Cauchy-Green strain tensor $\mathbf{B}$:

$$\mathrm{I}_1 = \mathrm{tr}\ \mathbf{B} \ , \quad 2\mathrm{I}_2 = (\mathrm{tr}\ \mathbf{B})^2 - \mathrm{tr}\ (\mathbf{B}^2) \ . \tag{2}$$

Because these materials are incompressible, the third invariant of $\mathbf{B}$ is

$$\mathrm{I}_3 = \det \mathbf{B} = 1 \ . \tag{3}$$

The components of $\mathbf{B}$, in a rectangular Cartesian-coordinate system, are

$$B_{ij} = (\partial x_i / \partial X_A)(\partial x_j / \partial X_A) \ , \tag{4}$$

where $x_i, (i = 1, 2, 3)$, are the coordinates at time t of the point whose coordinates are $X_A, (A = 1, 2, 3)$, in the undeformed reference configuration.

The corresponding constitutive equation for the symmetric Cauchy stress tensor $\mathbf{T}$ is

$$\mathbf{T} = -p_* \mathbf{1} + C\mathbf{B} - D\mathbf{B}^{-1} \ , \tag{5}$$

where

$$p_* = p - D\mathrm{I}_2 \tag{6}$$

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is an undetermined pressure.

It will be here assumed that the strong ellipticity conditions hold [6], i.e. that

$$C \geq 0 \text{ and } D > 0 \ , \text{ or } C > 0 \text{ and } D \geq 0 \ . \tag{7}$$

In order to be consistent with the linearized elasticity theory, we have [7]

$$C + D = \mu \ , \tag{8}$$

where μ is the shear modulus of linearized elasticity.

In the special case when $D = 0$, the elastic material is called "neo-Hookean".

3. Equations for motion superimposed on static deformation

Suppose now that these materials are first subjected to a static finite homogeneous deformation defined by

$$\mathbf{x} = \mathbf{F}\mathbf{X} \ , \quad x_i = F_{iA} X_A \ , \tag{9}$$

where F_{iA} is a constant deformation gradient. The corresponding constant left Cauchy-Green strain tensor is $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and its principal invariants $\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3$ are constants. On this state of deformation, we superpose a time-dependent displacement taking the particle at $\mathbf{x}$ in the static deformation to

$$\overline{\mathbf{x}} = \overline{\mathbf{x}}(\mathbf{x}, t) = \overline{\mathbf{x}}(\mathbf{F}\mathbf{X}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) \ . \tag{10}$$

Let $\overline{\mathbf{F}} = \partial \overline{\mathbf{x}} / \partial \mathbf{X}$ be the deformation gradient, from the undeformed state, of the time-dependent deformation resulting from the static deformation and superimposed displacement $\mathbf{u}(\mathbf{x}, t)$. The corresponding left Cauchy-Green strain tensor is $\overline{\mathbf{B}} = \overline{\mathbf{F}}\ \overline{\mathbf{F}}^T$ and its principal invariants are denoted $\overline{\mathrm{I}}_1, \overline{\mathrm{I}}_2, \overline{\mathrm{I}}_3$. In the absence of body forces, the equations of motion for this time-dependent deformation are

$$\overline{\rho}\ \ddot{\overline{\mathbf{x}}} = \mathrm{div}_{\overline{\mathbf{x}}} \overline{\mathbf{T}} \ , \quad \overline{\rho}\, \ddot{\overline{x}}_i = \partial \overline{T}_{ij} / \partial \overline{x}_j \ , \tag{11}$$

where $\overline{\rho}$ and $\overline{\mathbf{T}}$ respectively denote the mass density and Cauchy stress tensor at time t and position $\overline{\mathbf{x}}$. Using now the state of homogeneous static deformation as reference state, the Piola-Kirchhoff stress tensor is $\overline{\mathbf{P}}$ given by

$$\overline{\mathbf{P}} = (\rho/\overline{\rho}) \overline{\mathbf{T}} (\partial \mathbf{x} / \partial \overline{\mathbf{x}})^T \ , \quad \overline{P}_{ik} = (\rho/\overline{\rho}) \overline{T}_{ij} (\partial x_k / \partial \overline{x}_j) \ , \tag{12}$$

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and the equations of motion (11) may also be written in the form

$$\rho\ddot{\overline{\mathbf{x}}} = \mathrm{div}_{\mathbf{x}}\overline{\mathbf{P}} \ , \quad \rho\,\ddot{\overline{x}}_i = \partial\overline{P}_{ik}/\partial x_k \ , \tag{13}$$

where ρ denotes the constant mass density in the state of static homogeneous deformation. Note that because of the incompressibility condition, we have

$$\overline{\rho} = \rho = \rho_0 \ . \tag{14}$$

4. Superposition of transverse plane waves propagating in different directions and polarized in the same principal direction in deformed Mooney-Rivlin materials

In [1], it has been shown that, in any direction $\mathbf{n}$ of a deformed Mooney-Rivlin material, two finite-amplitude linearly polarized transverse waves may propagate. The two polarization directions $\mathbf{a}$ and $\mathbf{b}$ are such that [2]

$$\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{b} = \mathbf{a}\cdot\mathbf{b} = 0 \quad , \quad \mathbf{n}\cdot\mathbf{a} = \mathbf{n}\cdot\mathbf{b} = 0 \ , \tag{15}$$

where $\mathbf{B}^{-1}$ is the inverse of the left Cauchy-Green strain tensor of the homogeneous static deformation (9). It has also been shown [2] that the superposition of these two waves is again an exact solution.

Here, we are interested in superposition of waves propagating in *different* directions.

We consider the superposition of two transverse plane waves propagating in different directions $\mathbf{n}$ and $\mathbf{n}'$, and linearly polarized in the same direction $\mathbf{a}$ in deformed Mooney-Rivlin materials. Thus, here, we seek solutions of the equations of motion of the form

$$\overline{\mathbf{x}} = \mathbf{x} + g(\eta,t)\mathbf{a} + g'(\eta',t)\mathbf{a} \ , \tag{16}$$

$$\overline{p} = p + q(\eta,\eta',t) \ , \tag{17}$$

where g, g', and q are functions to be determined, and we use the notation

$$\eta = \mathbf{n}\cdot\mathbf{x} \quad , \quad \eta' = \mathbf{n}'\cdot\mathbf{x} \ . \tag{18}$$

Note that $\mathbf{n}$ and $\mathbf{n}'$ are not taken to be orthogonal. It is just assumed that they are not parallel. Also, because the superposed waves are transverse, $\mathbf{a}$ is orthogonal to both $\mathbf{n}$ and $\mathbf{n}'$,

$$\mathbf{a}\cdot\mathbf{n} = 0 \quad , \quad \mathbf{a}\cdot\mathbf{n}' = 0 \ . \tag{19}$$

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We require that each individual wave ($g \neq 0, g' = 0$ and $g = 0, g' \neq 0$) be a solution. Hence, recalling (15), we have the propagation conditions

$$\mathbf{b} \cdot \mathbf{B}^{-1}\mathbf{a} = 0 \ , \quad \text{with} \quad \mathbf{b} = \mathbf{n} \times \mathbf{a} \ , \tag{20}$$

and

$$\mathbf{b}' \cdot \mathbf{B}^{-1}\mathbf{a} = 0 \ , \quad \text{with} \quad \mathbf{b}' = \mathbf{n}' \times \mathbf{a} \ . \tag{21}$$

It follows from these two propagation conditions that the unit vector $\mathbf{a}$ must be along $\mathbf{B}^{-1}\mathbf{b} \times \mathbf{B}^{-1}\mathbf{b}'$, hence along $\mathbf{B}(\mathbf{b} \times \mathbf{b}')$, thus along $\mathbf{Ba}$, which means that $\mathbf{a}$ is along a principal direction of the static strain. The unit vectors $\mathbf{n}$ and $\mathbf{n}'$ are then arbitrary in the principal plane orthogonal to $\mathbf{a}$. Hence, we also have

$$\mathbf{n} \cdot \mathbf{B}^{-1}\mathbf{a} = 0 \quad \text{and} \quad \mathbf{n}' \cdot \mathbf{B}^{-1}\mathbf{a} = 0 \ . \tag{22}$$

The deformation gradient tensor $\overline{\mathbf{F}}$ associated with the motion (16) is given by

$$\overline{\mathbf{F}} = (\mathbf{1} + g_\eta \ \mathbf{a} \otimes \mathbf{n} + g'_{\eta'} \ \mathbf{a} \otimes \mathbf{n}')\mathbf{F} \ . \tag{23}$$

The left Cauchy-Green strain tensor $\overline{\mathbf{B}} = \overline{\mathbf{F}}\,\overline{\mathbf{F}}^T$ associated with the motion (16), and its inverse $\overline{\mathbf{B}}^{-1}$ are given by

$$\overline{\mathbf{B}} = (\mathbf{1} + g_\eta \ \mathbf{a} \otimes \mathbf{n} + g'_{\eta'} \ \mathbf{a} \otimes \mathbf{n}')\mathbf{B}(\mathbf{1} + g_\eta \ \mathbf{n} \otimes \mathbf{a} + g'_{\eta'} \ \mathbf{n}' \otimes \mathbf{a}) \ , \tag{24}$$

$$\overline{\mathbf{B}}^{-1} = (\mathbf{1} - g_\eta \ \mathbf{n} \otimes \mathbf{a} - g'_{\eta'} \ \mathbf{n}' \otimes \mathbf{a})\mathbf{B}^{-1}(\mathbf{1} - g_\eta \ \mathbf{a} \otimes \mathbf{n} - g'_{\eta'} \ \mathbf{a} \otimes \mathbf{n}'). \tag{25}$$

For the principal invariants $\bar{\mathrm{I}}_1, \bar{\mathrm{I}}_2, \bar{\mathrm{I}}_3$ of $\overline{\mathbf{B}}$, we have, using (22),

$$\bar{\mathrm{I}}_1 = \mathrm{I}_1 + g_\eta^2 \ \mathbf{n} \cdot \mathbf{Bn} + g_{\eta'}^{'2} \ \mathbf{n}' \cdot \mathbf{Bn}' + 2g_\eta \ g'_{\eta'} \ \mathbf{n} \cdot \mathbf{Bn}' \ , \tag{26}$$

$$\bar{\mathrm{I}}_2 = \mathrm{I}_2 + g_\eta^2 \ \mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a} + g_{\eta'}^{'2} \ \mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a} + 2g_\eta \ g'_{\eta'} \ \mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a} \ (\mathbf{n} \cdot \mathbf{n}') \ , \tag{27}$$

$$\bar{\mathrm{I}}_3 = \mathrm{I}_3 \ . \tag{28}$$

The Cauchy stress tensor $\overline{\mathbf{T}}$ associated with the superposed motion (16) is given by

$$\overline{\mathbf{T}} = -\overline{p}_*\mathbf{1} + C\overline{\mathbf{B}} - D\overline{\mathbf{B}}^{-1} \ , \tag{29}$$

where

$$\overline{p}_* = \overline{p} - D\bar{\mathrm{I}}_2 \ . \tag{30}$$

Introducing (27) into (30), we obtain

$$\overline{p}_* = p + q(\eta, \eta', t) - D(\mathrm{I}_2 + g_\eta^2 \mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a} + g_{\eta'}^{'2} \mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a} + 2g_\eta \ g'_{\eta'} \ \mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a} \ (\mathbf{n} \cdot \mathbf{n}')) \ . \tag{31}$$

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The components of the stress tensor $\overline{\mathbf{T}}$ are functions of η, η' and t, or equivalently $\overline{\eta}$, $\overline{\eta}'$ and t. Thus, using $\eta = \mathbf{n}\cdot\mathbf{x}, \xi = \mathbf{a}\cdot\mathbf{x}$ and $\eta' = \mathbf{n}'\cdot\mathbf{x}$ as coordinates (note that η, ξ, η' are non Cartesian coordinates) and taking the dot product of the equations of motion (11) with $\mathbf{n}, \mathbf{a}, \mathbf{n}'$, we obtain

$$0 = \frac{\partial \overline{T}^{\eta\eta}}{\partial \overline{\eta}} + \frac{\partial \overline{T}^{\eta\eta'}}{\partial \overline{\eta}'} = \frac{\partial \overline{T}^{\eta\eta}}{\partial \eta} + \frac{\partial \overline{T}^{\eta\eta'}}{\partial \eta'} , \tag{32}$$

$$\rho\left(\frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 g'}{\partial t^2}\right) = \frac{\partial \overline{T}^{\xi\eta}}{\partial \overline{\eta}} + \frac{\partial \overline{T}^{\xi\eta'}}{\partial \overline{\eta}'} = \frac{\partial \overline{T}^{\xi\eta}}{\partial \eta} + \frac{\partial \overline{T}^{\xi\eta'}}{\partial \eta'} , \tag{33}$$

$$0 = \frac{\partial \overline{T}^{\eta'\eta}}{\partial \overline{\eta}} + \frac{\partial \overline{T}^{\eta'\eta'}}{\partial \overline{\eta}'} = \frac{\partial \overline{T}^{\eta'\eta}}{\partial \eta} + \frac{\partial \overline{T}^{\eta'\eta'}}{\partial \eta'} . \tag{34}$$

We now compute the contravariant components of $\overline{\mathbf{T}}$ entering these equations. Using the expression (29) of $\overline{\mathbf{T}}$, we obtain, for the component $\overline{T}^{\eta\eta}$,

$$\overline{T}^{\eta\eta} = \mathbf{n}\cdot\overline{\mathbf{T}}\mathbf{n} = -\overline{p}_* + C\mathbf{n}\cdot\overline{\mathbf{B}}\mathbf{n} - D\mathbf{n}\cdot\overline{\mathbf{B}}^{-1}\mathbf{n} , \tag{35}$$

and using the expressions (24) of $\overline{\mathbf{B}}$, and (25) of $\overline{\mathbf{B}}^{-1}$, we have

$$\overline{T}^{\eta\eta} = T^{\eta\eta} - q(\eta,\eta',t) + Dg_{\eta'}'^2\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a} - Dg_{\eta'}'^2\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}\ (\mathbf{n}\cdot\mathbf{n}')^2 . \tag{36}$$

Similarly, for the component $\overline{T}^{\xi\eta}$, we have

$$\overline{T}^{\xi\eta} = \mathbf{a}\cdot\overline{\mathbf{T}}\mathbf{n} = C\mathbf{a}\cdot\overline{\mathbf{B}}\mathbf{n} - D\mathbf{a}\cdot\overline{\mathbf{B}}^{-1}\mathbf{n} , \tag{37}$$

and hence

$$\overline{T}^{\xi\eta} = C(g_\eta\ \mathbf{n}\cdot\mathbf{B}\mathbf{n} + g_{\eta'}'\ \mathbf{n}'\cdot\mathbf{B}\mathbf{n}) + D(g_\eta\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a} + g_{\eta'}'\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}(\mathbf{n}\cdot\mathbf{n}')) . \tag{38}$$

For the component $\overline{T}^{\eta'\eta}$, we have

$$\overline{T}^{\eta'\eta} = \mathbf{n}'\cdot\overline{\mathbf{T}}\mathbf{n} = -\overline{p}_*(\mathbf{n}\cdot\mathbf{n}') + C\mathbf{n}'\cdot\overline{\mathbf{B}}\mathbf{n} - D\mathbf{n}'\cdot\overline{\mathbf{B}}^{-1}\mathbf{n} , \tag{39}$$

and hence

$$\overline{T}^{\eta'\eta} = T^{\eta'\eta} - q(\eta,\eta',t)(\mathbf{n}\cdot\mathbf{n}') + Dg_\eta g_{\eta'}'\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}((\mathbf{n}\cdot\mathbf{n}')^2 - 1) . \tag{40}$$

For the component $\overline{T}^{\xi\eta'}$, we have

$$\overline{T}^{\xi\eta'} = \mathbf{a}\cdot\overline{\mathbf{T}}\mathbf{n}' = C\mathbf{a}\cdot\overline{\mathbf{B}}\mathbf{n}' - D\mathbf{a}\cdot\overline{\mathbf{B}}^{-1}\mathbf{n}' , \tag{41}$$

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and hence

$$\overline{T}^{\xi\eta'} = C(g_\eta\ \mathbf{n}\cdot\mathbf{B}\mathbf{n}' + g'_{\eta'}\ \mathbf{n}'\cdot\mathbf{B}\mathbf{n}') + D(g_\eta\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}(\mathbf{n}\cdot\mathbf{n}') + g'_{\eta'}\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})\ . \quad (42)$$

For the component $\overline{T}^{\eta'\eta'}$, we have

$$\overline{T}^{\eta'\eta'} = \mathbf{n}'\cdot\overline{\mathbf{T}}\mathbf{n}' = -\overline{p}_* + C\mathbf{n}'\cdot\overline{\mathbf{B}}\mathbf{n}' - D\mathbf{n}'\cdot\overline{\mathbf{B}}^{-1}\mathbf{n}'\ , \quad (43)$$

and hence

$$\overline{T}^{\eta'\eta'} = T^{\eta'\eta'} - q(\eta,\eta',t) + Dg_\eta^2\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a} - Dg_\eta^2\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}(\mathbf{n}\cdot\mathbf{n}')^2\ . \quad (44)$$

For the sake of completeness, we also compute the component $\overline{T}^{\xi\xi}$. We have

$$\overline{T}^{\xi\xi} = \mathbf{a}\cdot\overline{\mathbf{T}}\mathbf{a} = -\overline{p}_* + C\mathbf{a}\cdot\overline{\mathbf{B}}\mathbf{a} - D\mathbf{a}\cdot\overline{\mathbf{B}}^{-1}\mathbf{a}\ , \quad (45)$$

and hence

$$\begin{aligned}\overline{T}^{\xi\xi} &= T^{\xi\xi} - q(\eta,\eta',t) + C(g_\eta^2\ \mathbf{n}\cdot\mathbf{B}\mathbf{n} + 2g_\eta\ g'_{\eta'}\ \mathbf{n}\cdot\mathbf{B}\mathbf{n}' + g'^2_{\eta'}\ \mathbf{n}'\cdot\mathbf{B}\mathbf{n}') \\ &+ D\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}(g_\eta^2\ + 2g_\eta\ g'_{\eta'}\ (\mathbf{n}\cdot\mathbf{n}') + g'^2_{\eta'})\ . \quad (46)\end{aligned}$$

Introducing (36) and (40) into the equation of motion (32), we obtain

$$-q_\eta - q_{\eta'}(\mathbf{n}\cdot\mathbf{n}') + Dg_\eta\ g'_{\eta'\eta'}\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}((\mathbf{n}\cdot\mathbf{n}')^2 - 1) = 0\ . \quad (47)$$

Introducing (38) and (42) into the equation of motion (33), we obtain

$$(C\mathbf{n}\cdot\mathbf{B}\mathbf{n} + D\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})g_{\eta\eta} + (C\mathbf{n}'\cdot\mathbf{B}\mathbf{n}' + D\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a})g'_{\eta'\eta'} = \rho(g_{tt} + g'_{tt})\ , \quad (48)$$

and introducing (40) and (44) into the equation of motion (34), we obtain

$$-q_{\eta'} - q_\eta(\mathbf{n}\cdot\mathbf{n}') + Dg'_{\eta'}\ g_{\eta\eta}\ \mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}((\mathbf{n}\cdot\mathbf{n}')^2 - 1) = 0\ . \quad (49)$$

The equation (48) may also be written in the form

$$g_{tt} + g'_{tt} = v_{\mathbf{n}}^2 g_{\eta\eta} + v_{\mathbf{n}'}^2 g'_{\eta'\eta'}\ , \quad (50)$$

where $v_{\mathbf{n}}^2$ and $v_{\mathbf{n}'}^2$ are, respectively, the squared wave speed of the wave propagating in the direction $\mathbf{n}$, and polarized in the principal direction $\mathbf{a}$, and

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the squared wave speed of the wave propagating in the direction $\mathbf{n}'$, and polarized in the principal direction $\mathbf{a}$, given by [2]

$$\rho v_{\mathbf{n}}^2 = C\mathbf{n}\cdot\mathbf{B}\mathbf{n} + D\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}\ ,\ \rho v_{\mathbf{n}'}^2 = C\mathbf{n}'\cdot\mathbf{B}\mathbf{n}' + D\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}\ . \tag{51}$$

Because η and η' are independent variables, if we differentiate (50) with respect to η, and with respect to η', we obtain two uncoupled equations

$$g_{tt\eta} = v_{\mathbf{n}}^2 g_{\eta\eta\eta} \quad , \quad g'_{tt\eta'} = v_{\mathbf{n}'}^2 g'_{\eta'\eta'\eta'}. \tag{52}$$

If we integrate $(52)_1$ with respect to η, and integrate $(52)_2$ with respect to η', we obtain

$$g_{tt} - v_{\mathbf{n}}^2 g_{\eta\eta} = k(t) \quad , \quad g'_{tt} - v_{\mathbf{n}'}^2 g'_{\eta'\eta'} = -k(t)\ . \tag{53}$$

Here, for simplicity, we choose $k(t) = 0$, so that (53) reduces to two vibrating string equations

$$g_{tt} - v_{\mathbf{n}}^2 g_{\eta\eta} = 0 \quad , \quad g'_{tt} - v_{\mathbf{n}'}^2 g'_{\eta'\eta'} = 0\ . \tag{54}$$

Solving the equations (47) and (49) for q_η and $q_{\eta'}$ yields

$$q_\eta = D\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}(g'_{\eta'}\ g_{\eta\eta}\ \cos\alpha - g_\eta\ g'_{\eta'\eta'}\ )\ , \tag{55}$$

$$q_{\eta'} = D\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}(g_\eta\ g'_{\eta'\eta'}\ \cos\alpha - g'_{\eta'}\ g_{\eta\eta}\ )\ , \tag{56}$$

where α is the angle between the unit vectors $\mathbf{n}$ and $\mathbf{n}'$. The expressions for the first order derivatives of $q(\eta,\eta',t)$ must be compatible, i.e. $q_{\eta\eta'} = q_{\eta'\eta}$, thus,

$$g_\eta\ g'_{\eta'\eta'\eta'} = g'_{\eta'}\ g_{\eta\eta\eta}\ . \tag{57}$$

This compatibility condition may also be written in the form

$$\frac{g'_{\eta'\eta'\eta'}}{g'_{\eta'}} = \frac{g_{\eta\eta\eta}}{g_\eta} = -l(t)\ , \tag{58}$$

or, alternatively,

$$g_{\eta\eta\eta} = -l(t)g_\eta \quad , \quad g'_{\eta'\eta'\eta'} = -l(t)g'_{\eta'}\ , \tag{59}$$

where $l(t)$ is an arbitrary function of time. Integrating $(59)_1$ with respect to η', and $(59)_2$ with respect to η, and taking $l(t)$ to be constant and positive, $l(t) = \kappa^2$, we obtain

$$g_{\eta\eta} + \kappa^2 g = 0 \quad , \quad g'_{\eta'\eta'} + \kappa^2 g' = 0\ . \tag{60}$$

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The function $g(\eta, t)$ is governed by $(54)_1$ and $(60)_1$, and is thus given by

$$g(\eta, t) = a \cos \kappa(\eta - v_{\mathbf{n}} t + \varphi) + b \cos \kappa(\eta + v_{\mathbf{n}} t + \psi) \ , \tag{61}$$

where a, b, φ, ψ are arbitrary constants. The function $g'(\eta', t)$ is governed by $(54)_2$ and $(60)_2$, and is thus given by

$$g'(\eta', t) = a' \cos \kappa(\eta' - v_{\mathbf{n}'} t + \varphi') + b' \cos \kappa(\eta' + v_{\mathbf{n}'} t + \psi') \ , \tag{62}$$

where a', b', φ', ψ' are arbitrary constants.

Note that the waves (61) and (62) have the same wave number κ, thus, the same wavelength, and because $v_{\mathbf{n}}$ and $v_{\mathbf{n}'}$ are in general different, the frequencies $\omega = \kappa v_{\mathbf{n}}$, $\omega' = \kappa v_{\mathbf{n}'}$ are different. Hence, the superposition is, in general, not time-harmonic.

Using (55), (56), and (60), we obtain, for the additional pressure $q(\eta, \eta', t)$ due to the wave, up to an arbitrary additive function of time,

$$q(\eta, \eta', t) = D\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{a}(g_\eta g'_{\eta'} \cos \alpha + \kappa^2 g g') \ , \tag{63}$$

with $g(\eta, t)$ and $g'(\eta', t)$ given by (61) and (62). Note that this additional pressure due to the wave is the result of an interaction between the two waves $g(\eta, t)$ and $g'(\eta', t)$.

Thus, in this Section, we have shown that two transverse waves propagating in *different* directions $\mathbf{n}$ and $\mathbf{n}'$, and polarized in the same direction $\mathbf{a}$ may be superposed in deformed Mooney-Rivlin materials, provided $\mathbf{a}$ is along a principal direction of the static strain. The propagation directions $\mathbf{n}$, $\mathbf{n}'$ are then arbitrary in the principal plane orthogonal to $\mathbf{a}$. Although the theory is non linear and no assumptions have been made neither on the size of the static deformation nor on the amplitude of the waves, we showed that such a superposition is an exact solution of the equations of motion. The functions $g(\eta, t)$ and $g'(\eta', t)$ are of sinusoidal type and are given by (61) and (62), where κ is an arbitrary constant, and the additional pressure $q(\eta, \eta', t)$ due to the wave is given by (63).

Remark 1 : special case $v_{\mathbf{n}} = v_{\mathbf{n}'}$

When $\mathbf{n} \cdot \mathbf{B}\mathbf{n} = \mathbf{n}' \cdot \mathbf{B}\mathbf{n}'$ the two wave speeds $v_{\mathbf{n}}$, $v_{\mathbf{n}'}$ given by (51) are equal. Because $\mathbf{n}$ and $\mathbf{n}'$ are in a principal plane, this occurs when $\mathbf{n}$ and $\mathbf{n}'$ are equally inclined with respect to a principal direction of the static strain. Take, for instance,

$$\mathbf{n} = \cos \frac{\alpha}{2}\mathbf{i} - \sin \frac{\alpha}{2}\mathbf{k} \quad , \quad \mathbf{n}' = \cos \frac{\alpha}{2}\mathbf{i} + \sin \frac{\alpha}{2}\mathbf{k} \quad , \quad \mathbf{a} = \mathbf{j} \ , \tag{64}$$

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where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the principal directions. Then, $v_{\mathbf{n}}^2 = v_{\mathbf{n}'}^2 \equiv v_\alpha^2$ given by

$$\rho v_\alpha^2 = C(\lambda_1^2 \cos^2 \frac{\alpha}{2} + \lambda_3^2 \sin^2 \frac{\alpha}{2}) + D\lambda_1^2\lambda_3^2 \ , \tag{65}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches in the directions $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Note that if v_1 and v_3 denote the wave speeds of the waves propagating along $\mathbf{i}$ and $\mathbf{k}$, respectively, and polarized along $\mathbf{j}$,

$$\rho v_1^2 = C\lambda_1^2 + D\lambda_1^2\lambda_3^2 \quad , \quad \rho v_3^2 = C\lambda_3^2 + D\lambda_1^2\lambda_3^2 \ , \tag{66}$$

then (65) may be written as

$$v_\alpha^2 = v_1^2 \cos^2 \frac{\alpha}{2} + v_3^2 \sin^2 \frac{\alpha}{2} \ . \tag{67}$$

Choosing then forward waves with the same amplitude A for g and g',

$$g(\eta, t) = A \cos \kappa(\eta - v_\alpha t + \varphi) \quad , \quad g'(\eta', t) = A \cos \kappa(\eta' - v_\alpha t + \varphi') \ , \tag{68}$$

we obtain, for the superposition,

$$g(\eta, t) + g'(\eta', t) = 2A \cos \kappa(\sin \frac{\alpha}{2} x_3 + \varphi_-) \cos \kappa(\cos \frac{\alpha}{2} x_1 - v_\alpha t + \varphi_+) \ , \tag{69}$$

where $x_1 = \mathbf{i} \cdot \mathbf{x}$, $x_3 = \mathbf{k} \cdot \mathbf{x}$ are Cartesian coordinates along the principal axes $\mathbf{i}$, $\mathbf{k}$, and $\varphi_\pm = \frac{1}{2}(\varphi' \pm \varphi)$. Such a solution represents a wave propagating along the principal axis $\mathbf{i}$ with speed $v \equiv v_\alpha / \cos \frac{\alpha}{2}$ and whose amplitude varies sinusoidally in the direction of the principal axis $\mathbf{k}$.

5. Superposition of transverse plane waves propagating in different directions and polarized in the same direction in deformed neo-Hookean materials

For deformed neo-Hookean materials, in any direction $\mathbf{n}$, a transverse wave may propagate with polarization direction $\mathbf{a}$ arbitrary orthogonal to $\mathbf{n}$, the wave speed being independent of $\mathbf{a}$ [2].

Here, we consider the superposition of two transverse homogeneous plane waves propagating in different directions $\mathbf{n}$ and $\mathbf{n}'$, and linearly polarized in the same direction $\mathbf{a}$ in deformed neo-Hookean materials. Thus, as in Section 4, we seek solutions of the equations of motion of the form (16) (17) with (19).

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The deformation gradient tensor $\overline{\mathbf{F}}$ and the left Cauchy-Green strain tensor $\overline{\mathbf{B}}$ associated with the motion (16) are still given by (23) and (24). The principal invariants $\overline{\mathrm{I}}_1, \overline{\mathrm{I}}_2, \overline{\mathrm{I}}_3$ of $\overline{\mathbf{B}}$ are also still given by (26), (27), (28).

The Cauchy stress tensor $\overline{\mathbf{T}}$ associated with the superposed motion (16) is given by

$$\overline{\mathbf{T}} = -\overline{p}\mathbf{1} + C\overline{\mathbf{B}} \ . \tag{70}$$

Here, the contravariant components of the stress tensor $\overline{\mathbf{T}}$ are given by

$$\overline{T}^{\eta\eta} = T^{\eta\eta} - q(\eta, \eta', t) \ , \tag{71}$$

$$\overline{T}^{\xi\eta} = T^{\xi\eta} + C(g_\eta \ \mathbf{n}\cdot\mathbf{B}\mathbf{n} + g'_{\eta'} \ \mathbf{n}'\cdot\mathbf{B}\mathbf{n}) \ , \tag{72}$$

$$\overline{T}^{\eta'\eta} = T^{\eta'\eta} - q(\eta, \eta', t)(\mathbf{n}\cdot\mathbf{n}') \ , \tag{73}$$

$$\overline{T}^{\xi\eta'} = T^{\xi\eta'} + C(g_\eta \ \mathbf{n}\cdot\mathbf{B}\mathbf{n}' + g'_{\eta'} \ \mathbf{n}'\cdot\mathbf{B}\mathbf{n}') \ , \tag{74}$$

$$\overline{T}^{\eta'\eta'} = T^{\eta'\eta'} - q(\eta, \eta', t) \ , \tag{75}$$

$$\begin{aligned}\overline{T}^{\xi\xi} &= T^{\xi\xi} - q(\eta, \eta', t) + C(2g_\eta \ \mathbf{a}\cdot\mathbf{B}\mathbf{n} + g_\eta^2 \ \mathbf{n}\cdot\mathbf{B}\mathbf{n} + 2g'_{\eta'} \ \mathbf{a}\cdot\mathbf{B}\mathbf{n}' \\ &\quad + g'^2_{\eta'} \ \mathbf{n}'\cdot\mathbf{B}\mathbf{n}' + 2g_\eta \ g'_{\eta'} \ \mathbf{n}\cdot\mathbf{B}\mathbf{n}') \ .\end{aligned} \tag{76}$$

Introducing these components of the stress tensor $\overline{\mathbf{T}}$ into the equations of motion (32), (33), (34), we obtain

$$-q_\eta - q_{\eta'}(\mathbf{n}\cdot\mathbf{n}') = 0 \ , \tag{77}$$

$$C\mathbf{n}\cdot\mathbf{B}\mathbf{n} \ g_{\eta\eta} + C\mathbf{n}'\cdot\mathbf{B}\mathbf{n}' \ g'_{\eta'\eta'} = \rho(g_{tt} + g'_{tt}) \ , \tag{78}$$

$$-q_{\eta'} - q_\eta(\mathbf{n}\cdot\mathbf{n}') = 0 \ . \tag{79}$$

It follows from (77), and (79) that $q_\eta = q_{\eta'} = 0$, and hence we conclude that the additional pressure $q(\eta, \eta', t)$ due to the wave has to be a function of time t alone, and thus may be taken to be zero.

The equation (78) may also be written in the form

$$g_{tt} + g'_{tt} = v^2_{\mathbf{n}} g_{\eta\eta} + v^2_{\mathbf{n}'} g'_{\eta'\eta'} \ , \tag{80}$$

where $v^2_{\mathbf{n}}$ and $v^2_{\mathbf{n}'}$ are, respectively, the squared wave speeds of the waves propagating in the direction $\mathbf{n}$, and in the direction $\mathbf{n}'$, and are given by [2]

$$\rho v^2_{\mathbf{n}} = C\mathbf{n}\cdot\mathbf{B}\mathbf{n} \quad , \quad \rho v^2_{\mathbf{n}'} = C\mathbf{n}'\cdot\mathbf{B}\mathbf{n}' \ . \tag{81}$$

Because η and η' are independent variables, if we differentiate (80) with respect to η, and with respect to η', we obtain two uncoupled equations

$$g_{tt\eta} = v^2_{\mathbf{n}} g_{\eta\eta\eta} \quad , \quad g'_{tt\eta'} = v^2_{\mathbf{n}'} g'_{\eta'\eta'\eta'} . \tag{82}$$

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If we integrate $(82)_1$ with respect to η, and integrate $(82)_2$ with respect to η', we obtain

$$g_{tt} - v_{\mathbf{n}}^2 g_{\eta\eta} = k(t) \quad , \quad g'_{tt} - v_{\mathbf{n}'}^2 g'_{\eta'\eta'} = -k(t) \ . \tag{83}$$

Thus, we obtain two equations similar to the vibrating string equation but with a right-hand side depending on time t. The general solutions of (83) are

$$g(\eta, t) = h(\eta - v_{\mathbf{n}} t) + l(\eta + v_{\mathbf{n}} t) + K(t) \ , \tag{84}$$
$$g'(\eta', t) = h'(\eta' - v_{\mathbf{n}'} t) + l'(\eta' + v_{\mathbf{n}'} t) - K(t) \ , \tag{85}$$

with $K''(t) = k(t)$. Thus, we obtain two uncoupled equations (84), (85) for g and g'. Although the theory is non linear and no assumption has been made on the amplitudes, the two waves do not interact. Here, the polarization direction $\mathbf{a}$ is arbitrary, and the directions of propagation $\mathbf{n}$ and $\mathbf{n}'$ are arbitrary in the plane orthogonal to $\mathbf{a}$.

If we take $K(t) = 0$, we may in particular, choose for g and g' sinusoidal solutions,

$$g(\eta, t) = a \cos \kappa(\eta - v_{\mathbf{n}} t + \varphi) + b \cos \kappa(\eta + v_{\mathbf{n}} t + \psi) \ , \tag{86}$$
$$g'(\eta', t) = a' \cos \kappa'(\eta' - v_{\mathbf{n}'} t + \varphi') + b' \cos \kappa'(\eta' + v_{\mathbf{n}'} t + \psi) \ . \tag{87}$$

Here, contrary to the case of Mooney-Rivlin materials, the waves (86) (87) may have different wave numbers κ and κ'. In general, they also have different frequencies $\omega = \kappa v_{\mathbf{n}}$, $\omega' = \kappa' v_{\mathbf{n}'}$.

Remark 2 : special case $\kappa' = \kappa$ and $v_{\boldsymbol{n}} = v_{\boldsymbol{n}'}$

In order to compare with *Remark 1* (Section 4), we choose $\kappa' = \kappa$. When $\mathbf{n} \cdot \mathbf{B}\mathbf{n} = \mathbf{n}' \cdot \mathbf{B}\mathbf{n}'$, the two wave speeds $v_{\mathbf{n}}$, $v_{\mathbf{n}'}$ given by (81) are equal. Introducing the unit vectors $\mathbf{p}, \mathbf{q}$ along the internal and external bisectors of $\mathbf{n}$ and $\mathbf{n}'$, we have

$$\mathbf{n} = \cos \frac{\alpha}{2} \mathbf{p} - \sin \frac{\alpha}{2} \mathbf{q} \quad , \quad \mathbf{n}' = \cos \frac{\alpha}{2} \mathbf{p} + \sin \frac{\alpha}{2} \mathbf{q} \ , \tag{88}$$

where α is the angle between $\mathbf{n}$ and $\mathbf{n}'$. It follows that $\mathbf{n} \cdot \mathbf{B}\mathbf{n} = \mathbf{n}' \cdot \mathbf{B}\mathbf{n}'$ occurs if

$$\mathbf{p} \cdot \mathbf{B}\mathbf{q} = 0 \ . \tag{89}$$

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Because, $\mathbf{p}\cdot\mathbf{q}=0$, this means that $\mathbf{p}$ and $\mathbf{q}$ are along the principal axes of the elliptical section of the ellipsoid associated with $\mathbf{B}$ ($\mathbf{x}\cdot\mathbf{B}\mathbf{x}=1$) by the plane of $\mathbf{n}$ and $\mathbf{n}'$. Thus, when $\mathbf{n}$ and $\mathbf{n}'$ are given by (88) where $\mathbf{p}$ and $\mathbf{q}$ are along these axes, $v_{\mathbf{n}}^2=v_{\mathbf{n}'}^2\equiv v_\alpha^2$ given by

$$\rho v_\alpha^2=v_{\mathbf{p}}^2\cos^2\frac{\alpha}{2}+v_{\mathbf{q}}^2\sin^2\frac{\alpha}{2}\ , \tag{90}$$

where $v_{\mathbf{p}}^2$ and $v_{\mathbf{q}}^2$ are the squared wave speeds of the waves propagating along $\mathbf{p}$ and $\mathbf{q}$, respectively,

$$\rho v_{\mathbf{p}}^2=C\mathbf{p}\cdot\mathbf{B}\mathbf{p}\quad,\quad \rho v_{\mathbf{q}}^2=C\mathbf{q}\cdot\mathbf{B}\mathbf{q}\ . \tag{91}$$

Choosing then, as in *Remark 1* (Section 4), forward waves with the same amplitude A for g and g',

$$g(\eta,t)=A\cos\kappa(\eta-v_\alpha t+\varphi)\quad,\quad g'(\eta',t)=A\cos\kappa(\eta'-v_\alpha t+\varphi')\ , \tag{92}$$

we obtain, for the superposition,

$$g(\eta,t)+g'(\eta',t)=2A\cos\kappa(\sin\frac{\alpha}{2}\mathbf{q}\cdot\mathbf{x}+\varphi_-)\cos\kappa(\cos\frac{\alpha}{2}\mathbf{p}\cdot\mathbf{x}-v_\alpha t+\varphi_+)\ , \tag{93}$$

where $\varphi_\pm=\frac{1}{2}(\varphi'\pm\varphi)$. Such a solution represents a wave propagating along $\mathbf{p}$ with speed $v\equiv v_\alpha/\cos\frac{\alpha}{2}$ and whose amplitude varies sinusoidally in the direction of $\mathbf{q}$. Here, $\mathbf{p}$ and $\mathbf{q}$ are two arbitrary unit vectors such that $\mathbf{p}\cdot\mathbf{q}=\mathbf{p}\cdot\mathbf{B}\mathbf{q}=0$. In the slightly different context of special Blatz-Ko materials (a compressible counterpart of the neo-Hookean model), such a solution has been used [8] for the construction of a finite-amplitude Love wave.

6. Superposition of transverse homogeneous plane waves propagating and polarized in different directions in deformed neo-Hookean materials

Here we show that, in deformed neo-Hookean materials, two transverse homogeneous plane waves propagating and polarized in different directions may be superposed when the direction of propagation of one of the wave is orthogonal to the polarization direction of the other. So, we seek solutions of the equations of motion in the form

$$\overline{\mathbf{x}}=\mathbf{x}+g(\eta,t)\mathbf{a}+g'(\eta',t)\mathbf{a}'\ , \tag{94}$$

$$\overline{p}=p+q(\eta,\eta',t)\ , \tag{95}$$

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where g, g', and q are functions to be determined, and $\mathbf{n}$, $\mathbf{n}'$, $\mathbf{a}$, $\mathbf{a}'$ are unit vectors such that

$$\mathbf{n}\cdot\mathbf{a}=0\ ,\quad \mathbf{n}'\cdot\mathbf{a}'=0\ ,\quad \mathbf{n}\cdot\mathbf{a}'=0\ . \tag{96}$$

The functions g and g' are associated with transverse waves propagating along $\mathbf{n}$ and $\mathbf{n}'$, respectively, and polarized along $\mathbf{a}$ and $\mathbf{a}'$, respectively. The direction of propagation of one of these waves ($\mathbf{n}$) is orthogonal to the polarization direction of the other ($\mathbf{a}'$). Thus, $\mathbf{n}$ and $\mathbf{n}'$ are arbitrary, $\mathbf{a}$ is arbitrary orthogonal to $\mathbf{n}$, while $\mathbf{a}'$ is orthogonal to $\mathbf{n}$ and $\mathbf{n}'$.

For the motion (94), using the notations $\eta=\mathbf{n}\cdot\mathbf{x}$ and $\eta'=\mathbf{n}'\cdot\mathbf{x}$, we have for the deformation gradient tensor $\overline{\mathbf{F}}$ and for the left Cauchy-Green strain tensor $\overline{\mathbf{B}}=\overline{\mathbf{F}}\,\overline{\mathbf{F}}^T$

$$\overline{\mathbf{F}}=(\mathbf{1}+g_\eta\mathbf{a}\otimes\mathbf{n}+g'_{\eta'}\mathbf{a}'\otimes\mathbf{n}')\mathbf{F}\ , \tag{97}$$

$$\overline{\mathbf{B}}=(\mathbf{1}+g_\eta\mathbf{a}\otimes\mathbf{n}+g'_{\eta'}\mathbf{a}'\otimes\mathbf{n}')\mathbf{B}(\mathbf{1}+g_\eta\mathbf{n}\otimes\mathbf{a}+g'_{\eta'}\mathbf{n}'\otimes\mathbf{a}')\ . \tag{98}$$

Here, contrary to Sections 4 and 5, we found more convenient to use the Piola-Kirchhoff stress tensor $\overline{\mathbf{P}}$ instead of the Cauchy stress tensor $\overline{\mathbf{T}}$.

The Piola-Kirchhoff stress tensor defined by (12) may here be written as

$$\begin{aligned}\overline{\mathbf{P}}=&-(p+q(\eta,\eta',t))(\mathbf{1}-g_\eta\mathbf{n}\otimes\mathbf{a}-g'_{\eta'}\mathbf{n}'\otimes\mathbf{a}'+g_\eta g'_{\eta'}(\mathbf{n}'\cdot\mathbf{a})\mathbf{n}\otimes\mathbf{a}')\\&+C(\mathbf{B}+g_\eta\mathbf{a}\otimes\mathbf{B}\mathbf{n}+g'_{\eta'}\mathbf{a}'\otimes\mathbf{B}\mathbf{n}')\ .\end{aligned} \tag{99}$$

Introducing $\overline{\mathbf{P}}$ into the equations of motion (13), we obtain

$$\rho(g_{tt}\mathbf{a}+g'_{tt}\mathbf{a}')=-q_\eta\mathbf{n}-q_{\eta'}(\mathbf{n}'-(\mathbf{n}'\cdot\mathbf{a})g_\eta\ \mathbf{n})+C\mathbf{n}\cdot\mathbf{B}\mathbf{n}g_{\eta\eta}\mathbf{a}+C\mathbf{n}'\cdot\mathbf{B}\mathbf{n}'g'_{\eta'\eta'}\mathbf{a}'\ . \tag{100}$$

Using $(\mathbf{a},\mathbf{a}',\mathbf{n})$ as a basis, the reciprocal basis is $(\mathbf{a}^*,\mathbf{a}'^*,\mathbf{n})$, given by

$$\sin^2\gamma\ \mathbf{a}^*=\mathbf{a}-\cos\gamma\ \mathbf{a}'\quad,\quad \sin^2\gamma\ \mathbf{a}'^*=\mathbf{a}'-\cos\gamma\ \mathbf{a}\ , \tag{101}$$

where γ is the angle between $\mathbf{a}$ and $\mathbf{a}'$. Taking the dot product of the equations of motion (100) with the reciprocal basis $(\mathbf{a}^*,\mathbf{a}'^*,\mathbf{n})$, the equations of motion read

$$\rho g_{tt}=-q_{\eta'}\mathbf{n}'\cdot\mathbf{a}^*+C\mathbf{n}\cdot\mathbf{B}\mathbf{n}\ g_{\eta\eta}\ , \tag{102}$$

$$\rho g'_{tt}=-q_{\eta'}\mathbf{n}'\cdot\mathbf{a}'^*+C\mathbf{n}'\cdot\mathbf{B}\mathbf{n}'\ g'_{\eta'\eta'}\ , \tag{103}$$

$$0=-q_\eta-q_{\eta'}(\mathbf{n}\cdot\mathbf{n}'-(\mathbf{n}'\cdot\mathbf{a})g_\eta)\ . \tag{104}$$

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Taking the derivative of (102) with respect to η', and of (103) with respect to η, we obtain

$$q_{\eta'\eta'} = 0 \quad , \quad q_{\eta'\eta} = 0 \ . \tag{105}$$

Thus, $q_{\eta'}$ is a function of time only,

$$q_{\eta'} = A(t) \ . \tag{106}$$

Hence, using (104), we conclude that $q(\eta, \eta', t)$ is given, up to an arbitrary additive function of time, by

$$q(\eta, \eta', t) = -A(t)(\mathbf{n} \cdot \mathbf{n}')\eta + A(t)(\mathbf{n}' \cdot \mathbf{a})g + A(t)\eta' \ . \tag{107}$$

Using

$$\mathbf{n}' \cdot \mathbf{a}^* = \frac{\sin\alpha}{\sin\gamma} \quad , \quad \mathbf{n}' \cdot \mathbf{a}'^* = -\frac{\sin\alpha}{\sin\gamma}\cos\gamma \ , \tag{108}$$

where α is the angle between $\mathbf{n}$ and $\mathbf{n}'$, and recalling (106), the equations of motion (102) and (103) may now be written as

$$g_{tt} - v_{\mathbf{n}}^2 g_{\eta\eta} = K(t)\sin\alpha \ , \tag{109}$$

$$g'_{tt} - v_{\mathbf{n}'}^2 g'_{\eta'\eta'} = -K(t)\sin\alpha\cos\gamma \ . \tag{110}$$

Here,

$$K(t) = \frac{-A(t)}{\rho\sin\gamma} \ , \tag{111}$$

and $v_{\mathbf{n}}^2$ and $v_{\mathbf{n}'}^2$ are the squared wave speeds of the waves propagating in the directions $\mathbf{n}$ and $\mathbf{n}'$, respectively, and are given by (81). Using

$$\mathbf{n}' \cdot \mathbf{a} = \sin\alpha\sin\gamma \ , \tag{112}$$

and (111), the additional pressure (107) due to the wave may now be written as

$$q(\eta, \eta', t) = \rho K(t)\sin\gamma\{(\cos\alpha)\eta - (\sin\alpha\sin\gamma)g - \eta'\} \ . \tag{113}$$

Because $K(t)$ is arbitrary, we may choose $K(t) = 0$, then the additional pressure $q = 0$, and the equations (109) and (110) for g and g' reduce to two vibrating string equations

$$g_{tt} - v_{\mathbf{n}}^2 g_{\eta\eta} = 0 \quad , \quad g'_{tt} - v_{\mathbf{n}'}^2 g'_{\eta'\eta'} = 0 \ . \tag{114}$$

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Considering, for instance, sinusoidal wave solutions of $(114)_1$ and $(114)_2$, propagating with speeds $+v_{\mathbf{n}}$ and $+v_{\mathbf{n}'}$, respectively, the displacement field reads

$$\mathbf{u} = \lambda \mathbf{a} \cos \kappa(\eta - v_{\mathbf{n}} t + \varphi) + \lambda' \mathbf{a}' \cos \kappa'(\eta' - v_{\mathbf{n}'} t + \varphi') \ . \tag{115}$$

In (115), the two terms have, in general, different wave numbers κ, κ' and different frequencies $\omega = \kappa v_{\mathbf{n}}$, $\omega' = \kappa' v_{\mathbf{n}'}$. If κ and κ' are chosen such that $\kappa v_{\mathbf{n}} = \kappa' v_{\mathbf{n}'}$, then the motion is time-harmonic and at each point $\mathbf{x}$ the extremity of the displacement field vector $\mathbf{u}$ describes an ellipse.

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References

[1] P. Currie, M. Hayes, Longitudinal and transverse waves in finite elastic stain, Hadamard and Green materials, J. Inst. Maths Applics, 5 (1969) 140–161.

[2] Ph. Boulanger, M. Hayes, Finite-amplitude waves in deformed Mooney-Rivlin materials, Q. Jl Mech. Appl. Math. 45 (1992) 575–593.

[3] Ph. Boulanger, M. Hayes, Further properties of finite-amplitude plane waves in deformed Mooney-Rivlin materials, Q. Jl Mech. Appl. Math. 48 (1995) 427–464.

[4] Ph. Boulanger, M. Hayes, Finite-Amplitude Waves in Mooney-Rivlin and Hadamard Materials, in Topics in Finite Elasticity, CISM Courses and Lectures no. 424, Springer, Wien, (2001) 131–167.

[5] M. Mooney, A Theory of large elastic deformation, J. Appl. Phys. 11 (1940) 582–592.

[6] R.W. Ogden, Waves in isotropic elastic materials of Hadamard, Green, or harmonic type, J. Mech. Phys. Solids 18 (1970) 149–163.

[7] F. John, Plane elastic waves of finite amplitude. Hadamard materials and harmonic materials, Communications on Pure and Applied Mathematics 19 (1966) 309–341.

[8] E. Rodrigues Ferreira, Ph. Boulanger, M. Destrade, Large-amplitude Love waves, Q. Jl Mech. Appl. Math. 61 (2008) 353–371.

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